p-kernels occurring in an isogeny class of p-divisible groups

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Abstract

We give a criterion which allows to determine, in terms of the combinatorics of the root system of the general linear group, which p-kernels occur in an isogeny class of p-divisible groups over an algebraically closed field of positive characteristic. As an application we obtain a criterion for the non-emptiness of certain affine Deligne-Lusztig varieties associated to the general linear group.

1 Introduction

This article studies the relationship between two invariants of a p-divisible group $\mathscr{G}$ over an algebraically closed field of characteristic $p > 0$: The first is the isogeny class of $\mathscr{G}$ which is encoded in its Newton polygon and the second is the isomorphism class of the kernel of multiplication by p on $\mathscr{G}$. Once certain numerical invariants of $\mathscr{G}$ are fixed, both these invariants can only take on finitely many values. In this article, we give a computable criterion, in terms of the combinatorics of the root system of the general linear group, which determines which pairs of these invariants can occur together for some $\mathscr{G}$. That is we determine which p-kernels can occur in any isogeny class of p-divisible groups. We also consider the analogous question in equal characteristic.

This question is motivated by our interest in the stratifications of suitable moduli spaces of abelian varieties or p-divisible groups obtained by decomposing these spaces according to the two invariants described above. For example, on a Rapoport-Zink space (c.f. [RZ]), one can define the Ekedahl-Oort stratification by decomposing the space according to the isomorphism class of the p-kernel of the universal p-divisible group and our criterion allows to determine which of these strata are non-empty. Similarly, on a moduli spaces of abelian varieties with suitable extra structure in positive characteristic, one obtains two stratifications, the Newton polygon stratifications and the Ekedahl-Oort stratification and we would like to understand which strata of these two stratifications intersect each other. However, in this context one encounters not just p-divisible groups, but p-divisible groups with additional structure such as a pairing. For applications to such stratifications it would thus be necessary to obtain generalizations of the results of this article for p-divisible groups with such additional structure. It seems natural to expect that in such a setting the analogues of our results should hold with the group GL_h replaced by an arbitrary reductive group. The author intends to treat this question in a follow-up article.

As an another application of our results, in Section 6 we give a criterion for the non-emptiness of affine Deligne-Lusztig varieties for the group GL_h in the situation where the involved Hodge cocharacter is minuscule.

Throughout, we work with Dieudonné modules instead of p-divisible groups. We work over a fixed algebraically closed field k of characteristic p and work either over the Witt ring $\mathcal{O} = W(k)$ or $\mathcal{O} = k[[t]]$ whose uniformizer p or t we denote by ϵ. We use the following language: A Dieudonné module is a finite free module over $\mathcal{O}$ together with suitably semilinear endomorphisms F and V satisfying $FV = FV = \epsilon$. A 1-truncated Dieudonné module is a finite-dimensional vector space over k together with suitably semilinear endomorpism F and V satisfying $\ker F = \operatorname{im} V$ and $\operatorname{im} F = \ker V$. To each Dieudonné module M one can associate

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its truncation $M/\epsilon M$. By a lift of a 1-truncated Dieudonné module Z we mean a Dieudonné module M together with an isomorphism $M/\epsilon M \cong Z$. To each Dieudonné module M we associate the Newton polygon obtained via covariant Dieudonné theory. Then we answer the above question by determining for a given 1-truncated Dieudonné module Z and Newton polygon $\mathcal{P}$ whether there exists a lift of Z with Newton polygon $\mathcal{P}$.

For the sake of simplicity, in this introduction we restrict ourselves to the case that $\mathcal{P}$ is the straight Newton polygon with slope $n/(n+m)$ and endpoint $(n+m, n)$ for some non-negative coprime integers n and m. For the result for arbitrary Newton polygons see Theorem 5.4. To state our result, we will need the following:

Let $h := n+m$ and $G := \mathrm{GL}_{h,\mathcal{O}}$. Let $T \subset G$ be the torus of diagonal matrices and $B \subset G$ the Borel subgroup of upper triangular matrices. Let $W \cong S_n$ be the Weyl group of G with respect to T and $S = \{(i, i+1) \mid 1 \leq i \leq h-1\}$ the generating system of W induced by B. Let $\mu \in X_*(T)$ be the cocharacter $t \mapsto (t, \ldots, t, 1, \ldots, 1)$ where t occurs with multiplicity m. Let I be the type $S \setminus \{(m, m+1)\}$. We denote by $W_I \subset W$ the subgroup generated by I and by ${}^I W \subset W$ the set of left reduced elements with respect to W_I. There exists a natural bijection between isomorphism classes of 1-truncated Dieudonné modules Z satisfying $\mathrm{rk}_k Z = h$ and $\mathrm{rk}_k F(Z) = n$ and elements of ${}^I W \subset W$ (c.f. Subsection 2.2). For $w \in {}^I W$ we denote the corresponding 1-truncated Dieudonné module by Z_w.

Let $\mathcal{I} \subset G(\mathcal{O})$ be the preimage of $B(k)$ under the projection $G(\mathcal{O}) \to G(k)$. Let $\tilde{W}$ be the extended Weyl group of G. We denote the canonical inclusion $X_*(T) \hookrightarrow \tilde{W}$ by $\lambda \mapsto \epsilon^\lambda$. For $\lambda\colon t \mapsto (t^{\lambda_1}, \ldots, t^{\lambda_h}) \in X_*(T)$ we let η_λ be the unique permutation $\eta \in W$ such that $\lambda_{\eta(1)} \leq \ldots \lambda_{\eta(h)}$ and $\eta(i) \leq \eta(i')$ for any $i \leq i'$ such that $\lambda_i = \lambda_{i'}$. Finally, we let $x_{n,m} \in \tilde{W}$ be the matrix of Frobenius on the minimal Dieudonné module $H_{n,m}$ (c.f. Definition 2.1). Then our result is:

Theorem 1.1 (c.f. Theorem 5.4)**.** *Let $w \in {}^I W$. The following are equivalent:*

(i) The 1-truncated Dieudonné module Z_w admits a lift with Newton polygon $\mathcal{P}$.

(ii) There exist $\lambda \in X_*(T)$ satisfying $\epsilon^{-\lambda} x_{n,m} \epsilon^\lambda \in W \epsilon^\mu W$ as well as $y \in W$ such that $w w_0 w_{0,I} \epsilon^\mu \in \mathcal{I} y \mathcal{I} \eta_\lambda^{-1} \epsilon^{-\lambda} x_{n,m} \epsilon^\lambda \eta_\lambda \mathcal{I} y^{-1} \mathcal{I}$.

Let $\mathcal{Z}$ denote the center of G. The group $X_*(\mathcal{Z}) \subset X_*(T)$ acts on the set

$$\{\lambda \in X_*(T) \mid \epsilon^{-\lambda} x_{n,m} \epsilon^\lambda \in W \epsilon^\mu W\}$$

by addition. By Lemma 5.5, this action has finitely many orbits. In this way the existence quantifier in *(ii)* ranges over a finite set. Hence condition *(ii)* is computable.

