Jordan groups, conic bundles and abelian varieties

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Abstract

A group G is called *Jordan* if there is a positive integer $J = J_G$ such that every finite subgroup $\mathcal{B}$ of G contains a commutative subgroup $\mathcal{A} \subset \mathcal{B}$ such that $\mathcal{A}$ is normal in $\mathcal{B}$ and the index $[\mathcal{B} : \mathcal{A}]$ is at most J (V. L. Popov). In this paper, we deal with Jordan properties of the groups $\mathrm{Bir}(X)$ of birational automorphisms of irreducible smooth projective varieties X over an algebraically closed field of characteristic zero. It is known (Yu. Prokhorov and C. Shramov) that $\mathrm{Bir}(X)$ is Jordan if X is *non-uniruled*. On the other hand, the second-named author proved that $\mathrm{Bir}(X)$ is *not* Jordan if X is birational to a product of the projective line $\mathbb{P}^1$ and a positive-dimensional abelian variety.

We prove that $\mathrm{Bir}(X)$ is Jordan if (the uniruled variety) X is a *conic bundle* over a *non-uniruled* variety Y but is *not* birational to $Y \times \mathbb{P}^1$. (Such a conic bundle exists if and only if $\dim(Y) \geqslant 2$.) When Y is an abelian surface, this Jordan property result gives an answer to a question of Prokhorov and Shramov.

1. Introduction

In this paper we deal with the groups of birational and biregular automorphisms of algebraic varieties in characteristic *zero*.

If X is an irreducible algebraic variety over a field K of characteristic zero, then we write $\mathcal{O}_X$ for the structure sheaf of X. We write $\mathrm{Aut}(X) = \mathrm{Aut}_K(X)$ (respectively, $\mathrm{Bir}(X) = \mathrm{Bir}_K(X)$) for the group of its biregular (respectively, birational) automorphisms and $K(X)$ for the field of rational functions on X. We have

$$\mathrm{Aut}(X) \subset \mathrm{Bir}(X) = \mathrm{Aut}_K(K(X)),$$

where $\mathrm{Aut}_K(K(X))$ is the group of K-linear field automorphisms of $K(X)$. We write id_X for the identity automorphism of X, which is the identity element of the groups $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$. If n is a positive integer, then we write $\mathbb{P}^n_K$ (or just $\mathbb{P}^n$ when it does not cause confusion) for the n-dimensional projective space over K.

If X is smooth projective, then we write $q(X)$ for its *irregularity*. For example, if X is an abelian variety, then $q(X) = \dim(X)$.

In what follows we write k for an algebraically closed field of characteristic zero. We write $\cong$ and $\sim$ for an isomorphism and birational isomorphism of algebraic varieties, respectively. If $\mathcal{A}$ is a finite commutative group, then we define its rank to be its smallest possible number of generators and denote it by $\mathrm{rk}(\mathcal{A})$. If $\mathcal{B}$ is a finite group, then we write $|\mathcal{B}|$ for its order.

1.1 Jordan groups

Recall (Popov [Pop11, Pop14]) the following definitions that were motivated by the classical theorem of Jordan about finite subgroups of the complex matrix group $\mathrm{GL}(n, \mathbb{C})$ [CR62, Section 36].

DEFINITION 1.1. Let G be a group.

– We say that G is *Jordan* if there is a positive integer $J = J_G$ such that every finite subgroup $\mathcal{B}$ of G contains a commutative subgroup $\mathcal{A} \subset \mathcal{B}$ such that $\mathcal{A}$ is normal in $\mathcal{B}$ and the index $[\mathcal{B} : \mathcal{A}]$ is at most J.

– We say that G has *finite subgroups of bounded rank* if there is a positive integer $m = m_G$ such that any finite abelian subgroup $\mathcal{A}$ of G can be generated by at most m elements [MiRT15, PS14].

– We call a Jordan group G *strongly Jordan* if there is a positive integer $m = m_G$ such that any finite abelian subgroup $\mathcal{A}$ of G can be generated by at most m elements [MiRT15]. In other words, G is strongly Jordan if it is Jordan and has finite abelian subgroups of bounded rank.

– We say that G is *bounded* [Pop14, PS14] if there is a positive integer $C = C_G$ such that the order of every finite subgroup of G does not exceed C. (A bounded group is Jordan and even strongly Jordan.)

One may introduce similar definitions for families of groups [Pop11, PS14].

DEFINITION 1.2. Let $\mathcal{G}$ be a family of groups.

– We say that $\mathcal{G}$ is *uniformly Jordan* if there is a positive integer $\mathcal{J}$ such that each $G \in \mathcal{G}$ is Jordan with $J_G = \mathcal{J}$. We say that $\mathcal{G}$ is *uniformly strongly Jordan* if there are positive integers $\mathcal{J}$ and $\mathcal{M}$ such that each $G \in \mathcal{G}$ is strongly Jordan with $J_G = \mathcal{J}$ and $m_G = \mathcal{M}$.

– We say that $\mathcal{G}$ is *uniformly bounded* if there is a positive integer C such that the order of every finite subgroup of every G from $\mathcal{G}$ does *not* exceed C. See [PS14, Remark 2.9].

Remark 1.3. In the terminology of [PS14, Section 2], a family $\mathcal{G}$ is uniformly strongly Jordan if and only if it is uniformly Jordan and has finite subgroups of uniformly bounded rank.

1.2 Jordan properties of Bir(X) and Aut(X)

Let X be an irreducible quasiprojective variety over k. There is the natural group embedding

$$\mathrm{Aut}(X) \hookrightarrow \mathrm{Bir}(X)$$

that allows us to view $\mathrm{Aut}(X)$ as a subgroup of $\mathrm{Bir}(X)$. In particular, the Jordan property and the strong Jordan property for $\mathrm{Bir}(X)$ imply the same property for $\mathrm{Aut}(X)$. However, the converse is not necessarily true. More precisely, we have the following:

– It is known that $\mathrm{Aut}(X)$ is Jordan if $\dim(X) \leqslant 2$ [Pop11, Zar15, BZ15].

– It is also known (Popov [Pop11]) that $\mathrm{Bir}(X)$ is Jordan if $\dim(X) \leqslant 2$ and X is *not* birational to a product $E \times \mathbb{P}^1$ of an elliptic curve E and the projective line $\mathbb{P}^1$. (The Jordan property of the two-dimensional Cremona group $\mathrm{Bir}(\mathbb{P}^2)$ was established earlier by J.-P. Serre [Ser09].)

– In the remaining case $\mathrm{Bir}(E \times \mathbb{P}^1)$ is *not* Jordan. More generally, if A is an abelian variety of *positive dimension* over k and n is a positive integer, then $\mathrm{Bir}(A \times \mathbb{P}^n)$ is *not* Jordan [Zar14]. Notice that $\mathrm{Bir}(A)$ coincides with $\mathrm{Aut}(A)$ and is strongly Jordan [Pop11]. Actually, if $\mathbf{A}_d$ is the family of groups $\mathrm{Bir}(A)$ when A runs through the set of all d-dimensional abelian varieties over k, then $\mathbf{A}_d$ is uniformly strongly Jordan [PS14, Corollary 2.15].

– In higher dimensions, a recent result of Meng and Zhang [MZ15] asserts that $\mathrm{Aut}(X)$ is Jordan if X is *projective*.

For groups of birational automorphisms in higher dimensions, Prokhorov and Shramov proved the following strong result [PS14, Theorem 1.8].

