Birationally rigid Fano threefold hypersurfaces

Ivan Cheltsov and Jihun Park

Abstract. We prove that every quasi-smooth weighted Fano threefold hypersurface in the 95 families of Fletcher and Reid is birationally rigid.

Keywords: Fano hypersurface; weighted projective space; birationally rigid; birational involution.

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1 Introduction

1.1 Birational rigidity and Main Theorem

Let V be a smooth projective variety. If its canonical class K_V is pseudo-effective, then Minimal Model Program produces a birational model W of the variety V, so-called *minimal model*, which has mild singularities (terminal and $\mathbb{Q}$-factorial) and the canonical class K_W of which is nef. This has been verified in dimension 3 and in any dimension for varieties of general type (see [3, Theorem 1.1]). Meanwhile, if the canonical class K_V is not pseudo-effective, then Minimal Model Program yields a birational model U of V, so-called *Mori fibred space*. It also has terminal and $\mathbb{Q}$-factorial singularities and it admits a fiber structure $\pi\colon U\to Z$ of relative Picard rank 1 such that the divisor $-K_U$ is ample on fibers. This has been proved in all dimensions (see [3, Corollary 1.3.3]).

Mori fibred spaces, alongside the minimal models, represent the terminal objects in Minimal Model Program. If the canonical class is pseudo-effective and its minimal models exist, then they are unique up to flops. However, this is not the case when the canonical class is not pseudo-effective, since Mori fibred spaces are usually not unique terminal objects in Minimal Model Program. Nevertheless, some Mori fibred spaces behave very much the same as minimal models. To distinguish them, Corti introduced

Definition 1.1.1 ([24, Definition 1.3]). Let $\pi\colon U\to Z$ be a Mori fibred space. It is called *birationally rigid* if for a birational map $\xi\colon U\dashrightarrow U'$ to a Mori fibred space $\pi'\colon U'\to Z'$ there exist a birational automorphism $\tau : U\dashrightarrow U$ and a birational map $\sigma : Z\dashrightarrow Z'$ such that the birational map $\xi\circ\tau$ induces an isomorphism between the generic fibers of the Mori fibrations $\pi\colon U\to Z$ and $\pi'\colon U'\to Z'$ and the diagram

$$\begin{array}{ccccc} U & \overset{\tau}{\dashrightarrow} & U & \overset{\xi}{\dashrightarrow} & U' \\ \downarrow{\scriptstyle\pi} & & & & \downarrow{\scriptstyle\pi'} \\ Z & & \overset{\sigma}{\dashrightarrow} & & Z' \end{array}$$

commutes.

Fano varieties of Picard rank one with at most terminal $\mathbb{Q}$-factorial singularities are the basic examples of Mori fibred spaces. For them, Definition 1.1.1 can be simplified as follows:

Definition 1.1.2. Let V be a Fano variety of Picard rank 1 with at most terminal $\mathbb{Q}$-factorial singularities. Then the Fano variety V is called *birationally rigid* if the following property holds.

- If there is a birational map $\xi\colon V\dashrightarrow U$ to a Mori fibred space $U\to Z$, then the Fano variety V is biregular to U (and hence Z must be a point).

If, in addition, the birational automorphism group of V coincides with its biregular automorphism group, then V is called *birationally super-rigid*.

Birationally rigid Fano varieties behave very much like *canonical models*. Their birational geometry is very simple. In particular, they are non-rational. The first example of a birationally rigid Fano variety is due to Iskovskikh and Manin. In 1971, they proved

Theorem 1.1.3 ([33])**.** *A smooth quartic hypersurface in* $\mathbb{P}^4$ *is birationally super-rigid.*

In fact, Iskovskikh and Manin only proved that smooth quartic hypersurfaces in $\mathbb{P}^4$ do not admit any non-biregular birational automorphisms and, therefore, they are non-rational. In late nineties, Corti observed in [23] that their proof implies Theorem 1.1.3. Inspired by this observation, Pukhlikov generalized Theorem 1.1.3 as

Theorem 1.1.4 ([44])**.** *A general hypersurface of degree* $n \geq 4$ *in* $\mathbb{P}^n$ *is birationally super-rigid.*

Shortly after Theorem 1.1.4 was proved, Reid suggested to Corti and Pukhlikov that they should generalize Theorem 1.1.3 for singular threefolds. Together they proved

Theorem 1.1.5 ([25])**.** *Let* X *be a quasi-smooth hypersurface of degree* d *with only terminal singularities in weighted projective space* $\mathbb{P}(1, a_1, a_2, a_3, a_4)$*, where* $d = \sum a_i$*. Suppose that* X *is a general hypersurface in this family. Then* X *is birationally rigid.*

The singular threefolds in Theorem 1.1.5 have a long history. In 1979 Reid discovered the 95 families of $K3$ surfaces in three dimensional weighted projective spaces (see [45]). After this, Fletcher, who was a Ph.D. student of Ried, announced the 95 families of weighted Fano threefold hypersurfaces in his Ph.D. dissertation in 1988. These are quasi-smooth hypersurfaces of degrees d with only terminal singularities in weighted projective spaces $\mathbb{P}(1, a_1, a_2, a_3, a_4)$, where $d = \sum a_i$. The 95 families are determined by the quadruples of non-decreasing positive integers (a_1, a_2, a_3, a_4). All Reid's 95 families of $K3$ surfaces arise as anticanonical divisors of the Fano threefolds in Fletcher's 95 families . Because of this, the latter 95 families are often called the 95 families of Fletcher and Reid.

It is quite often that we need to know the non-rationality of an explicitly given Fano variety (which does not follow from the non-rationality of a general member in its family).

Example 1.1.6. Recently Prokhorov classified all finite simple subgroups in the birational automorphism group $\mathrm{Bir}(\mathbb{P}^3)$ of the three-dimensional projective space. Up to isomorphism, A_5, $\mathrm{PSL}_2(\mathbb{F}_7)$, A_6, A_7, $\mathrm{PSL}_2(\mathbb{F}_8)$ and $\mathrm{PSU}_4(\mathbb{F}_2)$ are all non-abelian finite simple subgroups in $\mathrm{Bir}(\mathbb{P}^3)$ ([43, Theorem 1.3]). Prokhorov's proof implies more. Up to conjugation, the group $\mathrm{Bir}(\mathbb{P}^3)$ contains a unique subgroup isomorphic to $\mathrm{PSL}_2(\mathbb{F}_8)$ and exactly two subgroups isomorphic to $\mathrm{PSU}_4(\mathbb{F}_2)$. For the alternating group A_7, he proved that $\mathrm{Bir}(\mathbb{P}^3)$ contains exactly one such subgroup provided that the threefold

$$\sum_{i=0}^{6} x_i = \sum_{i=0}^{6} x_i^2 = \sum_{i=0}^{6} x_i^3 = 0 \subset \mathrm{Proj}\Big(\mathbb{C}[x_0, \ldots, x_6]\Big) \cong \mathbb{P}^6 \tag{1.1.7}$$

is not rational. This threefold is the unique complete intersection of a quadric and a cubic hypersurfaces in $\mathbb{P}^5$ that admits a faithful action of A_7. Back in nineties Iskovskikh and Pukhlikov proved that a general threefold in this family is birationally rigid (see [34]). The threefold (1.1.7) is smooth. However, it does not satisfy the generality assumptions imposed in [34]. It is in 2012 that Beauville proved that the threefold (1.1.7) is not rational (see [2]). It is still unknown whether it is birationally rigid or not.

It took more than ten years to prove Theorem 1.1.4 for *every* smooth hypersurface in $\mathbb{P}^n$ of degree $n \geq 4$, which was conjectured in [44]. This was done by de Fernex who proved

Theorem 1.1.8 ([27]). *Every smooth hypersurface of degree $n \geq 4$ in $\mathbb{P}^n$ is birationally super-rigid.*

The goal of this paper is to prove Theorem 1.1.5 for *all* quasi-smooth hypersurfaces in each of the 95 families of Fletcher and Reid, which was conjectured in [25]. To be precise, we prove

Main Theorem. *Let X be a quasi-smooth hypersurface of degree d with only terminal singularities in the weighted projective space $\mathbb{P}(1, a_1, a_2, a_3, a_4)$, where $d = \sum a_i$. Then X is birationally rigid.*

Since birational rigidity implies non-rationality, we immediately obtain

Corollary 1.1.9. *Let X be a quasi-smooth hypersurface of degree d with only terminal singularities in the weighted projective space $\mathbb{P}(1, a_1, a_2, a_3, a_4)$, where $d = \sum a_i$. Then X is not rational.*

In addition, the proof of Main Theorem shows

Theorem 1.1.10. *Every quasi-smooth hypersurface in the families of the 95 families of Fletcher and Reid whose general members are birationally super-rigid is birationally super-rigid.*

The families corresponding to Theorem 1.1.10 are those in the list of Fletcher and Reid with entry numbers No. 1, 3, 10, 11, 14, 19, 21, 22, 28, 29, 34, 35, 37, 39, 49, 50, 51, 52, 53, 55, 57, 59, 62, 63, 64, 66, 67, 70, 71, 72, 73, 75, 77, 78, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94 and 95 (see Section 1.4).

The 95 families of Fletcher and Reid contain the family (No. 1) of quartic hypersurfaces in $\mathbb{P}^4$ and the family (No. 3) of hypersurfaces of degree 6 in $\mathbb{P}(1, 1, 1, 1, 3)$, i.e., double covers of $\mathbb{P}^3$ ramified along sextic surfaces. However, we do not consider these two families in the present paper since every smooth quartic threefold and every smooth double covers of $\mathbb{P}^3$ ramified along sextic surfaces (see [31]) are already proved to be birationally super-rigid.

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1.2 How to prove Main Theorem

In this section we present the synopsis of our proof of Main Theorem. Before we proceed, we introduce a terminology that is frequently used in birational geometry as well as in the present paper.

Definition 1.2.1. Let U be a normal $\mathbb{Q}$-factorial variety and $\mathcal{M}_U$ a mobile linear system (a linear system without a fixed component) on U. Let a be a non-negative rational number. An irreducible subvariety Z of U is called a center of non-canonical singularities (or simply non-canonical center) of the log pair $(U, a\mathcal{M}_U)$ if there is a birational morphism $h\colon W \to U$ and an h-exceptional divisor $E_1 \subset W$ such that

$$K_W + ah_*^{-1}(\mathcal{M}_U) = h^*(K_U + a\mathcal{M}_U) + \sum_{i=1}^{m} c_i E_i,$$

where each E_i is an h-exceptional divisor, $c_1 < 0$ and $h(E_1) = Z$.

The following result is known as the classical Nöther–Fano inequality.

Theorem 1.2.2 ([23, Theorem 4.2])**.** *Let X be a terminal $\mathbb{Q}$-factorial Fano variety with* $\mathrm{Pic}(X) \cong \mathbb{Z}$*.*

- *If the log pair $(X, \frac{1}{n}\mathcal{M})$ has canonical singularities for every positive integer n and every mobile linear subsystem $\mathcal{M}$ in $|-nK_X|$, then X is birationally super-rigid.*
- *If for every positive integer n and every mobile linear system $\mathcal{M}$ in $|-nK_X|$ there exists a birational automorphism τ of X such that the log pair $(X, \frac{1}{n_\tau}\tau(\mathcal{M}))$ has canonical singularities, where n_τ is the positive integer such that $\tau(\mathcal{M})$ is contained in $|-n_\tau K_X|$, then X is birationally rigid.*

The Nöther-Fano inequality will be the master key to the proof of Main Theorem.
To prove Main Theorem, we take the following steps in order.

Step 1. We suppose that a given hypersurface X from the 95 families has a mobile linear system $\mathcal{M}$ in $|-nK_X|$ for some positive integer n such that the log pair $(X, \frac{1}{n}\mathcal{M})$ is not canonical. Then we must have a center of non-canonical singularities of the pair $(X, \frac{1}{n}\mathcal{M})$. A center of non-canonical singularities of the log pair $(X, \frac{1}{n}\mathcal{M})$ can be, *a priori*, one of the following:

$$\begin{cases} \text{a smooth point,} \\ \text{an irreducible curve,} \\ \text{a singular point} \end{cases}$$

on the Fano threefold X.

Step 2. We prove that a smooth point of X cannot be a center of non-canonical singularities of the pair $(X, \frac{1}{n}\mathcal{M})$. This will be done in Section 2.1 (Theorem 2.1.10).

Step 3. In Section 2.2 we show that a curve contained in the smooth locus of X cannot be a center (Theorem 2.2.4). Then Theorem 2.2.1 implies that a singular point of X must be a center.

Step 4. For a given singular point of the hypersurface X we prove that either

- it cannot be a center of non-canonical singularities of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$ (the job proving this part will be called *excluding*) or
- there exists a birational automorphism τ of X such that $\tau(\mathcal{M})$ is contained in $|-n_\tau K_X|$ for some positive integer $n_\tau < n$ (the job proving this part will be called *untwisting*).

With using induction on n, it then follows from Theorem 1.2.2 that the given hypersurface X is birationally rigid. Step 4 will be done mainly in Section 5.2. However, to exclude or untwist singular points, we will need several pieces of machinery, some of which are light and some of which are heavy. These machines will be assembled from Section 3.2 to Section 4.3. In fact, the machines for excluding are relatively simple to use, so that they could be introduced in Section 3.2. Meanwhile, the machines for untwisting are complicated to assemble. It will be carried out one by one from Section 3.3 to Section 4.3. Before using these machines in practical situation, i.e., before reading the tables in Section 5.2, we require the reader to be acquainted with the manual for the machinery provided in Section 5.1.

Theorem 1.1.10 can be proved by excluding all the singular points of X as a center. Fifty families out of the 95 families are those considered in Theorem 1.1.10. In Section 5.2, we are immediately able to notice that a singular point of X cannot be a center if the hypersurface X belongs to one of the families considered in Theorem 1.1.10. Such families have the underlined entry numbers in their tables in Section 5.2.

1.3 Notations

Let us describe the notations we will use in the rest of the present paper. Unless otherwise mentioned, these notations are fixed from now until the end of the paper.

- In the weighted projective space $\mathbb{P}(1, a_1, a_2, a_3, a_4)$, we assume that $a_1 \leq a_2 \leq a_3 \leq a_4$. For weighted homogeneous coordinates, we always use x, y, z, t and w with weights $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$, $\mathrm{wt}(t) = a_3$ and $\mathrm{wt}(w) = a_4$.
- $f_m(x_{i_1}, \ldots, x_{i_k})$, $g_m(x_{i_1}, \ldots, x_{i_k})$ and $h_m(x_{i_1}, \ldots, x_{i_k})$ are quasi-homogeneous polynomials of degree m in variables $x_{i_1}, \ldots, x_{i_k}$ in the given weighted projective space $\mathbb{P}(1, a_1, a_2, a_3, a_4)$.
- If a monomial appears individually in a quasi-homogeneous polynomial, then the monomial is assumed not to be contained in any other terms. For example, in the polynomial $w^2 + t^3 + wf_6(x, y, z, t) + f_{12}(x, y, z, t)$, the polynomial f_{12} does not contain the monomial t^3.
- In each family, we always let X be a quasi-smooth hypersurface of degree d in the weighted projective space $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ with only terminal singularities, where $d = \sum_{i=1}^{4} a_i$. We also use X_d, instead of X, in order to indicate the degree d of X.
- On the threefold X, a given mobile linear system is denoted by $\mathcal{M}$.

- For a given mobile linear system $\mathcal{M}$, we always assume that $\mathcal{M} \sim_{\mathbb{Q}} -nK_X$.
- S_x is the surface on the hypersurface X cut by the equation $x = 0$.
- S_y is the surface on the hypersurface X cut by the equation $y = 0$.
- S_z is the surface on the hypersurface X cut by the equation $z = 0$.
- S_t is the surface on the hypersurface X cut by the equation $t = 0$.
- S_w is the surface on the hypersurface X cut by the equation $w = 0$.
- L_{tw} is the one-dimensional stratum on $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ defined by $x = y = z = 0$, and the other one-dimensional strata are labelled similarly.
- $O_y := [0 : 1 : 0 : 0 : 0]$.
- $O_z := [0 : 0 : 1 : 0 : 0]$.
- $O_t := [0 : 0 : 0 : 1 : 0]$.
- $O_w := [0 : 0 : 0 : 0 : 1]$.
- When we consider a singular point of type $\frac{1}{r}(1, a, r - a)$ on X, the weighted blow up of X at the singular point with weights $(1, a, r - a)$ will be denoted by $f\colon Y \to X$ unless otherwise stated.
- A is the pull-back of $-K_X$ by f.
- B is the anticanonical class of Y.
- E is the exceptional divisor of f.
- S is the proper transform of S_x by f.
- $\mathcal{M}_Y$ is the proper transform of the linear system $\mathcal{M}$ by f.
- When we have a curve C on X, its proper transform on Y will be always denoted by $\tilde{C}$. For instance, $\tilde{L}_{tw}$ is the proper transform of the curve L_{tw} on X (if it is contained in X) by the weighted blow up f.

1.4 The 95 families of Fletcher and Ried

We list the 95 families of Fletcher and Ried for the convenience of the reader (see [29, Table 5]). Here, $X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$ is a quasi-smooth hypersurface of degree d in the projective space $\mathbb{P}(1, a_1, a_2, a_3, a_4)$. The entry numbers of the list are originally given in the lexicographic order of (d, a_1, a_2, a_3, a_4).

No. 01. $X_4 \subset \mathbb{P}(1,1,1,1,1)$	**No. 02.** $X_5 \subset \mathbb{P}(1,1,1,1,2)$	**No. 03.** $X_6 \subset \mathbb{P}(1,1,1,1,3)$
No. 04. $X_6 \subset \mathbb{P}(1,1,1,2,2)$	**No. 05.** $X_7 \subset \mathbb{P}(1,1,1,2,3)$	**No. 06.** $X_8 \subset \mathbb{P}(1,1,1,2,4)$
No. 07. $X_8 \subset \mathbb{P}(1,1,2,2,3)$	**No. 08.** $X_9 \subset \mathbb{P}(1,1,1,3,4)$	**No. 09.** $X_9 \subset \mathbb{P}(1,1,2,3,3)$
No. 10. $X_{10} \subset \mathbb{P}(1,1,1,3,5)$	**No. 11.** $X_{10} \subset \mathbb{P}(1,1,2,2,5)$	**No. 12.** $X_{10} \subset \mathbb{P}(1,1,2,3,4)$

No. 13. $X_{11} \subset \mathbb{P}(1,1,2,3,5)$ **No. 14.** $X_{12} \subset \mathbb{P}(1,1,1,4,6)$ **No. 15.** $X_{12} \subset \mathbb{P}(1,1,2,3,6)$
No. 16. $X_{12} \subset \mathbb{P}(1,1,2,4,5)$ **No. 17.** $X_{12} \subset \mathbb{P}(1,1,3,4,4)$ **No. 18.** $X_{12} \subset \mathbb{P}(1,2,2,3,5)$
No. 19. $X_{12} \subset \mathbb{P}(1,2,3,3,4)$ **No. 20.** $X_{13} \subset \mathbb{P}(1,1,3,4,5)$ **No. 21.** $X_{14} \subset \mathbb{P}(1,1,2,4,7)$
No. 22. $X_{14} \subset \mathbb{P}(1,2,2,3,7)$ **No. 23.** $X_{14} \subset \mathbb{P}(1,2,3,4,5)$ **No. 24.** $X_{15} \subset \mathbb{P}(1,1,2,5,7)$
No. 25. $X_{15} \subset \mathbb{P}(1,1,3,4,7)$ **No. 26.** $X_{15} \subset \mathbb{P}(1,1,3,5,6)$ **No. 27.** $X_{15} \subset \mathbb{P}(1,2,3,5,5)$
No. 28. $X_{15} \subset \mathbb{P}(1,3,3,4,5)$ **No. 29.** $X_{16} \subset \mathbb{P}(1,1,2,5,8)$ **No. 30.** $X_{16} \subset \mathbb{P}(1,1,3,4,8)$
No. 31. $X_{16} \subset \mathbb{P}(1,1,4,5,6)$ **No. 32.** $X_{16} \subset \mathbb{P}(1,2,3,4,7)$ **No. 33.** $X_{17} \subset \mathbb{P}(1,2,3,5,7)$
No. 34. $X_{18} \subset \mathbb{P}(1,1,2,6,9)$ **No. 35.** $X_{18} \subset \mathbb{P}(1,1,3,5,9)$ **No. 36.** $X_{18} \subset \mathbb{P}(1,1,4,6,7)$
No. 37. $X_{18} \subset \mathbb{P}(1,2,3,4,9)$ **No. 38.** $X_{18} \subset \mathbb{P}(1,2,3,5,8)$ **No. 39.** $X_{18} \subset \mathbb{P}(1,3,4,5,6)$
No. 40. $X_{19} \subset \mathbb{P}(1,3,4,5,7)$ **No. 41.** $X_{20} \subset \mathbb{P}(1,1,4,5,10)$ **No. 42.** $X_{20} \subset \mathbb{P}(1,2,3,5,10)$
No. 43. $X_{20} \subset \mathbb{P}(1,2,4,5,9)$ **No. 44.** $X_{20} \subset \mathbb{P}(1,2,5,6,7)$ **No. 45.** $X_{20} \subset \mathbb{P}(1,3,4,5,89)$
No. 46. $X_{21} \subset \mathbb{P}(1,1,3,7,10)$ **No. 47.** $X_{21} \subset \mathbb{P}(1,1,5,7,8)$ **No. 48.** $X_{21} \subset \mathbb{P}(1,2,3,7,9)$
No. 49. $X_{21} \subset \mathbb{P}(1,3,5,6,7)$ **No. 50.** $X_{22} \subset \mathbb{P}(1,1,3,7,11)$ **No. 51.** $X_{22} \subset \mathbb{P}(1,1,4,6,11)$
No. 52. $X_{22} \subset \mathbb{P}(1,2,4,5,11)$ **No. 53.** $X_{24} \subset \mathbb{P}(1,1,3,8,12)$ **No. 54.** $X_{24} \subset \mathbb{P}(1,1,6,8,9)$
No. 55. $X_{24} \subset \mathbb{P}(1,2,3,7,12)$ **No. 56.** $X_{24} \subset \mathbb{P}(1,2,3,8,11)$ **No. 57.** $X_{24} \subset \mathbb{P}(1,3,4,5,12)$
No. 58. $X_{24} \subset \mathbb{P}(1,3,4,7,10)$ **No. 59.** $X_{24} \subset \mathbb{P}(1,3,6,7,8)$ **No. 60.** $X_{24} \subset \mathbb{P}(1,4,5,6,9)$
No. 61. $X_{25} \subset \mathbb{P}(1,4,5,7,9)$ **No. 62.** $X_{26} \subset \mathbb{P}(1,1,5,7,13)$ **No. 63.** $X_{26} \subset \mathbb{P}(1,2,3,8,13)$
No. 64. $X_{26} \subset \mathbb{P}(1,2,5,6,13)$ **No. 65.** $X_{27} \subset \mathbb{P}(1,2,5,9,11)$ **No. 66.** $X_{27} \subset \mathbb{P}(1,5,6,7,9)$
No. 67. $X_{28} \subset \mathbb{P}(1,1,4,9,14)$ **No. 68.** $X_{28} \subset \mathbb{P}(1,3,4,7,14)$ **No. 69.** $X_{28} \subset \mathbb{P}(1,4,6,7,11)$
No. 70. $X_{30} \subset \mathbb{P}(1,1,4,10,15)$ **No. 71.** $X_{30} \subset \mathbb{P}(1,1,6,8,15)$ **No. 72.** $X_{30} \subset \mathbb{P}(1,2,3,10,15)$
No. 73. $X_{30} \subset \mathbb{P}(1,2,6,7,15)$ **No. 74.** $X_{30} \subset \mathbb{P}(1,3,4,10,13)$ **No. 75.** $X_{30} \subset \mathbb{P}(1,4,5,6,15)$
No. 76. $X_{30} \subset \mathbb{P}(1,5,6,8,11)$ **No. 77.** $X_{32} \subset \mathbb{P}(1,2,5,9,16)$ **No. 78.** $X_{32} \subset \mathbb{P}(1,4,5,7,16)$
No. 79. $X_{33} \subset \mathbb{P}(1,3,5,11,14)$ **No. 80.** $X_{34} \subset \mathbb{P}(1,3,4,10,17)$ **No. 81.** $X_{34} \subset \mathbb{P}(1,4,6,7,17)$
No. 82. $X_{36} \subset \mathbb{P}(1,1,5,12,18)$ **No. 83.** $X_{36} \subset \mathbb{P}(1,3,4,11,18)$ **No. 84.** $X_{36} \subset \mathbb{P}(1,7,8,9,12)$
No. 85. $X_{38} \subset \mathbb{P}(1,3,5,11,19)$ **No. 86.** $X_{38} \subset \mathbb{P}(1,5,6,8,19)$ **No. 87.** $X_{40} \subset \mathbb{P}(1,5,7,8,20)$
No. 88. $X_{42} \subset \mathbb{P}(1,1,6,14,21)$ **No. 89.** $X_{42} \subset \mathbb{P}(1,2,5,14,21)$ **No. 90.** $X_{42} \subset \mathbb{P}(1,3,4,14,21)$
No. 91. $X_{44} \subset \mathbb{P}(1,4,5,13,22)$ **No. 92.** $X_{48} \subset \mathbb{P}(1,3,5,16,24)$ **No. 93.** $X_{50} \subset \mathbb{P}(1,3,5,16,24)$
No. 94. $X_{54} \subset \mathbb{P}(1,4,5,18,27)$ **No. 95.** $X_{66} \subset \mathbb{P}(1,5,6,22,33)$.

2 Smooth points and curves

2.1 Excluding smooth points

In this section we show that smooth points of X cannot be non-canonical centers of the log pair $\big(X, \frac{1}{n}\mathcal{M}\big)$.

Let $X \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$ be a quasi-smooth weighted hypersurface of degree $d = \sum a_i$ with terminal singularities. Suppose that a smooth point p on X is a center of non-canonical singularities of the log pair $\big(X, \frac{1}{n}\mathcal{M}\big)$. Then we obtain

$$\mathrm{mult}_p(\mathcal{M}^2) > 4n^2$$

by [24, Corollary 3.4].

Let s be an integer not greater than $\frac{4}{-K_X^3}$. Suppose that we have a divisor H in $|-sK_X|$ such that

- it passes through the point p,
- it contains no 1-dimensional component of the base locus of the linear system $\mathcal{M}$ that passes through the point p.

Then we can obtain the following contradictory inequality:

$$-sn^2K_X^3 = H \cdot \mathcal{M}^2 \geq \mathrm{mult}_p(H) \cdot \mathrm{mult}_p(\mathcal{M}^2) > 4n^2.$$

In order to show that a center of non-canonical singularities of the log pair $\big(X, \frac{1}{n}\mathcal{M}\big)$ cannot be a smooth point, we mainly try to find such a divisor.

Before we proceed, set $\widehat{a}_2 = \mathrm{lcm}\{a_1, a_3, a_4\}$, $\widehat{a}_3 = \mathrm{lcm}\{a_1, a_2, a_4\}$ and $\widehat{a}_4 = \mathrm{lcm}\{a_1, a_2, a_3\}$.

Lemma 2.1.1. *Suppose that the hypersurface X satisfies one of the following:*

- *X does not pass through the point O_w and $d \cdot \widehat{a}_4 \leq 4a_1a_2a_3a_4$;*
- *X does not pass through the point O_t and $d \cdot \widehat{a}_3 \leq 4a_1a_2a_3a_4$;*
- *X does not pass through the point O_z and $d \cdot \widehat{a}_2 \leq 4a_1a_2a_3a_4$.*

Then a smooth point of X cannot be a non-canonical center of the log pair $\big(X, \frac{1}{n}\mathcal{M}\big)$.

Proof. For simplicity we suppose that the hypersurface X satisfies the first condition. The proofs for the other cases are the same.

Let $\pi_4 : X \to \mathbb{P}(1, a_1, a_2, a_3)$ be the regular projection centered at the point O_w. The linear system $|\mathcal{O}_{\mathbb{P}(1,a_1,a_2,a_3)}(\widehat{a}_4)|$ is base point free. Choose a general member in the linear system $|\mathcal{O}_{\mathbb{P}(1,a_1,a_2,a_3)}(\widehat{a}_4)|$ that passes through the point $\pi_4(p)$. Then its pull-back by the finite morphism π_4 can play the role of the divisor H in the explanation at the beginning. □

The condition above is satisfied by all the families except the families

No. 2, 5, 12, 13, 20, 23, 25, 33, 40, 58, 61, 76.