Now we explain our argument:

Given a isosimple Dieudonné module M of slope $n/(n+m)$, we obtain a filtration $(G^j Z)_{j \in \mathbb{Z}}$ on $Z := M/\epsilon M$ such that for all $j \in \mathbb{Z}$ we have $F(G^j Z) = G^{j+n} Z \cap F(Z)$ and $V(G^j Z) = G^{j+m} Z \cap V(Z)$ by embedding M into the minimal Dieudonné module $M_{n,m}$ (c.f. Subsection 4.2). Conversely, given such a filtration on a 1-truncated Dieudonné module Z we can construct a lift of Z which is isoclinic of slope $n/(n+m)$ (c.f. Subsection 4.3). Hence, in order to determine whether a given Z admits such a lift, it suffices to determine whether there exists such a filtration on Z, which we call a compatible filtration of type (n, m) (c.f. Subsection 4.1).

To determine whether there exists a compatible filtration on Z of type (n, m), we first consider the associated graded situation: Given a compatible filtration $(G^j Z)_{j \in \mathbb{Z}}$ of type (n, m), one obtains the graded 1-truncated Dieudonné module $\oplus_{j \in \mathbb{Z}} G^j Z / G^{j+1} Z$ on which F and V act as morphisms of degree n and m respectively. Following an idea of Chen and Viehmann, in Subsection 3.2, we classify such graded 1-truncated Dieudonné modules in terms of cocharacters $\lambda \in X_*(T)$ satisfying $\epsilon^{-\lambda} x_{n,m} \epsilon^\lambda \in W \epsilon^\mu W$.

Then, by comparing compatible filtrations to the associated gradings, we obtain the following criterion for the existence of a compatible filtration:

Theorem 1.2 (c.f. Theorem 5.3)**.** *Let M be a Dieudonné module of rank h such that M/FM has length m. The following are equivalent:*

(i) On the truncation $Z = M/\epsilon M$ there exists a compatible filtration of type (n, m).

(ii) There exists $\lambda \in X_*(T)$ satisfying $\epsilon^{-\lambda} x_{n,m} \epsilon^\lambda \in W \epsilon^\mu W$ such that the matrix of $F\colon M \to M$ with respect to some $\mathcal{O}$-basis of M lies in $\mathcal{I} \eta_\lambda^{-1} \epsilon^{-\lambda} x_{n,m} \epsilon^\lambda \eta_\lambda \mathcal{I}$.

Then by combining the above steps we obtain Theorem 1.1.

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2 Preliminaries

2.1 Setup

Throughout, we will work with the following setup and notation:

- k is an algebraically closed field of characteristic $p > 0$.
- $\mathcal{O}$ is either the Witt ring $W(k)$ or the ring $k[[t]]$.
- For $a \in k$, we let $[a] \in \mathcal{O}$ be either the canonical lift of a in $W(k)$ or the image of a under the inclusion $k \hookrightarrow k[[t]]$.
- $\epsilon \in \mathcal{O}$ is the uniformizer p or t accordingly.
- L is the function field of $\mathcal{O}$.
- $v\colon L \to \mathbb{Z}$ is the valuation normalized such that $v(\epsilon) = 1$.
- $\sigma\colon \mathcal{O} \to \mathcal{O}$ is either the canonical lift $W(k) \to W(k)$ of Frobenius or the automorphism $k[[t]] \to k[[t]]$ fixing t and sending $a \in k$ to a^p.
- A Dieudonné module is a finite free $\mathcal{O}$-module together with a σ-linear endomorphism F and a σ^{-1}-linear endomorphism V satisfying $FV = VF = \epsilon$. (In the equicharacteristic case, such an object is usually called an effective and minuscule local GL_h-shtuka.)
- $F_k\colon k \to k,\ x \mapsto x^p$ is the Frobenius automorphism.
- A 1-truncated Dieudonné module is a finite-dimensional k-vector space together with an F_k-linear endomorphism F and an F_k^{-1}-linear endomorphism V such that $\operatorname{im} F = \ker V$ and $\ker V = \operatorname{im} V$.
- To a Dieudonné module M we associate the 1-truncated Dieudonné module $M/\epsilon M$.
- By the Newton polygon of a Dieudonné module M we mean the Newton polygon obtained via covariant Dieudonné theory. That is a Dieudonné module is isoclinic of slope r/s for integers $r, s \geq 0$ if and only if it is isogenous to a Dieudonné module on which $\epsilon^{-r}F^s$ is an automorphism.
- We write Newton polygons in the form $\mathcal{P} = (\nu_1, \ldots, \nu_N)$ where $\nu_1 \leq \ldots \leq \nu_N$ are the slopes occurring in $\mathcal{P}$ with multiplicities.
- For $\nu \in \mathbb{Q}^{\geq 0}$ we denote by n_ν and m_ν the unique non-negative coprime integers such that $\nu = n_\nu/(n_\nu + m_\nu)$.

We will often work with respect to given integers $0 \leq d \leq h$. Then we use the following:

- G is the group scheme $\mathrm{GL}_{h,\mathcal{O}}$.
- $T \subset G$ is the canonical torus of diagonal matrices.
- $B \subset G$ is the canonical Borel subgroup of upper triagonal matrices.
- $\mathcal{I} \subset G(\mathcal{O})$ is the preimage of $B(k)$ under the projection $G(\mathcal{O}) \to G(k)$.
- $G(\mathcal{O})_1$ is the kernel of the projection $G(\mathcal{O}) \to G(k)$.
- $W \cong S_h$ is the Weyl group of G with respect to T which we identify with the set of monomial matrices with entries in $\{0, 1\}$ in either $G(k)$ or $G(\mathcal{O})$.
- $S = \{(i, i+1) \mid 1 \leq i \leq h-1\} \subset W$ is the set of simple reflections induced by B.
- $I \subset S$ is the type $S \setminus \{(h-d-1, h-d)\}$.
- $W_I \subset W$ is the subgroup generated by I.
- ${}^IW$ is the set of left reduced elements with respect to I, that is the set of elements w which have minimal length in $W_I w$.

- w_0 is the longest element in W.
- $w_{0,I}$ is the longest element in W_I.
- We denote by $\tilde{W} \cong X_*(T) \rtimes W$ the extended Weyl group of G, which we identify with the group of monomial matrices in $G(\mathcal{O})$ with entries in $\{0\} \cup p^{\mathbb{Z}}$.
- For $\lambda \in X_*(T)$, we denote by $\epsilon^\lambda := \lambda(p)$ its image in $\tilde{W}$.
- We denote the cocharacter $\lambda \in X_*(T)$ which sends $t \in \mathbb{G}_m$ to the diagonal matrix with entries $(t^{\lambda_1}, \ldots, t^{\lambda_h})$ by $(\lambda_1, \ldots, \lambda_h)$.
- $\mu \in X_*(T)$ is the cocharacter $(1, \ldots, 1, 0, \ldots, 0)$ where the entry 1 has multiplicity $h - d$.
- We say that a Dieudonné module M has Hodge polygon given by μ if $\operatorname{rk}_{\mathcal{O}} M = h$ and M/FM has length d.
- We denote again by σ the automorphism of $G(\mathcal{O})$ induced by $\sigma \colon \mathcal{O} \to \mathcal{O}$.
- To an element $g \in G(\mathcal{O})\epsilon^\mu G(\mathcal{O})$ we associate the Dieudonné module $M_g := (\mathcal{O}^h, g\sigma)$. This gives a bijection between $G(\mathcal{O})$-σ-conjugacy classes in $G(\mathcal{O})\epsilon^\mu G(\mathcal{O})$ (i.e. orbits under the action $G(\mathcal{O}) \times G(\mathcal{O}) \to G(\mathcal{O}), (g,h) \mapsto gh\sigma(h)^{-1}$) and isomorphism classes of Dieudonné modules with Hodge polygon given by μ.