THEOREM 1.4. (i) *If X is non-uniruled, then $\mathrm{Bir}(X)$ is Jordan.*

(ii) *If X is non-uniruled and $q(X) = 0$, then $\mathrm{Bir}(X)$ is bounded.*

Remark 1.5. Prokhorov and Shramov [PS14, Remark 6.9] noticed that if X is non-uniruled, then $\mathrm{Bir}(X)$ has finite subgroups of bounded rank. This means that in the non-uniruled case $\mathrm{Bir}(X)$ is *strongly Jordan.*

In addition, in dimension 3, Prokhorov and Shramov proved [PS16, PS14] the following:

– If $q(X) = 0$, then $\mathrm{Bir}(X)$ is *Jordan.*

– If X is rationally connected, then $\mathrm{Bir}(X)$ is strongly Jordan. Even better, if X varies in the set of rationally connected threefolds, then the corresponding family of groups $\mathrm{Bir}(X)$ is *uniformly strongly Jordan* [PS16, Theorems 1.7 and 1.10].

Actually, they proved all these assertions in arbitrary dimension d, assuming that the well-known conjecture of A. Borisov, V. Alexeev and L. Borisov (BAB conjecture [Bor96]) about the boundedness of families of d-dimensional Fano varieties with terminal singularities is valid in dimension d.

In light of their results it remains to investigate Jordan properties of $\mathrm{Bir}(X)$ when X is *uniruled* with $q(X) > 0$. According to Prokhorov and Shramov [PS14, Question 9.1], it is natural to start with a *conic bundle* X over an *abelian surface* A when X is *not* birational to a product $A \times \mathbb{P}^1$.

In this work, we prove that in this case $\mathrm{Bir}(X)$ is Jordan. Actually, we prove the following more general statement.

THEOREM 1.6. *Let X be an irreducible smooth projective variety of dimension $d \geqslant 3$ over k, and let $f \colon X \to Y$ be a surjective morphism over k from X to a $(d-1)$-dimensional abelian variety Y over k. Let us assume that the generic fiber of f is a genus zero smooth projective irreducible curve $\mathcal{X}_f$ over $k(Y)$ without $k(Y)$-points. Then $\mathrm{Bir}(X)$ is strongly Jordan.*

We deduce Theorem 1.6 from the following more general statement.

THEOREM 1.7. *Let $d \geqslant 3$ be a positive integer, and let X and Y be smooth irreducible projective varieties over k of dimension d and $d-1$, respectively. Let $f \colon X \to Y$ be a surjective morphism whose generic fiber is a genus zero smooth projective irreducible curve $\mathcal{X}_f$ over $k(Y)$ without $k(Y)$-points. Assume that Y is non-uniruled. Then $\mathrm{Bir}(X)$ is strongly Jordan.*

The following assertion is a variant of Theorem 1.7.

THEOREM 1.8. *Let $d \geqslant 3$ be a positive integer, and let X and Y be smooth irreducible projective varieties over k of dimension d and $d-1$, respectively. Let $f : X \to Y$ be a surjective morphism. Suppose that there exists a nonempty open subset U of Y such that for all $y \in U(k)$ the corresponding fiber X_y of f is k-isomorphic to the projective line over k (that is, the general fiber satisfies $X_y \cong \mathbb{P}_k^1$).*

Assume that Y is non-uniruled and f does not admit a rational section $Y \dashrightarrow X$. Then $\mathrm{Bir}(X)$ is strongly Jordan.

The paper is organized as follows. Section 2 deals with Jordan properties of groups. In Section 3 we recall the basic properties of conic bundles over non-uniruled varieties. In Section 4 we describe finite subgroups of the automorphisms group of a conic without rational points. We prove the main results of the paper in Section 5.

2. Group theory

We will need the following useful result of Anton Klyachko [PS14, Lemma 2.8]. (See also [Zar15, Lemma 6.2].)

LEMMA 2.1. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be families of groups such that $\mathcal{G}_1$ is uniformly bounded and $\mathcal{G}_2$ is uniformly strongly Jordan. Let $\mathcal{G}$ be a family of groups G such that G sits in an exact sequence*

$$\{1\} \to G_1 \to G \to G_2 \to \{1\},$$

where $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$. Then $\mathcal{G}$ is uniformly Jordan.

Actually, we will use the following slight refinement of Lemma 2.1.

LEMMA 2.2. *In the notation and assumptions of Lemma 2.1, the family $\mathcal{G}$ is uniformly strongly Jordan.*

Proof. The assertion follows readily from Lemma 2.1 combined with the result [PS14, Lemma 2.7] about extensions of groups with uniformly bounded ranks. $\square$

COROLLARY 2.3. *Let d be a positive integer. Let $\mathcal{G}_1$ be a uniformly bounded family of groups. Let $\mathbf{A}_d$ be the family of groups $\mathrm{Bir}(A)$, where A runs through the set of all d-dimensional abelian varieties over k. Let $\mathcal{G}$ be a family of groups G such that there exists an exact sequence*

$$\{1\} \to G_1 \to G \to G_2 \to \{1\},$$

where $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathbf{A}_d$. Then $\mathcal{G}$ is uniformly Jordan.

Proof. One has only to recall that $\mathcal{G}_2 := \mathbf{A}_d$ is uniformly strongly Jordan [PS14, Corollary 2.15] and apply Lemma 2.2. $\square$

LEMMA 2.4. *Let G be a strongly Jordan group, and let H be a subgroup of G. Suppose that there exists a positive integer N_H such that every periodic element in H has order that does not exceed N_H. Then H is bounded.*

Proof. Let J_G be the Jordan index of H. We know that there is a positive integer m_G such that every finite abelian subgroup in G is generated by at most m_G elements.

Let $\mathcal{B}$ be a finite subgroup of H. Clearly, $\mathcal{B}$ is a subgroup of G as well. Then $\mathcal{B}$ contains a finite abelian subgroup $\mathcal{A}$ with index $[\mathcal{B} : \mathcal{A}] \leqslant J_G$. The abelian group $\mathcal{A}$ is generated by at most m_G elements, each of which has order at most N_H. This implies $|\mathcal{A}| \leqslant N_H^{m_G}$ and therefore

$$|\mathcal{B}| \leqslant J_H \cdot |\mathcal{A}| \leqslant J_G \cdot N_H^{m_G}. \qquad \square$$

Recall [MiRT15] that the matrix group $\mathrm{GL}(n, \mathbb{C})$ is *strongly* Jordan and each of its finite abelian subgroups is generated by at most n elements. This implies that for any field K of characteristic zero, the matrix group $\mathrm{GL}(n, K)$ is a strongly Jordan group with Jordan index $J_{\mathrm{GL}(n,\mathbb{C})}$ (see [Pop14, Section 1.2.2]); in addition, each of its finite abelian subgroups is generated by at most n elements. Combining this observation with Lemma 2.4 and the last formula of its proof, we obtain the following assertion that may be of independent interest.