No quasi-smooth hypersurface in the families No. 23, 40, 61, 76 contains the curve L_{tw}. Using suitable coordinate changes, we may write its defining equation as

$$tw^2 + w(tg_{a_4}(x,y,z) + g_d(x,y,z)) + x_it^3 + t^2g_{d-2a_3}(x,y,z) + tg_{d-a_3}(x,y,z) + g_d(x,y,z) = 0,$$

where $x_i = y$ for the families No. 23, 61 and $x_i = z$ for the families No. 40, 76.

There are two kinds of quasi-smooth hypersurfaces in each family of No. 5, 12, 13, 20, 25, 33, 58. The first kind are those that do not contain the curve L_{tw}. The second are those that contain the curve L_{tw}. After appropriate coordinate changes, every quasi-smooth hypersurface of the first kind in each family can be defined by

$$wt^2 + t(wg_{a_3}(x,y,z) + g_d(x,y,z)) + x_iw^2 + wg_{d-a_4}(x,y,z) + g_d(x,y,z) = 0,$$

where $x_i = y$ for the families No. 13, 25 and $x_i = z$ for the families No. 5, 12, 20, 33, 58.

Lemma 2.1.2. *Suppose that the hypersurface X satisfies the following conditions:*

- $d \cdot \widehat{a}_4 \leq 4a_1a_2a_3a_4$;
- $d \cdot \widehat{a}_3 \leq 4a_1a_2a_3a_4$.

In addition, we suppose that the curve L_{tw} is not contained in the hypersurface X. Then a smooth point of X cannot be a center of non-canonical singularities of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$.

Proof. Let $\mathcal{H}_4$ be the linear system consisting of the divisors in $|O_X(\hat{a}_4)|$ that pass through the point p and $\mathcal{H}_3$ be the linear system consisting of the divisors in $|O_X(\hat{a}_3)|$ that pass through the point p.

Let $\pi_4 : X \dashrightarrow \mathbb{P}(1,a_1,a_2,a_3)$ be the projection centered at the point O_w and $\pi_3 : X \dashrightarrow \mathbb{P}(1,a_1,a_2,a_4)$ the projection centered at the point O_t. The linear systems $|\mathcal{O}_{\mathbb{P}(1,a_1,a_2,a_3)}(\widehat{a}_4)|$ and $|\mathcal{O}_{\mathbb{P}(1,a_1,a_2,a_4)}(\widehat{a}_3)|$ are base point free. The pull-backs of a general member in $|\mathcal{O}_{\mathbb{P}(1,a_1,a_2,a_3)}(\widehat{a}_4)|$ that passes through the point $\pi_4(p)$ and a general member in the linear system $|\mathcal{O}_{\mathbb{P}(1,a_1,a_2,a_4)}(\widehat{a}_3)|$ that passes through the point $\pi_3(p)$ show that a general member either in $\mathcal{H}_4$ or in $\mathcal{H}_3$ can serve as the divisor H in the explanation at the beginning unless the point p belongs to both a curve contracted by π_4 and a curve contracted by π_3.

Suppose that the point p belongs to both a curve contracted by π_4 and a curve contracted by π_3. Let $A_1, A_2, \cdots, A_k$ (resp. $B_1, B_2, \cdots, B_m$) be quasi-homogeneous polynomials of degree $\hat{a}_4$ (resp. $\hat{a}_3$) that generate the linear system $\mathcal{H}_4$ (resp. $\mathcal{H}_3$).

Any irreducible curve except L_{tw} cannot be contracted both by π_4 and by π_3. Therefore, the base locus of $\mathcal{H}_4$ has no common 1-dimensional component with the base locus of $\mathcal{H}_3$ around the point p since we do not have the curve L_{tw} on X. This shows that the base locus of the linear system $\mathcal{H}$ generated by quasi-homogeneous polynomials $A_1^{\hat{a}_3}, A_2^{\hat{a}_3}, \cdots, A_k^{\hat{a}_3}, B_1^{\hat{a}_4}, B_2^{\hat{a}_4}, \cdots, B_m^{\hat{a}_4}$ of degree $\hat{a}_4\hat{a}_3$ has no 1-dimensional component passing through the point p. Therefore, for a general member H' of the linear system $\mathcal{H}$, we have

$$-\hat{a}_3\hat{a}_4n^2K_X^3 = H' \cdot \mathcal{M}^2 \geq \mathrm{mult}_p(H') \cdot \mathrm{mult}_p(\mathcal{M}^2) \geq \min\{\hat{a}_3, \hat{a}_4\} \cdot \mathrm{mult}_p(\mathcal{M}^2) > 4n^2 \min\{\hat{a}_3, \hat{a}_4\},$$

which implies $d\hat{a}_3\hat{a}_4 > \min\{\hat{a}_3, \hat{a}_4\}a_1a_2a_3a_4$. This contradicts our condition. □

The conditions above are satisfied by the families

$$\text{No. } 23, 40, 61, 76.$$

Also, the members of the first kind in the families No. 5, 12, 13, 20, 25, 33, 58, i.e., those that do not contain L_{tw}, meet these conditions.

The members of the second kind in the families No. 5, 12, 13, 20, 25, 33, 58, i.e., those that contain L_{tw}, and the family No. 2 remain.

We deal with the family No. 2 in the end of this section. Instead, we first consider the members of the second kind in the families No. 5, 12, 13, 20, 25, 33, 58, i.e., those that contain L_{tw}. These members are not covered by Lemma 2.1.2. Since these members are the ones that contain the curve L_{tw}, the defining polynomials of X do not contain the monomial t^2w. Therefore, using coordinate changes, we may assume that the polynomial is given by

$$w^2z + w(tg_{a_3}(x,y,z) + g_{2a_3}(x,y,z)) + t^3y + t^2h_{d-2a_3}(x,y,z) + th_{d-a_3}(x,y,z) + h_d(x,y,z) = 0$$

for the families No. 12, 20,

$$w^2y + w(tg_{a_3}(x,y,z) + g_{2a_3}(x,y,z)) + t^3z + t^2h_{d-2a_3}(x,y,z) + th_{d-a_3}(x,y,z) + h_d(x,y,z) = 0$$

for the families No. 5, 13, 25, 33, 58. Note that for the family No. 5 the coefficients of w^2 and t^3 cannot coincide, i.e., we cannot assume that the hypersurface X is defined by

$$w^2y + w(tg_2 + g_4) + t^3y + t^2h_3 + th_5 + h_7 = 0.$$

In such a case, the hypersurface is not quasi-smooth at the point defined by $x = y = z = w^2 + t^3 = 0$.

Lemma 2.1.3. *Suppose that the hypersurface X satisfies the following conditions:*

- $d \cdot \widehat{a}_4 \leq 4a_1a_2a_3a_4$;
- $d \cdot \widehat{a}_3 \leq 4a_1a_2a_3a_4$.

Suppose that the curve L_{tw} is contained in the hypersurface X. If a smooth point of X is a non-canonical center of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$, then the point lies on the curve L_{tw}.

Proof. The proof of Lemma 2.1.2 immediately shows the statement. □

Lemma 2.1.4. *Suppose that the curve L_{tw} is contained in the hypersurface X. In addition, we suppose that $a_3 > 1$, $(a_3, a_4) = 1$, $a_3a_4 > d$, and there are non-negative integers m_1 and m_2 such that $m_1a_1 + m_2a_2 = a_3a_4$. Then any smooth point on L_{tw} cannot be a center of non-canonical singularities of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$ if $-a_3a_4K_X^3 \leq 4$.*

Proof. Suppose that the hypersurface X is defined by $F(x,y,z,t,w) = 0$. Let p be a smooth point on the curve L_{tw}. Then there are non-zero constants λ and μ such that the surface cut by $\lambda t^{a_4} + \mu w^{a_3} = 0$ contains the point p. We then consider the linear system $\mathcal{H}$ on X generated by $x^{a_3a_4}$, $y^{m_1}z^{m_2}$, and $\lambda t^{a_4} + \mu w^{a_3}$. The base locus of this linear system consists of the locus cut by

$$x = y^{m_1}z^{m_2} = \lambda t^{a_4} + \mu w^{a_3} = 0.$$

The degree d of F is smaller than a_3a_4 by the condition and the polynomial $\lambda t^{a_4}+\mu w^{a_3}$ is irreducible since $(a_3,a_4)=1$. Therefore, neither $F(0,0,z,t,w)$ nor $F(0,y,0,t,w)$ can divide $\lambda t^{a_4}+\mu w^{a_3}$ and vice versa. Therefore, the base locus of the linear system $\mathcal{H}$ is of dimension at most 0. Then a general member of this linear system is able to play the role of the divisor H in the explanation at the beginning. □

Combining Lemmas 2.1.3 and 2.1.4, we can conclude that any smooth point cannot be a center of non-canonical singularities of the log pair $\big(X,\frac{1}{n}\mathcal{M}\big)$ for the families No. 33 and 58.

Lemma 2.1.5. *For the families No.* 13, 25*, a smooth point of* X *cannot be a center of non-canonical singularities of the log pair* $\big(X,\frac{1}{n}\mathcal{M}\big)$.

Proof. The following method works for both the families exactly in the same way. For this reason, we demonstrate the method only for the family No. 25.

Suppose that the log pair $\big(X,\frac{1}{n}\mathcal{M}\big)$ is not canonical at some smooth point p. Then Lemma 2.1.3 shows that the point p must lie on the curve L_{tw}.

Consider the pencil $|-K_X|$. Its base locus consists of two reduced and irreducible curves. One is the curve L_{tw} and the other is the curve C defined by the equations

$$x=y=t^3-cz^4=0,$$

where c is a non-zero constant. Note that the curve L_{tw} is quasi-smooth everywhere and C is quasi-smooth outside the point O_w. They intersect only at the point O_w. Choose a general member H in the pencil $|-K_X|$. Then the log pair $\big(X,H+\frac{1}{n}\mathcal{M}\big)$ is not log canonical at the point p. By Inversion of Adjunction ([38, Theorem 5.50]), we see that the log pair $\big(H,\frac{1}{n}\mathcal{M}|_H\big)$ is not log canonical at the point p.

Let D_y be the divisor on H defined by the equation $y=0$. Then $D_y=L_{tw}+C$. We have the following intersection numbers on the surface H:

$$L_{tw}^2=-\frac{11}{28},\quad C^2=-\frac{2}{7},\quad C\cdot L_{tw}=\frac{3}{7},\quad D_y\cdot L_{tw}=\frac{1}{28},\quad D_y\cdot C=\frac{1}{7}.$$

Indeed, we can obtain these intersection numbers directly from the polynomials defining the curves. On the other hand, we are also able to obtain them from the singularity types of the K3 surface H. Note that H is a $K3$ surface with A_3 and A_6 singularities at the points O_t and O_w, respectively. For instance, the curve L_{tw} is a smooth rational curve on the K3 surface H passing through one A_3-singular point and one A_6-singular point, and hence the self-intersection number L_{tw}^2 is obtained by $-2+\frac{3}{4}+\frac{6}{7}$. The A_3-singular point contributes to the self-intersection number by $\frac{3}{4}$ and the A_6-singular point by $\frac{6}{7}$ (see Remark 2.1.6 below for more detail).

Let M be a general member in the mobile linear system $\mathcal{M}$ and then put

$$M_H:=\frac{1}{n}M\big|_H=aL_{tw}+bC+\Delta,$$

where a and b are non-negative rational numbers and Δ is an effective divisor whose support contains neither L_{tw} nor C. We then obtain

$$\frac{1}{7}=C\cdot M_H=aL_{tw}\cdot C+bC^2+\Delta\cdot C\geq\frac{3a}{7}-\frac{2b}{7}.$$

On the other hand, we obtain

$$\frac{5}{28} = M_H^2 = aL_{tw} \cdot D_y + bC \cdot D_y + \Delta \cdot D_y \geq \frac{a}{28} + \frac{b}{7}.$$

Combining these two inequalities we see that $a \leq 1$. Therefore, the log pair $(H, L_{tw}+bC+\Delta)$ is not log canonical at the point p, and hence the log pair $(L_{tw}, (bC+\Delta)|_{L_{tw}})$ is not log canonical at the point p. Consequently, we see that

$$\mathrm{mult}_p \left((bC+\Delta)|_{L_{tw}}\right) > 1.$$

However,

$$(bC+\Delta) \cdot L_{tw} = (M_H - aL_{tw}) \cdot L_{tw} = \frac{1}{28} + \frac{11a}{28} \leq \frac{3}{7}.$$

This completes the proof. □

Remark 2.1.6. Let p be an A_n-singular point on a normal surface Σ. Suppose that a smooth curve C on Σ passes through the point p. Let $\phi : \bar{\Sigma} \to \Sigma$ be the minimal resolution of the point p. Then we have (-2)-curves $E_1, \cdots, E_n$ over the point p whose intersection matrix is

$$(E_i \cdot E_j) = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}.$$

The log pair (Σ, C) is purely log terminal by Inversion of Adjunction ([38, Theorem 5.50]). Therefore, the proper transform $\bar{C}$ by ϕ intersects transversally only one of E_i's and it should be either E_1 or E_n ([35, Theorem 9.6]). We may assume that it is E_n. We then obtain

$$\phi^*(C) = \bar{C} + \frac{1}{n+1}\left(E_1 + 2E_2 + \cdots + (n-1)E_{n-1} + nE_n\right).$$

Therefore,

$$C^2 = \bar{C}^2 + \frac{n}{n+1}.$$

Lemma 2.1.7. *For the families No.* 12, 20*, any smooth point of* X *cannot be a center of non-canonical singularities of the log pair* $\left(X, \frac{1}{n}\mathcal{M}\right)$.

Proof. The method for its proof is the same as that of Lemma 2.1.5 with some slight difference. The only difference is that two base locus curves of the pencil $|-K_X|$ intersect at the point O_t. For this reason, we omit the proof. □

Lemma 2.1.8. *For the family No.* 5*, a smooth point of* X_7 *cannot be a center of non-canonical singularities of the log pair* $\left(X_7, \frac{1}{n}\mathcal{M}\right)$.

Proof. Suppose that the log pair $\left(X_7, \frac{1}{n}\mathcal{M}\right)$ is not canonical at a smooth point p. Then Lemma 2.1.3 shows that the point p lies on the curve L_{tw}.

Consider the 2-dimensional linear system $|-K_{X_7}|$. Its base locus consists of the reduced and irreducible curve L_{tw}. The curve L_{tw} is a quasi-smooth curve passing through the singular points O_t and O_w.

Let H be the surface cut by the equation $z = \lambda x + \mu y$ with general complex numbers λ and μ. It is a $K3$ surface with A_1 and A_2 singularities at the points O_t and O_w, respectively. It also contains the rational curve L_{tw}. The self-intersection number of L_{tw} on H is $-\frac{5}{6}$ $(= -2 + \frac{1}{2} + \frac{2}{3})$. Let D_y be the divisor on H defined by the equation $y = 0$. Then we can easily see that $D_y = L_{tw} + R$, where R is the curve defined by the equation

$$y = z - \lambda x = \lambda t^3 + x h_5(x, t, w) = 0.$$

The two curves L_{tw} and R meet only at the point O_w.

Let M be a general member of the linear system $\mathcal{M}$ and then write

$$M_H := \frac{1}{n} M\big|_H = aL_{tw} + \Delta,$$

where a is a non-negative rational number and Δ is an effective divisor whose support does not contain the curve L_{tw}. The log pair $\left(X_7, H + \frac{1}{n}\mathcal{M}\right)$ is not log canonical at the point p. By Inversion of Adjunction ([38, Theorem 5.50]), we see that the log pair $\left(H, \frac{1}{n}\mathcal{M}|_H\right)$ is not log canonical at the point p. We then obtain

$$1 = R \cdot M_H = aL_{tw} \cdot R + \Delta \cdot R \geq a.$$

Therefore, the log pair $(H, L_{tw} + \Delta)$ is not log canonical at the point p, and hence the log pair $(L_{tw}, \Delta|_{L_{tw}})$ is not log canonical at the point p. Consequently, we see that

$$\mathrm{mult}_p\left(\Delta|_{L_{tw}}\right) > 1.$$

However,

$$\Delta \cdot L_{tw} = (M_H - aL_{tw}) \cdot L_{tw} = \frac{1}{6} + \frac{5a}{6} \leq 1.$$

This completes the proof. □

Finally, we deal with smooth points on quasi-smooth hypersurfaces in the family No. 2.

Lemma 2.1.9. *For the family No. 2, a smooth point of X_5 cannot be a center of non-canonical singularities of the log pair $\left(X_5, \frac{1}{n}\mathcal{M}\right)$.*

Proof. This case has been resolved completely in [25]. For the convenience of the reader we reproduce the proof from p.211 in [25].

By suitable coordinate change we may assume that the hypersurface X_5 in $\mathbb{P}(1,1,1,1,2)$ is given by

$$w^2 x + w f_3 + f_5 = 0,$$

where f_m is a quasi-homogeneous polynomial of degree m in variables x, y, z and t.

Suppose that the log pair $\left(X_5, \frac{1}{n}\mathcal{M}\right)$ is not canonical at some smooth point p. Then the point p must lie on the curve L contracted by the projection $\pi_4 : X_5 \dashrightarrow \mathbb{P}^3$ centered at the point O_w. By an additional coordinate change, we may assume that the curve L is defined by the equations $x = y = z = 0$, i.e., $L = L_{tw}$.

Let H be a general element in $|-K_{X_5}|$ containing the curve L_{tw}. Then the surface H is a $K3$ surface with an A_1 singularity at the point O_w. The self-intersection number of L_{tw} on H is $-\frac{3}{2}$.

We write

$$\mathcal{M}_H := \frac{1}{n}\mathcal{M}\big|_H = aL_{tw} + \mathcal{L},$$

where a is a non-negative rational number and $\mathcal{L}$ is a mobile linear system on H whose base locus does not contain the curve L_{tw}.

Choose another curve R that is contacted by the projection π_4. Note that such a curve is given by a point on the zero set in $\mathbb{P}^3$ defined by $x = f_3 = f_6 = 0$. Then we see that the intersection number L_{tw} and R is $\frac{1}{2}$. We then obtain

$$\frac{1}{2} = R \cdot \mathcal{M}_H = aL_{tw} \cdot R + \mathcal{L} \cdot R \geq \frac{a}{2},$$

and hence $a \leq 1$.

The log pair $\left(X_5, H + \frac{1}{n}\mathcal{M}\right)$ is not log canonical at the point p. By Inversion of Adjunction ([38, Theorem 5.50]), we see that the log pair $(H, \mathcal{M}_H)$ is not log canonical at the point p. We then obtain from [24, Theorem 3.1]

$$4(1-a) < \mathcal{L}^2 = (\mathcal{M}_H - aL_{tw})^2 = \mathcal{M}_H^2 - 2a\mathcal{M}_H \cdot L_{tw} + a^2 L_{tw}^2 = \frac{5}{2} - a - \frac{3a^2}{2}.$$

However, this inequality cannot be satisfied with any value of a. This completes the proof. □

In summary, we have verified

Theorem 2.1.10. *A smooth point on X cannot be a center of non-canonical singularities of the log pair $(X, \frac{1}{n}\mathcal{M})$.*

2.2 Excluding curves

We now show that an irreducible curve on X can not be a center of non-canonical singularities of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$ provided that no point on this curve is a center of non-canonical singularities of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$. Indeed, the proof comes from [25, pp. 206-207] and it is based on the following local result of Kawamata:

Theorem 2.2.1 ([36, Lemma 7])**.** *Let (U, p) be a germ of a threefold terminal quotient singularity of type $\frac{1}{r}(1, a, r-a)$, where $r \geq 2$ and a is coprime to r and let $\mathcal{M}_U$ be a mobile linear system on U. Suppose that $(U, \lambda\mathcal{M}_U)$ is not canonical at p for a positive rational number λ. Let $f\colon W \to U$ be the weighted blowup at the point p with weights $(1, a, r-a)$. Then*

$$\mathcal{M}_W = f^*(\mathcal{M}_U) - mE$$

for some positive rational number $m > \frac{1}{r\lambda}$, where E is the exceptional divisor of f and $\mathcal{M}_W$ is the proper transform of $\mathcal{M}_U$. In particular,

$$K_W + \lambda\mathcal{M}_W = f^*(K_U + \lambda\mathcal{M}_U) + \left(\frac{1}{r} - \lambda m\right)E,$$

where $\frac{1}{r} - \lambda m < 0$*, and hence the point* p *is a center of non-canonical singularities of the log pair* $(U, \lambda\mathcal{M}_U)$*.*

Note that, in this theorem, we do not assume that the point p is a center of non-canonical singularities of the log pair $(U, \lambda\mathcal{M}_U)$. A log pair may not be canonical at a point that is not a center of non-canonical singularities of the log pair. For example, consider the linear system $\mathcal{M}_{\mathbb{C}^3}$ generated by z_1^2 and z_2^2 on $\mathbb{C}^3$, where (z_1, z_2, z_3) is the standard coordinate system for $\mathbb{C}^3$. Then the log pair $(\mathbb{C}^3, \mathcal{M}_{\mathbb{C}^3})$ is not canonical at the origin. The line $z_1 = z_2 = 0$ is a center of non-canonical center of the log pair $(\mathbb{C}^3, \mathcal{M}_{\mathbb{C}^3})$. However, the origin is not a center of non-canonical center of the log pair $(\mathbb{C}^3, \mathcal{M}_{\mathbb{C}^3})$.

Theorem 2.2.1 and the mobility of the linear system $\mathcal{M}$ imply the following global properties.

Corollary 2.2.2 ([25, Lemma 5.2.1])**.** *Let* Λ *be a center of non-canonical singularities of the log pair* $\left(X, \frac{1}{n}\mathcal{M}\right)$*. In case when* Λ *is a singular point of type* $\frac{1}{r}(1, a, r-a)$*, let* $f\colon Y \to X$ *be the weighted blow up at* Λ *with weights* $(1, a, r-a)$*. In case when* Λ *is a smooth curve contained in the smooth locus of* X*, let* $f\colon Y \to X$ *be the blow up along* Λ*. Then the* 1*-cycle* $(-K_Y)^2 \in N_1(Y)$ *lies in the interior of the Mori cone of* Y*:*

$$(-K_Y)^2 \in \mathrm{Int}(\overline{\mathrm{NE}(Y)}).$$

Corollary 2.2.3 ([25, Corollary 5.2.3])**.** *Under the same notations as in Corollary 2.2.2, we have* $H \cdot (-K_Y)^2 > 0$ *for a non-zero nef divisor* H *on* Y*.*

Let L be an irreducible curve on X. Suppose that L is a center of non-canonical singularities of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$. Then it follows from Theorem 2.2.1 that every singular point of X contained in L (if any) must be a center of non-canonical singularities of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$. Later we will show that for a given singular point of X either it cannot be a center of non-canonical singularities of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$ or it can be untwisted by a birational involution (see Definition 3.3.1). Moreover, it will be done regardless of the fact that the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$ is canonical outside of this singular point. Therefore it is enough to exclude only irreducible curves contained in the smooth locus of X.

Suppose that L is contained in the smooth locus of X. Pick two general members H_1 and H_2 in the mobile linear system $\mathcal{M}$. Then we obtain

$$-n^2K_X^3 = -K_X \cdot H_1 \cdot H_2 \geq (\mathrm{mult}_L(\mathcal{M}))^2(-K_X \cdot L) > -n^2K_X \cdot L.$$

since we have $\mathrm{mult}_L(\mathcal{M}) > n$. Therefore, $-K_X \cdot L < -K_X^3$.

Since the curve L is contained in the smooth locus of X, we have $-K_X \cdot L \geq 1$. Therefore the curve L can exists only on the hypersurface X with $-K_X^3 > 1$ as a curve of degree less than $-K_X^3$. Such conditions can be satisfied only in the following cases:

- quasi-smooth hypersurface of degree 5 in $\mathbb{P}(1,1,1,1,2)$ with a curve L of degree 1 or 2;
- quasi-smooth hypersurface of degree 6 in $\mathbb{P}(1,1,1,2,2)$ with a curve L of degree 1;
- quasi-smooth hypersurface of degree 7 in $\mathbb{P}(1,1,1,2,3)$ with a curve L of degree 1.

Let $f\colon Y \to X$ be the blow up of the ideal sheaf of the curve L. Then Y is smooth whenever the curve L is smooth. As explained in [25, page 207] (it is independent of generality), in each of the three cases listed above, there exists a non-zero nef divisor M on Y such that $M \cdot K_Y^2 \leqslant 0$. Corollary 2.2.3 therefore shows that the curve L must be singular. Consequently, the curve L must be an irreducible curve of degree 2 in a quasi-smooth hypersurface of degree 5 in $\mathbb{P}(1,1,1,1,2)$. More precisely, the curve L has either an ordinary double point (which implies that Y has an ordinary double point on the exceptional divisor E) or L has a cusp (which implies that Y has an isolated double point that is locally given by $x^2+y^2+z^2+t^3=0$ in $\mathbb{C}^4$). In both the cases, we can proceed exactly as explained in [25, page 207] (the very end of the proof of [25, Theorem 5.1.1]) to obtain a contradiction.

In summary, so far we have proved

Theorem 2.2.4. *An irreducible curve contained in the smooth locus of X cannot be a center of non-canonical singularities of the log pair $\big(X, \frac{1}{n}\mathcal{M}\big)$.*

At this stage, we are therefore able to draw a conclusion that if the log pair $\big(X, \frac{1}{n}\mathcal{M}\big)$ is not canonical, then at least one singular point of X must be a non-canonical center of $\big(X, \frac{1}{n}\mathcal{M}\big)$.

3 Singular points

3.1 Cyclic quotient singular points

Let (U,p) be a germ of a cyclic quotient singular point of type $\frac{1}{r}(1,a,r-a)$, where r and a are relatively prime positive integers with $a<r$. We have an orbifold chart $\pi:(\hat{U},0)\to(U,p)$, where $\hat{U}$ is an open neighbourhood of the origin in $\mathbb{C}^3$ and the morphism π is the quotient map by the group action of $\mathbb{Z}/r\mathbb{Z}$. We say that functions z_1, z_2, z_3 on U induce local parameters at the point p if their pull-backs $\pi^*(z_1)$, $\pi^*(z_2)$, $\pi^*(z_3)$ are eigen-coordinate functions around the origin in $\hat{U}\subset\mathbb{C}^3$, corresponding to the weights 1, a, $r-a$.

Let $\tilde{f}:(\tilde{U},E)\to(U,p)$ be the local weighted blow up at the point p with weights $(1,a,r-a)$, where E is the exceptional divisor. The multiplicity of an effective Weil divisor D on U at the point p is defined by the number $\frac{m}{r}$ such that

$$\tilde{f}^*(D)=\tilde{D}+\frac{m}{r}E,$$

where $\tilde{D}$ is the proper transform of D by $\tilde{f}$. An analytic function $g(z_1,z_2,z_3)$ on U defines a divisor D on U. The vanishing order (or the multiplicity) of g at the point p is defined by the multiplicity of the divisor D at the point p. The multiplicity can be also obtained in the following way. The functions z_1, z_2, z_3 induce local parameters at the point p, so that we could assume that their pull-backs $\pi^*(z_1)$, $\pi^*(z_2)$, $\pi^*(z_3)$ are eigen-coordinate functions on $\mathbb{C}^3$ locally around the origin, corresponding to the weights 1, a, $r-a$, respectively. Counting the multiplicities of $\pi^*(z_1)$, $\pi^*(z_2)$, $\pi^*(z_3)$ as 1, a, $r-a$, respectively, we see that the multiplicity of g at the point p coincides with the number

$$\frac{1}{r}\mathrm{mult}_0(\pi^*(g(z_1,z_2,z_3)))$$

(see [42, Lemma 3.2.1]).