2.2 Classification of 1-truncated Dieudonné modules

Fix integers $0 \leq d \leq h$. We call a 1-truncated Dieudonné module Z *of numerical type* (d,h) ifit satisfies $\operatorname{rk}_k Z = h$ and $\operatorname{rk}_k F(Z) = h - d$. Any 1-truncated Dieudonné module of numerical type (d,h) can be lifted to a Dieudonné module with Hodge polygon given by μ. Furthermore, one can check that for two elements $g_1, g_2 \in G(\mathcal{O})\epsilon^\mu G(\mathcal{O})$ the truncations $M_{g_1}/\epsilon M_{g_1}$ and $M_{g_2}/\epsilon M_{g_2}$ are isomorphic as 1-truncated Dieudonné modules if and only if g_2 is $G(\mathcal{O})$-σ-conjugate to an element of $\mathrm{G}(\mathcal{O})_1\epsilon^\mu G(\mathcal{O})_1$. Hence isomorphism classes of 1-truncated Dieudonné modules of numerical type (d,h) correspond to the $\mathrm{G}(\mathcal{O})$-σ-conjugacy classes in $G(\mathcal{O})_1\backslash G(\mathcal{O})\epsilon^\mu G(\mathcal{O})/G(\mathcal{O})_1$. By [Vie, Theorem 1.1] the set $\{ww_0w_{0,I}\epsilon^\mu \mid w \in {}^I W\}$ gives a set of representatives for these conjugacy classes. Thus the 1-truncated Dieudonné modules $Z_w := M_{ww_0w_{0,I}\epsilon^\mu}/\epsilon M_{ww_0w_{0,I}\epsilon^\mu}$ for $w \in {}^I W$ are representatives for the isomorphism classes of 1-truncated Dieudonné modules.

2.3 Minimal Dieudonné modules

For coprime non-negative integers n and m, the minimal Dieudonné module $H_{n,m}$ of slope $n/(n+m)$ is defined as follows (c.f. [Oor1]): It is the free $\mathcal{O}$-module with basis $e_1, \ldots, e_{n+m}$. For $i > n+m$, we write $i = a(n+m) + b$ for unique integers $a > 1$ and $1 \leq b \leq n+m$ and define $e_i := \epsilon^a e_b$. Then F and V are defined by $F(e_i) = e_{i+n}$ and $V(e_i) = e_{i+m}$ for all $i \geq 1$.

Let Φ be the σ-semilinear automorphism of $H_{n,m}$ which fixes the e_i. Then $\Phi\pi = \pi\Phi$, $F = \Phi\pi^n$ and $V = \Phi^{-1}\pi^m$.

Definition 2.1. Let n and m be coprime non-negative integers. We define $x_{n,m} \in \tilde{W}$ to be the matrix of $F \colon H_{n,m} \to H_{n,m}$ with respect to the basis $(e_h, \ldots, e_1)$.

3 Graded 1-truncated Dieudonné modules

Throughout this section we fix coprime non-negative integers n and m and let $h := n+m$ and $d := n$.

By a grading of a vector space we will always mean a $\mathbb{Z}$-grading. For a graded vector space $X = \oplus_{j\in\mathbb{Z}} X^j$ we will call the elements of the X^j the homogenous elements of X. For $i \in \mathbb{Z}$, we say that an additive homomorphism $X \to X'$ between graded vector spaces is *of degree* i if it sends every homogenous element of degree j to a homogenous element of degree $j+i$.

Definition 3.1. A *graded* 1*-truncated Dieudonné module* is a 1-truncated Dieudonné module Z together with a grading $Z = \oplus_{j\in\mathbb{Z}} Z^j$ such that F and V send homogenous elements of Z to homogenous elements.

A morphism of graded 1-truncated Dieudonné modules is a morphism of 1-truncated Dieudonné modules of degree zero.

Definition 3.2. A *graded 1-truncated Dieudonné module of type (n,m)* over k is a graded 1-truncated Dieudonné module (Z,F,V) such that F is of degree n and V is of degree m.

Lemma 3.3. *Let $Z = \oplus_{j\in\mathbb{Z}} Z^j$ be a graded 1-truncated Dieudonné module of type (n,m). There exists an integer c such that $\mathrm{rk}_k Z = c(n+m)$ and such that for every $j \in \mathbb{Z}$ we have*

$$\sum_{i\equiv j \quad (n+m)} \mathrm{rk}_k Z^i = c.$$

Proof. For $j \in \mathbb{Z}$ let $Z(j) := \oplus_{i\equiv j \ (n+m)} Z^i$. The fact that Z is graded of type (n,m) implies that for each j we have a short exact sequence

$$0 \to Z(j-m)/(Z(j-m)\cap F(Z)) \xrightarrow{V} Z(j) \xrightarrow{F} Z(j+n)\cap F(Z) \to 0.$$

Using $Z(j-m) = Z(j+n)$ this implies $\mathrm{rk}_k Z(j) = \mathrm{rk}_k Z(j+n)$. Since n and $n+m$ are coprime, iterating this fact yields the claim.

□

3.1 Classification in terms of semimodules

Definition 3.4 (c.f. [Oor2, (1.7)] and [dJO, Section 6])**.** A *beginning of a semi-module of type (n,m)* is a subset $C \subset \mathbb{Z}$ such that for each $i \in \mathbb{Z}$ the equivalence class $i + (n+m)\mathbb{Z}$ contains exactly one element of C and for each $i \in C$ either $i+n \in C$ or $i-m \in C$.

Lemma 3.5. *Let $Z = \oplus_{j\in\mathbb{Z}} Z^j$ a graded 1-truncated Dieudonné module of type (n,m) of rank h. Then $C_Z := \{j \in \mathbb{Z} \mid Z^j \neq 0\}$ is a beginning of a semi-module of type (n,m).*

Proof. This follows from the definition of 1-truncated Dieudonné modules of type (n,m) together with Lemma 3.3. □

Construction 3.6. Let C be a beginning of a semi-module of type (n,m). We construct a graded 1-truncated Dieudonné module Z_C of type (n,m) and of rank h as follows:

Let Z be the free k-vector space with basis $(e_j)_{j\in C}$. Endow Z_C with the grading for which each e_j is homogenous of degree j. We define F and V as follows: Let $j \in C$. If $j+n \in C$ we let $F(e_j) := e_{j+n}$ and $V(e_{j+n}) := 0$. Otherwise $j-m \in C$ and we let $V(e_{j-m}) := e_j$ and $F(e_j) = 0$. Then by a direct verification Z_C has the required properties.