THEOREM 2.5. *Let K be a field of characteristic zero and n a positive integer. Suppose that H is a subgroup of $\mathrm{GL}(n, K)$ and N is a positive integer such that every periodic element in H has order at most N. Then there exists a positive integer $\mathbf{N} = \mathbf{N}(n, N)$ that depends only on n and N, such that every finite subgroup in H has order at most $\mathbf{N}$. In particular, H is bounded.*

3. Conic bundles

Let $f \colon X \to Y$ be a surjective morphism of smooth irreducible projective varieties of positive dimension over k. Since X and Y are projective, f is a *projective* morphism. It is well known that there is an open Zariski dense subset $U = U(f)$ of Y such that the restriction $f \colon f^{-1}(U) \to U$ is smooth [Har77, III.10, Corollary 10.7] and flat [Mum66, Lecture 8, 2°], [GD65, Theorem 6.9.1]. Thus the *generic fiber* $\mathcal{X} := \mathcal{X}_f$ is a smooth projective variety over $k(Y)$ and all its irreducible components have dimension $\dim(X) - \dim(Y)$ [Har77, III.9, Corollary 9.6 and III.10, Proposition 10.1]. In addition, if y is a closed point of U, then the corresponding fiber X_y of f is a smooth projective variety over the field $k(y) = k$ and all its irreducible components have dimension $\dim(X) - \dim(Y)$.

Notice that a *dominant* f defines the field embedding

$$f^* \colon k(Y) \hookrightarrow k(X)$$

that is the identity map on k. Further, we will identify $k(Y)$ with its image in $k(X)$. The field of rational functions of $\mathcal{X}_f$ coincides with $k(X)$, and the group of birational automorphisms $\mathrm{Bir}_{k(Y)}(\mathcal{X}_f)$ coincides with the (sub)group

$$\mathrm{Aut}(k(X)/k(Y)) \subset \mathrm{Aut}(k(X)/k) = \mathrm{Bir}_k(X) \tag{3.1}$$

that consists of all automorphisms of the field $k(X)$ leaving invariant every element of $k(Y)$.

We say that X is a *conic bundle* over Y if the generic fiber $\mathcal{X} := \mathcal{X}_f$ is an absolutely irreducible genus zero curve over $k(Y)$. (See [Sar81, Sar83].) In particular, $\dim(X) - \dim(Y) = 1$ and therefore the general fiber of f is a (smooth projective) curve.

Remark 3.1. As usual, by the *general fiber* of f we mean the fiber X_y of f over a point y from some nonempty open subset of Y. If the generic fiber is an irreducible smooth projective curve, then there is an open nonempty subset U of Y such that for all closed points $y \in U$, the corresponding (closed) fibers X_y are irreducible smooth projective curves over $k(y) = k$ as well [GD66, Corollary 9.5.6, Proposition 9.7.8]. The semi-continuity theorem [Har77, Chapter III, Theorem 12.8], [Mum74, II.5, Corollary, p. 47] implies that the general fiber has genus zero if and only if the same is true for the generic fiber. Thus, in our setting, the condition that the generic fiber is a smooth irreducible curve of genus zero is equivalent to the same condition for the general fiber.

Remark 3.2. If the genus zero curve $\mathcal{X}_f$ has a $k(Y)$-rational point, then it is biregular to the projective line over $k(Y)$ [HS00, Theorem A.4.3.1]. This implies that X is k-birational to $Y \times \mathbb{P}^1$. It follows from [Zar14] that if Y is an abelian variety (of positive dimension), then $\mathrm{Bir}(X)$ is *not* Jordan.

EXAMPLE 3.3. Let us consider a smooth projective plane quadric

$$\mathcal{X}_q = \left\{ a_1 T_1^2 + a_2 T_2^2 + a_3 T_3^2 = 0 \right\} \subset \mathbb{P}^2{}_{k(Y)}$$

over the field $K := k(Y)$ where all a_i are *nonzero* elements of $k(Y)$ such that the nondegenerate ternary quadratic form

$$q(T) = a_1 T_1^2 + a_2 T_2^2 + a_3 T_3^2$$

in $T = (T_1, T_2, T_3)$ is *anisotropic* over $k(Y)$, that is, $q(T) \neq 0$ if $T_i \in k(Y)$ for all i and at least one of them is not zero in $k(Y)$. (It follows from [Kah90, Theorem 1], see also [Kah91], that such a form exists if and only if $2^{\dim(Y)} \geqslant 3$, that is, if and only if $\dim(Y) \geqslant 2$.) Clearly, $\mathcal{X}_q$ is an absolutely irreducible smooth projective curve of genus zero over K that does *not* have K-points.

We want to construct a conic bundle with generic fiber $\mathcal{X}_q$ without K-rational point. First, let us consider the field $K(\mathcal{X}_q)$ of the rational functions on $\mathcal{X}_q$. It is finitely generated over K and has transcendence degree 1 over it. This implies that $K(\mathcal{X}_q)$ is finitely generated over k and has transcendence degree $\dim(Y) + 1$ over it. Since $\mathcal{X}_q$ is absolutely irreducible over K, the latter is algebraically closed in $K(\mathcal{X}_q)$.

By Hironaka's results, there is an irreducible smooth projective variety X over k of dimension $\dim(Y) + 1$ with $k(X) = K(\mathcal{X}_q)$ such that the dominant rational map $f \colon X \to Y$ induced by the field embedding $k(Y) = K \subset K(\mathcal{X}_q) = k(X)$ is actually a morphism. Clearly, the generic fiber $\mathcal{X}_f$ is a smooth projective variety, all whose irreducible components have dimension 1. Since K is algebraically closed in $K(\mathcal{X}_q) = k(X)$, the curve $\mathcal{X}_f$ is absolutely irreducible over K [GD61, Corollary 4.3.7 and Remark 4.3.8]. On the other hand, the field $K(\mathcal{X}_f)$ of rational functions on the K-curve $\mathcal{X}_f$ coincides with $k(X)$ by the very definition of the generic fiber. Since $K(\mathcal{X}_f) = k(X) = K(\mathcal{X}_q)$, the K-curves $\mathcal{X}_f$ and $\mathcal{X}_q$ are birational. Taking into account that both curves are smooth projective and absolutely irreducible over K, we conclude that $\mathcal{X}_f$ and $\mathcal{X}_q$ are biregularly isomorphic over K. This implies that $\mathcal{X}_f$ has genus zero and has no $k(Y)$-rational points. Thus $f \colon X \to Y$ is the conic bundle we wanted to construct.

LEMMA 3.4. *Let X and Y be smooth irreducible projective varieties of positive dimension over k and $f \colon X \to Y$ a surjective morphism such that the general fiber $F_y = f^{-1}(y)$ is isomorphic to $\mathbb{P}^1_k$. Let us identify $k(Y)$ with its image in $k(X)$. Assume additionally that Y is non-uniruled. (For example, Y is an abelian variety.)*

Then every k-linear automorphism σ of the field $k(X)$ leaves invariant $k(Y)$; that is,

$$\sigma(k(Y)) = k(Y) \quad \forall \sigma \in \operatorname{Aut}(k(X)).$$

In addition, there is exactly one birational automorphism u_Y of Y whose action on $k(Y)$ coincides with σ.
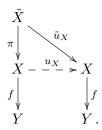
Proof. There is a birational automorphism u_X of X that induces σ on $k(X)$. Let $\pi \colon \tilde{X} \to X$ be a resolution of indeterminancy of u_X; that is, we consider a smooth irreducible projective k-variety $\tilde{X}$ and birational morphisms

$$\pi, \tilde{u}_X \colon \ \tilde{X} \to X$$

that enjoy the following properties:

- The map $\pi^{-1} \colon X \dashrightarrow \tilde{X}$ is an isomorphism outside the indeterminancy locus of u_X.

– The following diagram commutes:

$$
\begin{array}{ccc}
\tilde{X} & & \\
\pi \downarrow & \searrow^{\tilde{u}_X} & \\
X & \dashrightarrow^{u_X} & X \\
f \downarrow & & \downarrow f \\
Y & & Y\,.
\end{array}
$$

Consider morphisms $\tilde{f} = f \circ \pi \colon \tilde{X} \to Y$ and $g = f \circ \tilde{u}_X \colon \tilde{X} \to Y$.