In the present paper, it is crucial to obtain the multiplicities of various quasi-homogeneous polynomials $G(x,y,z,t,w)$ at a singular point p on a given quasi-smooth hypersurface X. At the point p, we can always see that three, say z_1, z_2 and z_3, of the homogenous coordinates x, y, z, t, w induce local parameters at the point p. Locally around the point p, the quasi-homogeneous polynomial $G(x,y,z,t,w)$ induces a function $g(z_1,z_2,z_3)$ as a formal power series in variables z_1, z_2, z_3. The vanishing order (or the multiplicity) of G at the point p is defined by the multiplicity of $g(z_1,z_2,z_3)$ at the point p which is equal to the number $\frac{1}{r}\mathrm{mult}_0(\pi^*(g(z_1,z_2,z_3)))$ with counting the multiplicities of $\pi^*(z_1)$, $\pi^*(z_2)$, $\pi^*(z_3)$ as 1, a, $r-a$, respectively, as before.

3.2 Excluding singular points

This section provides the methods we apply when we exclude the singular points as centers of non-canonical singularities of the log pair $(X,\frac{1}{n}\mathcal{M})$.

Let p be a singular point of type $\frac{1}{r}(1,a,r-a)$ on X, where r and a are relatively prime positive integers with $a<r$. As mentioned in Section 1.3, the weighted blow up of X at the point p with weights $(1,a,r-a)$ will be denoted by $f\colon Y\to X$. Its exceptional divisor and the anticanonical divisor of Y will be denoted by E and B, respectively. We denote by $\mathcal{M}_Y$ the proper transform of the linear system $\mathcal{M}$ by the weighted blow up f. The pull-back of $-K_X$

will be denoted by A. The surface S is the proper transform of the surface on X cut by the equation $x = 0$.

Since the Picard group of X is generated by $-K_X$, the surface S is always irreducible. The surface S can be assumed to be $\mathbb{Q}$-linearly equivalent to B if one of the following conditions holds:

- $a_1 = 1$;
- $d - 1$ is not divisible by r.

If $a_1 > 1$ and $d - 1$ is divisible by r, then it is easy to check that S is always $\mathbb{Q}$-linearly equivalent to either B or $B - E$.

Before we explain how to show that p is not a center of non-canonical singularities of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$, let us prove the following statement slightly modified from Corollary 2.2.2.

Lemma 3.2.1. *Suppose that p is a center of non-canonical singularities of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$. Then the 1-cycle $B \cdot S \in N_1(Y)$ lies in the interior of the Mori cone of Y:*

$$B \cdot S \in \mathrm{Int}(\overline{\mathrm{NE}(Y)}).$$

Proof. It follows from Theorem 2.2.1 that

$$\mathcal{M}_Y \sim_{\mathbb{Q}} nB - \epsilon E$$

for some positive rational number ϵ. Since

$$S \cdot \mathcal{M}_Y = S \cdot (nB - \epsilon E)$$

is an effective 1-cycle, the 1-cycle $B \cdot S$ must lie in the interior of the Mori cone of Y because the 1-cycles $S \cdot E$ and $S \cdot B$ are not proportional in $N_1(Y)$ and the 1-cycle $S \cdot E$ generates the extremal ray contracted by f. □

We have two kinds of singular points on X. The singular points with $B^3 \leq 0$ are one kind and the singular points with $B^3 > 0$ are the other kind. Those with $B^3 \leq 0$ will be excluded as centers of non-canonical singularities of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$. Meanwhile, those with $B^3 > 0$ will be either excluded or untwisted (see Definition 3.3.1).

To exclude singular points with $B^3 \leq 0$, we mainly apply the following lemma. It is a slightly modified version of [25, Lemma 5.4.3].

Lemma 3.2.2. *Suppose that $B^3 \leq 0$ and there is an index i such that*

- *there is a surface T on Y such that $T \sim_{\mathbb{Q}} a_i A - \frac{m}{r} E$ with $a_i \geq m > 0$;*
- *the intersection $\Gamma = S \cap T$ consists of irreducible curves that are numerically proportional to each other;*
- *$T \cdot \Gamma \leq 0$.*

Then the point p is not a center of non-canonical singularities of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$.

Proof. Let $\Gamma = \sum e_i \tilde{C}_i$, where $e_i > 0$ and $\tilde{C}_i$'s are distinct irreducible and reduced curves. Let $\tilde{R}$ be the extremal ray of the Mori cone $\overline{\mathrm{NE}(Y)}$ of Y contracted by $f\colon Y \to X$.

Since the curves $\tilde{C}_i$ are numerically proportional to each other, each irreducible curve $\tilde{C}_i$ defines the same ray in the Mori cone of Y. None of the curves $\tilde{C}_i$ is contained in E because $T \cdot \tilde{C}_i \leq 0$ and $T \cdot E^2 < 0$. Therefore, the ray $\tilde{Q}$ defined by Γ cannot be $\tilde{R}$.

We first claim that the ray $\tilde{Q}$ is an extremal ray of $\overline{\mathrm{NE}(Y)}$, so that the Mori cone $\overline{\mathrm{NE}(Y)}$ could be spanned by $\tilde{R}$ and $\tilde{Q}$.

Since $\tilde{C}_i \not\subset E$ for each i, we have $E \cdot \tilde{C}_i \geq 0$. Therefore,

$$a_i B \cdot \Gamma \leq T \cdot \Gamma \leq 0,$$

where the first inequality follows from $a_i \geq m$.

If the surface T is nef, then $T \cdot \Gamma = 0$ and hence Γ is in the boundary of $\overline{\mathrm{NE}(Y)}$. Therefore, the ray $\tilde{Q}$ is an extremal ray of $\overline{\mathrm{NE}(Y)}$.

Suppose that the surface T is not nef and that the ray $\tilde{Q}$ is not an extremal ray. Then there is a curve $\tilde{C}$ with $T \cdot \tilde{C} < 0$ that generates a ray between $\tilde{Q}$ and the extremal ray other than $\tilde{R}$ since $T \cdot \tilde{R} = -\frac{m}{r} E \cdot \tilde{R} > 0$. It follows from $\tilde{C} \not\subset E$ that

$$S \cdot \tilde{C} \leq B \cdot \tilde{C} = \frac{1}{a_i}\left(T - \frac{a_i - m}{r} E\right) \cdot \tilde{C} < 0,$$

and hence $\tilde{C} \subset S \cap T$. Therefore, the curve $\tilde{C}$ must be one of the component of Γ, and hence it generates the ray $\tilde{Q}$. This is a contradiction. Therefore, $\tilde{Q}$ must be the extremal ray of $\overline{\mathrm{NE}(Y)}$ other than $\tilde{R}$.

If $B \cdot S \in \mathrm{Int}(\overline{\mathrm{NE}(Y)})$, then the ray

$$\tilde{Q} = \mathbb{R}_+ \left[S \cdot \left(a_i B + \frac{a_i - m}{r} E \right) \right]$$

cannot be a boundary of $\overline{\mathrm{NE}(Y)}$. Therefore, Lemma 3.2.1 implies that the point p cannot be a center of non-canonical singularities of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$. □

Remark 3.2.3. The condition $T \cdot \Gamma \leq 0$ is equivalent to the inequality

$$ra(r-a)a_i^2 A^3 \leq km^2,$$

where $k = 1$ if $S \sim_{\mathbb{Q}} B$; $k = r + 1$ otherwise.

We have singular points with $B^3 \leq 0$ to which we cannot apply Lemma 3.2.2 in a simple way. Such singular points are dealt with in a special way in [25, Subsections 5.7.2 and 5.7.3]. However, we are dealing with every quasi-smooth hypersurface, not only a general one and the method of [25] is too complicated for us to analyze the irreducible components of the intersections Γ, which is inevitable for our purpose. We here present another method that enables us to avoid such difficulty.

Lemma 3.2.4. *Suppose that there is a nef divisor T on Y with $T \cdot S \cdot B \leq 0$ and $T \cdot S \cdot A > 0$. Then the point p cannot be a center of non-canonical singularities of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$.*

Proof. Suppose that p is a center of non-canonical singularities of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$. Then it follows from Theorem 2.2.1 that

$$\frac{1}{n}\mathcal{M}_Y = f^*\left(\frac{1}{n}\mathcal{M}\right) - mE$$

with some rational number $m > \frac{1}{r}$. The intersection of the surface S and a general surface M_Y in the mobile linear system $\mathcal{M}_Y$ gives us an effective 1-cycle. However,

$$T \cdot S \cdot M_Y = nT \cdot S \cdot (A - mE) < nT \cdot S \cdot \left(A - \frac{1}{r}E\right) = nT \cdot S \cdot B \leq 0,$$

where the first inequality follows from $0 < T \cdot S \cdot A \leq \frac{1}{r}T \cdot S \cdot E$. This contradicts the condition that T is nef. □

Remark 3.2.5. For the divisor T equivalent to $cA - \frac{m}{r}E = cB + \frac{c-m}{r}E$ with some positive integers c and m, the condition $T \cdot S \cdot B \leq 0$ is equivalent to the inequality

$$ra(r-a)cA^3 \leq km,$$

where $k = 1$ if $S \sim_{\mathbb{Q}} B$; $k = r+1$ otherwise. The condition $T \cdot S \cdot A > 0$ is always satisfied by any divisor T equivalent to $cA - \frac{m}{r}E$ with positive integers c.

To apply Lemma 3.2.4, we construct a nef divisor T in $|cB + bE|$ for some integers $c \geq 0$ and $b \leq \frac{c}{r}$. To construct a nef divisor T the following will be useful.

Lemma 3.2.6. *Let $\mathcal{L}_X$ be a mobile linear subsystem in $|-cK_X|$ for some positive integer c. Denote the proper transforms of the base curves of the linear system $\mathcal{L}_X$ on Y by $\tilde{C}_1, \ldots, \tilde{C}_s$ (if any). Let T be the proper transform of a general surface in $\mathcal{L}_X$. Then the following hold.*

- *The divisor T belongs to $|cB + bE|$ for some integer b not greater than $\frac{c}{r}$.*
- *The divisor T is nef if $T \cdot \tilde{C}_i \geqslant 0$ for every i. In particular, it is nef if the base locus of $\mathcal{L}_X$ contains no curves.*

Proof. Since $T \sim_{\mathbb{Q}} cA - \frac{m}{r}E$ for some non-negative integer m and $B \sim_{\mathbb{Q}} A - \frac{1}{r}E$, we obtain $T \sim_{\mathbb{Q}} cB + \frac{c-m}{r}E$. The number $b := \frac{c-m}{r}$ must be an integer because the divisor class group of Y is generated by B and E.

Suppose that T is not nef. Then there exists a curve $\tilde{C} \subset Y$ such that $T \cdot \tilde{C} < 0$, which implies that the curve $\tilde{C}$ is contained in the base locus of the proper transform of the linear system $\mathcal{L}_X$. Since $E \cong \mathbb{P}(1, a, r-a)$, $\mathcal{O}_E(E) = \mathcal{O}_E(-r)$ and $b \leq \frac{c}{r}$, the divisor $T|_E$ is nef, and hence $\tilde{C} \not\subset E$. We then draw an absurd conclusion that $\tilde{C}$ is one of the curves $\tilde{C}_1, \ldots, \tilde{C}_s$. □

With Lemma 3.2.4 we can easily exclude the singular points that are taken special cares in [25, 5.7.2 and 5.7.3]. However, in spite of our new methods, we encounter special cases that cannot be excluded by the methods proposed so far. To deal with these special cases, we apply the following two lemmas.

Lemma 3.2.7. *Suppose that the surface S is $\mathbb{Q}$-linearly equivalent to B and there is a normal surface T on Y such that the support of the 1-cycle $S|_T$ consists of curves on T whose intersection form is negative-definite. Then the singular point p cannot be a center of non-canonical singularities of the pair $\left(X, \frac{1}{n}\mathcal{M}\right)$.*

Proof. Put $S|_T = \sum c_i\tilde{C}_i$, where c_i's are positive numbers and $\tilde{C}_i$'s are distinct irreducible and reduced curves on the normal surface T. Suppose that the point p is a center of non-canonical singularities of the log pair $(X, \frac{1}{n}\mathcal{M})$. Then we have

$$K_Y + \frac{1}{n}\mathcal{M}_Y + cE = f^*\left(K_X + \frac{1}{n}\mathcal{M}\right) \sim_{\mathbb{Q}} 0,$$

where c is a positive constant. Therefore, we obtain $\mathcal{M}_Y + ncE \sim_{\mathbb{Q}} nS$, and hence

$$(\mathcal{M}_Y + ncE)\,\big|_T \sim_{\mathbb{Q}} n\sum c_i\tilde{C}_i.$$

We may write the left-hand side as

$$(\mathcal{M}_Y + ncE)\,\big|_T = \sum a_j\tilde{D}_j + \sum b_i\tilde{C}_i,$$

where each $\tilde{D}_j$ is an irreducible curves on T different from $\tilde{C}_i$ and a_j, b_i are positive rational numbers. Note that $\sum a_j\tilde{D}_j$ cannot be a zero divisor because $\mathcal{M}_Y$ is a mobile linear system. We then obtain

$$\sum a_j\tilde{D}_j + \sum_{nc_i-b_i<0} -(nc_i-b_i)\tilde{C}_i \sim_{\mathbb{Q}} \sum_{nc_i-b_i>0}(nc_i-b_i)\tilde{C}_i.$$

Therefore,

$$\left(\sum a_j\tilde{D}_j + \sum_{nc_i-b_i<0} -(nc_i-b_i)\tilde{C}_i\right)\cdot\left(\sum_{nc_i-b_i>0}(nc_i-b_i)\tilde{C}_i\right) = \left(\sum_{nc_i-b_i>0}(nc_i-b_i)\tilde{C}_i\right)^2.$$

However, since the divisor $\sum\tilde{C}_i$ is negative-definite and $\sum_{nc_i-b_i>0}(nc_i-b_i)\tilde{C}_i$ cannot be a zero divisor on T, the equality is absurd. □

Lemma 3.2.8. *Suppose that there is a one-dimensional family of irreducible curves $\tilde{C}_\lambda$ on Y with $E\cdot\tilde{C}_\lambda > 0$ and $-K_Y\cdot\tilde{C}_\lambda \leq 0$. Then the singular point p cannot be a center of non-canonical singularities of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$.*

Proof. We have

$$K_Y + \frac{1}{n}\mathcal{M}_Y = f^*\left(K_X + \frac{1}{n}\mathcal{M}\right) + cE$$

with a negative number c. Suppose that there is a one-dimensional family of curves $\tilde{C}_\lambda$ on Y with $E\cdot\tilde{C}_\lambda > 0$ and $-K_Y\cdot\tilde{C}_\lambda \leq 0$. Then for each member $\tilde{C}_\lambda$, we have

$$\mathcal{M}_Y\cdot\tilde{C}_\lambda = -nK_Y\cdot\tilde{C}_\lambda + cnE\cdot\tilde{C}_\lambda \leq cnE\cdot\tilde{C}_\lambda < 0,$$

and hence the curve $\tilde{C}_\lambda$ is contained in the base locus of the linear system $\mathcal{M}_Y$. This is a contradiction since the linear system $\mathcal{M}_Y$ is mobile. □

Notice that Lemmas 3.2.4, 3.2.7 and 3.2.8 do not require B^3 to be non-positive. Therefore, these lemmas can be applied to exclude the singular points with $B^3 > 0$.

For example, the lemma below, which follows from Lemma 3.2.8, excludes all the singular points with $B^3 > 0$, except O_z in the family No. 62, that appear in Theorem 1.1.10. The exception, the singular point O_z in the family No. 62, can be also treated in the same way as Lemma 3.2.9. The only difference is that the variable z plays the role of t in Lemma 3.2.9.

Lemma 3.2.9. *Suppose that the hypersurface X is given by a quasi-homogeneous equation*

$$w^2 + x_i t^k + w f_{d-a_4}(x, x_i, x_j, t) + f_d(x, x_i, x_j, t) = 0$$

of degree d, where one of the variables x_i and x_j is y and the other is z. Let a_i and a_j be the weights of the variables x_i and x_j, respectively. If $2a_4 = 3a_3 + a_i$, then the singular point O_t cannot be a center of non-canonical singularities of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$.

Proof. The singular point O_t is of type $\frac{1}{a_3}(1, a_j, a_4 - a_3)$. Local parameters at O_t are induced by x, x_j, w with multiplicities $\frac{1}{a_3}$, $\frac{a_j}{a_3}$, $\frac{a_4-a_3}{a_3}$.

Let T be the proper transform of the surface S_{x_i} on X cut by the equation $x_i = 0$. Due to the monomial w^2, we see that the surface S_{x_i} has multiplicity $\frac{2(a_4-a_3)}{a_3}$ at the point O_t. Therefore, the surface T belongs to $|a_i B - E|$ since $2a_4 = 3a_3 + a_i$.

Let C_λ be the curve on the surface S_{x_i} defined by

$$\begin{cases} x_i = 0, \\ x_j = \lambda x^{a_j} \end{cases}$$

for a sufficiently general complex number λ. Then the curve C_λ is a curve of degree d in $\mathbb{P}(1, a_3, a_4)$ defined by the equation

$$w^2 + w f_{d-a_4}(x, 0, \lambda x^{a_j}, t) + f_d(x, 0, \lambda x^{a_j}, t) = 0.$$

Then

$$-K_Y \cdot \tilde{C}_\lambda = a_j B^2 \cdot (a_i B - E) = a_1 a_2 A^3 - \frac{2a_j(a_4 - a_3)}{a_3^3} E^3 = \frac{2}{a_3} - \frac{2}{a_3} = 0.$$

If the curve $\tilde{C}_\lambda$ is reducible, it consists of two irreducible components that are numerically equivalent since the two components of the curve C_λ are symmetric with respect to the biregular quadratic involution of X defined by

$$[x : y : z : t : w] \mapsto [x : y : z : t : -f_{d-a_4}(x, y, z, t) - w].$$

Then each component of $\tilde{C}_\lambda$ intersects $-K_Y$ trivially. Consequently, Lemma 3.2.8 implies the statement. □

3.3 Untwisting singular points

Excluding methods are introduced in the previous section. Now we explain how to deal with singular points of X that require some treatments by birational automorphisms of X. For us to prove Main Theorem, for a given singular point either it should be excluded as a center of non-canonical singularities of the log pair $(X, \frac{1}{n}\mathcal{M})$ or it should be untwisted as a center of non-canonical singularities of the log pair $(X, \frac{1}{n}\mathcal{M})$. Untwisting is defined as follows:

Definition 3.3.1. Let τ be a birational automorphism of X. Suppose that a singular point p of X is a center of non-canonical singularities of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$. We say that the birational automorphism τ *untwists* the point p (as a center of non-canonical singularities of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$) if

- the birational automorphism τ is not biregular;
- there exists a biregular in codimension one birational automorphism τ_Y of Y that fits the commutative diagram

$$\begin{array}{ccc} Y & \overset{\tau_Y}{\dashrightarrow} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ X & \underset{\tau}{\dashrightarrow} & X. \end{array}$$

In fact, this is a special case of a Sarkisov link of Type II (cf. [25, Definition 3.1.4]). The reason why such a birational automorphism is said to untwist a singular point is that it improves the singularities of the mobile linear system $\mathcal{M}$. This improvement results from the following property of such a birational automorphism.

Lemma 3.3.2. *Suppose that a singular point p of X is a center of non-canonical singularities of the log pair $(X\frac{1}{n}\mathcal{M})$ and that there exists a birational automorphism τ of X that untwists the point p as a center of non-canonical singularities of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$. Then $\tau(\mathcal{M}) \subset |-n_\tau K_X|$ for some positive integer $n_\tau < n$.*

Proof. Put $\tau_Y = f^{-1}\circ\tau\circ f$. Then τ_Y is biregular in codimension one and τ_Y is not biregular. In particular, τ_Y acts on the Picard group $\mathrm{Pic}(Y)$. Then $\tau_Y(B) = B$ since $B = -K_Y$. However, $\tau_Y(E) \neq E$ since τ_Y is biregular in codimension one. Indeed, if $\tau_Y(E) = E$, then τ is also biregular in codimension one. Then [23, Proposition 3.5] implies that τ is biregular since $\mathrm{Pic}(X) \cong \mathbb{Z}$.

On the other hand, we have

$$f^*(\mathcal{M}) = \mathcal{M}_Y + mE,$$

for some positive rational number m. Furthermore, $m > \frac{n}{r}$ by Theorem 2.2.1. Since τ_Y acts on $\mathrm{Pic}(Y)$, there are rational numbers a, b, c, d such that $a, c > 0$ and

$$\begin{cases} \tau_Y(A) = aA - bE, \\ \tau_Y(E) = cA - dE. \end{cases}$$

Since $\tau_Y(B) = B$, we obtain

$$A - \frac{1}{r}E = \tau_Y\left(A - \frac{1}{r}E\right) = \tau_Y(A) - \frac{1}{r}\tau_Y(E) = \left(a - \frac{c}{r}\right)A - \left(b - \frac{d}{r}\right)E,$$

and hence $a - \frac{c}{r} = 1$. We then see

$$\tau_Y(\mathcal{M}_Y) = \tau_Y(nA - mE) = n\tau_Y(A) - m\tau_Y(E) = (na - mc)A - (nb - md)E.$$

Since

$$na - mc = na - m(ar - r) = na - mr(a-1) < na - n(a-1) = n,$$

we obtain $\tau(\mathcal{M}) \subset |-n_\tau K_X|$ with $n_\tau < n$. This proves the statement. □

Thus, to complete the proof of Main Theorem after Theorems 2.1.10 and 2.2.4, it is enough to show that every singular point of X either is not a center of non-canonical singularities of the log pair $(X\frac{1}{n}\mathcal{M})$ or can be untwisted by some appropriate birational automorphism of X. This follows from Theorem 1.2.2 and Lemma 3.3.2 with induction on n. The appropriate birational automorphisms to untwist singular points are introduce in the following chapter.

Remark 3.3.3. The proof of Lemma 3.3.2 shows that in order to find a birational automorphism of X untwisting the center p, it is enough to find a biregular in codimension one birational automorphism τ_Y of Y such that $\tau_Y(E) \neq E$. Indeed, this untwisting birational automorphism is defined by $\tau = f \circ \tau_Y \circ f^{-1}$.

As in [25], in the case when a singular point of X is untwisted by some birational automorphism of X, it can be untwisted by a very explicit birational involution. Since X has only finitely many singular points, there are finitely many such involutions for a given hypersurface X. These birational automorphisms generate a subgroup, denoted by Γ_X, in the birational automorphism group $\mathrm{Bir}(X)$. Using [23, Theorem 4.2] instead of Theorem 1.2.2, we prove

Theorem 3.3.4. *Let X be a quasi-smooth hypersurface of degrees d with only terminal singularities in the weighted projective space $\mathbb{P}(1, a_1, a_2, a_3, a_4)$, where $d = \sum a_i$. Then the birational automorphism group of X is generated by the subgroup Γ_X and the biregular automorphism group of X.*

In the case when X is a general hypersurface in its family, Theorem 3.3.4 is proved in [25] (see [25, Remark 1.4]).

4 Birational involutions

4.1 Quadratic involution

In many cases, explicit birational automorphisms arise from generically 2-to-1 rational maps of X onto appropriate 3-dimensional weighted projective spaces. The birational automorphism constructed by interchanging the two points on a generic fiber of the generically 2-to-1 rational map is called a quadratic involution.

Lemma 4.1.1 ([25, Theorem 4.9])**.** *Suppose that the hypersurface X is given by*

$$x_{i_3}x_{i_4}^2 + f_e x_{i_4} + g_d = 0, \tag{4.1.2}$$

where x_{i_4}, x_{i_3} are two of the coordinates and f_e, g_d are quasi-homogeneous polynomials of degrees e and d not involving x_{i_4}. In addition, suppose that the polynomial f_e is not divisible by x_{i_3}. Then interchanging the roots of the equation with respect to x_{i_4} defines a birational involution $\tau_{O_{x_{i_4}}}$ of X. The involution $\tau_{O_{x_{i_4}}}$ untwists the point $O_{x_{i_4}}$ as a center of non-canonical singularities of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$.

Proof. If the the polynomial f_e is not divisible by x_{i_3}, the equations $x_{i_3} = f_e = g_d = 0$ define a finitely many lines passing through the point $O_{x_{i_4}}$. The statement then follows from the proof of [25, Theorem 4.9]. □

Now we suppose that f_e in (4.1.2) is divisible by x_{i_3}. Then we are able to write $f_e = 2x_{i_3}g$ for some polynomial g not involving x_{i_4}. Therefore, we obtain

$$x_{i_3}x_{i_4}^2 + f_e x_{i_4} + g_d = x_{i_3}\left(x_{i_4}^2 + 2gx_{i_4}\right) + g_d = x_{i_3}\left(x_{i_4} + g\right)^2 - x_{i_3}g^2 + g_d.$$

Using the change of coordinate $x_{i_4} + g \mapsto x_{i_4}$, we see that the singular point $O_{x_{i_4}}$ on the hypersurface of X defined by (4.1.2) with f_e divisible by x_{i_3} can be excluded by the following lemma.

Lemma 4.1.3. *Suppose that the hypersurface X is given by*

$$x_{i_3}x_{i_4}^2 + x_{i_3}g_e(x, x_{i_1}, x_{i_2}, x_{i_3}) + h_d(x, x_{i_1}, x_{i_2}) = 0,$$

where x_{i_k}'s are the coordinates of $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ different from x. If the weights of x_{i_1}, x_{i_2} are less than the weight of x_{i_4}, then the singular points $O_{x_{i_4}}$ cannot be a center of non-canonical singularities of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$.

Proof. Note that the quasi-homogeneous polynomial h_d must be irreducible. Indeed, if it is reducible, then we may write $h_d(x, x_{i_1}, x_{i_2}) = g_{d_1}(x, x_{i_1}, x_{i_2})g_{d_2}(x, x_{i_1}, x_{i_2})$ for some non-constant polynomials g_{d_1} and g_{d_2}. Then, the hypersurface X is not quasi-smooth at the points defined by $x_{i_3} = x_{i_4}^2 + g_e(x, x_{i_1}, x_{i_2}, x_{i_3}) = g_{d_1}(x, x_{i_1}, x_{i_2}) = g_{d_2}(x, x_{i_1}, x_{i_2}) = 0$.

Let T be the proper transform on Y of the surface $S_{x_{i_3}}$ cut by $x_{i_3} = 0$. The singular point $O_{x_{i_4}}$ is of type $\frac{1}{a_{i_4}}(1, a_{i_1}, a_{i_2})$. Since local parameters at $O_{x_{i_4}}$ are induced by x, x_{i_1}, x_{i_2} whose multiplicities are $\frac{1}{a_{i_4}}$, $\frac{a_{i_1}}{a_{i_4}}$,$\frac{a_{i_2}}{a_{i_4}}$, respectively, and the polynomial $h_d(x, x_{i_1}, x_{i_2})$ cannot be zero, the surface cut by $x_{i_3} = 0$ has multiplicity $\frac{d}{a_{i_4}}$ at $O_{x_{i_4}}$. Therefore, the surface T belongs to $|a_{i_3}B - 2E|$ since $a_{i_3} + 2a_{i_4} = d$.

Let C_λ be the curve on the surface $S_{x_{i_3}}$ defined by

$$\begin{cases} x_{i_3} = 0, \\ x_{i_2} = \lambda x^{a_{i_2}} \end{cases}$$

for a sufficiently general complex number λ. Then the curve C_λ is a curve of degree d in $\mathbb{P}(1, a_{i_1}, a_{i_4})$ defined by equation

$$h_d(x, x_{i_1}, \lambda x^{a_{i_2}}) = 0.$$

To obtain a one-dimensional family of irreducible curves on Y that is required for Lemma 3.2.8, we claim that every curve on T intersects B non-negatively. To this end, we consider the linear system $\mathcal{L}$ on X given by the monomials $x^{a_{i_1}+a_{i_2}}$, $x_{i_1}x_{i_2}$, $x^{a_{i_1}}x_{i_2}$, $x^{a_{i_2}}x_{i_1}$. The proper transform of a surface in $\mathcal{L}$ is equivalent to $(a_{i_1} + a_{i_2})B$. The base locus of the proper transform $\mathcal{L}_Y$ of the linear system $\mathcal{L}$ consists of the proper transform of the curve cut by $x = x_{i_1} = 0$ and the proper transform of the curve cut by $x = x_{i_2} = 0$.