Proposition 3.7. *The assignments $Z = \oplus_{j\in\mathbb{Z}} Z^j \mapsto C_Z$ and $C \mapsto Z_C$ give mutually inverse bijections between the set of isomorphism classes of 1-truncated Dieudonné modules of type (n,m) and of rank h and the set of beginnings of semi-modules of type (n,m).*

Proof. The identity $C = C_{Z_C}$ follows directly from the definition of Z_C.

It remains to prove that each Z is isomorphic to Z_{C_Z} as a graded 1-truncated Dieudonné module. To see this, start with an element $j_0 \in C_Z$ and a non-zero element $f_0 \in Z^{j_0}$. We iteratively construct a sequence of pairs $(j_s \in C_Z, f_s \in Z^{j_s}\setminus\{0\})$ as follows: If $j_k+n \in C$ we let $j_{k+1} := j_k+n$ and $f_{j+1} := F(f_j)$. Otherwise we let $j_{k+1} := j_k - m$ and $f_{j+1} \in Z^{j_k-m}$ the unique element such that $V(f_{j+1}) = f_j$.

By construction, for $s \geq 0$, the element $j_s \in C_Z$ is the unique element of C_Z in $j_s+sn+h\mathbb{Z}$. Thus $j_h = j_0$ and hence $f_h = \lambda f_0$ for some $\lambda \in k^*$. Pick $\mu \in k^*$ such that $\mu^{p^{m-n}}\lambda = \mu$. In C_Z there a m elements j satisfying $j+n \in C_Z$ and n elements j satisfying $j-m \in C_Z$ (c.f. [dJO, Section 6]). Hence by replacing f_0 by μf_0 in the above construction we obtain a sequence such that $f_h = f_0$. Then for each $j \in C_Z$ we let $e_j := f_k$ for the unique $0 \leq k < h$ such that $j_k = j$. The resulting basis $(e_j)_{j\in C_Z}$ of Z gives an isomorphism $Z \cong Z_{C_Z}$ of graded 1-truncated Dieudonné modules. □

3.2 Classification in terms of cocharacters

Now we show that 1-truncated Dieudonné modules of type (n,m) and rank h can also be classified by certain cocharacters $\lambda \in X_*(T)$. The idea behind this classification is due to Chen and Viehmann (c.f. [CV]).

Construction 3.8. Let $\lambda = (\lambda_1, \ldots, \lambda_h) \in X_*(T)$ be a cocharacter satisfying $\epsilon^{-\lambda} x_{n,m} \epsilon^{\lambda} \in W\epsilon^{\mu}W$. We construct a graded 1-truncated Dieudonné module of type (n, m) as follows: As in Definition 2.1, we consider the Dieudonné module $M_{n,m}$ with the basis $(e_{n+m}, \ldots, e_1)$. For $1 \leq j \leq h$ let $f_j := \epsilon^{\lambda_j} e_{h+1-j}$. The f_j form a $\mathcal{O}$-basis of a submodule $M \subset M_{n,m}$ and the matrix of F with respect to this basis is $\epsilon^{-\lambda} x_{n,m} \epsilon^{\lambda}$. Hence the assumption $\epsilon^{-\lambda} x_{n,m} \epsilon^{\lambda} \in W\epsilon^{\mu}W$ means that M is a sub-Dieudonné module of $M_{n,m}$ with Hodge polygon given by μ. Let $Z := M/\epsilon M$ with basis $(\bar{f}_j := f_j + \epsilon M)_{1 \leq j \leq h}$. Equipping Z with the grading for which each $\bar{f}_j$ is homogenous of degree $h+1-j+h\lambda_j$ makes Z into a graded 1-truncated Dieudonné module of type (n, m) which we denote by Z_λ.

Proposition 3.9. *The assignment $\lambda \mapsto Z_\lambda$ gives a bijection from the set*

$$\{\lambda \in X_*(T) \mid \epsilon^{-\lambda} x_{n,m} \epsilon^{\lambda} \in W\epsilon^{\mu}W\}$$

to the set of isomorphism classes of 1-truncated Dieudonné modules of type (n, m) and rank h.

Proof. Let Z be a 1-truncated Dieudonné module of type (n, m) and rank h. By Proposition 3.7 we may assume that $Z = Z_C$ for some beginning of a semi-module C. For each $1 \leq j \leq h$ let λ_j be the unique integer such that $h + 1 - j + h\lambda_j \in C$. Let M be the the sub-$\mathcal{O}$-module of $H_{n,m}$ spanned by $\{e_j \mid j \in C\} = \{\epsilon^{\lambda_j} e_{h+1-j} \mid 1 \leq j \leq h\}$. The fact that C is the beginning of a semi-module of type (n, m) implies that M is a sub-Dieudonné module of $H_{n,m}$. Furthermore, the assignment $e_j \in M \mapsto e_j \in Z_C$ for $i \in C$ induces an isomorphism $M/\epsilon M \cong Z_C$ of 1-truncated Dieudonné modules. This implies that M has Hodge polygon given by μ which in turn is equivalent to $\epsilon^{-\lambda} x_{n,m} \epsilon^{\lambda} \in W\epsilon^{\mu}W$. It follows from the above that $Z_C \cong Z_\lambda$ as graded 1-truncated Dieudonné modules. Thus the map in question is surjective. As for the injectivity, it follows directly from Construction 3.8 that λ can be recovered from the grading on Z_λ. □

4 Compatible filtrations on 1-truncated Dieudonné modules

4.1 Definitions

By a decreasing filtration $(G^j X)_{j \in \mathbb{Z}}$ on a finite-dimensional vector space X we mean a family of subspaces such that $G^j X \supset G^{j+1} X$ for all $j \in \mathbb{Z}$, such that $G^j X = X$ for all small enough j and such that $G^j X = 0$ for all large enough j. Given two descending filtrations $(G^j X)_{j \in \mathbb{Z}}$ and $(\mathrm{G}^j X')_{j \in \mathbb{Z}}$ on two such vector spaces X and X' and an integer i, we call an additive homomorphism $h \colon X \to X'$ filtered of degree i if $h(G^j X) \subset G^{j+i} X$ for all $j \in \mathbb{Z}$.

Lemma 4.1. *Let n and m be coprime non-negative integers and Z a 1-truncated Dieudonné module over k. Let $(G^j Z)_{j \in \mathbb{Z}}$ a descending filtration on Z such that F is filtered of degree n and such that V is filtered of degree m. The following two conditions are equivalent:*

(i) The vector space $\operatorname{gr} Z := \oplus_j G^j Z / G^{j+1} Z$ together with the graded semilinear endomorphisms of degree n and m induced by F and V is a graded 1-truncated Dieudonné module of type (n, m).