Let $\Sigma_1 \subset X$, $\Sigma_2 \subset X$ be the loci of indeterminancy of π^{-1} and $\tilde{u}_X^{-1}$, respectively.

Since $\operatorname{codim}_X(\Sigma_1) \geqslant 2$ and $\operatorname{codim}_X(\Sigma_2) \geqslant 2$, we obtain that $\operatorname{codim}_Y(f(\Sigma_1)) \geqslant 1$ and $\operatorname{codim}_Y(f(\Sigma_2)) \geqslant 1$. (Recall that $\dim(X) = \dim(Y) + 1$.)

This implies that there is a nonempty open subset $U \subset Y \setminus (f(\Sigma_1) \cup f(\Sigma_2))$ such that

$$
\tilde{F}_y := \tilde{f}^{-1}(y) = \pi^{-1}(F_y) \cong F_y \cong \mathbb{P}^1
$$
$$
G_y := g^{-1}(y) = u_X^{-1}(F_y) \cong F_y \cong \mathbb{P}^1
$$

for all $y \in U(k)$.

Since Y is non-uniruled, $g(\tilde{F}_y)$ and $\tilde{f}(G_y)$ are points for every $y \in U(k)$ (see, for example, [Kol96, IV, Proposition 1.3, (1.3.4)]).

It follows from Kawamata's lemma [Iit82, Lemma 10.7] applied (twice) to the morphisms
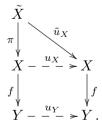
$$
\tilde{f}, g \colon \ \tilde{X} \to Y
$$

that there exist rational maps $h_1, h_2 \colon Y \dashrightarrow Y$ such that

$$
g = h_1 \circ \tilde{f}, \quad \tilde{f} = h_2 \circ g \,.
$$

This implies that h_1 and h_2 are mutually inverse birational automorphisms of Y. Let us put

$$
u_Y := h_1 \in \operatorname{Bir}(Y) \,.
$$

Then u_Y may be included into the commutative diagram

$$
\begin{array}{ccc}
\tilde{X} & & \\
\pi \downarrow & \searrow^{\tilde{u}_X} & \\
X & \dashrightarrow^{u_X} & X \\
f \downarrow & & \downarrow f \\
Y & \dashrightarrow^{u_Y} & Y\,.
\end{array}
$$

For the corresponding embeddings $k(Y) \hookrightarrow k(X)$ of fields of rational functions, we have $f^* \circ (u_Y)^* = \sigma \circ f^*$; thus

$$
\sigma(k(Y)) = k(Y) \,. \qquad \square
$$

Remark 3.5. Lemma 3.4 follows from the theorem on *maximal rational connected fibrations* [Kol96, IV, Theorem 5.5]. However, our particular case is much easier, so we decided to provide a simple proof rather than to use the powerful theory.

The next statement follows immediately from Lemma 3.4.

COROLLARY 3.6. *Keeping the notation and assumptions of Lemma 3.4, the map*

$$u_X \mapsto u_Y$$

gives rise to the group homomorphism $\mathrm{Bir}(X) \to \mathrm{Bir}(Y)$, *whose kernel is*

$$\mathrm{Aut}(k(X)/k(Y)) = \mathrm{Bir}_{k(Y)}(\mathcal{X}_f)$$

(see (3.1)), where $\mathcal{X}_f$ *is the generic fiber of* f. *In particular, we get an exact sequence of groups*

$$\{1\} \to \mathrm{Bir}_{k(Y)}(\mathcal{X}_f) \hookrightarrow \mathrm{Bir}(X) \to \mathrm{Bir}(Y).$$

Remark 3.7. The special case of Corollary 3.6 when Y is an abelian surface may be deduced from [Sar81, Corollary 1.7].

COROLLARY 3.8. *Keeping the notation and assumptions of Lemma 3.4, suppose that* $\mathrm{Bir}(\mathcal{X}_f)$ *is bounded. Then* $\mathrm{Bir}(X)$ *is strongly Jordan.*

Proof. By results of Prokhorov and Shramov (see Theorem 1.4 and Remark 1.5 above), the group $\mathrm{Bir}(Y)$ is strongly Jordan, because Y is *non-uniruled*. More precisely, they proved in [PS14, Corollary 6.8] that the group $\mathrm{Bir}(Y)$ is Jordan provided that Y is non-uniruled. In [PS14, Remark 6.9] they claim that, actually, if Y is non-uniruled, then $\mathrm{Bir}(Y)$ has finite subgroups of bounded rank (and therefore is strongly Jordan). The proof has to be based on the same arguments as the proof of [PS14, Corollary 6.8] but is not given. Let us provide the needed mild modifications of the proof of [PS14, Corollary 6.8] in order to obtain that non-uniruledness of Y implies having finite subgroups of bounded rank. Indeed, for Y non-uniruled and for a finite commutative subgroup $G \subset \mathrm{Bir}(Y)$ one has the following (using [PS14, Proposition 6.2] and its notation, only replacing X by Y and X_{nu} by Y_{nu}):

– We have $Y_{\mathrm{nu}} \sim Y$ (since Y is not uniruled), and the group G_{nu} is isomorphic to G. In particular, G_{nu} is also finite and commutative.

– There are short exact sequences [PS14, 6.5]

$$1 \to G_{\mathrm{alg}} \to G \to G_N \to 1,$$
$$1 \to G_L \to G_{\mathrm{alg}} \to G_{\mathrm{ab}} \to 1,$$

where

(1) the finite groups G_{alg} and G_N are commutative;
(2) we have $\mathrm{rk}(G_{\mathrm{ab}}) \leqslant m_1$, where m_1 depends only on $q(Y)$;
(3) we have $|G_L| \leqslant n = n(Y)$; that is, G_L is finite and its order is bounded from above by a number n that depends on Y but not on G [PS14, Lemma 5.2];
(4) it follows from statements (2) and (3) combined with the second exact sequence above that $\mathrm{rk}(G_{\mathrm{alg}}) \leqslant n + m_1$;
(5) the order $|G_N|$ is bounded from above by a number $b := b(Y)$ that depends only on Y [PS14, Corollary 2.14].

It follows from the exact sequences that

$$\mathrm{rk}(G) = \mathrm{rk}(G_{\mathrm{nu}}) \leqslant \mathrm{rk}(G_{\mathrm{alg}}) + b(Y) \leqslant (n + m_1) + b(Y) =: m(Y),$$

where the bound $m(Y)$ depends on Y but does not depend on G. This proves that $\mathrm{Bir}(Y)$ has finite subgroups of bounded rank.

So, we know that $\mathrm{Bir}(Y)$ is strongly Jordan. Now, the desired result follows readily from Corollary 3.6 combined with Lemma 2.2. □

In the next section we prove that $\mathrm{Bir}(\mathcal{X}_f)$ is bounded if $\mathcal{X}_f$ is *not* the projective line over $k(Y)$.

4. Linear algebra

Throughout this section K is a field of characteristic zero that contains *all* roots of unity. Let V be a vector space over K of finite positive dimension d. We write 1_V for the identity automorphism of V. As usual, $\mathrm{End}_K(V)$ stands for the algebra of K-linear operators in V and

$$\mathrm{Aut}_K(V) = \mathrm{End}_K(V)^*$$

for the group of linear invertible operators in V. We write

$$\det = \det{}_V \colon \mathrm{End}_K(V) \to K$$

for the determinant map. It is well known that $\mathrm{Aut}_K(V)$ consists of all elements of $\mathrm{End}_K(V)$ with *nonzero* determinant and that

$$\det \colon \mathrm{Aut}_K(V) \to K^*$$

is a group homomorphism.