Suppose that we have a curve R on T such that $B \cdot R < 0$. Since the linear system $\mathcal{L}_Y$ is free outside the proper transforms of the curve cut by $x = x_{i_1} = 0$ and the curve by $x = x_{i_2} = 0$, one of the proper transforms must contain the curve R. Therefore, the curve R on the surface T should be either the proper transform $\tilde{L}_{24}$ of the curve L_{24} defined by $x = x_{i_1} = 0$ and $x_{i_3} = 0$ or the proper transform $\tilde{L}_{14}$ of the curve L_{14} defined by $x = x_{i_2} = 0$ and $x_{i_3} = 0$. However, since $E \cdot \tilde{L}_{24} = \frac{1}{a_{i_2}}$ and $E \cdot \tilde{L}_{14} = \frac{1}{a_{i_1}}$, we obtain

$$B \cdot \tilde{L}_{24} = \left(A - \frac{1}{a_{i_4}}E\right) \cdot \tilde{L}_{24} = \frac{1}{a_{i_2}a_{i_4}} - \frac{1}{a_{i_4}a_{i_2}} = 0;$$
$$B \cdot \tilde{L}_{14} = \left(A - \frac{1}{a_{i_4}}E\right) \cdot \tilde{L}_{14} = \frac{1}{a_{i_1}a_{i_4}} - \frac{1}{a_{i_4}a_{i_1}} = 0.$$

This verifies the claim. Then from the equation

$$-K_Y \cdot \tilde{C}_\lambda = a_{i_2}B^2 \cdot (a_{i_3}B - 2E) = 0$$

we obtain a one-dimensional family of irreducible curves on Y that is required for Lemma 3.2.8. It then follows from Lemma 3.2.8 that $O_{x_{i_4}}$ cannot be a center of non-canonical singularities of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$. □

Theorem 4.1.4. *Suppose that the weights a_3, a_4 are relatively prime and $2a_3 + a_4 = d$. In addition, the equation of the hypersurface X does not involves the monomial wt^2. Then the singular point O_t cannot be a center of non-canonical singularities of the log pair $(X, \frac{1}{n}\mathcal{M})$.*

Proof. We first note that the singular point O_t of the hypersurface X is of type $\frac{1}{a_3}(1, a_1, a_2)$. The hypersurface X may be assumed to be defined by the equation

$$x_it^3 + t^2g_{a_4}(x, y, z) + twg_{a_3}(x, y, z) + tg_{a_3+a_4}(x, y, z)+$$
$$+w^2g_{d-2a_4}(x, y, z) + wg_{2a_3}(x, y, z) + g_d(x, y, z) = 0,$$

where x_i is either y or z. We let x_j be z if x_i is y and vice versa. By a suitable coordinate change (if necessary), we may assume that the polynomial g_{d-2a_4} contains the monomial x_j.

Suppose that the singular point O_t is a center of non-canonical singularities of the log pair $(X, \frac{1}{n}\mathcal{M})$. Consider the linear system $\mathcal{L}$ on X generated by x^e and x_j, where e is the weight of x_j. The proper transform of each member of $\mathcal{L}$ is $\mathbb{Q}$-linearly equivalent to eB. The base locus of the linear system $\mathcal{L}$ consists of the curve cut by $x = x_j = 0$. It consists of the curve L_{tw} and its residual curve R. Note that the residual curve R cannot pass through the point O_t since we have the monomial $x_i t^3$. Therefore,

$$B \cdot \tilde{L}_{tw} = eB^3 + K_X \cdot R = \frac{2ea_3 + ea_4}{a_1a_2a_3a_4} - \frac{e}{a_1a_2a_3} - \frac{3ea_3}{a_1a_2a_3a_4} = -\frac{e}{a_1a_2a_4}.$$

Let T be the proper transform of the surface on X cut by the equation $x_i = 0$. In addition, let $\tilde{S}_\lambda$ be the proper transform of the surface on X cut by the equation $x_j - \lambda x^e = 0$ for a general constant λ. The intersection 1-cycle of the surface on X cut by the equation $x_i = 0$ and the surface on X cut by the equation $x_j - \lambda x^e = 0$ is defined in $\mathbb{P}(1, a_3, a_4)$ by the equation

$$t^2 g_{a_4}(x, 0, \lambda x^e) + twg_{a_3}(x, 0, \lambda x^e) + tg_{a_3+a_4}(x, 0, \lambda x^e) +$$
$$+ w^2 g_{d-2a_4}(x, 0, \lambda x^e) + wg_{2a_3}(x, 0, \lambda x^e) + g_d(x, 0, \lambda x^e) = 0$$

if $x_i = y$,

$$t^2 g_{a_4}(x, \lambda x^e, 0) + twg_{a_3}(x, \lambda x^e, 0) + tg_{a_3+a_4}(x, \lambda x^e, 0) +$$
$$+ w^2 g_{d-2a_4}(x, \lambda x^e, 0) + wg_{2a_3}(x, \lambda x^e, 0) + g_d(x, \lambda x^e, 0) = 0$$

if $x_i = z$. Since g_{d-2a_4} contains the monomial x_j, the equation in both the cases is divisible by x^e but not by x^{e+1}. This implies that

$$T \cdot \tilde{S}_\lambda = e\tilde{L}_{tw} + \tilde{R}_\lambda = (2a_3 - a_4)\tilde{L}_{tw} + \tilde{R}_\lambda,$$

where $\tilde{R}_\lambda$ is the residual curve. The multiplicity of the surface cut by $x_i = 0$ along E is determined by the monomial $w^2 x_j$. It is $\frac{e+2e'}{a_3}$, where e' is the weight of x_i, since the multiplicity of w is $\frac{e'}{a_3}$ and that of x_j is $\frac{e}{a_3}$. Therefore, the surface T is equivalent to $e'B - E$ since $e + e' = a_1 + a_2 = a_3$. Then

$$B \cdot \tilde{R}_\lambda = eB^2 \cdot T - (2a_3 - a_4)B \cdot \tilde{L}_{tw} = \frac{a_1a_2 + ea_3 - ea_4}{a_1a_2a_4} = 0$$

since $a_4 = e' + a_3$ and $ee' = a_1a_2$. We then obtain a contradiction from Lemma 3.2.8. □

4.2 Elliptic involution

Another way to obtain an involution is from an elliptic fibration with a section and the group structure on its generic fiber. We can, roughly speaking, construct the involution by sending every point to its inverse point with respect to the group structure. The involution constructed in this way is called an elliptic involution.

Proposition 4.2.1. *Let $\pi : W \to \Sigma$ be an elliptic fibration over a normal surface Σ with a section F. Then there is a birational involution τ_W of W such that it induces the elliptic involution with respect to the point $C \cap F$ on a general fiber C.*

Proof. Let W_ζ be the (scheme) fiber of π over a generic point ζ of Σ. Then W_ζ is a smooth geometrically irreducible curve over the rational function field $\mathbb{K}$ of Σ over $\mathbb{C}$, which is birational to a cubic curve on $\mathbb{P}^2_{\mathbb{K}}$. Since F is a section of π, it defines a $\mathbb{K}$-rational point of the curve W_ζ. We denote this point by F_ζ. Thus, W_ζ is an elliptic curve defined over $\mathbb{K}$. To be precise, W_ζ has a group structure such that the $\mathbb{K}$-rational point F_ζ is its identity and all the group operations are morphisms defined over $\mathbb{K}$ (see, for example, [47, Theorem 3.6 in Chapter III]). This group structure gives an involution τ_{W_ζ} of W_ζ that sends every $\mathbb{K}$-rational point to its inverse. By construction, the involution τ_{W_ζ} is a biregular automorphism of the curve W_ζ defined over $\mathbb{K}$ that leaves the point F_ζ fixed. Since the rational function field of W over $\mathbb{C}$ and the rational function filed of W_ζ over $\mathbb{K}$ are naturally isomorphic as $\mathbb{C}$-algebras, the involution τ_{W_ζ} defines a $\mathbb{C}$-algebra involution of the rational function field of W that leaves the subfield $\mathbb{K}$ fixed. Therefore, it induces a birational involution $\tau_W \in \mathrm{Bir}(W)$ such that the diagram

$$\begin{array}{ccc} W & \overset{\tau_W}{\dashrightarrow} & W \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \Sigma & \end{array}$$

commutes. □

Taken the Weierstrass equation of an elliptic curve into consideration, an elliptic involution can be also regarded as a quadratic involution. Because its expression in polynomials becomes extremely complicated after weighted blow ups and log flips (see (4.2.11)), it is difficult to see the virtue of an elliptic involution from the point of view of a quadratic involution.

In this section, we deal with the singular point O_t on each quasi-smooth hypersurface in the families No. 23, 40, 44, 61, 76, and the singular point O_z on each quasi-smooth hypersurface in the families No. 20, 36. Also, the singular points of type $\frac{1}{2}(1,1,1)$ on each quasi-smooth hypersurface in the family No. 7 are treated. These singular points on general hypersurfaces in such families are untwisted by birational involutions induced by the elliptic fibration models in [25, 4.10]. This section deals with these singular points on *every* quasi-smooth hypersurface in the families mentioned above with the more *geometric* point of view.

Before we proceed, we divide the family No. 7, quasi-smooth hypersurfaces X_8 of degree 8 in $\mathbb{P}(1,1,2,2,3)$, into two types.

Proposition 4.2.2. *Let X_8 be a quasi-smooth hypersurface of degree 8 in $\mathbb{P}(1,1,2,2,3)$. Then it may be assumed to be defined by an equation of one of the following forms*

$$\begin{aligned} &\textit{Type I:}\\ &tw^2 + wg_5(x,y,z) - zt^3 - t^2g_4(x,y,z) - tg_6(x,y,z) + g_8(x,y,z) = 0;\\ &\textit{Type II:}\\ &(z+f_2(x,y))w^2 + wf_5(x,y,z,t) - zt^3 - t^2f_4(x,y,z) - tf_6(x,y,z) + f_8(x,y,z) = 0. \end{aligned} \tag{4.2.3}$$

In the latter equation, the quasi-homogeneous polynomial f_5 must contain either xt^2 or yt^2.

Proof. Let $F(x,y,z,t,w)$ be a quasi-homogeneous polynomial of degree 8 that defines the hypersurface X_8. The hypersurface X_8 has exactly four singular points of type $\frac{1}{2}(1,1,1)$. They correspond to the four solutions to the equation $F(0,0,z,t,0) = 0$ and they are located along the curve L_{zt}. Let p be one of the singular points. By a coordinate change, we may assume that p is the point O_t. Then the polynomial F does not contain the monomial t^4. Therefore, we may write

$$F(x,y,z,t,w) = w^2A_2(x,y,z,t) + w\left(2t^2B_1(x,y,z) + 2tB_3(x,y,z) + B_5(x,y,z)\right) + \\ + t^3B_2(x,y,z) - t^2B_4(x,y,z) - tB_6(x,y,z) + B_8(x,y,z),$$

where $A_i(x,y,z,t)$ is a quasi-homogeneous polynomial of degree i in x, y, z, t and $B_j(x,y,z)$ is a quasi-homogeneous polynomial of degree j in variables x, y, z.

Now we have two kinds of possibility for $A_2(x,y,z,t)$. The first possibility is that $A_2(x,y,z,t)$ contains the monomial t (this is a general case). In this case, we may assume that $A_2(x,y,z,t) = t$ by the coordinate change $A_2(x,y,z,t) \mapsto t$. Note that

$$t\left(w^2 + 2wtB_1(x,y,z) + 2wB_3(x,y,z)\right)$$

$$= t\left(w + tB_1(x,y,z) + B_3(x,y,z)\right)^2 - t\left(tB_1(x,y,z) + B_3(x,y,z)\right)^2.$$

By the coordinate change $w + tB_1(x,y,z) + B_3(x,y,z) \mapsto w$, we may assume that

$$F(x,y,z,t,w) = tw^2 + wB_5(x,y,z) + \\ + t^3B_2(x,y,z) - t^2B_4(x,y,z) - tB_6(x,y,z) + B_8(x,y,z).$$

The second possibility is that $A_2(x,y,z,t)$ does not contain the monomial t (this is a special case). In this case, it must contain the monomial z since X_8 is quasi-smooth at O_w. We may then write $A_2(x,y,z,t) = z + f_2(x,y)$.

Since X_8 is quasi-smooth at O_t, B_2 must contain the monomial z. Therefore, by the coordinate change $B_2(x,y,z) \mapsto -z$, we see that the quasi-homogeneous polynomial F can be written in either Type I or Type II.

In the equation of Type II, the quasi-homogeneous polynomial f_5 must contain either xt^2 or yt^2. If not, then the hypersurface X_8 is not quasi-smooth at the point $[0:0:0:1:1]$. □

The hypersurface in the family No. 7 defined by the equation of Type II may have an involution that untwists the singular point O_t. Since its construction is quite complicated, we explain the method in a separate section.

First, we consider the following six families and their singular point O_t.

- No. 7 (Type I), $X_8 \subset \mathbb{P}(1,1,2,2,3)$;
- No. 23, $X_{14} \subset \mathbb{P}(1,2,3,4,5)$;
- No. 40, $X_{19} \subset \mathbb{P}(1,3,4,5,7)$;
- No. 44, $X_{20} \subset \mathbb{P}(1,2,5,6,7)$;
- No. 61, $X_{25} \subset \mathbb{P}(1,4,5,7,9)$;
- No. 76, $X_{30} \subset \mathbb{P}(1,5,6,8,11)$.

For these six families, we may assume that the hypersurface X is defined by the equation

$$tw^2 + wg_{d-a_4}(x,y,z) - x_i t^3 - t^2 g_{d-2a_3}(x,y,z) - tg_{d-a_3}(x,y,z) + g_d(x,y,z) = 0, \qquad (4.2.4)$$

where x_i is either y or z.

Put $y = \lambda_1 x^{a_1}$ and $z = \lambda_2 x^{a_2}$. We then consider the curve C_{λ_1,λ_2} defined by

$$\begin{aligned} &tw^2 + wg_{d-a_4}(x, \lambda_1 x^{a_1}, \lambda_2 x^{a_2}) - \lambda_i x^{a_i} t^3 \\ &- t^2 g_{d-2a_3}(x, \lambda_1 x^{a_1}, \lambda_2 x^{a_2}) - tg_{d-a_3}(x, \lambda_1 x^{a_1}, \lambda_2 x^{a_2}) + g_d(x, \lambda_1 x^{a_1}, \lambda_2 x^{a_2}) = 0, \end{aligned} \qquad (4.2.5)$$

where $i = 1$ if $x_i = y$; $i = 2$ if $x_i = z$, in $\mathbb{P}(1, a_3, a_4)$. From now let x_j be the variable such that $\{x_i, x_j\} = \{y, z\}$. If $x_i = y$, then put $a_i = a_1$ and $\lambda_i = \lambda_1$. If $x_i = z$, then put $a_i = a_2$ and $\lambda_i = \lambda_2$. Also we define a_j and λ_j in the same manner.

Theorem 4.2.6. *Let X be a quasi-smooth hypersurface in the families No. 7 (Type I), 23, 40, 44, 61, 76. If the singular point O_t is a center of non-canonical singularities of the log pair $(X, \frac{1}{n}\mathcal{M})$, then it is untwisted by a birational involution.*

Proof. Let $\pi\colon X \dashrightarrow \mathbb{P}(1, a_1, a_2)$ be the rational map induced by

$$[x : y : z : t : w] \mapsto [x : y : z].$$

It is a morphism outside of the point O_t and the point O_w. Moreover, the map is dominant. Its general fiber is an irreducible curve birational to an elliptic curve. To see this, on the hypersurface X, consider the surface cut by $y = \lambda_1 x^{a_1}$ and the surface cut by $z = \lambda_2 x^{a_2}$, where λ_1 and λ_2 are sufficiently general complex numbers. Then the intersection of these two surfaces is the curve C_{λ_1,λ_2} defined by (4.2.5). From the equation we can easily see that the curve C_{λ_1,λ_2} is irreducible and reduced. Furthermore, plugging $x = 1$ into (4.2.5), we see that the curve is birational to an elliptic curve. The curve C_{λ_1,λ_2} is a general fiber of the map π.

Let $\mathcal{H}$ be the linear subsystem of $|-a_2K_X|$ generated by the monomials of degree a_2 in the variables x, y, z. Its proper transform $\mathcal{H}_Y$ on Y coincides with $|-a_2K_Y|$.

Let $g\colon W \to Y$ be the weighted blow up at the point over O_w with weight $(1, a_1, a_2)$ and let F be its exceptional divisor. Let $\hat{E}$ be the proper transform of the exceptional divisor E by the morphism g. Let $\mathcal{H}_W$ be the proper transform of the linear system $\mathcal{H}$ by the morphism $f \circ g$. We then see that $\mathcal{H}_W = |-a_2K_W|$. We also see that $-K_W^3 = 0$.

We first claim that the divisor class $-K_W$ is nef. Indeed, the base curve of the linear system $|-a_2K_W|$ is given by the proper transform of the curve C cut by the equation $x = z = 0$ on X. If the curve is irreducible then its proper transform $\hat{C}$ on W intersects $-K_W$ trivially since $-K_W^3 = 0$. Suppose that the curve C is reducible. It then consists of two irreducible components. Moreover, one of the components must be L_{yw}. Note that it passes through the point O_w. Its proper transform $\hat{L}_{yw}$ on W passes though the singular point of index a_1 on the exceptional divisor F. We then obtain

$$-K_W \cdot \hat{L}_{yw} = -K_X \cdot L_{yw} - \frac{1}{a_4}F \cdot \hat{L}_{yw} = \frac{1}{a_1a_4} - \frac{1}{a_4a_1} = 0.$$

Since $-K_W \cdot \hat{C} = 0$, the proper transform of the other component of C intersects $-K_W$ trivially. Therefore, the divisor class $-K_W$ is nef.

The linear system $|-mK_W|$ is free for sufficiently large m by Log Abundance ([37]). Hence, it induces an elliptic fibration $\eta\colon W \to \mathbb{P}(1, a_1, a_2)$. Moreover, we have proved the existence of a commutative diagram

$$\begin{array}{ccc} Y & \xleftarrow{\ g\ } & W \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \eta} \\ X & \overset{}{\underset{\pi}{\dashrightarrow}} & \mathbb{P}(1, a_1, a_2). \end{array}$$

We immediately see from (4.2.5) that the divisor F is a section of the elliptic fibration η and the divisor $\hat{E}$ is a multi-section of the elliptic fibration η. Therefore, by Proposition 4.2.1, we can construct a birational involution $\tau_W \in \mathrm{Bir}(W)$ from the reflection of the generic fiber of η with respect to the section F. The involution τ_W is biregular in codimension one because K_W is η-nef ([38, Corollary 3.54]). In particular, τ_W acts on $\mathrm{Pic}(W)$.

Put $\tau_Y = g \circ \tau_W \circ g^{-1}$ and $\tau = f \circ \tau_Y \circ f^{-1}$.

We have $\tau_W(F) = F$ by our construction. Therefore, τ_Y is also biregular in codimension one. In order to show that the point O_t is untwisted by τ, it is enough to verify $\tau_Y(E) \neq E$ by Remark 3.3.3. For this verification, we suppose that $\tau_Y(E) = E$ and then we look for a contradiction.

First, note that $\tau_Y(E) = E$ immediately implies $\tau_W(\hat{E}) = \hat{E}$. It also implies that the involution τ is biregular in codimension one. Furthermore, the involutions τ, τ_Y, τ_W induce the identity maps on the Picard groups of X, Y, W, respectively, since $\tau(-K_X) = -K_X$, $\tau_Y(E) = E$ and $\tau_W(F) = F$. Therefore, it follows from [23, Proposition 2.7] that they are all biregular.

Let S_{λ_i} be the surface on the hypersurface X cut by the equation $x_i = \lambda_i x^{a_i}$ with a general complex number λ_i. It is a normal surface (see Remark 4.2.7 below). However, it is not quasi-smooth possibly at the point O_t and the point O_{x_j}. The surface S_{λ_i} is τ-invariant by our construction. Moreover, the projection $\pi\colon X \dashrightarrow \mathbb{P}(1, a_1, a_2)$ induces a rational map $\pi_{\lambda_i}\colon S_{\lambda_i} \dashrightarrow \mathbb{P}(1, a_j) \cong \mathbb{P}^1$. The rational map $\pi_{\lambda_i}\colon S_{\lambda_i} \dashrightarrow \mathbb{P}^1$ is given by the pencil of the curves on the surface $S_{\lambda_i} \subset \mathbb{P}(1, a_j, a_3, a_4)$ cut by the equations

$$\delta x^{a_j} = \epsilon x_j,$$

where $[\delta : \epsilon] \in \mathbb{P}^1$. Its base locus is cut out on S_{λ_i} by $x = x_j = 0$, which implies that the base locus of the pencil consists of two points O_t and O_w. The map π_{λ_i} is defined outside of the points O_w and O_t.

Denote by $\hat{S}_{\lambda_i}$ the proper transform of the surface S_{λ_i} by the birational morphism $f \circ g$. Then $\hat{S}_{\lambda_i}$ is a normal surface that belongs to $|-a_i K_W|$. Moreover, the morphism $f \circ g$ induces a birational morphism $\gamma\colon \hat{S}_{\lambda_i} \to S_{\lambda_i}$. Furthermore, we have a commutative diagram

$$\begin{array}{ccccc} & & \hat{S}_{\lambda_i} & & \\ & {\scriptstyle \gamma}\swarrow & & \searrow{\scriptstyle \hat{\pi}_{\lambda_i}} & \\ S_{\lambda_i} & & \underset{\pi_{\lambda_i}}{\dashrightarrow} & & \mathbb{P}^1, \end{array}$$

where $\hat{\pi}_{\lambda_i}$ is the morphism induced by the elliptic fibration $\eta\colon W \to \mathbb{P}(1, a_1, a_2)$. In particular, a general fiber of $\hat{\pi}_{\lambda_i}$ is a smooth elliptic curve.

Let $\sigma\colon \bar{S}_{\lambda_i} \to \hat{S}_{\lambda_i}$ be the minimal resolution of singularities of the normal surface $\hat{S}_{\lambda_i}$. Then we have a commutative diagram

$$\begin{array}{ccc} \hat{S}_{\lambda_i} & \xleftarrow{\ \sigma\ } & \bar{S}_{\lambda_i} \\ {\scriptstyle\gamma}\downarrow & \searrow^{\hat{\pi}_{\lambda_i}} & \downarrow{\scriptstyle\bar{\pi}_{\lambda_i}} \\ S_{\lambda_i} & \underset{\pi_{\lambda_i}}{\dashrightarrow} & \mathbb{P}^1, \end{array}$$

where $\bar{\pi}_{\lambda_i} = \hat{\pi}_{\lambda_i} \circ \sigma$. Then $\bar{\pi}_{\lambda_i}$ is also an elliptic fibration.

The surface $\hat{S}_{\lambda_i}$ is τ_W-invariant by our construction. Let $\hat{\tau}_{\lambda_i}$ be the restriction of the involution τ_W to the surface $\hat{S}_{\lambda_i}$. Then it is a biregular involution of the surface $\hat{S}_{\lambda_i}$ since τ_W is biregular. Put $\hat{E}_{\lambda_i} = \hat{E}|_{\hat{S}_{\lambda_i}}$ and $F_{\lambda_i} = F|_{\hat{S}_{\lambda_i}}$. Then $\hat{E}_{\lambda_i}$ and F_{λ_i} are reduced $\hat{\tau}_{\lambda_i}$-invariant curves. Moreover, the curve F_{λ_i} is irreducible and is a section of the elliptic fibration $\hat{\pi}_{\lambda_i}$. The curve $\hat{E}_{\lambda_i}$ is a multi-section of the elliptic fibration $\hat{\pi}_{\lambda_i}$.

Put $\bar{\tau}_{\lambda_i} = \sigma^{-1} \circ \hat{\tau}_{\lambda_i} \circ \sigma$. Then $\bar{\tau}_{\lambda_i}$ is biregular because $\hat{\tau}_{\lambda_i}$ is biregular and σ is the minimal resolution of singularities, i.e., $\bar{S}_{\lambda_i}$ is a minimal model over $\hat{S}_{\lambda_i}$ ([38, Corollary 3.54]). Let $\bar{E}_{\lambda_i}$ and $\bar{F}_{\lambda_i}$ be the proper transforms of $\hat{E}_{\lambda_i}$ and $\hat{F}_{\lambda_i}$ by the birational morphism σ, respectively. These are $(\gamma\circ\sigma)$-exceptional. Denote the other $(\gamma\circ\sigma)$-exceptional curves (if any) by $G_1, \ldots, G_r$. Again, $\bar{F}_{\lambda_i}$ is a section of the elliptic fibration $\bar{\pi}_{\lambda_i}$ and $\bar{E}_{\lambda_i}$ is a multi-section of the elliptic fibration $\bar{\pi}_{\lambda_i}$.

Let $\bar{C}_{\lambda_i}$ be a general fiber of the map $\bar{\pi}_{\lambda_i}$. Then $\bar{C}_{\lambda_i}$ is $\bar{\tau}_{\lambda_i}$-invariant. Furthermore, $\bar{\tau}_{\lambda_i}|_{\bar{C}_{\lambda_i}}$ is given by the reflection with respect to the point $\bar{F}_{\lambda_i} \cap \bar{C}_{\lambda_i}$. On the other hand, the curve $\bar{E}_{\lambda_i}$ is $\bar{\tau}_{\lambda_i}$-invariant. Then Lemma 4.2.8 below implies that the divisor $\bar{E}_{\lambda_i} - a_i\bar{F}_{\lambda_i}$ must be numerically equivalent to a $\mathbb{Q}$-linear combination of curves on $\bar{S}_{\lambda_i}$ that lie in the fibers of $\bar{\pi}_{\lambda_i}$.

Let C_x be the curve on S_{λ_i} cut by the equation $x = 0$. It is defined by the equation

$$tw^2 + wh_{d-a_4}(x_j) + h_d(x_j, t) = 0$$

in $\mathbb{P}(a_j, a_3, a_4)$. It can be reducible. We write $C_x = \sum_{k=1}^{\ell} m_k C_k$, where C_k's are the irreducible components of C_x. Denote by $\hat{C}_k$ be the proper transform of C_k by γ. Put $\hat{C}_x = \sum_{k=1}^{\ell} m_k \hat{C}_k$. Then all the curves $\hat{C}_k$ lie in the same fiber of the elliptic fibration $\hat{\pi}_{\lambda_i}$

Let $\bar{C}_k$ be the proper transform of $\hat{C}_k$ by σ. Then all the curves $\bar{C}_k$ lie in the same fiber of the elliptic fibration $\bar{\pi}_{\lambda_i}$. In addition, the fiber containing $\bar{C}_k$'s does not carry any other non-$(\gamma \circ \sigma)$-exceptional curve.

We also claim that every other fiber of $\bar{\pi}_{\lambda_i}$ contains exactly one irreducible and reduced curve that is not $(\gamma \circ \sigma)$-exceptional. For this claim, it is enough to show that for a general complex number λ_i, the curve C_{λ_1,λ_2} is always irreducible and reduced for every value of λ_j. Suppose that this is not true. Then, for a general complex number λ_i there is a complex number λ_j such that the curve C_{λ_1,λ_2} is reducible. Therefore there is a one-dimensional family of reducible curves C_{λ_1,λ_2} with general λ_i and some λ_j depending on λ_i. Denote the general curve in this one-dimensional family by C. Since the defining equation (4.2.5) contains tw^2, it cam splits into at most three irreducible components. Furthermore, one of them must be the curve C_1 defined by either

$$y - \lambda_1 x^{a_1} = z - \lambda_2 x^{a_2} = w + f_{a_4}(x, t) = 0$$

or

$$y - \lambda_1 x^{a_1} = z - \lambda_2 x^{a_2} = w^2 + w g_{a_4}(x,t) + g_{2a_4}(x,t) = 0$$

for some quasi-homogeneous polynomials $f_{a_4}(x,t)$, $g_{a_4}(x,t)$ and $g_{2a_4}(x,t)$. We then obtain

$$\begin{aligned} B \cdot \tilde{C}_1 &= \left(A - \frac{1}{a_3}E\right) \cdot \tilde{C}_1 \\ &= -K_X \cdot C_1 - \frac{1}{a_3} E \cdot C_1, \\ &= \frac{k a_1 a_2 a_4}{a_1 a_2 a_3 a_4} - \frac{k}{a_4} = 0 \end{aligned}$$

where $k = 1$ for $w + f_{a_4}(x,t) = 0$ and $k = 2$ for $w^2 + w g_{a_4}(x,t) + g_{2a_4}(x,t) = 0$. By Lemma 3.2.8, the point O_t cannot be a center of non-canonical singularities of the log pair $(X, \frac{1}{n}\mathcal{M})$. Therefore, since O_t is a center, every other fiber of $\bar{\pi}_{\lambda_i}$ contains exactly one irreducible and reduced curve that is not $(\gamma \circ \sigma)$-exceptional.