(ii) For all $j \in \mathbb{Z}$ we have $F(G^j Z) = G^{j+n} Z \cap F(Z)$ and $V(G^j Z) = G^{j+m} Z \cap V(Z)$.

Proof. This follows from a direct verification. □

Definition 4.2. Let n and m be coprime non-negative integers and Z a 1-truncated Dieudonné module over k. A *compatible filtration of type (n, m)* on Z is a decreasing filtration $E = (\mathrm{G}^j Z)_{j \in \mathbb{Z}}$ by k-submodules such that F is filtered of degree n, such that V is filtered of degree m and such that the equivalent conditions of Lemma 4.1 are satisfied.

For such an E, we denote by $\operatorname{gr}_E(Z)$ the associated graded 1-truncated Dieudonné module from Lemma 4.1.

Example 4.3. Let n and m be coprime non-negative integers and $Z = \oplus_{i \in \mathbb{Z}} Z^i$ a graded 1-truncated Dieudonné module of type (n, m). Then the filtration E given by $G^j(Z) := \oplus_{i \geq j} Z^i$ is a compatible filtration of type (n, m). The associated graded 1-truncated Dieudonné module $\operatorname{gr}_E Z$ is canonically isomorphic to Z.

Definition 4.4. Let $\mathcal{P} = (\nu_1, \dots, \nu_N)$ a Newton polygon. Let Z be a 1-truncated Dieudonné module. A *compatible filtration with Newton polyon* $\mathcal{P}$ on Z is a filtration $0 = Z_0 \subset Z_1 \dots \subset Z_N = Z$ by sub-1-truncated Dieudonné modules such that the subquotients Z_i/Z_{i-1} are 1-truncated Dieudonné modules of rank $n_{\nu_i} + m_{\nu_i}$ together with compatible filtrations E_i on the Z_i/Z_{i-1} of type (n_{ν_i}, m_{ν_i}).

4.2 Compatible filtrations associated to Dieudonné modules

In this subsection, for a Dieudonné module M with Newton polygon $\mathcal{P}$ we construct a compatible filtration with Newton polygon $\mathcal{P}$ on $M/\epsilon M$. The idea behind this construction is originally due to Manin (c.f. [Man, Section III.5]) and was also used by de Jong and Oort in [dJO] and by Oort in [Oor2].

Construction 4.5. Let n and m be coprime non-negative integers. Let M be an isosimple Dieudonné module of slope $n/(n+m)$. We define a compatible filtration of type (n, m) on the 1-truncated Dieudonné module $Z := M/pM$ as follows:

By the slope assumption there exists an embedding $M \hookrightarrow H_{n,m}$. We choose such an embedding and let $M^j := M \cap \pi^j H_{n,m}$ for all $j \geq 0$. The fact that $F = \pi^n \Phi$ and $V = \pi^m \Phi^{-1}$ on $H_{n,m}$ implies that $F(M^j) = M^{j+n} \cap F(M)$ and $V(M^j) = M^{j+m} \cap V(M)$ for all $j \in \mathbb{Z}$. These two identities imply that $G^j(Z) := (M^j + \epsilon M)/\epsilon M \subset Z$ defines a compatible filtration E_M of type (n, m) on Z. Since M is isosimple, the vector spaces Z and $\operatorname{gr}_E Z$ have rank $n + m$.

Remark 4.6. By [dJO, Section 5.6] a different choice of embedding $M \hookrightarrow H_{n,m}$ in Construction 4.5 yields to a filtration which differs from the given one only by a shift of the indexing of the filtration.

Construction 4.7. Let M be a Dieudonné module and $Z := M/\epsilon M$. Let $\mathcal{P}$ be the Newton polygon of M. We define a compatible filtration on Z as follows: We start with the slope filtration of M (c.f. e.g. [Zin, Corollary 13]) and refine it to a filtration $0 = M_0 \subset M_1 \subset \dots \subset M_N = M$ by sub-Dieudonné modules such that each M_i/M_{i-1} is isosimple. For $0 \leq i \leq N$ let $Z_i := M_i/\epsilon M_i$. Then Construction 4.5 applied to the Dieudonné modules M_i/M_{i-1} yields compatible filtrations E_i on $Z_i/Z_{i-1} \cong (M_i/M_{i-1})/\epsilon(M_i/M_{i-1})$. Alltogether we obtain a compatible filtration with Newton polygon $\mathcal{P}$.

4.3 Lifts associated to compatible filtrations

In Construction 4.7, we associate to each Dieudonné module M a compatible filtration E_M on $M/\epsilon M$ with the same Newton polygon as M. In this subsection we show that conversely, for each 1-truncated Dieudonné module Z together with a compatible filtration E on Z with Newton polygon $\mathcal{P}$ there exists a Dieudonné module M lifting Z which has Newton polygon $\mathcal{P}$.

Construction 4.8. Let $\mathcal{P} = (\nu_1, \dots, \nu_N)$ be a Newton polygon. Let Z be a 1-truncated Dieudonné module and $E = ((Z_i)_{0 \leq i \leq N}, (E_i)_{1 \leq i \leq N})$ a compatible filtration with Newton polygon $\mathcal{P}$ on Z. We construct a Dieudonné module M lifting Z as follows:

For each $1 \leq i \leq N$ let C_i be the beginning of a semi-module of type $(n_i, m_i) := (n_{\nu_i}, m_{\nu_i})$ associated to $\operatorname{gr}_{E_i}(Z_i/Z_{i-1})$. By Proposition 3.7 we can choose isomorphisms $\operatorname{gr}_{E_i}(Z_i/Z_{i-1}) \cong Z_{C_i}$ of graded 1-truncated Dieudonné modules and hence obtain bases $(e^i_j)_{j \in C_i}$ of the $\operatorname{gr}_{E_i}(Z_i/Z_{i-1})$. In the following by a pair (i, j) we always mean such a pair satisfying $1 \leq i \leq N$ and $j \in C_i$. For each pair (i, j) let $f^i_j \in Z_i$ be a lift of e^i_j.

Let M be the free $\mathcal{O}$-module with basis $(g^j_i)_{(i,j)}$. We make M into a Dieudonné module by defining the image of g^i_j under F and V by a nested double induction, with the outer induction being increasing on i and the inner induction being decreasing on j. For pairs (i, j) and (i', j') we let $(i, j) \prec (i', j')$ if and only if either the conditions $i = i'$ and $j > j'$ or the condition $i < i'$ is satisfied.