Since K has characteristic zero and contains all roots of unity, every periodic (K-linear) automorphism $u \in \mathrm{Aut}_K(V)$ of V admits a basis of V that consists of eigenvectors of u, because u is semisimple and all its eigenvalues lie in K.

Let

$$\phi \colon V \times V \to K$$

be a nondegenerate symmetric K-bilinear form that is *anisotropic*, that is, $\phi(v, v) \neq 0$ for all *nonzero* $v \in V$. The form ϕ defines the *involution of the first kind*

$$\sigma = \sigma_\phi \colon \mathrm{End}_K(V) \to \mathrm{End}_K(V)$$

characterized by the property

$$\phi(ux, y) = \phi(x, \sigma(u)y) \quad \forall\, x, y \in V$$

(see [KMRT98, Chapter 1]). It is known [KMRT98, 1.2, Corollary 2.2, and Proposition 2.19] that

$$\det(u) = \det(\sigma(u)) \quad \forall\, u \in \mathrm{End}_K(V)\,.$$

We write $\mathrm{GO}(V, \phi) \subset \mathrm{Aut}_K(V)$ for the (sub)group of *similitudes* of ϕ. In other words, a K-linear automorphism u of V lies in $\mathrm{GO}(V, \phi)$ if and only if there exists a

$$\mu = \mu(g) \in K^*$$

such that

$$\phi(ux, uy) = \mu \cdot \phi(x, y) \quad \forall\, x, y \in V\,.$$

If this is the case, then

$$\sigma(u)u = \mu \cdot 1_V\,.$$

Clearly,

$$K^* \cdot 1_V \subset \mathrm{GO}(V, \phi)\,.$$

We have

$$\mathrm{SO}(V, \phi) \subset \mathrm{O}(V, \phi) \subset \mathrm{GO}(V, \phi)\,,$$

where

$$O(V, \phi) = \{u \in \mathrm{Aut}_K(V) \,|\, \phi(ux, uy) = \phi(x, y) \; \forall\, x, y \in V\},$$

while $SO(V, \phi)$ consists of all elements of $O(V, \phi)$ with determinant 1. (Recall that elements of $O(V, \phi)$ have determinant 1 or -1. In particular, $SO(V, \phi)$ is a normal subgroup of index 2 in $O(V, \phi)$.) Clearly,

$$O(V, \phi) = \{u \in \mathrm{Aut}_K(V) \,|\, \sigma(u)u = 1_V\}.$$

It is also clear that

$$GO(V, \phi) \to K^*, \quad u \mapsto \mu(u)$$

is a group homomorphism, whose kernel coincides with $O(V, \phi)$; in particular, $O(V, \phi)$ is a *normal* subgroup of $GO(V, \phi)$. It is well known (and may be easily checked) that

$$O(V, \phi) \bigcap [K^* \cdot 1_V] = \{\pm 1_V\};$$

in addition, if $d = \dim(V)$ is *odd*, then

$$SO(V, \phi) \bigcap [K^* \cdot 1_V] = \{1_V\}.$$

We denote by $PGO(V, \phi)$ the quotient group $GO(V, \phi)/(K^* \cdot 1_V)$.

Remark 4.1. The importance of the group $PGO(V, \phi)$ is explained by the following result [EKM08, Corollary 69.6]. Let

$$q(v) := \phi(v, v)$$

be the corresponding quadratic form on V, and let

$$X_q \subset \mathbb{P}(V)$$

be the projective quadric defined by the equation $q(v) = 0$, which is a smooth projective irreducible $(d - 2)$-dimensional variety over K. Then the groups $\mathrm{Aut}(X_q)$ and $PGO(V, \phi)$ are isomorphic.

Remark 4.2. Restricting the surjection

$$GO(V, \phi) \twoheadrightarrow GO(V, \phi)/(K^* \cdot 1_V) = PGO(V, \phi)$$

to the subgroup $O(V, \phi)$, we get a group homomorphism

$$O(V, \phi) \to PGO(V, \phi),$$

whose kernel is a finite subgroup $\{\pm 1_V\}$. This implies that if u is an element of $O(V, \phi)$ whose image in $PGO(V, \phi)$ has finite order, then u itself has finite order.

LEMMA 4.3. *Let u be an element of finite order in $O(V, \phi)$. Then $u^2 = 1_V$.*

Proof. Let λ be an eigenvalue of u. Then λ is a root of unity and therefore lies in K. This implies that there is a (nonzero) eigenvector $x \in V$ with $ux = \lambda x$. Since $u \in O(V, \phi)$,

$$\phi(ux, ux) = \phi(x, x).$$

Since $ux = \lambda x$,

$$\phi(ux, ux) = \phi(\lambda x, \lambda x) = \lambda^2 \phi(x, x),$$

and therefore $\lambda^2 \phi(x, x) = \phi(x, x)$. Since ϕ is anisotropic, $\phi(x, x) \neq 0$, and therefore $\lambda^2 = 1$. In other words, every eigenvalue of u^2 is 1 and, since u is semisimple, $u^2 = 1_V$. $\qquad\square$

COROLLARY 4.4. *Let G be a finite subgroup of $\mathrm{O}(V,\phi)$. If G does not coincide with $\{1_V\}$, then it is a commutative group of exponent 2 whose order divides 2^d. If, in addition, G lies in $\mathrm{SO}(V,\phi)$, then its order divides 2^{d-1}.*

Proof. By Lemma 4.3, every $u \in G$ satisfies $u^2 = 1_V$. This implies that G is commutative. In addition, G is a 2-group; that is, its order is a power of 2. The commutativeness of G implies that there is a basis of V such that the matrices of all elements of G become diagonal with respect to this basis. Since all the diagonal entries are either 1 or -1, the order of G does not exceed 2^d and therefore divides 2^d. If, in addition, all elements of G have determinant 1, then the order of G does not exceed 2^{d-1} and, therefore, divides 2^{d-1}. $\square$

COROLLARY 4.5. *Let $\mathbf{u}$ be an element of finite order in $\mathrm{PGO}(V,\phi)$. Then $\mathbf{u}^4 = 1$.*

Proof. Choose an element u of $\mathrm{GO}(V,\phi)$ such that its image in $\mathrm{PGO}(V,\phi)$ coincides with $\mathbf{u}$. Then there is a $\mu \in K^*$ such that

$$\phi(ux, uy) = \mu\phi(x,y) \quad \forall\, x,y \in V\,.$$

This implies that $u_2 := \mu^{-1}u^2$ lies in $\mathrm{O}(V,\phi)$. Clearly, the image $\bar{u}_2 \in \mathrm{PGO}(V,\phi)$ of u_2 coincides with $\mathbf{u}^2$ and therefore has finite order. By Remark 4.2, u_2 has finite order. It follows from Lemma 4.3 that $u_2^2 = 1_V$. This implies that $\mathbf{u}^2$ has order 1 or 2, and therefore the order of $\mathbf{u}$ divides 4. $\square$