Since every fiber of $\bar{\pi}_{\lambda_i}$ (with scheme structure) is numerically equivalent to each other and the divisor $\bar{E}_{\lambda_i} - a_i \bar{F}_{\lambda_i}$ is numerically equivalent to a $\mathbb{Q}$-linear combination of curves that lie in the fibers of $\bar{\pi}_{\lambda_i}$, we obtain

$$\bar{E}_{\lambda_i} - a_i \bar{F}_{\lambda_i} \sim_{\mathbb{Q}} \sum_{k=1}^{\ell} \bar{c}_k \bar{C}_k + \sum_{k=1}^{r} g_k G_k$$

for some rational numbers $\bar{c}_1, \ldots, \bar{c}_\ell$, $g_1, \ldots, g_r$. On the other hand, the intersection form of the curves $\bar{E}_{\lambda_i}$, $\bar{F}_{\lambda_i}$, $G_1, \ldots, G_r$ is negative-definite since these curves are $\gamma \circ \sigma$-exceptional. This implies

$$0 > \left(\bar{E}_{\lambda_i} - a_i \bar{F}_{\lambda_i} - \sum_{k=1}^{r} g_k G_k\right)^2 = \left(\sum_{k=1}^{\ell} \bar{c}_k \bar{C}_k\right)^2.$$

Therefore, $\bar{c}_k \neq 0$ for some k. On the other hand, we have

$$\sum_{k=1}^{\ell} \bar{c}_k C_k \sim_{\mathbb{Q}} 0$$

on the surface S_{λ_i}. In particular, the intersection form of the curve(s) C_k's is degenerate on the surface S_{λ_i}. This however contradicts Lemma 4.2.9 below.

The obtained contradiction shows that $\tau_Y(E) \neq E$. In particular, the involution τ is not biregular. Since the involution τ_Y is biregular in codimension one, the involution τ meets the conditions in Definition 3.3.1. Therefore, the birational involution τ untwists the singular point O_t. □

Remark 4.2.7. Each affine piece of the surface S_{λ_i} is the quotient of a hypersurface in $\mathbb{C}^3$ by a finite group action. Since the surface S_{λ_i} has only isolated singularities, so does the hypersurface. Therefore, the hypersurface is normal, and hence its quotient by a finite group action is also normal. Consequently, the original surface S_{λ_i} is normal. For the same reason, a hypersurface in a weighted projective space is normal if it is smooth in codimension 1.

The lemma below originates from Bogomolov and Tschinkel ([4], [5]).

Lemma 4.2.8. *Let Σ be a smooth surface with an elliptic fibration $\pi : \Sigma \to B$ over a smooth curve B. Let N be a section of π and M be a multi-section of degree $m \geq 1$. Suppose that the surface Σ has an involution τ satisfying the following:*

(1) a general fiber E is τ-invariant:

(2) M is τ-invariant:

(3) $\tau|_E$ is given by the reflection on the elliptic curve E with respect to the point $N \cap E$.

Then the divisor $M - mN$ is numerically equivalent to a $\mathbb{Q}$-linear combination of curves that lie in the fibres of π.

Proof. The divisor $(M - mN)|_E$ on the elliptic curve E belongs to $\mathrm{Pic}^0(E)$. The conditions (2) and (3) imply that $\tau((M - mN)|_E) = (M - mN)|_E$. On the other hand, the condition (3) shows $\tau((M - mN)|_E) = -(M - mN)|_E$. Consequently, the divisor

$$(M - mN)|_E \in \mathrm{Pic}^0(E).$$

is 2-torsion. Then [46, Theorem 1.1] verifies the statement. □

Lemma 4.2.9. *Let S_{λ_i} be the surface on X cut by the equation $x_i = \lambda_i x^{a_i}$ for a general complex number λ_i. Let $C_x = \sum_{k=1}^{\ell} m_k C_k$ be the divisor on S_{λ_i} cut by the equation $x = 0$. Then the intersection form of the curves C_k's on the surface S_{λ_i} is non-degenerate.*

Proof. Suppose that it is not a case. This immediately implies that $\ell \geq 2$. It cannot happen in the families No. 44, 61 and 76 since the polynomial g_d must contain a power of x_j, i.e., the curve C_x is irreducible.

The curve C_x is defined by

- $tw^2 + ay^5w + y^4(bt^2 + cy^2t + dy^4) = 0$ in $\mathbb{P}(1, 2, 3)$ for the family No. 7 (Type I);
- $tw^2 + az^3w + bz^2t^2 = 0$ in $\mathbb{P}(3, 4, 5)$ for the family No. 23;
- $tw^2 + ay^4w + by^3t^2 = 0$ in $\mathbb{P}(3, 5, 7)$ for the family No. 40.

The curve C_x must consist of two irreducible components C_1 and C_2, i.e., $\ell = 2$, except the case when $a = b = d = 0$ and $c \neq 0$ in the family No. 7 (Type I). This exceptional case will be considered separately at the end.

By our assumption, the intersection matrix of C_1 and C_2 on the surface S_{λ_i} is singular.

Suppose that the curve C_x is reduced. Then

$$\begin{pmatrix} C_1^2 & C_1 \cdot C_2 \\ C_1 \cdot C_2 & C_2^2 \end{pmatrix} = \begin{pmatrix} C_x \cdot C_1 - C_1 \cdot C_2 & C_1 \cdot C_2 \\ C_1 \cdot C_2 & C_x \cdot C_2 - C_1 \cdot C_2 \end{pmatrix},$$

and hence we have

$$C_1 \cdot C_2 = \frac{(C_x \cdot C_1)(C_x \cdot C_2)}{C_x^2} = \frac{2}{a_j d} \quad (\text{resp. } \frac{a_3 + a_4}{a_j a_3 d})$$

if $a = 0$, $b \neq 0$ (resp. $a \neq 0$, $b = 0$). Note that the intersection numbers by the curve C_x can be obtained easily because it is in $|\mathcal{O}_{S_{\lambda_i}}(1)|$.

Meanwhile, since the surface S_{λ_i} is not quasi-smooth at the point O_t and possibly at the point O_{x_j}, we have some difficulty to find the numbers $C_1 \cdot C_2$ without assuming that the matrix is singular. In order to compute the intersection number $C_1 \cdot C_2$ on the surface S_{λ_i} directly, we consider the divisor C_t (resp. C_w) cut by the equation $t = 0$ (resp. $w = 0$) on the surface S_{λ_i} in case when $a = 0$ (resp. $a \neq 0$).

Consider the case when $a = 0$, $b \neq 0$. We may assume that the curve C_1 is defined by the equation $x = t = 0$ in $\mathbb{P}(1, a_j, a_3, a_4)$. Since the divisor C_t contains the curve C_1, we can write $C_t = mC_1 + R$, where R is a curve whose support does not contain the curve C_1. From the intersection numbers

$$(C_1 + C_2) \cdot C_1 = C_x \cdot C_1 = \frac{1}{a_j a_4}, \quad (mC_1 + R) \cdot C_1 = C_t \cdot C_1 = \frac{a_3}{a_j a_4}$$

we obtain

$$C_1 \cdot C_2 = \frac{1}{a_j a_4} - C_1^2 = \frac{m - a_3}{m a_j a_4} + \frac{1}{m} R \cdot C_1 \geq \frac{m - a_3}{m a_j a_4} + \frac{1}{m} (R \cdot C_1)_{O_w},$$

where $(R \cdot C_1)_{O_w}$ is the local intersection number of the curves C_1 and R at the point O_w. Note that the curves C_1 and R always meet at the point O_w at which the surface S_{λ_i} is quasi-smooth. They may also intersect at the point O_{x_j}. However, we do not care about the intersection at the point O_{x_j}. The local intersection at the point O_w will be enough for our purpose.

For the family No. 7 (Type I), we are considering the case when $a = d = 0$ and $b \neq 0$. In such a case, if $c \neq 0$, then the curves C_1 and C_2 intersect at a smooth point of S_{λ_i} and hence $C_1 \cdot C_2 \geq 1$. If $c = 0$, then the conditions imply that the defining equation of X_8 must contain either xy^7 or zy^6. Therefore, we can conclude that $m = 1$ or 2, depending on the existence of the monomials xy^7, xy^4w in the defining equation of X_8, and that the local intersection number $(R \cdot C_1)_{O_w}$ is at least $\frac{4}{3}$. For the family No. 23, we see that $m = 2$ and $C_1 \cdot R = \frac{3}{5}$. For the family No. 40, we can easily see that m can be 1, 3, or 4 , depending on the existence of the monomials xy^6 and wy^3x^3 in the defining equation, and that the local intersection number $(R \cdot C_1)_{O_w}$ is at least $\frac{3}{7}$. In all the cases, we see $C_1 \cdot C_2 > \frac{2}{a_j d}$. It is a contradiction.

Consider the case when $a \neq 0$, $b = 0$. We may assume that the curve C_1 is defined by the equation $x = w = 0$ in $\mathbb{P}(1, a_j, a_3, a_4)$. Since we have the monomial of the form $x_j^s w$ in each defining equation, the surface S_{λ_i} is quasi-smooth at the point O_{x_j}. Furthermore, by changing the coordinate w in suitable ways for the hypersurface X, we may assume that we have neither xy^6 nor x^2y^4t for the family No. 40 and that we have neither x^2z^4 nor xz^3t for the family No. 23 by changing the coordinate function w. For the family No. 7 (Type I), we may assume that none of the monomials xy^7, ty^6, xy^5t appear in the defining equation of X_8.

Since the divisor C_w contains the curve C_1, we can write $C_w = mC_1 + R$, where R is a curve whose support does not contain the curve C_1. From the intersection numbers

$$(C_1 + C_2) \cdot C_1 = C_x \cdot C_1 = \frac{1}{a_j a_3}, \quad (mC_1 + R) \cdot C_1 = C_w \cdot C_1 = \frac{a_4}{a_j a_3}$$

we obtain

$$C_1 \cdot C_2 = \frac{1}{a_j a_3} - C_1^2 = \frac{m - a_4}{m a_j a_3} + \frac{1}{m} R \cdot C_1 \geq \frac{m - a_4}{m a_j a_3} + \frac{1}{m} (R \cdot C_1)_{O_{x_j}},$$

where $(R\cdot C_1)_{O_{x_j}}$ is the local intersection number of the curves C_1 and R at the point O_{x_j}. Similarly as in the previous case, they may also intersect at the point O_t. We do not care about the intersection at the point O_t. As before, the local intersection at the point O_{x_j} will be big enough.

For the family No. 7 (Type I), we have $b=c=d=0$ and $a\neq 0$. Note that the point O_y is a smooth point of the surface S_{λ_i}. We see that m can be 1 or 2 , depending on the existence of the monomial xy^3t^2 in the defining equation, and that the local intersection number $(R\cdot C_1)_{O_y}$ is at least 2. For the family No. 23, we see that $m=2$ and $C_1\cdot R=1$. For the family No. 40, we see that m can be 3 or 4 , depending on the existence of the monomial $x^3y^2t^2$ in the defining equation, and that the local intersection number $(R\cdot C_1)_{O_y}$ is at least $\frac{2}{3}$. In all the cases, we see $C_1\cdot C_2>\frac{a_3+a_4}{a_ja_3d}$. Again we have obtained a contradiction.

Suppose that the curve C_x is not reduced. Then $C_x=C_1+2C_2$, where C_1 is defined by $x=t=0$ and C_2 is defined by $x=w=0$. We then have

$$\begin{pmatrix} C_1^2 & C_1\cdot C_2\\ C_1\cdot C_2 & C_2^2\end{pmatrix}=\begin{pmatrix} C_x\cdot C_1-2C_1\cdot C_2 & C_1\cdot C_2\\ C_1\cdot C_2 & C_x\cdot C_2-\frac{1}{2}C_1\cdot C_2\end{pmatrix},$$

and hence we have

$$C_1\cdot C_2=\frac{2(C_x\cdot C_1)(C_x\cdot C_2)}{C_x\cdot(C_1+4C_2)}=\frac{2}{a_j(a_3+4a_4)}.$$

In this case, the curves C_1 and C_2 intersect at the point O_{x_j}. The surface S_{λ_i} is not quasi-smooth at the point O_{x_j}, i.e., the defining equation of X contains the monomial of the form $x_j^sx_i$. If it is quasi-smooth there, then we obtain an absurd identity $C_1\cdot C_2=\frac{1}{a_j}$ from a direct computation. Note that we do not have the monomial of the form x_j^sw. Furthermore, we may assume that we do not have xy^6 (resp. xy^7) for the family No. 40 (resp. No. 7) by changing the coordinate function z.

Since the divisor C_t contains the curve C_1, we can write $C_t=mC_1+R$, where R is a curve whose support does not contain the curve C_1. From the intersection numbers

$$(C_1+2C_2)\cdot C_1=C_x\cdot C_1=\frac{1}{a_ja_4},\quad (mC_1+R)\cdot C_1=C_t\cdot C_1=\frac{a_3}{a_ja_4}$$

we obtain

$$C_1\cdot C_2=\frac{1}{2}\left(\frac{1}{a_ja_4}-C_1^2\right)=\frac{1}{2}\left(\frac{m-a_3}{ma_ja_4}+\frac{1}{m}R\cdot C_1\right)\geq\frac{1}{2}\left(\frac{m-a_3}{ma_ja_4}+\frac{1}{m}(R\cdot C_1)_{O_w}\right),$$

where $(R\cdot C_1)_{O_w}$ is the local intersection number of the curves C_1 and R at the point O_w.

As in the first case, $m=1$ or 2, depending on the existence of the monomials xy^7, xy^4w in the defining equation of X_8, and $(R\cdot C_1)_{O_w}\geq\frac{4}{3}$ for the family No. 7 (Type I). We also obtain $m=2$ and $C_1\cdot R=\frac{3}{5}$ for the family No. 23. For the family No. 40, we obtain $m=3$ or 4, depending on the existence of the monomial wy^3x^3 in the defining equation, and $(R\cdot C_1)_{O_w}\geq\frac{3}{7}$. In all the cases, we see $C_1\cdot C_2>\frac{2}{a_j(a_3+4a_4)}$. It is a contradiction again.

We now consider the exceptional case $a=b=d=0$ and $c\neq 0$ in the family No. 7 (Type I). The curve C_x is defined by

$$t(w-\alpha_1y^3)(w-\alpha_2y^3)=0$$

in $\mathbb{P}(1,2,3)$. It consists of three irreducible components L, C_1 and C_2. The curve L is defined by $x=t=0$ in $\mathbb{P}(1,1,2,3)$ and the curve C_k by

$$x = w - \alpha_k y^3 = 0$$

in $\mathbb{P}(1,1,2,3)$. The curves L and C_k intersect at the point defined by $x=t=w-\alpha_k y^3=0$. At this point the surface S_{λ_i} is smooth. We then have

$$(L+C_1+C_2)\cdot L = \frac{1}{3}, \quad (L+C_1+C_2)\cdot C_1 = (L+C_1+C_2)\cdot C_2 = \frac{1}{2}, \quad L\cdot C_1 = L\cdot C_2 = 1.$$

The intersection matrix of the curves L, C_1 and C_2 on the surface S_{λ_i}

$$\begin{pmatrix} -\frac{5}{3} & 1 & 1 \\ 1 & -\frac{1}{2} - C_1\cdot C_2 & C_1\cdot C_2 \\ 1 & C_1\cdot C_2 & -\frac{1}{2} - C_1\cdot C_2 \end{pmatrix}$$

is non-singular regardless of the value of $C_1 \cdot C_2$. This completes the proof. □

Now, we consider the following two families and their singular point O_z.

- No. 20, $X_{13} \subset \mathbb{P}(1,1,3,4,5)$;
- No. 36, $X_{18} \subset \mathbb{P}(1,1,4,6,7)$.

Before we proceed, we put a remark here. The proof of Theorem 4.2.6 works verbatim to treat these two cases. Indeed, we are able to obtain elliptic fibrations right after taking weighted blow ups at the point O_z and at the point O_w with the corresponding weights. We however follow another way that has evolved from [25, Section 4.10], instead of applying the same method. This can enhance our understanding of the involutions described in this section with various points of view.

We have two types of hypersurfaces in the family No. 20. One is the hypersurfaces whose defining equations contain the monomial tz^3 (Type I) and the other is the hypersurfaces not containing the monomial tz^3 (Type II).

We first consider both X_{13} of Type I in the family No. 20 and X_{18} in the family No. 36 at the same time. Note that the defining equation of X_{18} always contains the monomial tz^3.

We may then assume that these hypersurfaces X are defined by the equation

$$zw^2 + wf_{d-a_4}(x,y,t) - tz^3 - z^2 f_{d-2a_2}(x,y,t) - zf_{d-a_2}(x,y,t) + f_d(x,y,t) = 0. \qquad (4.2.10)$$

We can define an involution τ_z of X as follows:

$$[x:y:z:t:w] \mapsto \left[x : y : \frac{f_{d-a_4}^2(u+f_{d-a_2}) - f_d^2}{f_{d-a_4}uw + f_{d-a_4}^2 zt + f_d u} : t : \frac{-f_{d-a_4}u(u+f_{d-a_2}) - f_d(uw+f_{d-a_4}zt)}{f_{d-a_4}uw + f_{d-a_4}^2 zt + f_d u}\right], \qquad (4.2.11)$$

where $u = w^2 - tz^2 - zf_{d-2a_2} - f_{d-a_2}$. Indeed, the involution is obtained by the following way. We have a birational map ϕ from X to a hypersurface Z of degree $6a_4$ in $\mathbb{P}(1,a_1,2a_4,a_3,3a_4)$ defined by

$$[x:y:z:t:w] \mapsto [x:y:u:t:v],$$

where $v = uw + f_{d-a_4}zt + f_{d-a_4}f_{d-2a_2}$. Note that we have

$$\begin{pmatrix} f_{d-a_4} & u \\ f_d & v \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} = -\begin{pmatrix} f_d \\ f_{d-a_4}(u+f_{d-a_2}) \end{pmatrix}.$$

The hypersurface Z is defined by the equation

$$v^2 - f_{d-a_4}f_{d-2a_2}v = u^3 + u^2 f_{d-a_2} - (f_{d-2a_2}f_d + f_{d-a_4}^2 t)u + (-f_{d-a_4}^2 f_{d-a_2} + f_d^2)t.$$

Therefore the hypersurface Z has a biregular involution ι defined by

$$[x : y : u : t : v] \mapsto [x : y : u : t : f_{d-a_4}f_{d-2a_2} - v].$$

The birational involution of X is obtained by

$$\tau_z = \phi^{-1} \circ \iota \circ \phi.$$

To see that it is a birational involution in detail, refer to [25, Section 4.10]. However, it can be a biregular automorphism under a certain condition. For example, if the polynomial f_{d-a_4} is identically zero, then the involution becomes biregular. Indeed, it is the biregular involution

$$[x : y : z : t : w] \mapsto [x : y : z : t : -w].$$

Moreover, the converse is true.

Lemma 4.2.12. *The involution τ_z is biregular if and only if the polynomial f_{d-a_4} is identically zero.*

Proof. Suppose that f_{d-a_4} is not a zero polynomial. Consider the surface cut by the equation $u = 0$. It is easy to check that on this surface the involution becomes the map

$$[x : y : z : t : w] \mapsto \left[x : y : -z - \frac{f_{d-2a_2}}{t} : t : w\right].$$

Therefore, unless the polynomial f_{d-2a_2} is either identically zero or divisible by t, the involution τ_t cannot be biregular since it contracts the curve defined by $u = t = 0$ to a point.

If the polynomial f_{d-2a_2} is identically zero, then on the surface cut by $z = 0$, the involution becomes the map

$$[x : y : z : t : w] \mapsto \left[x : y : -\frac{2f_d}{u} : t : -w\right],$$

and hence the involution τ_z cannot be biregular. It contracts the curve defined by $u = z = 0$ to a point.

Finally, suppose that the polynomial $f_{d-2a_2}(x, y, t)$ is divisible by t. In this case, we consider the surface cut by the equation $t = 0$. On this surface the involution τ_z becomes

$$[x : y : z : t : w] \mapsto \left[x : y : -z - \frac{2f_d}{u} : t : -w\right].$$

It shows that the involution τ_z cannot be biregular because it contracts the curve defined by $t = u = 0$. ∎

Theorem 4.2.13. *Let X be a quasi-smooth hypersurface in the families No. 20 (Type I) and 36. If the singular point O_z is a center of non-canonical singularities of the log pair $(X, \frac{1}{n}\mathcal{M})$, then it is untwisted by the birational involution τ_z.*

Proof. We suppose that f_{d-a_4} is identically zero. Then the polynomial f_d must be a non-zero irreducible polynomial since X is quasi-smooth.

Set

$$u = w^2 - tz^2 - zf_{d-2a_2} - f_{d-a_2}$$

and then let T be the proper transform of the surface given by the equation $u = 0$. We can immediately check that the surface T belongs to the linear system $|2a_4B|$.

Choose a general point $[1 : \mu_1 : \mu_2]$ on the curve defined by the equation $f_d = 0$ in $\mathbb{P}(1, a_1, a_3)$. Then let C_{μ_1,μ_2} be the curve defined by the equations

$$u = y - \mu_1 x^{a_1} = t - \mu_2 x^{a_3} = 0$$

in $\mathbb{P}(1, a_1, a_2, a_3, a_4)$. This curve lies on the hypersurface X by our construction. If the curve is irreducible, then we have

$$B \cdot \tilde{C}_{\mu_1,\mu_2} = (A - \frac{1}{a_2}E) \cdot \tilde{C}_{\mu_1,\mu_2} = \frac{2}{a_2} - \frac{1}{a_2}E \cdot \tilde{C}_{\mu_1,\mu_2} = 0$$

since $E\cdot\tilde{C}_{\mu_1,\mu_2} = 2$. If C_{μ_1,μ_2} is reducible, then it can have at most two irreducible components. Furthermore, each component $C_{\mu_1,\mu_2,i}$ is defined by

$$w - h(x, z) = y - \mu_1 x^{a_1} = t - \mu_2 x^{a_3} = 0$$

in $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ for some polynomial h. This shows

$$B \cdot \tilde{C}_{\mu_1,\mu_2,i} = \left(A - \frac{1}{a_2}E\right) \cdot \tilde{C}_{\mu_1,\mu_2,i} = \frac{1}{a_2} - \frac{1}{a_2}E \cdot \tilde{C}_{\mu_1,\mu_2,i} = 0$$

since $E \cdot \tilde{C}_{\mu_1,\mu_2,i} = 1$.

Since O_z is a center, this is a contradiction by Lemma 3.2.8. Therefore, f_{d-a_4} is not identically zero, and hence τ_z is a non-biregular involution by Lemma 4.2.12.

Note that τ_z leaves the point O_w fixed. On the threefold W obtained by the weighted blow ups at O_z and O_w as in the proof of Theorem 4.2.6, the lift τ_W of the involution τ_z leaves the exceptional divisor over O_w fixed. For the same reason as in the proof of Theorem 4.2.6, the involution τ_W is biregular in codimension one, so is the lift τ_Y of the involution τ_z to Y.

Consequently, the involution τ_z untwists the singular point O_z. □

Now we consider X_{13} of Type II, i.e., its defining equation does not contain the monomial tz^3.

Theorem 4.2.14. *Let X_{13} be a quasi-smooth hypersurface of degree 13 in $\mathbb{P}(1, 1, 3, 4, 5)$ in the family No. 20 (Type II). Then the singular point O_z cannot be a center of non-canonical singularities of the log pair $(X_{13}, \frac{1}{n}\mathcal{M})$.*

Proof. Since X_{13} is of Type II, we may assume that the hypersurface X_{13} is defined by the equation

$$zw^2 + w(f_8(x,y,t) + at^2) - yz^4 - z^3 f_4(x,y) - z^2 f_7(x,y,t) - z f_{10}(x,y,t) + f_{13}(x,y,t) = 0,$$

where a is a constant. Note that the polynomial f_{13} must contain the monomial xt^3; otherwise X_{13} would not be quasi-smooth.

Let $\tilde{S}_y$ be the proper transform of the surface S_y. Let $\mathcal{L}$ be the linear system on X_{13} generated by x^5, xt and w.

First we consider the case where $a = 0$. The base locus of the linear system $|-K_{X_{13}}|$ consists of the curve cut by $x = y = 0$. The curve has two irreducible components. One is the curve L_{zt} and the other is the curve L_{tw}. We see that

$$S \cdot \tilde{S}_y = \tilde{L}_{tw} + 2\tilde{L}_{zt}.$$

Note that the curve L_{tw} does not pass through the point O_z. We obtain

$$B \cdot \tilde{L}_{zt} = \frac{1}{2} B \cdot S \cdot \tilde{S}_y - \frac{1}{2} A \cdot \tilde{L}_{tw} = \frac{1}{2} A^3 - \frac{4}{54} E^3 - \frac{1}{40} = -\frac{1}{4}.$$

For the proper transform $\tilde{S}_{\lambda,\mu}$ of a general member in $\mathcal{L}$, we have

$$\tilde{S}_y \cdot \tilde{S}_{\lambda,\mu} = \tilde{L}_{zt} + \tilde{R}_{\lambda,\mu},$$

where $\tilde{R}_{\lambda,\mu}$ is the residual curve and it sweeps the surface $\tilde{S}_y$. We then obtain

$$B \cdot \tilde{R}_{\lambda,\mu} = B \cdot \tilde{S}_y \cdot \tilde{S}_{\lambda,\mu} - B \cdot \tilde{L}_{zt} = 5A^3 - \frac{8}{27} E^3 + \frac{1}{4} = 0.$$

It then follows from Lemma 3.2.8 that the singular point O_z cannot be a center of non-canonical singularities of the log pair $(X_{13}, \frac{1}{n}\mathcal{M})$.

Now we consider the case where $a \neq 0$. By a coordinate change we may assume that $a = 1$. The base locus of the linear system $|-K_{X_{13}}|$ consists of the curve cut by $x = y = 0$. The curve has two irreducible components. One is L_{zt} and the other is the curve L defined by

$$x = y = zw + t^2 = 0.$$

The curves $\tilde{L}$ and $\tilde{L}_{zt}$ intersect the exceptional divisor E at a smooth point. We have $S \cdot \tilde{S}_y = \tilde{L}_{zt} + \tilde{L}$ and

$$B \cdot \tilde{L}_{zt} = A \cdot \tilde{L}_{zt} - \frac{1}{3} E \cdot \tilde{L}_{zt} = -\frac{1}{4}, \quad B \cdot \tilde{L} = A \cdot \tilde{L} - \frac{1}{3} E \cdot \tilde{L} = \frac{2}{15} - \frac{1}{3} = -\frac{1}{5}.$$

For the proper transform $\tilde{S}_{\lambda,\mu}$ of a general member in $\mathcal{L}$, we have

$$\tilde{S}_y \cdot \tilde{S}_{\lambda,\mu} = \tilde{L}_{zt} + \tilde{R}_{\lambda,\mu},$$

where $\tilde{R}_{\lambda,\mu}$ is the residual curve and it sweeps the surface $\tilde{S}_y$. Note that the curve $\tilde{R}_{\lambda,\mu}$ does not contain the curve $\tilde{L}_{zt}$ since the defining polynomial of X_{13} contains either xt^3 or wt^2. Therefore,

$$B \cdot \tilde{R}_{\lambda,\mu} = B \cdot \tilde{S}_y \cdot \tilde{S}_{\lambda,\mu} - B \cdot \tilde{L}_{zt} = 5A^3 - \frac{8}{27} E^3 + \frac{1}{4} = 0.$$

Then the statement immediately follows from Lemma 3.2.8. □

Remark 4.2.15. Note that Theorem 4.2.6 can be proved in the same way that we apply to Theorems 4.2.13. The involution of X for the singular point O_t is defined as follows:

$$[x:y:z:t:w]\mapsto$$
$$\left[x:y:z:\frac{g_{d-a_4}^2(v+g_{d-a_3})-g_d^2}{g_{d-a_4}vw+g_{d-a_4}^2x_it+g_dv}:\frac{-g_{d-a_4}v(v+g_{d-a_3})-g_d(vw+g_{d-a_4}x_it)}{g_{d-a_4}vw+g_{d-a_4}^2x_it+g_dv}\right],$$

where $v=w^2-x_it^2-tg_{d-2a_3}-g_{d-a_3}$. This birational involution is also extracted from [25, Section 4.10]. We are immediately able to check that it is biregular if and only if the polynomial g_{d-a_4} is identically zero.