First we define F: Consider a pair (i, j). If $j + n_i \in C_i$ then

$$F(f^i_j) = f^i_{j+n_i} + \sum_{(i',j') \prec (i,j+n_i)} a^{i'}_{j'} f^{i'}_{j'}$$

for certain $a^{i'}_{j'} \in k$. Then we let

$$F(g^i_j) := g^i_{j+n_i} + \sum_{(i',j') \prec (i,j+n_i)} [a^{i'}_{j'}] g^{i'}_{j'}.$$

Otherwise we have $j - m_i \in C_i$ and

$$f^i_j = V(f^i_{j-m_i}) + \sum_{(i',j') \prec (i,j)} b^{i'}_{j'} f^{i'}_{j'}$$

for certain $b^{i'}_{j'} \in k$. In this case we define

$$F(f^i_j) := \epsilon g^i_{j-m_i} + \sum_{(i',j') \prec (i,j)} [(b^{i'}_{j'})^p] F(g^{i'}_{j'}),$$

where the terms $F(g^{i'}_{j'})$ appearing are already defined by induction.

We define V dually: Consider a pair (i,j). If $j + m_i \in C_i$ then

$$V(f^i_j) = f^i_{j+m_i} + \sum_{(i',j') \prec (i,j+m_i)} c^{i'}_{j'} f^{i'}_{j'}$$

for certain $c^{i'}_{j'} \in k$. Then we let

$$V(g^i_j) := g^i_{j+m_i} + \sum_{(i',j') \prec (i,j+m_i)} [c^{i'}_{j'}] g^{i'}_{j'}.$$

Otherwise we have $j - n_i \in C_i$ and

$$f^i_j = F(f^i_{j-n_i}) + \sum_{(i',j') \prec (i,j)} d^{i'}_{j'} f^{i'}_{j'}$$

for certain $d^{i'}_{j'} \in k$. In this case we define

$$V(f^i_j) := \epsilon g^i_{j-n_i} + \sum_{(i',j') \prec (i,j)} [(d^{i'}_{j'})^{-p}] V(g^{i'}_{j'}),$$

where the terms $V(g^{i'}_{j'})$ appearing are already defined by induction.

We extend F and V to a σ- respectively a σ^{-1}-linear endomorphism of M.

Lemma 4.9. *Let M be a Dieudonné module and n and m non-negative integers such that the Newton polygon of M has endpoint $(cn, c(n+m))$ for some integer $c \geq 0$. Assume that there exists a function $v \colon M \setminus \{0\} \to \mathbb{Z}^{\geq 0}$ with the following properties:*

(i) $v(F(x)) = v(x) + n$ for all $x \in M$.

(ii) $v(V(x)) = v(x) + m$ for all $x \in M$.

Then M is isoclinic of slope $n/(n+m)$.

Proof. Let ν be a slope of M. There exists a non-zero Dieudonné submodule M' of M such that for all integers $a \geq 0$ we have $F^{a(n_\nu+m_\nu)}M' = p^{an_\nu}M'$. Let x be a non-zero element of M'. For some integer $a \geq 0$, write $F^{a(n_\nu+m_\nu)}(x) = p^{an_\nu}(x')$ for some $x' \in M'$. Then we get:

$$v(x) + a(n_\nu + m_\nu)n = v(F^{a(n_\nu+m_\nu)}(x)) = v(p^{an_\nu}(x')) = v(x') + an_\nu(n+m) \geq an_\nu(n+m)$$

By letting a go to infinity this inequality implies $\nu = n_\nu/(n_\nu + m_\nu) \leq n/(n+m)$. From this the claim follows by comparing the Newton polygon of M to the constant Newton polygon of slope $n/(n+m)$ with the same endpoint.

□

Proposition 4.10. *Let Z and E be as in Construction 4.8. For each $1 \leq i \leq N$ let $M_i \subset M$ be the $\mathcal{O}$-submodule spanned by $\{g^{i'}_j \mid i' \leq i, j \in C_{i'}\}$.*

(i) The module M from Construction 4.8 is a Dieudonné module, i.e. $FV = VF = \epsilon$.

(ii) *The assigment $g_j^i+\epsilon M \mapsto f_j^i$ gives an isomorphism $M/\epsilon M \cong Z$ of 1-truncated Dieudonné modules.*

(iii) *The M_i are Dieudonné submodules of M.*

(iv) *For each $1 \leq i \leq N$, the Dieudonné module M_i/M_{i-1} is isoclinic of slope $n_i/(n_i+m_i)$.*

(v) *The Dieudonné module M has Newton polygon $\mathcal{P}$.*

Proof. (i), (ii) and (iii) follow from the definition of M by the same double induction as in Construction 4.8.

(iv): We continue to use the notation from Construction 4.8. For $j \in C_i$ we denote $f_j^i + M_{i-1}$ by $\bar{f}_j^i$. These elements form a $\mathcal{O}$-basis of M_i/M_{i-1}. We define a function

$$v\colon M_i/M_{i-1} \setminus \{0\} \to \mathbb{Z}^{\geq 0}$$

by

$$v(\sum_{j\in C_i} a_j \bar{f}_j^i) := \min_{j\in C_i}((n_i+m_i)v(a_j)+j).$$

It follows from the definition of M that v satisfies the conditions of Lemma 4.9 for $n = n_i$ and $m = m_i$. Thus (iv) follows from Lemma 4.9.

(v) follows from (iv).

□

5 Existence of compatible flags

Let $\mathcal{P} = (\nu_1, \dots, \nu_N)$ be a Newton polygon. For $1 \leq i \leq N$ we denote (n_{ν_i}, m_{ν_i}) by (n_i, m_i) and let $h_i := n_i + m_i$ and $d_i := m_i$. For such i we let $G_i, T_i, W_i, \mathcal{I}_i, \mu_i$, etc., be the data from Subsection 2.1 associated to $(h,d) = (h_i, d_i)$. Let $h = \sum_i h_i$ and $\prod_{1\leq i\leq N} G_i \cong H \subset G = \mathrm{GL}_h$ be the Levi subgroup containing T corresponding to the decomposition $h = h_1 + \ldots + h_N$. We denote by $\tilde{W}_H := H(W(k)) \cap \tilde{W}$ (resp. W_H) the extended Weyl group (resp. the Weyl group) of H. Let $d := \sum_{1\leq i\leq N} n_i$.

Definition 5.1. Let $\lambda \in X_*(T)$. There is a unique permutation $\eta \in W_H$ with the following properties:

(i) For each $1 \leq i \leq N$ we have $\lambda_{\eta(h_1+\ldots+h_{i-1}+1)} \leq \lambda_{\eta(h_1+\ldots+h_{i-1}+2)} \leq \ldots \leq \lambda_{\eta(h_1+\ldots+h_i)}$.

(ii) For each $1 \leq j, j' \leq h$ such that $\lambda_j = \lambda_{j'}$ we have $j < j'$ if and only if $\eta(j) < \eta(j')$.

We denote this permutation η by η_λ.

Definition 5.2. Let $x_{\mathcal{P}} \in \tilde{W}_H$ be the matrix whose i-th block is given by x_{n_i,m_i} for each $1 \leq i \leq N$.