COROLLARY 4.6. *If $\mathcal{B}$ is a finite subgroup of $\mathrm{PGO}(V,\phi)$, then it sits in a short exact sequence*

$$\{1\} \to \mathcal{A}_1 \to \mathcal{B} \to \mathcal{A}_2 \to \{1\}\,,$$

where both $\mathcal{A}_1$ and $\mathcal{A}_2$ are finite elementary commutative 2-groups and $|\mathcal{A}_1|$ divides 2^{d-1}. In particular, each finite subgroup $\mathcal{B}$ of $\mathrm{PGO}(V,\phi)$ is a finite 2-group such that

$$[[\mathcal{B},\mathcal{B}],[\mathcal{B},\mathcal{B}]] = \{1\}\,.$$

Proof. Let $\mathcal{A}_1$ be the subgroup of all elements of $\mathcal{B}$ that are the images of elements of $\mathrm{O}(V,\phi)$. Since $\mathrm{O}(V,\phi)$ is normal in $\mathrm{GO}(V,\phi)$, the subgroup $\mathcal{A}_1$ is normal in $\mathcal{B}$. It follows from the proof of Corollary 4.5 that for each $\mathbf{u} \in \mathcal{B}$, its square $\mathbf{u}^2$ lies in $\mathcal{A}_1$. This implies that all the elements of the quotient $\mathcal{A}_2 := \mathcal{B}/\mathcal{A}_1$ have order 1 or 2. It follows that $\mathcal{A}_2$ is an elementary abelian 2-group. We get a short exact sequence

$$\{1\} \to \mathcal{A}_1 \to \mathcal{B} \to \mathcal{A}_2 \to \{1\}\,.$$

Let $\tilde{\mathcal{A}}_1$ be the preimage of $\mathcal{A}_1$ in $\mathrm{O}(V,\phi)$. Clearly, $\tilde{\mathcal{A}}_1$ is a subgroup of $\mathrm{O}(V,\phi)$ and $|\tilde{\mathcal{A}}_1| = 2\cdot|\mathcal{A}_1|$. On the other hand, it follows from Corollary 4.4 that $\tilde{\mathcal{A}}_1$ is an elementary abelian 2-group whose order divides 2^d. Since $\tilde{\mathcal{A}}_1$ maps *onto* $\mathcal{A}_1$, the latter is also an elementary abelian 2-group and its order divides $\frac{1}{2}2^d = 2^{d-1}$. Since

$$|\mathcal{B}| = |\mathcal{A}_1| \cdot |\mathcal{A}_2|\,,$$

the order of $\mathcal{B}$ is a power of 2; that is, $\mathcal{B}$ is a finite 2-group. On the other hand, since $\mathcal{A}_2 = \mathcal{B}/\mathcal{A}_1$ is abelian, $[\mathcal{B},\mathcal{B}] \subset \mathcal{A}_1$. Since $\mathcal{A}_1$ is abelian,

$$[[\mathcal{B},\mathcal{B}],[\mathcal{B},\mathcal{B}]] \subset [\mathcal{A}_1,\mathcal{A}_1] = \{1\}\,;$$

that is, $[[\mathcal{B},\mathcal{B}],[\mathcal{B},\mathcal{B}]] = \{1\}$. $\square$

In the case of *odd d* we can do better. Let us start with the following observation.

LEMMA 4.7. *Suppose that $d = 2\ell + 1$ is an odd integer that is greater than or equal to 3. Then every $u \in \mathrm{GO}(V, \phi)$ can be presented as*

$$u = \mu_0 \cdot u_0$$

with $u_0 \in \mathrm{SO}(V, \phi)$ and $\mu_0 \in K^*$

EXAMPLE 4.8. *If u is an element of $\mathrm{O}(V, \phi)$ with determinant -1, then*

$$u = (-1) \cdot (-u), \quad (-u) \in \mathrm{SO}(V, \phi) \,.$$

Proof of Lemma 4.7. Recall that there is a $\mu \in K^*$ such that

$$\phi(ux, uy) = \mu \cdot \phi(x, y) \quad \forall x, y \in V \,,$$

and therefore

$$\sigma(u)u = \mu \cdot 1_V \,.$$

Now, let $\gamma \in K^*$ be the determinant of u. Since $\det(\sigma(u)) = \det(u)$, we obtain

$$\mu^{2\ell+1} = \mu^d = \det(\mu \cdot 1_V) = \det(\sigma(u)u) = \det(\sigma(u))\det(u) = \gamma^2 \,.$$

This implies

$$\gamma^2 = \mu^d = \mu^{2\ell+1} \,.$$

Let us put

$$\mu_0 = \frac{\gamma}{\mu^\ell}, \quad u_0 = \mu_0^{-1} \cdot u \,.$$

Then

$$\mu_0^2 = \mu, \quad \gamma = \mu_0^{2\ell+1} = \mu_0^d \,,$$
$$u = \mu_0 \cdot u_0, \quad \det(u_0) = \mu_0^{-d} \cdot \det(u) = \gamma^{-1}\gamma = 1 \,.$$

We also have

$$\phi(u_0 x, u_0 y) = \phi(\mu_0^{-1}ux, \mu_0^{-1}uy) = \mu_0^{-2} \cdot \phi(ux, uy) = \mu^{-1}\phi(ux, uy) = \mu^{-1} \cdot \mu \cdot \phi(x, y) = \phi(x, y) \,.$$

This implies that $u_0 \in \mathrm{O}(V, \phi)$. Since $\det(u_0) = 1$,

$$u_0 \in \mathrm{SO}(V, \phi) \,. \qquad \square$$

COROLLARY 4.9. *Suppose that $d = 2\ell + 1$ is an odd integer that is greater than or equal to 3. Then the group homomorphism*

$$\mathrm{prod}\colon K^*1_V \times \mathrm{SO}(V, \phi) \to \mathrm{GO}(V, \phi), \quad (\mu_0 \cdot 1_V, u_0) \mapsto \mu_0 \cdot u_0$$

is a group isomorphism. In particular, the group $\mathrm{PGO}(V, \phi) = \mathrm{GO}(V, \phi)/(K^*1_V)$ is canonically isomorphic to $\mathrm{SO}(V, \phi)$.

Proof. Since d is odd,

$$\mathrm{SO}(V, \phi) \bigcap [K^* \cdot 1_V] = \{1_V\} \,,$$

which implies that prod is injective. Its surjectiveness follows from Lemma 4.7. $\qquad \square$

THEOREM 4.10. *Suppose that K is a field of characteristic zero that contains all roots of unity, $d \geqslant 3$ is an odd integer, V is a d-dimensional K-vector space and*

$$\phi\colon V \times V \to K$$

is a nondegenerate symmetric K-bilinear form that is anisotropic; that is, $\phi(v, v) \neq 0$ for all nonzero $v \in V$.

Let G be a finite subgroup in $\mathrm{PGO}(V, \phi)$. Then G is commutative, all its nonidentity elements have order 2 and the order of G divides 2^{d-1}.