4.3 Invisible elliptic involution

In this section we consider the singular point O_z on the hypersurfaces of a special type in the family No. 23 and the singular points of type $\frac{1}{2}(1,1,1)$ on the hypersurfaces of Type II in the family No. 7. The method we use here is almost the same as the one for Theorem 4.2.6. In the proof of Theorem 4.2.6, only with the weighted blow ups at the point O_t (or O_z) and the point O_w we can obtain an elliptic fibration with a section. However, in this special cases of the families No. 7 and 23, after these two weighted blow-ups, our elliptic fibrations still remain invisible. When we reach a threefold W with $-K_W^3=0$, instead of elliptic fibrations, we see several curves that intersect $-K_W$ negatively. Eventually, log-flips along these curves reveal elliptic fibrations with sections.

We first consider the singular point O_z on the hypersurface of the special type in the family No. 23. In general, every quasi-smooth hypersurface of degree 14 in $\mathbb{P}(1,2,3,4,5)$ can be defined by the equation

$$(t+by^2)w^2+y(t-\alpha_1y^2)(t-\alpha_2y^2)(t-\alpha_3y^2)+z^3(a_1w+a_2yz)+cz^2t^2+$$
$$+wf_9(x,y,z,t)+f_{14}(x,y,z,t)=0$$

for suitable constants b, α_1, α_2, α_3, and suitable polynomials $f_9(x,y,z,t)$, $f_{14}(x,y,z,t)$. Here, we will deal with the singular point O_z on this hypersurface. However, in the cases when at least one of the constants c, a_1 is non-zero, the singular point O_z can be easily excluded (see the table for the family No. 23 in Section 5.2). For this reason, we consider only the case when $a_1=c=0$. In this case, the defining equation must possess the monomial xtz^3. If not, then the hypersurface is not quasi-smooth at the point defined by $x=y=w=t^3+a_2z^4=0$. Consequently, it is the singular points O_z on the hypersurface X_{14} of degree 14 in $\mathbb{P}(1,2,3,4,5)$ defined by the equation

$$(t+by^2)w^2+y(t-\alpha_1y^2)(t-\alpha_2y^2)(t-\alpha_3y^2)+z^4y+xtz^3+wf_9(x,y,z,t)+f_{14}(x,y,z,t)=0,$$

where f_9 does not contain z^3 and f_{14} does not contain z^2t^2, that we should deal with here. By replacing $t-\alpha_3y^2$ by t, we may assume that X_{14} has a singular point at O_y without loss of generality. Note that by a suitable coordinate change with respect to t, we may assume that neither x^3w^2 nor xyw^2 appears in the defining equation. However we cannot change the coefficient term $(t+by^2)$ of w^2 into t by a coordinate change since we have already assumed that O_y is a singular point.

Theorem 4.3.1. *Suppose that the hypersurface X_{14} of degree 14 in $\mathbb{P}(1,2,3,4,5)$ is defined by the equation*

$$(t+by^2)w^2+yt(t-\alpha_1y^2)(t-\alpha_2y^2)+z^4y+xtz^3+wf_9(x,y,z,t)+f_{14}(x,y,z,t)=0$$

as explained just before. If the singular point O_z is a center of non-canonical singularities of the log pair $\left(X_{14},\frac{1}{n}\mathcal{M}\right)$, then there is a birational involution that untwists O_z.

Proof. Let $\mathcal{H}$ be the linear subsystem of $|-5K_{X_{14}}|$ generated by x^5, xy^2, x^3y and $yz+xt$. Note that the polynomial $yz+xt$ vanishes at the point O_z with multiplicity $\frac{5}{3}$ (see Remark 4.3.4 below). Let $\pi: X_{14} \dashrightarrow \mathbb{P}(1,2,5)$ be the rational map induced by

$$[x:y:z:t:w]\mapsto[x:y:yz+xt].$$

Then π is a morphism outside of the curves L_{zt} and L_{zw}. Moreover, the map π is dominant, which implies, in particular, that $\mathcal{H}$ is not composed from a pencil. Furthermore, its general fiber is an irreducible curve that is birational to an elliptic curve. To see this, we put $y=\lambda x^2$ and $yz+xt=\mu x^5$ with sufficiently general complex numbers λ and μ. On the hypersurface X_{14}, we take the intersection of the surface defined by $y=\lambda x^2$ and the surface defined by $yz+xt=\mu x^5$. This intersection is the same as the intersection of the surface defined by $y=\lambda x^2$ and the reducible surface defined by $x(\lambda xz+t-\mu x^4)=0$. Therefore, the intersection is the 1-cycle

$$(L_{zw}+2L_{zt})+(L_{zw}+C_{\lambda,\mu})=2L_{zw}+2L_{zt}+C_{\lambda,\mu},$$

where the curve $C_{\lambda,\mu}$ is defined by the equation

$$\begin{aligned}&(\mu x^3-\lambda z+b\lambda^2x^3)w^2+\lambda x^4(\mu x^3-\lambda z)(\mu x^3-\alpha_1\lambda^2x^3-\lambda z)(\mu x^3-\alpha_2\lambda^2x^3-\lambda z)+\mu x^4z^3+\\&+\frac{wf_9(x,\lambda x^2,z,\mu x^4-\lambda xz)+f_{14}(x,\lambda x^2,z,\mu x^4-\lambda xz)}{x}=0\end{aligned}\tag{4.3.2}$$

in $\mathbb{P}(1,3,5)$. The curve $C_{\lambda,\mu}$ is a general fiber of the map π. Setting $x=1$ in (4.3.2), we consider the curve defined by

$$\begin{aligned}&(\mu+b\lambda^2-\lambda z)w^2+\lambda(\mu-\lambda z)(\mu-\alpha_1\lambda^2-\lambda z)(\mu-\alpha_2\lambda^2-\lambda z)+\\&+\mu z^3+wf_9(1,\lambda,z,\mu-\lambda z)+f_{14}(1,\lambda,z,\mu-\lambda z)=0\end{aligned}\tag{4.3.3}$$

in $\mathbb{C}^2$. It is a smooth affine plane cubic curve. Moreover, for a general complex number λ, the curve (4.3.3) is always irreducible and reduced for every value of μ (see Lemma 4.3.5 below).

Let $\mathcal{H}_Y$ be the proper transform of the linear system $\mathcal{H}$ by the weighted blow up f. It is the linear system $|-5K_Y|$ because the linear system $\mathcal{H}$ consists exactly of the members of $|-5K_X|$ with multiplicity at least $\frac{5}{3}$ at O_z (see Remark 4.3.4 below). Let $g\colon W\to Y$ be the weighted blow up at the point over O_w with weight $(1,2,3)$ and $\mathcal{H}_W$ the proper transform of $\mathcal{H}_Y$ by the morphism g. Let $\hat{E}$ be the proper transform of E by the weighted blow up g and G be the exceptional divisor of g.

The linear system $\mathcal{H}_W$ coincides with the linear system $|-5K_W|$ since every member in $|-5K_Y|$ has multiplicity at least 1 at the point corresponding to O_w (see Remark 4.3.4 below).

The base locus of the linear system $\mathcal{H}$ is given by the equation $x = yz + xt = 0$. Therefore, it consists of L_{zw}, L_{zt} and the curve C cut by the equation $x = z = 0$. The curve C may not be irreducible. Indeed, C is irreducible if and only if $b \neq 0$. If $b = 0$, then C consists of two irreducible curves L_{yw} and R, where R is an irreducible curve passing through neither the point O_z nor the point O_w. Let $\hat{L}_{zw}$, $\hat{L}_{zt}$, $\hat{L}_{yw}$, $\hat{R}$ and $\hat{C}$ be the proper transforms of the curves L_{zw}, L_{zt}, L_{yw}, R and C, respectively, by the morphism $f \circ g$. We have

$$
\begin{aligned}
&-K_W \cdot \hat{L}_{zw} = -K_{X_{14}} \cdot L_{zw} - \frac{1}{3}\hat{E} \cdot \hat{L}_{zw} - \frac{1}{5}G \cdot \hat{L}_{zw} = -\frac{1}{6};\\
&-K_W \cdot \hat{L}_{zt} = -K_{X_{14}} \cdot L_{zt} - \frac{1}{3}\hat{E} \cdot \hat{L}_{zt} = -\frac{1}{4};\\
&-K_W \cdot \hat{L}_{yw} = -K_{X_{14}} \cdot L_{yw} - \frac{1}{5}G \cdot \hat{L}_{yw} = \frac{1}{30};\\
&-K_W \cdot \hat{R} = -K_{X_{14}} \cdot R > 0; \quad -K_W \cdot \hat{C} = -K_{X_{14}} \cdot C > 0.
\end{aligned}
$$

Therefore, the curves $\hat{L}_{zw}$ and $\hat{L}_{zt}$ are the only curves that intersect $-K_W$ negatively. The log pair $\left(W, \frac{1}{5}\mathcal{H}_W\right)$ is canonical, and hence the log pair $\left(W, \left(\frac{1}{5} + \epsilon\right)\mathcal{H}_W\right)$ is Kawamata log terminal for sufficiently small $\epsilon > 0$. Since

$$K_W + \left(\frac{1}{5} + \epsilon\right)\mathcal{H}_W \sim_{\mathbb{Q}} -\epsilon K_W,$$

the curves $\hat{L}_{zw}$ and $\hat{L}_{zt}$ are the only curves that intersect $K_W + \left(\frac{1}{5} + \epsilon\right)\mathcal{H}_W$ negatively. Therefore, there is a log flip $\chi : W \dashrightarrow U$ along the curves $\hat{L}_{zw}$ and $\hat{L}_{zt}$ ([48]). Let $\check{E}$ and $\check{G}$ be the proper transforms of the divisors $\hat{E}$ and G, respectively, by χ. The anticanonical divisor $K_U + \left(\frac{1}{5} + \epsilon\right)\mathcal{H}_U$ is nef, where $\mathcal{H}_U$ is the proper transform of $\mathcal{H}_W$ by the birational map χ that is an isomorphism in codimension one.

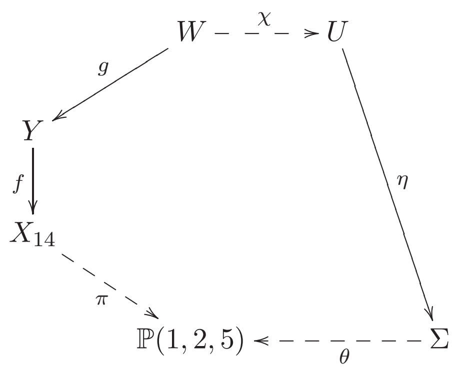
By Log Abundance ([37]), the linear system $|-mK_U|$ is free for sufficiently large m. Hence, it induces a dominant morphism $\eta\colon U \to \Sigma$ with connected fibers, where Σ is a normal variety. We claim that Σ is a surface and η is an elliptic fibration. For this claim, let $\hat{C}_{\lambda,\mu}$ be the proper transform of a general fiber $C_{\lambda,\mu}$ of the map π on the threefold W and let $\check{C}_{\lambda,\mu}$ be its proper transform on U. Then

$$-K_W \cdot \hat{C}_{\lambda,\mu} = -10K_W^3 - 2(-K_W) \cdot (\hat{L}_{zw} + \hat{L}_{zt}) = 0.$$

In particular, the curve $\hat{C}_{\lambda,\mu}$ is disjoint from the curves $\hat{L}_{zt}$ and $\hat{L}_{zw}$ because the base locus of the linear system $|-5K_W|$ contains the curves $\hat{L}_{zt}$ and $\hat{L}_{zw}$. Therefore,

$$-K_U \cdot \check{C}_{\lambda,\mu} = 0.$$

It implies that η contracts $\check{C}_{\lambda,\mu}$. Since we already proved that $C_{\lambda,\mu}$ is birational to an elliptic curve and $\mathcal{H}$ is not composed from a pencil, we can see that η is an elliptic fibration. Moreover, we have proved the existence of a commutative diagram



where θ is a birational map.

We see from (4.3.2) that the divisor $\check{G}$ is a section of the elliptic fibration η and $\check{E}$ is a 2-section of η. Let τ_U be the birational involution of the threefold U obtained from the elliptic fibration $\eta : U \to \Sigma$ with the section $\check{G}$ by Proposition 4.2.1. Then τ_U is biregular in codimension one because K_U is η-nef by our construction ([38, Corollary 3.54]).

Put $\tau_W = \chi^{-1} \circ \tau_U \circ \chi$, $\tau_Y = g \circ \tau_W \circ g^{-1}$ and $\tau = f \circ \tau_Y \circ f^{-1}$.

Since τ_U and χ are biregular in codimension one, so is the involution τ_W. Moreover, we have $\tau_W(G) = G$ since $\tau_U(\check{G}) = \check{G}$ by our construction. This implies that τ_Y is also biregular in codimension one.

In order to see that the point O_z is untwisted by τ, we have only to show that the involution τ is not biregular. To prove this, we suppose that τ is biregular and then look for a contradiction. Note that the proof of Lemma 3.3.2 shows that $\tau_Y(E) = E$ if τ is biregular. This is a key point from which we are able to derive a contradiction.

Let S_λ be the surface on the hypersurface X_{14} cut by the equation $y = \lambda x^2$ with a general complex number λ. It follows from the defining equation of the surface S_λ that the surface has only isolated singularities. Therefore, it is normal (see Remark 4.2.7). Moreover, the surface S_λ is τ-invariant by our construction. Let τ_λ be the restriction of τ to the surface S_λ. It is a birational involution of the surface S_λ since the surface is τ-invariant.

We have a rational map $\pi_\lambda\colon S_\lambda \dashrightarrow \mathbb{P}(1,5) \cong \mathbb{P}^1$ induced by the rational map $\pi\colon X_{14} \dashrightarrow \mathbb{P}(1,2,5)$. Note that the curves L_{zt} and L_{zw} are contained in S_λ. The rational map $\pi_\lambda\colon S_\lambda \dashrightarrow \mathbb{P}^1$ is given by the pencil $\mathcal{P}$ of the curves on the surface $S_\lambda \subset \mathbb{P}(1,3,4,5)$ cut by the equations

$$\delta x^4 = \epsilon(\lambda xz + t),$$

where $[\delta : \epsilon] \in \mathbb{P}^1$. Its base locus is cut out on S_λ by $x = t = 0$, which implies that the base locus of the pencil $\mathcal{P}$ is the curve L_{zw}.

The map π_λ is not defined only at the points O_w and O_z. To see this, plug in $t = \frac{\delta}{\epsilon}x^4 - \lambda xz$ into the defining equation of the surface S_λ (with general $[\delta : \epsilon] \in \mathbb{P}^1$), divide the resulting equation by x (removing the base curve L_{zw}), and put $x = 0$ into the resulting equation in x, z, and w (we know that the base locus of $\mathcal{P}$ is L_{zw}). This gives the system of equations $zw^2 = x = t = 0$, which means that the map π_λ is not defined only at the points O_w and O_z.

Let C_λ be a general fiber of the map π_λ. Then C_λ is given by (4.3.2) with a general complex number μ. As shown in the beginning, the fiber C_λ is an irreducible curve birational to a smooth elliptic curve. Let $\nu\colon \check{C}_\lambda \to C_\lambda$ be the normalization of the curve C_λ. It follows from

(4.3.2) that $\nu^{-1}(O_w)$ consists of a single point and $\nu^{-1}(O_z)$ consists of two distinct points. Note that we can consider the curves C_λ and $\check{C}_\lambda$ (and the map ν) to be defined over the function field $\mathbb{C}(\mu)$. In this case, $\nu^{-1}(O_z)$ consists of a single point of degree 2, i.e., a point splitting into two points over the algebraic closure of $\mathbb{C}(\mu)$.

Let $\hat{S}_\lambda$ be the proper transform of S_λ via $f \circ g$. Put $\hat{E}_\lambda = \hat{E}|_{\hat{S}_\lambda}$ and $\hat{G}_\lambda = G|_{\hat{S}_\lambda}$. Resolving the indeterminacy of the rational map π_λ through $\hat{S}_\lambda$, we obtain an elliptic fibration $\bar{\pi}_\lambda\colon \bar{S}_\lambda \to \mathbb{P}^1$. Thus, we have a commutative diagram

$$\begin{array}{ccc} & \bar{S}_\lambda & \\ {}^{\sigma}\swarrow & & \searrow^{\bar{\pi}_\lambda} \\ S_\lambda & \underset{\pi_\lambda}{\dashrightarrow} & \mathbb{P}^1, \end{array}$$

where σ is a birational map. Note that there exist exactly two σ-exceptional prime divisors that do not lie in the fibers of $\bar{\pi}_\lambda$. One is the proper transform of $\hat{E}_\lambda$ and the other is the proper transform of $\hat{G}_\lambda$. Let $\bar{E}_\lambda$ and $\bar{G}_\lambda$ be these two exceptional divisors, respectively. Then $\bar{G}_\lambda$ is a section of $\bar{\pi}_\lambda$ and $\bar{E}_\lambda$ is a 2-section of $\bar{\pi}_\lambda$. Denote the other σ-exceptional curves (if any) by $F_1, \ldots, F_r$.

Put $\bar{\tau}_\lambda = \sigma^{-1} \circ \tau_\lambda \circ \sigma$. Due to [28, Theorem 3.2], we may assume that $\bar{\tau}_\lambda$ is biregular and $\bar{S}_\lambda$ is smooth.

Let $\bar{C}_\lambda$ be the proper transform of the curve C_λ on $\bar{S}_\lambda$. Then $\bar{C}_\lambda \cong \check{C}_\lambda$, since $\bar{C}_\lambda$ is smooth. Moreover, the curve $\bar{C}_\lambda$ is $\bar{\tau}_\lambda$-invariant. Furthermore, $\bar{\tau}_\lambda|_{\bar{C}_\lambda}$ is given by the reflection with respect to the point $\bar{G}_\lambda \cap \bar{C}_\lambda$. On the other hand, the divisor $\bar{E}_\lambda$ must be $\bar{\tau}_\lambda$-invariant since $\tau_Y(E) = E$. Therefore, the divisor $\bar{E}_\lambda - 2\bar{G}_\lambda$ must be numerically equivalent to a $\mathbb{Q}$-linear combination of curves on $\bar{S}_\lambda$ that lie in the fibers of $\bar{\pi}_\lambda$ by Lemma 4.2.8.

Let $\bar{L}_{zt}$ and $\bar{L}_{zw}$ be the proper transforms of the curves L_{zt} and L_{zw} by σ, respectively. Then $\bar{L}_{zt}$ and $\bar{L}_{zw}$ lies in the same fiber of the elliptic fibration $\bar{\pi}_\lambda$. In the fiber containing $\bar{L}_{zt}$ and $\bar{L}_{zw}$, the other components are, if any, σ-exceptional since the fiber of π_λ over the point $[0:1]$ consists only of L_{zt} and L_{zw}. In addition, we see that every other fiber of $\bar{\pi}_\lambda$ contains exactly one irreducible reduced curve that is not σ-exceptional. Indeed, this immediately follows from Lemma 4.3.5 below. Since all fibers of $\bar{\pi}_\lambda$ (with scheme structure) are numerically equivalent and the divisor $\bar{E}_\lambda - 2\bar{G}_\lambda$ is numerically equivalent to a $\mathbb{Q}$-linear combination of curves that lie in the fibers of $\bar{\pi}_\lambda$, we obtain

$$\bar{E}_\lambda - 2\bar{G}_\lambda \sim_{\mathbb{Q}} c_{zt}\bar{L}_{zt} + c_{zw}\bar{L}_{zw} + \sum_{i=1}^{r} c_i F_i$$

for some rational numbers c_{zt}, c_{zw}, $c_1, \ldots, c_r$. The intersection form of the curves $\bar{E}_\lambda$, $\bar{G}_\lambda$, $F_1, \ldots, F_r$ is negative-definite since these curves are σ-exceptional. Therefore, $(c_{zt}, c_{zw}) \neq (0,0)$. On the other hand, we have

$$0 \sim_{\mathbb{Q}} c_{zt}L_{zt} + c_{zw}L_{zw}$$

on the surface S_λ. In particular, the intersection form of the curves L_{zw} and L_{zt} is degenerate on the surface S_λ.

Meanwhile, from the intersection numbers

$$(2L_{zt} + L_{zw}) \cdot L_{zw} = \frac{1}{15}, \quad (2L_{zt} + L_{zw}) \cdot L_{zt} = \frac{1}{12}$$

on the surface S_λ, we obtain

$$\begin{pmatrix} L_{zw}^2 & L_{zw}\cdot L_{zt} \\ L_{zw}\cdot L_{zt} & L_{zt}^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{15} - 2L_{zw}\cdot L_{zt} & L_{zw}\cdot L_{zt} \\ L_{zw}\cdot L_{zt} & \frac{1}{24} - \frac{1}{2}L_{zw}\cdot L_{zt} \end{pmatrix}.$$

The curves L_{zw} and L_{zt} intersect only at the point O_z. However, the surface S_λ is not quasi-smooth at the point O_z. To get the intersection number $L_{zw}\cdot L_{zt}$, we consider the divisor D_t on the surface S_λ cut by the equation $t=0$. We can immediately see that $D_t = 2L_{zw}+R$, where R is the residual curve. The curves L_{zw} and R intersect only at the point O_w at which the surface S_λ is quasi-smooth. Then we obtain $L_{zw}^2 = -\frac{4}{15}$ from the intersection numbers

$$(2L_{zw}+R)\cdot L_{zw} = \frac{4}{15}, \quad R\cdot L_{zw} = \frac{4}{5}.$$

Therefore, $L_{zw}\cdot L_{zt} = \frac{1}{6}$ and hence the intersection matrix is non-singular. This is a contradiction. It shows that $\tau_Y(E)\neq E$. In particular, the involution τ is not biregular. Since the involution τ_Y is biregular in codimension one, the involution τ meets the conditions in Definition 3.3.1. Consequently, the birational involution τ untwists the singular point O_z. □

Remark 4.3.4. Local parameters at O_z are induced by x, t, w whose multiplicities are $\frac{1}{3}$, $\frac{1}{3}$, and $\frac{2}{3}$. The monomial z^4y shows that y vanishes at the point O_z with multiplicity at least $\frac{2}{3}$. Furthermore, since

$$-y = xt + (t+by^2)w^2 + yt(t-\alpha_1y^2)(t-\alpha_2y^2) + wf_9(x,y,1,t) + f_{14}(x,y,1,t)$$

around O_z and xt vanishes at the point O_z with multiplicity $\frac{2}{3}$, the monomial y vanishes at the point O_z with multiplicity exactly $\frac{2}{3}$. Then the relation

$$-(y+xt) = (t+by^2)w^2 + yt(t-\alpha_1y^2)(t-\alpha_2y^2) + wf_9(x,y,1,t) + f_{14}(x,y,1,t)$$

around O_z shows that $yz+xt$ vanishes at the point O_z with multiplicity $\frac{5}{3}$.

The linear system $|-5K_X|$ is generated by w, xt, yz, x^2z, xy^2, x^3y and x^5. First of all, the last three monomials vanish at O_z with multiplicity $\frac{5}{3}$. In terms of the local parameters x, t, w, we have

$$yz = -xt + \text{higher degree terms}$$

locally around the point O_z. Furthermore, for any complex numbers α, β, δ, ϵ, we have

$$\alpha w + \beta xt + \delta yz + \epsilon x^2z = (\alpha w + \beta xt - \delta xt + \epsilon x^2) + \text{higher degree terms}$$

locally around the point O_z. For the monomial $\alpha w + \beta xt + \delta yz + \epsilon x^2z$ to have multiplicity bigger than $\frac{2}{3}$ at O_z, we must have $\alpha=\epsilon=0$ and $\beta=\delta$. Since $yz+xt$ vanishes at the point O_z with multiplicity $\frac{5}{3}$, we see that the linear system $\mathcal{H}$ consists exactly of the members of $|-5K_X|$ vanishing at O_z with multiplicity at least $\frac{5}{3}$.

Meanwhile, the variables x, y, z induce local parameters at the point O_w with multiplicities $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, respectively. Also, since t vanishes at the point O_w with multiplicity at least $\frac{4}{5}$, the polynomial $yz+xt$ vanishes at the point O_w with multiplicity 1. Therefore, every member in $\mathcal{H}$ vanishes at O_w with multiplicity at least 1.

Lemma 4.3.5. *Under the conditions of Theorem 4.3.1, for a general complex number λ, the curve* (4.3.2) *is irreducible and reduced for every complex number μ.*

Proof. Suppose that for a general complex number λ there is always μ such that the curve $C_{\lambda,\mu}$ is reducible. There is then a one-dimensional family of reducible curves $C_{\lambda,\mu}$ given by (4.3.2) with a general complex number λ and a complex number μ depending on λ. Denote a general curve in this one-dimensional family by C.

Since (4.3.2) always contains the monomial zw^2, the curve C must have an irreducible component C_1 that is defined by either

$$y - \lambda x = t - \mu x^4 + \lambda xz = w + h_5(x, z) = 0$$

or

$$y - \lambda x = t - \mu x^4 + \lambda xz = w^2 + wg_5(x, z) + g_{10}(x, z) = 0.$$

Then

$$-K_Y \cdot \tilde{C}_1 = -K_{X_{14}} \cdot C_1 - \frac{1}{3} E \cdot \tilde{C}_1 = \begin{cases} \dfrac{1 \cdot 2 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \dfrac{1}{3} = 0 \text{ for the former case,} \\ \\ \dfrac{1 \cdot 2 \cdot 4 \cdot 2 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \dfrac{2}{3} = 0 \text{ for the latter case} \end{cases}$$

and $\tilde{C}_1 \cdot E > 0$. By Lemma 3.2.8, the point O_z cannot be a center of non-canonical singularities of the log pair $(X_{14}, \frac{1}{n}\mathcal{M})$. This contradiction proves the statement. □

Now we go back to the hypersurfaces in the family No. 7 described in the previous section. The hypersurface X_8 of Type II is defined in $\mathbb{P}(1, 1, 2, 2, 3)$ by the equation of the type

$$(z + f_2(x, y))w^2 + wf_5(x, y, z, t) - zt^3 - t^2 f_4(x, y, z) - tf_6(x, y, z) + f_8(x, y, z) = 0. \quad (4.3.6)$$

Since f_5 must contain either xt^2 or yt^2, we write $f_5(x, y, z, t) = g_5(x, y, z, t) + a_1 xt^2 + a_2 yt^2$. Furthermore, we may assume that $a_1 = 1$ and $a_2 = 0$ by a suitable coordinate change.

By coordinate change $z + f_2(x, y) \mapsto z$, we may assume that our hypersurface X_8 is defined by

$$zw^2 + wf_5(x, y, z, t) - (z - f_2(x, y))t^3 - t^2 f_4(x, y, z) - tf_6(x, y, z) + f_8(x, y, z) = 0. \quad (4.3.7)$$

This assumption will help us understand, without any loss of generality, the intersection of the surface cut by $y = \lambda x$ and the surface cut by $z = \mu x^2$, where λ and μ are constants.

On the hypersurface X_8, consider the surface cut by $y = \lambda x$ and the surface cut by $z = \mu x^2$. Then the intersection of these two surfaces is the 1-cycle $L_{tw} + C_{\lambda,\mu}$, where the curve $C_{\lambda,\mu}$ is defined by the equation

$$\begin{aligned} &\mu xw^2 + wt^2 - (\mu - f_2(1, \lambda))xt^3 + \\ &+ \frac{wg_5(x, \lambda x, \mu x^2, t) - t^2 f_4(x, \lambda x, \mu x^2) - tf_6(x, \lambda x, \mu x^2) + f_8(x, \lambda x, \mu x^2)}{x} = 0 \end{aligned} \quad (4.3.8)$$

in $\mathbb{P}(1, 2, 3)$. For sufficiently general complex numbers λ and μ the curve $C_{\lambda,\mu}$ is birational to an elliptic curve. To figure this out, we plug in $x = 1$ into (4.3.8) so that we could see that the curve is birational to a double cover of $\mathbb{C}$ ramified at four distinct points.