Theorem 5.3. *Let M be a Dieudonné module with Hodge polygon given by μ. The following are equivalent:*

(i) *On the truncation $Z = M/\epsilon M$ there exists a compatible filtration with Newton polygon $\mathcal{P}$.*

(ii) *There exists $\lambda \in X_*(T)$ satisfying $\epsilon^{-\lambda} x_{\mathcal{P}} \epsilon^{\lambda} \in W\epsilon^{\mu}W$ such that the matrix of $F\colon M \to M$ with respect to some $\mathcal{O}$-basis of M lies in $\mathcal{I}\eta_\lambda^{-1}\epsilon^{-\lambda}x_{\mathcal{P}}\epsilon^{\lambda}\eta_\lambda\mathcal{I}$.*

Proof. Using σ-conjugation by elements of $G(\mathcal{O})$, which amounts to base change on M, one sees that (ii) is equivalent to saying that there exists such a λ such that the matrix of F with respect to some $\mathcal{O}$-basis of M lies in ${}^{n_\lambda}\mathcal{I}\epsilon^{-\lambda}x_{\mathcal{P}}\epsilon^{\lambda}$.

$(i) \Rightarrow (ii)$: Let $E = ((Z_i)_{0\leq i\leq N}, (E_i)_{1\leq i\leq N})$ be a compatible filtration of Newton polygon $\mathcal{P}$ on Z. Fix $1 \leq i \leq N$. By Proposition 3.9 there exists $\lambda^i \in X_*(T_i)$ satisfying $\epsilon^{-\lambda^i} x_{n_i,m_i} \epsilon^{\lambda^i} \in W_i \epsilon^{\mu_i} W_i$ such that $\mathrm{gr}_{E_i}(Z_i/Z_{i-1}) \cong Z_{\lambda^i}$. Let M^i and $(f_j^i)_{1\leq j\leq h_i}$ be the Dieudonné module together with its $\mathcal{O}$-basis from Construction 3.8 applied to $\lambda = \lambda^i$ such that $Z_{\lambda^i} = M^i/\epsilon M^i$ and the matrix of $F\colon M^i \to M^i$ with respect to $(f_j^i)_{1\leq j\leq h_i}$ is $\epsilon^{-\lambda^i} x_{n_i,m_i} \epsilon^{\lambda^i}$. Fix an isomorphism $M^i/\epsilon M^i \cong \mathrm{gr}_{E_i}(Z_i/Z_{i-1})$ and let $(\bar{f}_j^i)_{1\leq j\leq h_i}$ be the image of $(f_j^i)_{1\leq j\leq h_i}$ in $\mathrm{gr}_{E_i}(Z_i/Z_{i-1})$. Let M_i be the preimage of Z_i in M and for $1 \leq j \leq h_i$ let $\tilde{f}_j^i$ be lift of $\bar{f}_j^i$ to Z_i and g_j^i a lift of $\tilde{f}_j^i$ to M_i.

By comparing the definition of Z_{λ^i} and η_{λ^i} one sees that the subspaces appearing in the filtration E_i on Z_i/Z_{i-1} are those of the form $\sum_{1\leq j'\leq j} k\tilde{f}^i_{\eta_{\lambda^i}(j')} + Z_{i-1}$ for $1 \leq j \leq h_i$. This together with the fact that the matrix of $F\colon M^i \to M^i$ with respect to $(f^i_j)_{1\leq j\leq h_i}$ is $\epsilon^{-\lambda_i} x_{n_i,m_i}\epsilon^{\lambda_i}$ implies that the matrix of $F\colon M_i/M_{i-1} \to M_i/M_{i-1}$ with respect to the basis $(g^i_j)_{1\leq j\leq h_i}$ lies in ${}^{\eta_{\lambda_i}}\mathcal{I}\epsilon^{-\lambda_i}x_{n_i,m_i}\epsilon^{\lambda_i}$.

Now let $\lambda \in X_*(T)$ be the cocharacter whose factor in the i-th block of H is given by λ^i for each $1 \leq i \leq N$. From the definition of $x_{\mathcal{P}}$ and the corresponding property of the λ^i it follows that $\epsilon^{-\lambda}x_{\mathcal{P}}\epsilon^{\lambda} \in W\epsilon^{\mu}W$. Furthermore, from the definition of η_λ and the above it follows that the matrix of $F\colon M \to M$ with respect to the $\mathcal{O}$-basis $(f^i_j)_{i,j}$ lies in ${}^{\eta_\lambda}\mathcal{I}\epsilon^{-\lambda}x_{\mathcal{P}}\epsilon^{\lambda}$. This proves (ii).

$(ii) \Rightarrow (i)$: We reverse the above arguments: By assumption there exists a $\mathcal{O}$-basis of M with respect to which the matrix of F lies in ${}^{\eta_\lambda}\mathcal{I}\epsilon^{-\lambda}x_{\mathcal{P}}\epsilon^{\lambda}$. Write such a basis as $(f^1_1, f^1_2, \ldots, f^1_{h_i}, f^2_1, \ldots, f^N_{h_N})$. For $1 \leq i \leq N$ let $M_i := \sum_{i'\leq i,j} \mathcal{O}f^{i'}_j$ and Z_i the image of M_i in Z. The form of the matrix of F with respect to the basis $(f^i_j)_{(i,j)}$ implies that $F(M_i) \subset M_i$ for each i. Fix $1 \leq i \leq N$. Let λ^i (resp. η_{λ^i}) be the part of λ (resp. η_λ) in G_i. Then the matrix of F on M_i/M_{i-1} with respect to $(g^i_j)_{1\leq j\leq h_i}$ lies in ${}^{\eta_{\lambda^i}}\mathcal{I}_i\epsilon^{-\lambda^i}x_{n_i,m_i}\epsilon^{\lambda^i}$ which proves that M_i/M_{i-1} is a Dieudonné module with Hodge polygon given by μ_i and hence that Z_i/Z_{i-1} is a 1-truncated Dieudonné module of rank h_i.

For $1 \leq j \leq h_i$ let $\tilde{f}^i_j$ be the image of g^i_j in Z_i. As above we consider the graded 1-truncated Dieudonné module Z_{λ^i} with its canonical basis $(\bar{f}^i_j)_{1\leq i\leq j}$. Let $(G^j(Z_{\lambda^i}))_{j\in\mathbb{Z}}$ be the canonical filtration of type (n_i, m_i) associated to the grading on Z_{λ^i}. For $j \in \mathbb{Z}$ define $G^j(Z_i/Z_{i-1}) := \sum_{\{j'\colon \bar{f}^i_{j'}\in G^j(Z_{\lambda^i})\}} k\tilde{f}^i_{j'}$. Similar to the above one checks by comparison with Z_{λ^i} that this defines a compatible filtration E_i of type (n_i, m_i) on Z_i/Z_{i-1}. Alltogether we have constructed a compatible filtration with Newton polygon $\mathcal{P}$ on Z. □

Now we can prove our main result:

Theorem 5.4. *Let $w \in {}^IW$. The following are equivalent:*

(i) The 1-truncated Dieudonné module Z_w admits a lift with Newton polygon $\mathcal{P}$.

(ii) On Z_w there exists a compatible filtration with Newton polygon $\mathcal{P}$.