Proof. The result follows readily from Corollary 4.9 combined with Corollary 4.4. $\quad\square$

COROLLARY 4.11. *Suppose that K is a field of characteristic zero that contains all roots of unity, $d \geqslant 3$ an odd integer and V a d-dimensional K-vector space, and let $q \colon V \to K$ be a quadratic form such that $q(v) \neq 0$ for all nonzero $v \in V$. Let us consider the projective quadric $X_q \subset \mathbb{P}(V)$ defined by the equation $q = 0$; this is a smooth projective irreducible $(d-2)$-dimensional variety over K. Let $\mathrm{Aut}(X_q)$ be the group of biregular automorphisms of X_q. Let G be a finite subgroup in $\mathrm{Aut}(X_q)$. Then G is commutative, all its nonidentity elements have order 2, and the order of G divides 2^{d-1}.*

Proof. Let $\phi \colon V \times V \to K$ be the symmetric K-bilinear form such that

$$\phi(v, v) = q(v) \quad \forall\, v \in V\,.$$

Namely, for all $x, y \in V$

$$\phi(x, y) := \frac{q(x+y) - q(x) - q(y)}{2}\,.$$

Clearly, ϕ is nondegenerate. In the notation of [EKM08, Section 69],

$$\mathrm{GO}(q) = \mathrm{GO}(V, \phi), \quad \mathrm{PGO}(q) = \mathrm{PGO}(V, \phi)\,.$$

By [EKM08, Corollary 69.6], the groups $\mathrm{Aut}(X_q)$ and $\mathrm{PGO}(q)$ are isomorphic. Now the result follows from Theorem 4.10. $\quad\square$

COROLLARY 4.12. *Suppose that K is a field of characteristic zero that contains all roots of unity. Let $\mathcal{C}$ be a smooth irreducible projective genus zero curve over K that is not biregular to $\mathbb{P}^1$ over K.*

Let $\mathrm{Bir}_K(\mathcal{C})$ be the group of birational automorphisms of $\mathcal{C}$. Let G be a finite subgroup in $\mathrm{Bir}_K(\mathcal{C})$. Then G is commutative, all its nonidentity elements have order 2, and the order of G divides 4. In other words, if G is nontrivial, then it either is a cyclic group of order 2 or is isomorphic to a product of two cyclic groups of order 2.

Proof. Since $\mathcal{C}$ has genus zero, it is K-biregular to a smooth projective plane quadric

$$\mathcal{X} = \left\{ a_1 T_1^2 + a_2 T_2^2 + a_3 T_3^2 = 0 \right\} \subset \mathbb{P}^2\,,$$

where all a_i are *nonzero* elements of K. Since $\mathcal{C}$ is *not* biregular to $\mathbb{P}^1$, the set $\mathcal{X}(K)$ is *empty* ([HS00, Theorem A.4.3.1], [EKM08, Section 45, Proposition 45.1]), which means that the nondegenerate ternary quadratic form

$$q(T) = a_1 T_1^2 + a_2 T_2^2 + a_3 T_3^2$$

is *anisotropic*. We may view q as the quadratic form on the coordinate 3-dimensional K-vector space $V = K^3$. Then (in the notation of Corollary 4.11) $d = 3$ and $\mathcal{X} = X_q$. Since $\mathcal{X}$ is a smooth projective curve, its group $\mathrm{Bir}_K(\mathcal{X})$ of birational automorphisms coincides with the group $\mathrm{Aut}(\mathcal{X})$ of biregular automorphisms. Now, the result follows from Corollary 4.11. $\quad\square$

The rest of this section deals with the case of *even* d; its results will not be used elsewhere in the paper.

Theorem 4.13. *Suppose that K is a field of characteristic zero that contains all roots of unity, $d \geqslant 2$ is an even positive integer, V is a d-dimensional K-vector space, and*

$$\phi \colon V \times V \to K$$

is a nondegenerate symmetric K-bilinear form that is anisotropic; that is, $\phi(v,v) \neq 0$ for all nonzero $v \in V$. Then the group $\mathrm{PGO}(V, \phi)$ is bounded. More precisely, there is a positive integer $\mathbf{n} = \mathbf{n}(d)$ that depends only on d, such that every finite subgroup of $\mathrm{PGO}(V, \phi)$ has order dividing $2^{\mathbf{n}(d)}$.

Proof. We deduce Theorem 4.13 from Theorem 2.5. Let $\mathrm{Aut}_K(\mathrm{End}_K(V))$ be the group of automorphisms of the K-algebra $\mathrm{End}_K(V)$. We write $\mathcal{V}_2$ for $\mathrm{End}_K(V)$ viewed as the d^2-dimensional K-vector space and $\mathrm{Aut}_K(\mathcal{V}_2)$ for its group of K-linear automorphisms. We have

$$\mathrm{Aut}_K(\mathrm{End}_K(V)) \subset \mathrm{Aut}_K(\mathcal{V}_2).$$

Let us choose a basis $\{e_1, \ldots, e_{d^2}\}$ of $\mathcal{V}_2$. Such a choice gives us a group isomorphism

$$\mathrm{Aut}_K(\mathcal{V}_2) \cong \mathrm{GL}\left(d^2, K\right).$$

Let us consider a group homomorphism

$$\mathrm{Ad} \colon \ \mathrm{GO}(V, \phi) \subset \mathrm{Aut}_K(V) \to \mathrm{Aut}_K(\mathrm{End}_K(V)), \quad u \mapsto \{w \mapsto uwu^{-1} \ \forall \, w \in \mathrm{End}_K(V)\}$$

for all $u \in \mathrm{GO}(V, \phi) \subset \mathrm{Aut}_K(V)$. Clearly,

$$\ker(\mathrm{Ad}) = K^* \cdot 1_V.$$

This gives us an embedding

$$\mathrm{PGO}(V, \phi) = \mathrm{GO}(V, \phi)/\{K^* \cdot 1_V\} \hookrightarrow \mathrm{Aut}_K(\mathrm{End}_K(V)) \hookrightarrow \mathrm{Aut}_K(\mathcal{V}_2) \cong \mathrm{GL}\left(d^2, K\right).$$

This implies that $\mathrm{PGO}(V, \phi)$ is isomorphic to a subgroup of $\mathrm{GL}(d^2, K)$. By Corollary 4.5, every periodic element in $\mathrm{PGO}(V, \phi)$ has order dividing 4. This implies (thanks to the first Sylow theorem) that the order of every finite subgroup of $\mathrm{PGO}(V, \phi)$ is a power of 2. In other words, all finite subgroups in $\mathrm{PGO}(V, \phi)$ are 2-groups. Now the desired result follows from Theorem 2.5 (applied to $n = d^2$ and $N = 4$). $\qquad \square$

Combining Theorem 4.13, Corollary 4.6 and Remark 4.2, we obtain the following assertion.

Theorem 4.14. *Suppose that K is a field of characteristic zero that contains all roots of unity, $d \geqslant 2$ an even integer and V a d-dimensional K-vector space. Let $q \colon V \to K$ be a quadratic form such that $q(v) \neq 0$ for all nonzero $v \in V$. Let us consider the projective quadric $X_q \subset \mathbb{P}(V)$ defined by the equation $q = 0$, which is a smooth projective irreducible $(d-2)$-dimensional variety over K. Let $\mathrm{Aut}(X_q)$ be the group of biregular automorphisms of X_q.*

Then:

(i) *The group $\mathrm{Aut}(X_q)$ is bounded. More precisely, there is a positive integer $\mathbf{n} = \mathbf{n}(d)$ that depends only on d such that every finite subgroup of $\mathrm{Aut}(X_q)$ has order dividing $2^{\mathbf{n}(d)}$.*

(ii) *If $\mathcal{B}$ is a finite subgroup of $\mathrm{Aut}(X_q)$, then it is a finite 2-group that sits in a short exact sequence*

$$\{1\} \to \mathcal{A}_1 \to \mathcal{B} \to \mathcal{A}_2 \to \{1\},$$

where both $\mathcal{A}_1$ and $\mathcal{A}_2$ are finite elementary abelian 2-groups and $|\mathcal{A}_1|$ divides 2^{d-1}. In particular,

$$[[\mathcal{B}, \mathcal{B}], [\mathcal{B}, \mathcal{B}]] = \{1\}.$$

We hope to return to a classification of finite subgroups of $\mathrm{Aut}(X_q)$ (for even d) in a future publication.