Let $\mathcal{H}$ be the linear subsystem of $|-2K_{X_8}|$ generated by x^2, xy, y^2 and z. Let $\pi\colon X_8 \dashrightarrow \mathbb{P}(1,1,2)$ be the rational map induced by

$$[x:y:z:t:w] \mapsto [x:y:z].$$

It is a morphism outside of the curve L_{tw}. Moreover, the map is dominant. The curve $C_{\lambda,\mu}$ is a fiber of the map π. Its general fiber is an irreducible curve birational to an elliptic curve since the curve $C_{\lambda,\mu}$ with sufficiently general complex numbers λ and μ is birational to an elliptic curve.

Lemma 4.3.9. *Suppose that the hypersurface X_8 in the family No. 7 is defined by* (4.3.7)*. If the singular point O_t is a center of non-canonical singularities of the log pair $(X_8, \frac{1}{n}\mathcal{M})$, then for a general complex number λ, the curve $C_{\lambda,\mu}$ is always irreducible for every value of μ.*

Proof. Suppose that for a general complex number λ there is always μ such that the curve $C_{\lambda,\mu}$ is reducible. Since the base locus of $\mathcal{H}$ consists of the curve L_{tw}, there is a one-dimensional family of reducible curves $C_{\lambda,\mu}$ given by (4.3.8) with a general complex number λ and a complex number μ depending on λ. Denote a general curve in this one-dimensional family by C.

We claim that the curve C always has an irreducible component C_1 defined by

$$y-\lambda x = z-\mu x^2 = w - h_3(x,t) = 0$$

for some polynomial h_3. To prove the claim, write $g_5(x,y,z,t) = f_3(x,y,z)t + f_5(x,y,z)$, set $x=1$ for (4.3.8), and then obtain

$$\begin{aligned}&\mu w^2 + w\left(t^2 + f_3(1,\lambda,\mu)t + f_5(1,\lambda,\mu)\right) - \\ &-(\mu - f_2(1,\lambda))t^3 - f_4(1,\lambda,\mu)t^2 - f_6(1,\lambda,\mu)t + f_8(1,\lambda,\mu) = 0.\end{aligned}$$

Suppose that the claim is not a case. Then we must have $\mu = 0$ and the polynomial

$$w\left(t^2 + f_3(1,\lambda,0)t + f_5(1,\lambda,0)\right) + f_2(1,\lambda))t^3 - f_4(1,\lambda,0)t^2 - f_6(1,\lambda,0)t + f_8(1,\lambda,0)$$

must be reducible. Since λ is general, this implies that

$$wf_5(x,y,0,t) + f_2(x,y)t^3 - t^2 f_4(x,y,0) - tf_6(x,y,0) + f_8(x,y,0) = A(x,y,t,w)B(x,y,t,w),$$

for some non-constant polynomials $A(x,y,t,w)$ and $B(x,y,t,w)$. Since we may write

$$\begin{aligned}zw^2 + wf_5(x,y,z,t) - (z - f_2(x,y))t^3 - t^2 f_4(x,y,z) - tf_6(x,y,z) + f_8(x,y,z)\\ = zH(x,y,z,t,w) + A(x,y,t,w)B(x,y,t,w),\end{aligned}$$

for some non-constant polynomial $H(x,y,z,t,w)$, the hypersurface X_8 is not quasi-smooth at the points defined by $z = H(x,y,z,t,w) = A(x,y,t,w) = B(x,y,t,w) = 0$. This is a contradiction. Consequently, the reducible curve C splits into an irreducible curve C_1 defined by

$$y-\lambda x = z-\mu x^2 = w - h_3(x,t) = 0$$

for some polynomial h_3 and the curve C_2 (possibly reducible) defined by

$$y-\lambda x = z-\mu x^2 = t^2 - h_4(x,t,w) = 0$$

for some polynomial h_4.

Note that C_1 passes through the point O_t but C_2 does not. Then $-K_Y\cdot\tilde{C}_1=0$ and $\tilde{C}_1\cdot E>0$. By Lemma 3.2.8, the point O_t cannot be a center of non-canonical singularities of the log pair $(X_8,\frac{1}{n}\mathcal{M})$. This contradicts our condition. Therefore, for a general complex number λ, the curve $C_{\lambda,\mu}$ is always irreducible for every value of μ. □

For a general complex number λ, the curve $C_{\lambda,\mu}$ is always reduced for every value of μ. Indeed, if the curve is not reduced, then the proof shows that $\mu\neq 0$. Then the equation for the curve must contain xw^2 and wt^2. Hence, it must split into the form $(t^2+xw+\cdots)(w+\cdots)$. The polynomial of the type $(t^2+xw+\cdots)$ cannot be a square. Therefore, $C_{\lambda,\mu}$ is always reduced. Moreover, for a general complex number λ, the curve L_{tw} cannot be an irreducible component of the curve $C_{\lambda,\mu}$ for every value of μ.

Theorem 4.3.10. *Suppose that the hypersurface X_8 in the family No. 7 is defined by* (4.3.7). *If the singular point O_t is a center of non-canonical singularities of the log pair $(X_8,\frac{1}{n}\mathcal{M})$, then there is a birational involution that untwists the singular point O_t.*

Proof. Let $g\colon Z\to Y$ be the weighted blow up at the point over O_w with weight $(1,1,2)$ and let F be its exceptional divisor. The divisor F contains a singular point of Z that is of type $\frac{1}{2}(1,1,1)$. Let $h:W\to Z$ be the blow up at this singular point with the exceptional divisor G. Let $\hat{L}_{tw}$ and $\breve{L}_{tw}$ be the proper transforms of the curve L_{tw} by the morphism $f\circ g\circ h$ and by the morphism $f\circ g$, respectively. Also, let $\hat{E}$ and $\hat{F}$ be the proper transforms of the exceptional divisors E and F by the morphism $g\circ h$ and by the morphism h, respectively.

Let $\mathcal{H}_Y$ and $\mathcal{H}_W$ be the proper transforms of the linear system $\mathcal{H}$ by the morphism f and by the morphism $f\circ g\circ h$, respectively. We then see that $\mathcal{H}_Y=|-2K_Y|$ and $\mathcal{H}_W=|-2K_W|$. The base locus of the linear system $\mathcal{H}$ consists of the single curve L_{tw}. We have

$$-K_W\cdot\hat{L}_{tw}=-K_{X_8}\cdot L_{tw}-\frac{1}{2}\hat{E}\cdot\hat{L}_{tw}-\frac{1}{3}F\cdot\breve{L}_{tw}-\frac{1}{2}G\cdot\hat{L}_{tw}=-1.$$

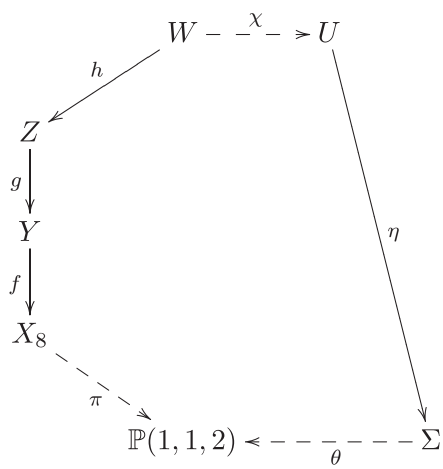
Therefore, the curve $\hat{L}_{tw}$ is the only curve that intersects $-K_W$ negatively.

By the same procedure as in the proof of Theorem 4.3.1, we construct a log flip $\chi:W\dashrightarrow U$ along the curve $\hat{L}_{tw}$ and a dominant morphism η of U into a normal variety Σ with connected fibers by the base-point-free linear system $|-mK_U|$ for sufficiently large m.

Let $\check{E}$, $\check{F}$ and $\check{G}$ be the proper transforms of the divisors $\hat{E}$, $\hat{F}$ and G, respectively, by χ. Let $\hat{C}_{\lambda,\mu}$ be the proper transform of a general fiber $C_{\lambda,\mu}$ of the map π on W and let $\check{C}_{\lambda,\mu}$ be its proper transform on U. We then see

$$-K_W\cdot\hat{C}_{\lambda,\mu}=-2K_W^3-(-K_W)\cdot\hat{L}_{tw}=0.$$

By the same reason as in the proof of Theorem 4.3.1, we see that η is an elliptic fibration and we obtain the following commutative diagram:



where θ is a birational map.

It follows from (4.3.8) that the divisors $\check{E}$ and $\check{G}$ are sections of the elliptic fibration η. Let τ_U be the birational involution of the threefold U that is induced by the reflection of the general fiber of η with respect to the section $\check{G}$. Then τ_U is biregular in codimension one because K_U is η-nef by our construction ([38, Corollary 3.54]).

Put $\tau_W = \chi^{-1} \circ \tau_U \circ \chi$, $\tau_Y = (g \circ h) \circ \tau_W \circ (g \circ h)^{-1}$ and $\tau = f \circ \tau_Y \circ f^{-1}$ as before. Then τ_W is also biregular in codimension one since χ is a log flip. Moreover, we have $\tau_W(G) = G$ since $\tau_U(\check{G}) = \check{G}$ by our construction. The image $\tau_U(\check{F})$ is an irreducible surface since τ_U is biregular in codimension one. The map $\pi \circ f \circ g$ sends F to the curve in $\mathbb{P}(1,1,2)$ defined by $z = 0$ and the log flip χ changes nothing on the intersection of G and $\hat{F}$. Therefore, the morphism η contracts $\check{F}$ to a curve and the image $\tau_U(\check{F})$ lies over this curve. Since $\check{F}$ intersects with the section $\check{G}$ along a curve and $\tau_U(\check{F})$ intersects with the section $\check{G}$ along the curve, $\tau_U(\check{F}) = \check{F}$ and $\tau_W(\hat{F}) = \hat{F}$. Consequently, τ_Y is biregular in codimension one.

We claim that the point O_t is untwisted by τ. For us to prove the claim, it is enough to show that $\tau_Y(E) \neq E$ due to Remark 3.3.3. For this end, we suppose that $\tau_Y(E) = E$ and look for a contradiction.

Let S_λ be the surface on the hypersurface X_8 cut by the equation $y = \lambda x$ with a general complex number λ. It is a $K3$ surface with only cyclic du Val singularities. The point O_t is a A_1 singular point of S_λ and the point O_w is a A_2 singular point of S_λ. Let τ_λ be the restriction of τ to the surface S_λ. It is a birational involution of the surface S_λ since the surface is τ-invariant by our construction.

The projection $\pi\colon X_8 \dashrightarrow \mathbb{P}(1,1,2)$ induces a rational map $\pi_\lambda\colon S_\lambda \to \mathbb{P}(1,2) \cong \mathbb{P}^1$. The rational map $\pi_\lambda\colon S_\lambda \dashrightarrow \mathbb{P}^1$ is given by the pencil of the curves on the surface $S_\lambda \subset \mathbb{P}(1,2,2,3)$ cut by the equations

$$\delta x^2 = \epsilon z,$$

where $[\delta : \epsilon] \in \mathbb{P}^1$. Its base locus is cut out on S_λ by $x = z = 0$. Therefore, the base locus is the curve L_{tw}. We can easily see from (4.3.8) that the map π_λ is not defined only at the points O_w and O_t.

Let $\hat{S}_\lambda$ be the proper transform of S_λ via $f \circ g \circ h$ and put $\hat{E}_\lambda = \hat{E}|_{\hat{S}_\lambda}$ and $\hat{G}_\lambda = G|_{\hat{S}_\lambda}$ as in the proof of Theorem 4.3.1. Resolving the indeterminacy of the rational map π_λ through $\hat{S}_\lambda$, we obtain an elliptic fibration $\bar{\pi}_\lambda \colon \bar{S}_\lambda \to \mathbb{P}^1$. Thus, we have a commutative diagram

$$\begin{array}{ccc} & \bar{S}_\lambda & \\ {}^{\sigma}\swarrow & & \searrow^{\bar{\pi}_\lambda} \\ S_\lambda & \overset{}{\underset{\pi_\lambda}{\dashrightarrow}} & \mathbb{P}^1, \end{array}$$

where σ is a birational morphism. There exist exactly two σ-exceptional prime divisors that do not lie in the fibers of $\bar{\pi}_\lambda$. One is the proper transform of $\hat{E}_\lambda$ and the other is the proper transform of $\hat{G}_\lambda$. Let $\bar{E}_\lambda$ and $\bar{G}_\lambda$ be these two exceptional divisors, respectively. Then $\bar{E}_\lambda$ and $\bar{G}_\lambda$ are sections of $\bar{\pi}_\lambda$. Denote the other σ-exceptional curves (if any) by $F_1, \ldots, F_r$.

Put $\bar{\tau}_\lambda = \sigma^{-1} \circ \tau_\lambda \circ \sigma$. We may assume that $\bar{\tau}_\lambda$ is biregular and $\bar{S}_\lambda$ is smooth by [28, Theorem 3.2].

By the same argument as in the proof of Theorem 4.3.1, the divisor $\bar{E}_\lambda - \bar{G}_\lambda$ is numerically equivalent to a $\mathbb{Q}$-linear combination of curves on $\bar{S}_\lambda$ that lie in the fibers of $\bar{\pi}_\lambda$. Observe that we use the assumption $\tau_Y(E) = E$ at this step.

Note that the equation $x = 0$ cuts out S_λ into a curve that splits as a union $L_{tw} + C_x$, where C_x is the curve defined by

$$x = w^2 - t^3 + azt^2 + bz^2t + cz^3 = 0$$

for some constants a, b, c in $\mathbb{P}(1, 2, 2, 3)$. The curve C_x is irreducible and reduced.

Let $\bar{L}_{tw}$ and $\bar{C}_x$ be the proper transforms of the curves L_{tw} and C_x by σ, respectively. Then $\bar{L}_{tw}$ and $\bar{C}_x$ lie in the same fiber of the elliptic fibration $\bar{\pi}_\lambda$ and they are the only non-σ-exceptional curves in this fiber. Moreover, every other fiber of $\bar{\pi}_\lambda$ contains exactly one irreducible and reduced curve that is not σ-exceptional because for a general complex number λ, the curve $C_{\lambda,\mu}$ is always irreducible and reduced for every value of μ by Lemma 4.3.9. Therefore, as before, we are able to obtain

$$\bar{E}_\lambda - \bar{G}_\lambda \sim_{\mathbb{Q}} c_{tw}\bar{L}_{tw} + c_x\bar{C}_x + \sum_{i=1}^{r} c_i F_i$$

for some rational numbers c_{tw}, c_x, $c_1, \ldots, c_r$. The intersection form of the curves $\bar{E}_\lambda$, $\bar{G}_\lambda$, $F_1, \ldots, F_r$ is negative-definite since these curves are σ-exceptional. Therefore, $(c_{tw}, c_x) \neq (0, 0)$. On the other hand, we have

$$0 \sim_{\mathbb{Q}} c_{tw}L_{tw} + c_xC_x$$

on the surface S_λ, and hence the intersection form of the curves L_{tw} and C_x is degenerate on the surface S_λ.

However, from the intersection numbers

$$(L_{tw} + C_x) \cdot L_{tw} = \frac{1}{6}, \quad (L_{tw} + C_x)^2 = \frac{2}{3}, \quad L_{tw} \cdot C_x = 1$$

on the surface S_λ, we obtain

$$\begin{pmatrix} L_{tw}^2 & L_{tw} \cdot C_x \\ L_{tw} \cdot C_x & C_x^2 \end{pmatrix} = \begin{pmatrix} -\frac{5}{6} & 1 \\ 1 & -\frac{1}{2} \end{pmatrix}.$$

This is a contradiction. The obtained contradiction verifies that $\tau_Y(E) \neq E$. This completes the proof. $\square$

5 Proof of Main Theorem

5.1 How to read the tables

The remaining job is to exclude or untwist all the singular points on quasi-smooth hypersurfaces in the 95 families. To execute this crucial job, we need to know how to read the tables in the next section. They carry all the information for excluding and untwisting the singular points.

For each family we present a table that carries

- the entry number (the underlined entry number means that the family corresponds to Theorem 1.1.10, i.e., birationally super-rigid family),
- the intersection number of the anticanonical divisor, i.e., $-K_X^3 = A^3$,
- a defining equation of the hypersurface X,
- its singularities,
- the sign of B^3,
- the linear system on Y containing the key surface T in the applied method,
- a defining equation for the surface $f(T)$ or generators of a linear system that contains $f(T)$ as a general member,
- terms that determine the multiplicity of the surface $f(T)$ at the given singular point.

When the table carries only a monomial or a binomial, instead of B^3, the linear system, the surface T and the vanishing order for the corresponding singular point(s), we apply the methods below with the squared symbols to the corresponding singular points. The monomial or the binomial plays an essential role in defining the involution untwisting the singular point.

The table shows which method is applied to each of the singular points by the symbols ⓑ, ⓝ, ⓢ, ⓕ, ⓟ, $\boxed{\tau}$, $\boxed{\tau_1}$, $\boxed{\epsilon}$, $\boxed{\epsilon_1}$, $\boxed{\epsilon_2}$, $\boxed{\iota}$ and $\boxed{\iota_1}$. The following explain the method corresponding to each of the symbols.

ⓑ : Apply Lemma 3.2.2.
The condition $T\cdot\Gamma \leq 0$ can be easily checked by the items in the table (see Remark 3.2.3). The condition on the 1-cycle Γ can be immediately checked. This can be done on the hypersurface X even though the cycle lies on the threefold Y. Indeed, in the cases where this method is applied, the surface T is given in such a way that the 1-cycle Γ has no component on the exceptional divisor E.

ⓝ : Apply Lemma 3.2.4.
The divisor T is given as the proper transform of a general member of the linear system generated by the monomial(s) in the slot for the item T of the table. Using Lemma 3.2.6 we check that the given divisor T is nef. The non-positivity of $T\cdot S\cdot B$ can be immediately verified from the items in the table and the positivity of $T \cdot S \cdot A$ is always guaranteed (see Remark 3.2.5).

ⓢ : Apply Lemma 3.2.7.
We take a general member H in the linear system generated by the polynomials given in the slot for the item T in the table. We can easily show that the surface H is normal by checking that it has only isolated singularities. The surface T is given as the proper transform of the surface H by the morphism f. The divisor on T cut out by the surface S is a reducible curve. We check that this reducible curve forms a negative-definite divisor on the normal surface T.

ⓕ : Apply Lemma 3.2.8.
We find a 1-dimensional family of irreducible curves $\tilde{C}_\lambda$ such that $-K_Y \cdot \tilde{C}_\lambda \leq 0$. We can find this family on the surface T that is given as the proper transform of a general member of the linear system generated by the polynomial(s) provided in the slot for the item T of the table.

ⓟ : Apply Lemma 3.2.9.
If the singular point O_t satisfies the conditions of Lemma 3.2.9, we can always find a 1-dimensional family of irreducible curves $\tilde{C}_\lambda$ on the given surface T such that $-K_Y \cdot \tilde{C}_\lambda \leq 0$, so that we could immediately exclude the singular point O_t.

As we see, the circled methods are applied to exclude singular points on X. The squared symbols below are the methods with which we can untwist the corresponding singular point if it is a center of non-canonical singularities of the log pair.

$\boxed{\tau}$: Apply Lemma 4.1.1 and Lemma 4.1.3
The given monomial in the table is the monomial $x_{i_3}x_{i_4}^2$ in Lemma 4.1.1 and Lemma 4.1.3 that plays a central role in defining the involution. If the hypersurface X is defined by the equation as in Lemma 4.1.1, the involution given by the quadratic equation is birational and untwists the given singular point. If the hypersurface X is defined by the equation as in Lemma 4.1.3, the involution given by the quadratic equation is biregular. In such a case, Lemma 4.1.3 excludes the corresponding singular point. Note that both the cases can always happen.

$\boxed{\tau 1}$: Apply Lemma 4.1.1, Lemma 4.1.3 and Theorem 4.1.4
This method is basically the same as the method $\boxed{\tau}$. The difference is that we may have no $x_{i_3}x_{i_4}^2$ in the defining equation. Such cases occur only when the corresponding singular point is O_t and $x_{i_3}x_{i_4}^2 = wt^2$. In cases, Theorem 4.1.4 excludes the singular point O_t. These three cases can always occur, i.e., the case when the defining equation has the monomial wt^2 with f_e not divisible by w, the case when the defining equation has the monomial wt^2 with f_e divisible by w and the case when the defining equation does not have the monomial wt^2.

$\boxed{\epsilon}$: Apply Theorem 4.2.6
This is for the singular point O_t of quasi-smooth hypersurfaces in the families No. 7 (Type I), 23, 40, 44, 61 and 76. The given binomial in the table is the binomial $tw^2 - x_it^3$ in (4.2.4) that plays a central role in defining the involution. The singular point O_t may not be a center of non-canonical singularities of the log pair in some situation. However, if it is a center, then it can be untwisted by an elliptic involution.

$\boxed{\epsilon 1}$: Apply Theorem 4.2.13.
This is for the singular point O_z of quasi-smooth hypersurfaces in the family No. 36.

$\boxed{\epsilon 2}$: Apply Theorem 4.2.13 and Theorem 4.2.14
This is for the singular point O_z of quasi-smooth hypersurfaces in the family No. 20.

$\boxed{\iota}$: Apply Theorem 4.3.10
This is for the singular points of type $\frac{1}{2}(1,1,1)$ on quasi-smooth hypersurfaces of Type II in the family No. 7.

$\boxed{\iota 1}$: Apply Theorem 4.3.1
This is for the singular point O_z of the special hypersurfaces in the family No. 23 described in Section 4.3.

In each table, we present a defining equation of the hypersurface in the family. For this we use the following notations and conventions.

- The Roman alphabets a, b, c, d, e with numeric subscripts or without subscripts are constants.
- The Greek alphabets α, β with numeric subscripts or without subscripts are constants.
- The same Roman alphabets with distinct numeric subscripts, e.g., a_1, a_2, a_3, in an equation are constants one of which is not zero.
- The same Greek alphabets with distinct numeric subscripts, e.g., α_1, α_2, α_3, in an equation are distinct constants.
- The singularity types are often given as a form $\frac{1}{r}(w^1_{x_{k_1}}, w^2_{x_{k_2}}, w^3_{x_{k_3}})$, where the subscript x_{k_i} is the homogeneous coordinate function which induces a local parameter corresponding to the weight $w^i_{x_{k_i}}$.

For each family, the defining equation of the hypersurface X must satisfies the following rules in order to be quasi-smooth (see [29] for more detail).

- If $a_i > 1$, it is relatively prime to the other weights and it divides d, then $x_i^{\frac{d}{a_i}}$ must appear in the defining equation.
- If $a_i > 1$, it is relatively prime to the other weights but it does not divide d, then $x_i^{\frac{d-a_j}{a_i}} x_j$ for some j must appear in the defining equation.
- If a_i and a_j are not relatively prime, then a reduce polynomial of degree d in x_i and x_j must appear in the defining equation.

In each table, the defining equation is written in the form

$$\text{key-monomial part} + wf_{d-a_4}(x,y,z,t) + f_d(x,y,z,t) \quad \text{if } d < 3a_4;$$

$$\text{key-monomial part} + w^2 f_{d-2a_4} + wf_{d-a_4}(x,y,z,t) + f_d(x,y,z,t) \quad \text{if } d = 3a_4,$$

where key-monomial part consists of the monomials that are required for quasi-smoothness and necessary for our methods of excluding or untwisting the singularities. If necessary, we expand $f_d(x,y,z,t)$ with respect to the variable t, i.e., instead of $f_d(x,y,z,t)$, we write

$$g_{d-a_3m}(x,y,z)t^m + g_{d-a_3m+a_3}(x,y,z)t^{m-1} + \cdots + g_d(x,y,z).$$

Note that we do not put all the monomials required for quasi-smoothness in the key-monomial part. We put only some of them that play roles for our methods of excluding or untwisting the singularities on the given hypersurface. To simplify the key monomial part as much as possible without loss of generality, we apply suitable coordinate changes, if necessary. It will not be too complicated to check that the given quasi-homogeneous polynomial represents every quasi-smooth hypersurface in the family.

5.2 The tables

To prove Main Theorem, we suppose that a given quasi-smooth hypersurface X from the 95 families has a mobile linear system $\mathcal{M}$ in $|-nK_X|$ for some positive integer n such that the log pair $(X, \frac{1}{n}\mathcal{M})$ is not canonical. Therefore, we have a center of non-canonical singularities of the pair $(X, \frac{1}{n}\mathcal{M})$. Theorems 2.1.10 and 2.2.4 show that if there is a center on X, then it must be a singular point.

In this section, we exclude or untwist every singular point on a given quasi-smooth hypersurface in each of the 95 families. To be precise, we prove

Theorem 5.2.1. *If a singular point on X is a center of non-canonical singularities of the log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$, then it can be untwisted by a birational involution of X.*

By verifying this theorem, we obtain a complete proof of Main Theorem from Theorem 1.2.2.

Proof. The proof is given mainly by the tables. Following the instruction in Section 5.1 with the extra explanation (if necessary) provided right after the table, we prove Theorem 5.2.1 for each family.

No. 2: $X_5 \subset \mathbb{P}(1,1,1,1,2)$ $A^3 = 5/2$

$tw^2 + wf_3(x,y,z,t) + f_5(x,y,z,t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{2}(1,1,1)$ $\boxed{\tau}$	tw^2				

No. 4: $X_6 \subset \mathbb{P}(1,1,1,2,2)$ $A^3 = 3/2$

$(t-\alpha_1 w)(t-\alpha_2 w)(t-\alpha_3 w) + wf_4(x,y,z,t) + f_6(x,y,z,t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_tO_w = 3 \times \frac{1}{2}(1,1,1)$ $\boxed{\tau}$	tw^2				

- We may assume that $\alpha_1 = 0$. To see how to treat the singular points of type $\frac{1}{2}(1,1,1)$, we have only to consider the singular point O_w. The other points can be dealt with in the same way.

No. 5: $X_7 \subset \mathbb{P}(1,1,1,2,3)$ $A^3 = 7/6$
$zw^2 + wf_4(x,y,z,t) + f_7(x,y,z,t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{3}(1,1,2)$ $\boxed{\tau}$	zw^2				
$O_t = \frac{1}{2}(1,1,1)$ $\boxed{\tau_1}$	wt^2				

No. 6: $X_8 \subset \mathbb{P}(1,1,1,2,4)$ $A^3 = 1$
$(w - \alpha_1 t^2)(w - \alpha_2 t^2) + wf_4(x,y,z,t) + f_8(x,y,z,t) = 0$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_tO_w = 2 \times \frac{1}{2}(1,1,1)$ $\boxed{\tau}$	wt^2				

• We may assume that $\alpha_1 = 0$. To see how to treat the singular points of type $\frac{1}{2}(1,1,1)$, we have only to consider the singular point O_t. The other point can be dealt with in the same way. After we set $\alpha_1 = 0$, by a suitable coordinate change with respect to w, we may assume that the monomials of types $t^3g_2(x,y,z)$, $t^2g_4(x,y,z)$ do not appear in the defining equation.

No. 7: $X_8 \subset \mathbb{P}(1,1,2,2,3)$ $A^3 = 2/3$
Type I : $tw^2 + wg_5(x,y,z) - zt^3 - t^2g_4(x,y,z) - tg_6(x,y,z) + g_8(x,y,z)$
Type II : $(z + f_2(x,y))w^2 + wf_5(x,y,z,t) - zt^3 - t^2f_4(x,y,z) - tf_6(x,y,z) + f_8(x,y,z)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{3}(1,1,2)$ $\boxed{\tau}$	tw^2				
$O_zO_t = 4 \times \frac{1}{2}(1,1,1)$ $\boxed{\epsilon}$	$tw^2 - zt^3$				Type I
$O_zO_t = 4 \times \frac{1}{2}(1,1,1)$ $\boxed{\iota}$					Type II

• For the singular points of type $\frac{1}{2}(1,1,1)$ we have only to consider one of them. The others can be untwisted or excluded in the same way. The singular point to be considered here may be assumed to be the point O_t by a suitable coordinate change.