(iii) There exists $\lambda \in X_*(T)$ satisfying $\epsilon^{-\lambda}x_{\mathcal{P}}\epsilon^{\lambda} \in W\epsilon^{\mu}W$ such that $ww_0w_{0,I}\epsilon^{\mu}$ is $G(\mathcal{O})$-σ-conjugate to an element of $\mathcal{I}\eta_\lambda^{-1}\epsilon^{-\lambda}x_{\mathcal{P}}\epsilon^{\lambda}\eta_\lambda\mathcal{I}$.

(iv) There exist $\lambda \in X_*(T)$ satisfying $\epsilon^{-\lambda}x_{\mathcal{P}}\epsilon^{\lambda} \in W\epsilon^{\mu}W$ as well as $y \in W$ such that $ww_0w_{0,I}\epsilon^{\mu} \in \mathcal{I}y\mathcal{I}\eta_\lambda^{-1}\epsilon^{-\lambda}x_{\mathcal{P}}\epsilon^{\lambda}\eta_\lambda\mathcal{I}y^{-1}\mathcal{I}$.

Proof. The implication $(i) \Rightarrow (ii)$ follows from Construction 4.7. The implication $(ii) \Rightarrow (i)$ follows from Proposition 4.7. The equivalence of (ii) and (iii) is a reformulation of Theorem 5.3 applied to the Dieudonné module $M_{ww_0w_{0,I}\epsilon^{\mu}}$.

The implication $(iii) \Rightarrow (iv)$ follows from the decomposition $G(\mathcal{O}) = \coprod_{y\in W}\mathcal{I}y\mathcal{I}$. If (iv) holds, there exists an element of $\mathcal{I}ww_0w_{0,I}\epsilon^{\mu}\mathcal{I}$ which is $G(\mathcal{O})$-σ-conjugate to an element of $\mathcal{I}\eta_\lambda^{-1}\epsilon^{-\lambda}x_{\mathcal{P}}\epsilon^{\lambda}\eta_\lambda\mathcal{I}$. By [Vie, Theorem 1.1], each element of $\mathcal{I}ww_0w_{0,I}\epsilon^{\mu}\mathcal{I}$ is $G(\mathcal{O})$-σ-conjugate to an element of $G(\mathcal{O})_1ww_0w_{0,I}\epsilon^{\mu}G(\mathcal{O})_1$. Using the fact that $G(\mathcal{O})_1$ is normal in $G(\mathcal{O})$ this implies (iii). □

Let $\mathcal{Z}$ be the center of H. Then $X_*(\mathcal{Z})$ acts on the set

$$X_*(T)^{\mathcal{P}} := \{\lambda \in X_*(T) \mid \epsilon^{-\lambda}x_{\mathcal{P}}\epsilon^{\lambda} \in W\epsilon^{\mu}W\}$$

by addition.

Lemma 5.5. *This action on $X_*(T)^{\mathcal{P}}$ has finitely many orbits.*

Proof. By looking at each block of H separately, we assume that $\mathcal{P} = (n/(n+m))$ for coprime non-negative integers n and m. Via Propositions 3.7 and 3.9, the set $X_*(T)^{\mathcal{P}}$ can be identified with the set of beginnings C of semimodules of type (n, m). Under this identification, an element $i \in X_*(\mathcal{Z}) \cong \mathbb{Z}$ sends $C \subset \mathbb{Z}$ to $C + i$. In this form the claim is [dJO, 6.3]. □

6 Non-emptiness of certain affine Deligne-Lusztig varieties

Fix $0 \leq d \leq h$. For $x \in \tilde{W}$ and $b \in G(L)$, we consider the associated affine Deligne-Lusztig variety (c.f. Rapoport [Rap]), which is the following set:

$$X_x(b) := \{g\mathcal{I} \in G(L)/\mathcal{I} \mid g^{-1}b\sigma(g) \in \mathcal{I}x\mathcal{I}\}$$

From Theorem 5.4 we get the follwing criterion for the non-emptiness of certain of the $X_x(b)$. Here we use again the objects defined in Section 5 with respect to the given Newton polygon $\mathcal{P}$. In case the Newton polygon $\mathcal{P}$ has a single slope, a different such criterion was previously given by Görtz, He and Nie in [GHN].

Theorem 6.1. *Let $x \in W\epsilon^\mu W$ and $b \in G(\mathcal{O})\epsilon^\mu G(\mathcal{O})$. Let $\mathcal{P}$ the Newton polygon of the Dieudonné module M_b. The following are equivalent:*

(i) The set $X_x(b)$ is non-empty.

(ii) There exist $\lambda \in X_*(T)$ satisfying $\epsilon^{-\lambda}x_{\mathcal{P}}\epsilon^\lambda \in W\epsilon^\mu W$ and $y \in W$ such that

$$x \in \mathcal{I}y\mathcal{I}\eta_\lambda^{-1}\epsilon^{-\lambda}x_{\mathcal{P}}\epsilon^\lambda\eta_\lambda\mathcal{I}y^{-1}\mathcal{I}.$$

Proof. $(i) \Rightarrow (ii)$: Let $g\mathcal{I} \in X_x(b)$ and $h := gb\sigma(g^{-1}) \in \mathcal{I}x\mathcal{I}$. Since $x \in W\epsilon^\mu W$ we obtain a Dieudonné module M_h with Hodge polygon given by μ and Newton polygon $\mathcal{P}$. Hence by Theroem 5.4 there exists a compatible filtration with Newton polygon $\mathcal{P}$ on $M_h/\epsilon M_h$. Hence by Theorem 5.3 applied to $M = M_h$ there exist λ as in (ii) and $r \in G(\mathcal{O})$ such that $rh\sigma(r)^{-1} \in \mathcal{I}\eta_\lambda\epsilon^{-\lambda}x_{\mathcal{P}}\epsilon^\lambda\eta_\lambda\mathcal{I}$. Using $G(\mathcal{O}) = \coprod_{y\in W}\mathcal{I}w\mathcal{I}$ this proves (ii).

$(ii) \Rightarrow (i)$: By (ii) there exists an element $h \in \mathcal{I}x\mathcal{I}$ which is $G(\mathcal{O})$-σ-conjugate to an element of $\mathcal{I}\eta_\lambda^{-1}\epsilon^{-\lambda}x_{\mathcal{P}}\epsilon^\lambda\eta_\lambda\mathcal{I}$. Hence by Theorem 5.4 the 1-truncated Dieudonné module $Z := M_h/\epsilon M_h$ has a lift M with Newton polygon $\mathcal{P}$. Since M and M_h have the same truncation, as discussed in Subsection 2.2 the matrix h' of $F\colon M \to M$ with respect to a suitable basis lies in $G(\mathcal{O})_1 hG(\mathcal{O})_1$. Since $G(\mathcal{O})_1 \subset \mathcal{I}$ we have $h' \in \mathcal{I}x\mathcal{I}$. Since $M_{h'} \cong M$ has Newton polygon $\mathcal{P}$ there exists $g \in G(L)$ such that $g^{-1}b\sigma(g) = h' \in \mathcal{I}x\mathcal{I}$. Thus $g\mathcal{I} \in X_x(b)$. □

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