5. Jordan properties of Bir

Proof of Theorem 1.7. Let us put $K = k(Y)$. Then $\mathrm{char}(K) = 0$. Since K contains the algebraically closed k, it contains all roots of unity. In the notation of Corollary 4.12, let us put $\mathcal{C} = \mathcal{X}_f$. Since $\mathcal{X}_f$ has no K-points, it is *not* biregular to $\mathbb{P}^1$ over K.

It follows from Corollary 4.12 that $\mathrm{Bir}(\mathcal{X}_f)$ is *bounded*. Now, Corollary 3.8 implies that $\mathrm{Bir}(X)$ is strongly Jordan. □

Proof of Theorem 1.8. It follows from Remark 3.1 that $f\colon X \to Y$ is a conic bundle. In particular, the generic fiber $\mathcal{X} = \mathcal{X}_f$ is an absolutely irreducible smooth projective genus zero curve over $K := k(Y)$.

Since each K-point of $\mathcal{X}$ gives rise to a rational section $Y \dashrightarrow X$ of f, there are no K-points on $\mathcal{X}$. Now, the result follows from Theorem 1.7. □

EXAMPLE 5.1. Let Y be a smooth irreducible projective variety over k of dimension at least 2. Let a_1, a_2, a_3 be nonzero elements of $k(Y)$ such that the ternary quadratic form

$$q(T) = a_1 T_1^2 + a_2 T_2^2 + a_3 T_3^2$$

is *anisotropic* over $k(Y)$. Example 3.3 gives us a smooth irreducible projective variety $\tilde{X}_q$ and a surjective regular map $f\colon \tilde{X}_q \to Y$ whose generic fiber is the quadric

$$\left\{ a_1 T_1^2 + a_2 T_2^2 + a_3 T_3^2 = 0 \right\} \subset \mathbb{P}^2_{k(Y)}$$

over $k(Y)$ without $k(Y)$-points. Now, Theorem 1.7 tells us that $\mathrm{Bir}(\tilde{X}_q)$ is *strongly Jordan* if Y is non-uniruled.

Remark 5.2. Recall (Example 3.3) that if $\dim(Y) \geqslant 2$, then there always exists an *anisotropic* ternary quadratic form over $k(Y)$. (A theorem of Tsen implies that such a form does *not* exists if $\dim(Y) = 1$.)

Proof of Theorem 1.6. Recall that an abelian variety Y does *not* contain rational curves; in particular, it is *not* uniruled. Now, Theorem 1.6 follows from Theorem 1.7. □

THEOREM 5.3. *Let $d \geqslant 3$ be an integer. Let $\mathcal{G}$ be the collection of groups $\mathrm{Bir}(X)$, where X runs through the set of smooth irreducible projective d-dimensional varieties that can be realized as conic bundles $f\colon X \to Y$ over a $(d-1)$-dimensional abelian variety Y but where X is not birational to $Y \times \mathbb{P}^1$. Then $\mathcal{G}$ is uniformly strongly Jordan.*

Proof. It follows from Remark 3.2 that the generic fiber $\mathcal{X}_f$ of f has *no* $k(Y)$-rational points. It follows from Corollary 4.12 that the collection of groups of the form $\mathrm{Bir}(\mathcal{X}_f)$ is uniformly bounded—actually, the order of every finite subgroup in $\mathrm{Bir}(\mathcal{X}_f)$ divides 4. Recall (Corollary 3.6) that there is an exact sequence

$$\{1\} \to \mathrm{Bir}(\mathcal{X}_f) \hookrightarrow \mathrm{Bir}(X) \to \mathrm{Bir}(Y).$$

Now, the result follows from Corollary 2.3. □

Remark 5.4. The condition that the k-varieties X and Y in Theorems 1.6, 1.7 and 1.8 are smooth is nonessential. Indeed, let X and Y be irreducible projective varieties of dimensions d and $d-1$,

respectively, endowed with a surjective morphism $f\colon X \to Y$. Let Y be non-uniruled. Due to the resolution of singularities (see, for example, [Kol07, Section 3.3]), one can always find two smooth projective irreducible varieties $\tilde{X}$ and $\tilde{Y}$, birational morphisms $\pi_X\colon \tilde{X} \to X$ and $\pi_Y\colon \tilde{Y} \to Y$ and a morphism $\tilde{f}\colon \tilde{X} \to \tilde{Y}$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\
\pi_X \downarrow & & \downarrow \pi_Y \\
X & \xrightarrow{f} & Y\,.
\end{array}
$$

Then the following properties are valid:

(1) The variety $\tilde{Y}$ is non-uniruled since it is birational to the non-uniruled Y (see, for example, [Deb01, Remark 4.2]).

(2) If the general fiber $F_u := f^{-1}(u)$ for $u \in U \subset Y$ is irreducible, then so is the general fiber $\tilde{F}_v := \tilde{f}^{-1}(v)$ of $\tilde{f}$ for $v \in V \subset \tilde{Y}$ (here U and V are some open dense subsets of Y and $\tilde{Y}$, respectively).

Indeed, there is an open dense $V' \subset \tilde{Y}$ such that the following hold:

– The morphism π_Y is an isomorphism of V' to $\pi_Y(V')$.
– For every $v \in V'$ the exceptional set S_X of morphism π_X intersects the fiber $\tilde{F}_v$ of $\tilde{f}$ only at an empty or finite set of points. This holds because $\dim(S_X) \leqslant \dim(\tilde{Y}) = d - 1$; hence the restriction of $\tilde{f}$ onto an irreducible component of S_X is either nondominant or generically finite.

Thus, $\tilde{F}_v \cap (\tilde{X} \setminus S_X)$ is open and dense in $\tilde{F}_v$ for any point $v \in V'$ (because every irreducible component of $\tilde{F}_v$ has dimension at least 1). On the other hand, via the morphism π_X, the intersection $\tilde{F}_v \cap (\tilde{X} \setminus S_X)$ is isomorphic to $F_{\pi_Y(v)} \cap (X \setminus \pi_X(S_X))$, which is an open and dense subset of the irreducible set $F_{\pi_Y(v)}$ if $\pi_Y(v) \in U$. Hence, for all $v \in V' \cap \pi_Y^{-1}(U)$ the fibers $\tilde{F}_v$ and $F_{\pi_Y(v)}$ are birational.

(3) If the generic fiber $\mathcal{X}_f$ of f is absolutely irreducible, so is the generic fiber $\mathcal{X}_{\tilde{f}}$ of $\tilde{f}$. Indeed, in this case the general fiber of f is irreducible (see [GD66, Proposition 9.7.8]). According to property (2), the general fiber of $\tilde{f}$ is also irreducible and, hence, so is $\mathcal{X}_{\tilde{f}}$ [GD66, Proposition 9.7.8].

(4) The general fiber $\tilde{F}_v$ and generic fiber $\mathcal{X}_{\tilde{f}}$ of $\tilde{f}$ are smooth since $\tilde{f}$ is a surjective morphism between smooth projective varieties.

It follows that if the generic (respectively, general) fiber of f is a genus zero curve, then the generic (respectively, general) fiber of $\tilde{f}$ is a smooth genus zero curve. According to Theorem 1.7 (respectively, Theorem 1.8), the group $\mathrm{Bir}(X) = \mathrm{Bir}(\tilde{X})$ has to be Jordan.

Added in proof Recently, the BAB conjecture was proved by Birkar [Bir16].

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