No. 8: $X_9 \subset \mathbb{P}(1,1,1,3,4)$ $A^3 = 3/4$
$zw^2 + wf_5(x,y,z,t) + f_9(x,y,z,t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{4}(1,1,3)$ $\boxed{\tau}$	zw^2				

No. 9: $X_9 \subset \mathbb{P}(1,1,2,3,3)$					$A^3 = 1/2$
$(w-\alpha_1 t)(w-\alpha_2 t)(w-\alpha_3 t) + z^3(a_1 t + a_2 yz) + w^2 f_3(x,y,z) + w f_6(x,y,z,t) + f_9(x,y,z,t)$					
Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_z = \frac{1}{2}(1_x, 1_y, 1_t)$ ⓑ	0	B	y	y	$a_1 \neq 0$
$O_z = \frac{1}{2}(1_x, 1_t, 1_w)$ ⓑ	0	$B-E$	y	w^3	$a_1 = 0$
$O_t O_w = 3 \times \frac{1}{3}(1,1,2)$ $\boxed{T}$	wt^2				

We may assume that neither $z^3 w$ nor xz^4 appears in the defining equation of X_9.

- If $a_1 \neq 0$, then the 1-cycle Γ for the singular point O_z is irreducible.

- Suppose that $a_1 = 0$. Then $a_2 \neq 0$. Then the 1-cycle Γ consists of three irreducible curves $\tilde{C}_i$, $i = 1,2,3$, each of which is the proper transform of the curve defined by

$$x = y = w - \alpha_i t = 0.$$

One can easily check that

$$B \cdot \tilde{C}_i = -\frac{1}{3}, \quad E \cdot \tilde{C}_i = 1$$

for each i. Therefore, these three curves are numerically equivalent to each other.

- For the singular points of type $\frac{1}{3}(1,1,2)$ we may assume that $\alpha_3 = 0$ and we have only to consider the singular point O_t. The others can be untwisted or excluded in the same way. Note that if $\alpha_3 = 0$ then we may assume that wt^2 is the only monomial in the defining equation of X_9 divisible by t^2.

No. 10: $X_{10} \subset \mathbb{P}(1,1,1,3,5)$					$A^3 = 2/3$
$w^2 + zt^3 + w f_5(x,y,z,t) + f_{10}(x,y,z,t)$					
Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t = \frac{1}{3}(1_x, 1_y, 2_w)$ ⓟ	$+$	$B-E$	z	w^2	

No. 11: $X_{10} \subset \mathbb{P}(1,1,2,2,5)$					$A^3 = 1/2$
$w^2 + \prod_{i=1}^{5}(t-\alpha_i z) + w f_5(x,y,z,t) + f_{10}(x,y,z,t)$					
Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_z O_t = 5 \times \frac{1}{2}(1_x, 1_y, 1_w)$ ⓑ	0	B	y	y	

- The curve defined by $x = y = 0$ is irreducible since the defining polynomial of X_{10} contains the monomial w^2 and a reduced polynomial $\prod_{i=1}^{5}(t-\alpha_i z)$ of degree 10. Therefore, the 1-cycle Γ is irreducible.

No. 12: $X_{10} \subset \mathbb{P}(1,1,2,3,4)$	$A^3 = 5/12$
$z(w-\alpha_1 z^2)(w-\alpha_2 z^2) + t^2(a_1 w + a_2 yt) + cz^2t^2 + wf_6(x,y,z,t) + f_{10}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{4}(1,1,3)$ $\boxed{\mathcal{T}}$	zw^2				
$O_t = \frac{1}{3}(1,1,2)$ $\boxed{\mathcal{T}_1}$	wt^2				
$O_zO_w = 2\times\frac{1}{2}(1_x,1_y,1_t)$ ⓑ	$-$	B	y	y	$c \neq 0,\ a_1 \neq 0$
$O_zO_w = 2\times\frac{1}{2}(1_x,1_y,1_t)$ ⓢ	$-$	B	$x,\ y$	x, y	$c \neq 0,\ a_1 = 0$
$O_zO_w = 2\times\frac{1}{2}(1_x,1_y,1_t)$ ⓕ	$-$	B	$x,\ y$	x, y	$c = 0$

By a coordinate change we assume that $\alpha_1 = 0$. Furthermore we may assume that the monomials z^3xt, z^3yt, z^4x^2, z^4xy, z^4y^2 do not appear in the defining equation by changing the coordinate w in an appropriate way. We may also assume that xt^3 is not contained in f_{10}.

- For the singular points of type $\frac{1}{2}(1,1,1)$ with $c \neq 0$ and $a_1 \neq 0$ the 1-cycle Γ is irreducible due to the monomials zw^2, t^2w and z^2t^2.

- For the singular points of type $\frac{1}{2}(1,1,1)$ with $c \neq 0$ and $a_1 = 0$ choose a general surface H in $|-K_{X_{10}}|$ and then let T be the proper transform of the surface H. The surface H is a K3 surface only with du Val singularities. The intersection of T with the surface S gives us a divisor consisting of two irreducible curves on the normal surface T. One is the proper transform of the curve L_{tw}. The other is the proper transform of the curve C defined by

$$x = y = w^2 - \alpha_2 z^2 w + czt^2 = 0$$

in $\mathbb{P}(1,1,2,3,4)$. Since we have

$$\tilde{L}_{tw}^2 = -\frac{7}{12}, \quad \tilde{L}_{tw}\cdot\tilde{C} = \frac{2}{3}, \quad \tilde{C}^2 = -\frac{5}{6}$$

the curves $\tilde{L}_{tw}$ and $\tilde{C}$ are negative-definite.

We remark here that the surface obtained from T by contracting the two curves $\tilde{L}_{tw}$ and $\tilde{C}$ is a K3 surface only with one E_8 singular point. Indeed, the surface T has one A_1 singular point on $\tilde{C}$, one A_3 singular point on $\tilde{L}_{tw}$ and the curves $\tilde{C}$, $\tilde{L}_{tw}$ intersect at one A_2 singular point tangentially on an orbifold chart. Therefore, on the minimal resolution of the surface T, the proper transforms of the curves $\tilde{C}$, $\tilde{L}_{tw}$ with the exceptional curves over three du Val points form the configuration of the -2-curves for an E_8 singular point.

- For the singular points of type $\frac{1}{2}(1,1,1)$ with $c = 0$ we may assume that $\alpha_1 = 0$ and we have only to consider the point O_z. The other singular point can be treated in the same way by a suitable coordinate change. The quasi-smoothness implies that $a_1 = 0$ and $a_2 \neq 0$. Let $Z_{\lambda,\mu}$ be the curve on X_{10} cut out by

$$\begin{cases} y = \lambda x, \\ w = \mu x^4 \end{cases}$$

for some sufficiently general complex numbers λ and μ. Then $Z_{\lambda,\mu} = L_{zt} + C_{\lambda,\mu}$, where $C_{\lambda,\mu}$ is an irreducible and reduced curve whose normalisation is an elliptic curve. Indeed, the curve $C_{\lambda,\mu}$ is defined by

$$y - \lambda x = w - \mu x^4 = \mu^2 x^7 z - \alpha_2 \mu x^3 z^3 + \lambda a_2 t^3 + \mu x^3 f_6(x, \lambda x, z, t) + f_{10}(x, \lambda x, z, t) = 0.$$

Then

$$\begin{cases} -K_Y \cdot (\tilde{L}_{zt} + \tilde{C}_{\lambda,\mu}) = 4B^3 = -\dfrac{1}{3}, \\ -K_Y \cdot \tilde{L}_{zt} = -K_X \cdot L_{zt} - \dfrac{1}{2} E \cdot \tilde{L}_{zt} = -\dfrac{1}{3}, \end{cases}$$

and hence $-K_Y \cdot \tilde{C}_{\lambda,\mu} = 0$.

No. 13: $X_{11} \subset \mathbb{P}(1,1,2,3,5)$ $A^3 = 11/30$

$yw^2 + t^2(a_1 w + a_2 zt) + z^3(b_1 w + b_2 zt + b_3 xz^2 + b_4 yz^2) + w f_6(x,y,z,t) + f_{11}(x,y,z,t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{5}(1,2,3)$ $\boxed{\tau}$	yw^2				
$O_t = \frac{1}{3}(1,1,2)$ $\boxed{\tau_1}$	wt^2				
$O_z = \frac{1}{2}(1,1,1)$ ⓑ	–	B	y	y	$a_1 \neq 0$, $b_1 \neq 0$ $a_1 b_2 - a_2 b_1 \neq 0$
$O_z = \frac{1}{2}(1_x, 1_y, 1_t)$ ⓢ	–	B	x, y	x, y	$a_1 \neq 0$, $b_1 \neq 0$ $a_1 b_2 - a_2 b_1 = 0$
$O_z = \frac{1}{2}(1_x, 1_y, 1_w)$ ⓕ	–	B	x, y	x, y	$a_1 \neq 0$ $b_1 = 0$, $b_2 \neq 0$
$O_z = \frac{1}{2}(1,1,1)$ ⓕ	–	B	y	y	$a_1 \neq 0$ $b_1 = 0$, $b_2 = 0$
$O_z = \frac{1}{2}(1_x, 1_y, 1_t)$ ⓢ	–	B	x, y	x, y	$a_1 = 0$ $b_1 \neq 0$
$O_z = \frac{1}{2}(1_x, 1_y, 1_w)$ ⓕ	–	B	x, y	x, y	$a_1 = 0$ $b_1 = 0$, $b_2 \neq 0$
$O_z = \frac{1}{2}(1,1,1)$ ⓕ	–	B	y	y	$a_1 = 0$ $b_1 = 0$, $b_2 = 0$

To exclude the singular point O_z we first suppose that $a_1 \neq 0$. We may then assume that $a_1 = 1$ and $a_2 = 0$.

- The conditions $b_1 \neq 0$ and $a_1 b_2 - a_2 b_1 \neq 0$ imply that both b_1 and b_2 are non-zero. In such a case the 1-cycle Γ is irreducible since we have the monomials $t^2 w$, $z^3 w$ and $z^4 t$.

- The conditions $b_1 \neq 0$ and $a_1 b_2 - a_2 b_1 = 0$ imply that $b_1 \neq 0$ and $b_2 = 0$. In such a case we take a general surface H from the pencil $|-K_{X_{11}}|$ and then let T be the proper transform of the surface. The intersection of T with the surface S defines a divisor consisting of two irreducible curves on the normal surface T. One is the proper transform of the curve L_{zt} on H. The other is the proper transform of the curve C defined by

$$x = y = t^2 + b_1 z^3 = 0$$

in $\mathbb{P}(1,1,2,3,5)$. Since

$$\tilde{L}_{zt}^2 = -\frac{4}{3}, \quad \tilde{L}_{zt}\cdot\tilde{C} = 1, \quad \tilde{C}^2 = -\frac{4}{5}$$

the curves $\tilde{L}_{zt}$ and $\tilde{C}$ on the normal surface T are negative-definite.

• In the case when $b_1 = 0$ and $b_2 \neq 0$ we may assume that $b_2 = 1$, $b_3 = b_4 = 0$ by a suitable coordinate change. Let $Z_{\lambda,\mu}$ be the curve on X_{11} cut out by

$$\begin{cases} y = \lambda x, \\ t = \mu x^3 \end{cases}$$

for some sufficiently general complex numbers λ and μ. Then $Z_{\lambda,\mu} = L_{zw} + C_{\lambda,\mu}$, where $C_{\lambda,\mu}$ is an irreducible and reduced curve. Then

$$\begin{cases} -K_Y\cdot(\tilde{L}_{zw} + \tilde{C}_{\lambda,\mu}) = 3B^3 = -\dfrac{2}{5}, \\ -K_Y\cdot\tilde{L}_{zw} = -K_X\cdot L_{zw} - \dfrac{1}{2}E\cdot\tilde{L}_{zw} = -\dfrac{2}{5}, \end{cases}$$

and hence $-K_Y\cdot\tilde{C}_{\lambda,\mu} = 0$.

• In the case when $b_1 = b_2 = 0$, we must have $b_3 \neq 0$ since X_{11} is quaisy-smooth. We may assume that $b_3 = 1$ and $b_4 = 0$ by a suitable coordinate change. Let Z_λ be the curve on the surface S_x defined by

$$\begin{cases} x = 0, \\ t = \lambda y^3 \end{cases}$$

for a sufficiently general complex number λ. Then $Z_\lambda = L_{zw} + C_\lambda$, where C_λ is an irreducible and reduced curve. Then

$$\begin{cases} -K_Y\cdot(\tilde{L}_{zw} + \tilde{C}_\lambda) = (B-E)(3B+E)B = -\dfrac{2}{5}, \\ -K_Y\cdot\tilde{L}_{zw} = -K_X\cdot L_{zw} - \dfrac{1}{2}E\cdot\tilde{L}_{zw} = -\dfrac{2}{5}, \end{cases}$$

and hence $-K_Y\cdot\tilde{C}_\lambda = 0$.

Now we suppose that $a_1 = 0$. Then $a_2 \neq 0$, so that we could assume that $a_2 = 1$.

• Suppose that $b_1 \neq 0$. Then by a suitable coordinate change we may assume that $b_1 = 1$ and $b_2 = 0$. We take a general surface H from the pencil $|-K_{X_{11}}|$ and then let T be the proper transform of the surface. The intersection of T with S gives us a divisor consisting of two irreducible curves on T. One is the proper transform of the curve L_{tw} on H. The other is the proper transform of the curve C defined by

$$x = y = t^3 + z^2 w = 0$$

in $\mathbb{P}(1,1,2,3,5)$. Since

$$\tilde{L}_{tw}^2 = -\frac{8}{15}, \quad \tilde{L}_{tw}\cdot\tilde{C} = \frac{3}{5}, \quad \tilde{C}^2 = -\frac{4}{5}$$

the curves $\tilde{L}_{tw}$ and $\tilde{C}$ form a negative-definite divisor on the normal surface T.

• We suppose that $b_1 = 0$ and $b_2 \neq 0$. We then let $Z_{\lambda,\mu}$ be the curve on X_{11} cut out by

$$\begin{cases} y = \lambda x, \\ t = \mu x^3 \end{cases}$$

for some sufficiently general complex numbers λ and μ. Then $Z_{\lambda,\mu} = L_{zw} + C_{\lambda,\mu}$, where $C_{\lambda,\mu}$ is an irreducible and reduced curve. We have

$$\begin{cases} -K_Y \cdot (\tilde{L}_{zw} + \tilde{C}_{\lambda,\mu}) = 3B^3 = -\frac{2}{5}, \\ -K_Y \cdot \tilde{L}_{zw} = -K_X \cdot L_{zw} - \frac{1}{2} E \cdot \tilde{L}_{zw} = -\frac{2}{5}, \end{cases}$$

and hence $-K_Y \cdot \tilde{C}_{\lambda,\mu} = 0$.

• Finally, we suppose that $b_1 = 0$ and $b_2 = 0$. Then b_3 must be non-zero since X_{11} is quasi-smooth. We may assume that $b_3 = 1$ and $b_4 = 0$ by a suitable coordinate change. Let Z_λ be the curve on the surface S defined by

$$\begin{cases} x = 0, \\ t = \lambda y^3 \end{cases}$$

for a sufficiently general complex number λ. Then $Z_\lambda = L_{zw} + C_\lambda$, where C_λ is an irreducible and reduced curve. We have

$$\begin{cases} -K_Y \cdot (\tilde{L}_{zw} + \tilde{C}_\lambda) = (B - E)(3B + E)B = -\frac{2}{5}, \\ -K_Y \cdot \tilde{L}_{zw} = -K_X \cdot L_{zw} - \frac{1}{2} E \cdot \tilde{L}_{zw} = -\frac{2}{5}, \end{cases}$$

and hence $-K_Y \cdot \tilde{C}_\lambda = 0$.

<u>**No. 14**</u>: $X_{12} \subset \mathbb{P}(1, 1, 1, 4, 6)$ $A^3 = 1/2$

$w^2 + t^3 + wf_6(x, y, z, t) + f_{12}(x, y, z, t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_tO_w = 1 \times \frac{1}{2}(1_x, 1_y, 1_z)$ ⓑ	0	B	y	y	

• The curve defined by $x = y = 0$ is irreducible because we have the monomials w^2 and t^3 in the quasi-homogenous polynomial defining X_{12}. Therefore, the 1-cycle Γ is irreducible.

No. 15: $X_{12} \subset \mathbb{P}(1, 1, 2, 3, 6)$ $A^3 = 1/3$

$(w - \alpha_1 z^3)(w - \alpha_2 z^3) + t^2 w + wf_6(x, y, z, t) + tg_9(x, y, z) + g_{12}(x, y, z)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition

$O_tO_w = 2 \times \frac{1}{3}(1,1,2)$ $\boxed{\tau}$	wt^2				
$O_zO_w = 2\times\frac{1}{2}(1_x, 1_y, 1_t)$ ⓑ	$-$	B	y	y	$\alpha_1\alpha_2 \neq 0$
$O_zO_w = 2\times\frac{1}{2}(1_x, 1_y, 1_t)$ ⓢ	$-$	B	x, y	x, y	$\alpha_1\alpha_2 = 0$

- To see how to deal with the singular points of type $\frac{1}{3}(1,1,2)$ we have only to consider the singular point O_t. The other point can be treated in the same way after a suitable coordinate change.

- The 1-cycle Γ for each singular point of type $\frac{1}{2}(1,1,1)$ with $\alpha_1\alpha_2 \neq 0$ is irreducible since $(w-\alpha_1 z^3)(w-\alpha_2 z^3) + t^2 w$ is irreducible.

- For the singular points of type $\frac{1}{2}(1,1,1)$ with $\alpha_1\alpha_2 = 0$, we suppose that $\alpha_2 = 0$. Then $\alpha_1 \neq 0$. We take a general surface H from the pencil $|-K_{X_{12}}|$. The intersection of its proper transform T and the surface S defines a divisor consisting of two irreducible curves on the normal surface T. One is the proper transform of the curve L_{zt}. The other is the proper transform of the curve C defined by

$$x = y = w - \alpha_1 z^3 + t^2 = 0.$$

From the intersection numbers

$$(\tilde{L}_{zt} + \tilde{C}) \cdot \tilde{L}_{zt} = -K_Y \cdot \tilde{L}_{zt} = -\frac{1}{3}, \quad (\tilde{L}_{zt} + \tilde{C})^2 = B^3 = -\frac{1}{6}$$

on the surface T, we obtain

$$\tilde{L}_{zt}^2 = -\frac{1}{3} - \tilde{L}_{zt} \cdot \tilde{C}, \quad \tilde{C}^2 = \frac{1}{6} - \tilde{L}_{zt} \cdot \tilde{C}$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{zt}^2 & \tilde{L}_{zt} \cdot \tilde{C} \\ \tilde{L}_{zt} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} - \tilde{L}_{zt} \cdot \tilde{C} & \tilde{L}_{zt} \cdot \tilde{C} \\ \tilde{L}_{zt} \cdot \tilde{C} & \frac{1}{6} - \tilde{L}_{zt} \cdot \tilde{C} \end{pmatrix}$$

is negative-definite since $\tilde{L}_{zt} \cdot \tilde{C} = 1$.

No. 16: $X_{12} \subset \mathbb{P}(1,1,2,4,5)$ $A^3 = 3/10$

$zw^2 + (t-\alpha_1 z^2)(t-\alpha_2 z^2)(t-\alpha_3 z^2) + wf_7(x,y,z,t) + f_{12}(x,y,z,t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{5}(1,1,4)$ $\boxed{\tau}$	zw^2				
$O_zO_t = 3\times\frac{1}{2}(1_x, 1_y, 1_w)$ ⓑ	$-$	B	y	y	

- The 1-cycle Γ for each singular point of type $\frac{1}{2}(1,1,1)$ is irreducible due to zw^2 and t^3.

No. 17: $X_{12} \subset \mathbb{P}(1,1,3,4,4)$ $A^3 = 1/4$

$(t-\alpha_1 w)(t-\alpha_2 w)(t-\alpha_3 w)+z^4+wf_8(x,y,z,t)+f_{12}(x,y,z,t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_tO_w = 3\times\frac{1}{4}(1,1,3)$ $\boxed{\tau}$	tw^2				

• To see how to deal with the singular points of type $\frac{1}{4}(1,1,3)$ we may assume that $\alpha_1 = 0$. We then consider the singular point O_w. The other points can be treated in the same way.

No. 18: $X_{12}\subset\mathbb{P}(1,2,2,3,5)$	$A^3 = 1/5$
$yw^2+t^4+\prod_{i=1}^{6}(y-\alpha_i z)+wf_7(x,y,z,t)+f_{12}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{5}(1,2,3)$ $\boxed{\tau}$	yw^2				
$O_yO_z = 6\times\frac{1}{2}(1_x,1_t,1_w)$ ⓑ	$-$	$2B$	$y-\alpha_i z$	zw^2	

• The 1-cycle Γ for each singular point of type $\frac{1}{2}(1,1,1)$ is irreducible due to yw^2 and t^4.

<u>**No. 19**</u>: $X_{12}\subset\mathbb{P}(1,2,3,3,4)$	$A^3 = 1/6$
$(w-\alpha_1y^2)(w-\alpha_2y^2)(w-\alpha_3y^2)+(z-\beta_1t)(z-\beta_2t)(z-\beta_3t)(z-\beta_4t)+w^2f_4(x,y,z,t)+wf_8(x,y,z,t)+f_{12}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_yO_w = 3\times\frac{1}{2}(1_x,1_z,1_t)$ ⓝ	$-$	$3B+E$	$xy,\ z,\ t$	$xy,\ z,\ t$	
$O_zO_t = 4\times\frac{1}{3}(1_x,2_y,1_w)$ ⓑ	0	$2B$	y	y	

• The divisor T for each singular point of type $\frac{1}{2}(1,1,1)$ is nef since the linear system generated by xy, z, t has no base curve.

• The 1-cycle Γ for each singular point of type $\frac{1}{3}(1,2,1)$ is irreducible since the curve cut by $x=y=0$ is irreducible.

No. 20: $X_{13}\subset\mathbb{P}(1,1,3,4,5)$	$A^3 = 13/60$
$zw^2+t^2(a_1w+a_2yt)-z^3(b_1t+b_2yz+b_3xz)+wf_8(x,y,z,t)+f_{13}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{5}(1,1,4)$ $\boxed{\tau}$	zw^2				
$O_t = \frac{1}{4}(1,1,3)$ $\boxed{\tau_1}$	wt^2				
$O_z = \frac{1}{3}(1,1,2)$ $\boxed{\epsilon_2}$	zw^2-tz^3				

<u>**No. 21**</u>: $X_{14}\subset\mathbb{P}(1,1,2,4,7)$	$A^3 = 1/4$

$w^2+z(t-\alpha_1z^2)(t-\alpha_2z^2)(t-\alpha_3z^2)+wf_7(x,y,z,t)+f_{14}(x,y,z,t)$					
Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t=\frac{1}{4}(1_x,1_y,3_w)$ ⓟ	$+$	$2B-E$	z	w^2	
$O_zO_t=3\times\frac{1}{2}(1_x,1_y,1_w)$ ⓑ	$-$	B	y	y	

- The curve defined by $x=y=0$ is irreducible, and hence the 1-cycle Γ for the singularities of type $\frac{1}{2}(1,1,1)$ is irreducible.

No. 22: $X_{14}\subset\mathbb{P}(1,2,2,3,7)$ $\quad A^3=1/6$					
$w^2+zt^4+h_{14}(y,z)+wf_7(x,y,z,t)+t^3g_5(x,y,z)+t^2g_8(x,y,z)+tg_{11}(x,y,z)+g_{14}(x,y,z)$					
Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t=\frac{1}{3}(1_x,2_y,1_w)$ ⓑ	0	$2B$	y	y	
$O_yO_z=7\times\frac{1}{2}(1_x,1_t,1_w)$ ⓑ	$-$	$2B$	$y-\alpha_iz$	w^2	

Note that the homogenous polynomial h_{14} cannot be divisible by z since the hypersurface X_{14} is quasi-smooth. Therefore, we may write

$$h_{14}(y,z)=\prod_{i=1}^{7}(y-\alpha_iz).$$

- The curve defined by $x=y=0$ is irreducible because we have the monomials w^2 and zt^4.

- The curves defined by $x=y-\alpha_iz=0$ are also irreducible for the same reason. Therefore, the 1-cycle Γ for each singular point is irreducible.

No. 23: $X_{14}\subset\mathbb{P}(1,2,3,4,5)$ $\quad A^3=7/60$					
$(t+by^2)w^2+y(t-\alpha_1y^2)(t-\alpha_2y^2)(t-\alpha_3y^2)+z^3(a_1w+a_2yz)+cz^2t^2+wf_9(x,y,z,t)+f_{14}(x,y,z,t)$					
Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w=\frac{1}{5}(1,2,3)$ $\boxed{\tau}$	tw^2				
$O_t=\frac{1}{4}(1,3,1)$ $\boxed{\epsilon}$	tw^2+yt^3				
$O_z=\frac{1}{3}(1_x,2_y,1_t)$ ⓑ	$-$	$2B$	y	y	$c\neq 0$, $a_1\neq 0$
$O_z=\frac{1}{3}(1_x,1_t,2_w)$ ⓢ	$-$	$2B$	x^2, y	x^2, z^2t^2	$c\neq 0$, $a_1=0$
$O_z=\frac{1}{3}(1_x,2_y,1_t)$ ⓕ	$-$	$2B$	y	y	$c=0$, $a_1\neq 0$
$O_z=\frac{1}{3}(1_x,1_t,2_w)$ $\boxed{\iota_1}$					$c=0$, $a_1=0$
$O_yO_t=3\times\frac{1}{2}(1_x,1_z,1_w)$ ⓝ	$-$	$3B+E$	xy, z	xy, z	$b\neq 0$
$O_yO_t=3\times\frac{1}{2}(1_x,1_z,1_w)$ ⓢ	$-$	$3B+E$	x^3, xy, z	xy, z	$b=0$

- For the singular point O_z with $c\neq 0$ and $a_1\neq 0$ the 1-cycle Γ is irreducible due to the monomials tw^2, z^3w and z^2t^2.

• For the singular point O_z with $c \neq 0$ and $a_1 = 0$ we may assume that $a_2 = 1$ and $c = 1$. We take a general surface H from the pencil $|-2K_{X_{14}}|$ and then let T be the proper transform of the surface. The surface H is normal. However, it is not quasi-smooth at the points O_z and O_t. The intersection of T with the surface S defines a divisor consisting of two irreducible curves on the normal surface T. One is the proper transform of the curve L_{zw} on H. The other is the proper transform of the curve C defined by

$$x = y = w^2 + z^2t = 0$$

in $\mathbb{P}(1, 2, 3, 4, 5)$. From the intersection numbers

$$(\tilde{L}_{zw} + \tilde{C}) \cdot \tilde{L}_{zw} = -K_Y \cdot \tilde{L}_{zw} = -\frac{1}{10}, \quad (\tilde{L}_{zw} + \tilde{C})^2 = 2B^3 = -\frac{1}{10}$$

on the surface T, we obtain

$$\tilde{L}_{zw}^2 = -\frac{1}{10} - \tilde{L}_{zw} \cdot \tilde{C}, \quad \tilde{C}^2 = -\tilde{L}_{zw} \cdot \tilde{C}$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{zw}^2 & \tilde{L}_{zw} \cdot \tilde{C} \\ \tilde{L}_{zw} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{10} - \tilde{L}_{zw} \cdot \tilde{C} & \tilde{L}_{zw} \cdot \tilde{C} \\ \tilde{L}_{zw} \cdot \tilde{C} & -\tilde{L}_{zw} \cdot \tilde{C} \end{pmatrix}$$

is negative-definite since $\tilde{L}_{zw} \cdot \tilde{C}$ is positive.

• For the singular point O_z with $c = 0$ and $a_1 \neq 0$ we may assume that $a_1 = 1$ and $a_2 = 0$. Furthermore, we may also assume that f_{14} does not contain the monomial xz^3t by changing the coordinate w in a suitable way. We then consider the surface S_w cut by the equation $w = 0$. Let Z_λ be the curve on the surface S_w defined by

$$\begin{cases} w = 0 \\ y = \lambda x^2 \end{cases}$$

for a sufficiently general complex number λ. Then $Z_\lambda = 2L_{zt} + C_\lambda$, where C_λ is an irreducible and reduced curve. We have

$$\begin{cases} -K_Y \cdot (2\tilde{L}_{zt} + \tilde{C}_\lambda) = 10B^3 = -\dfrac{1}{2}, \\ -K_Y \cdot \tilde{L}_{zt} = -K_X \cdot L_{zt} - \dfrac{1}{3}E \cdot \tilde{L}_{zt} = -\dfrac{1}{4}, \end{cases}$$

and hence $-K_Y \cdot \tilde{C}_\lambda = 0$.

• For the singular point O_z with $c = 0$ and $a_1 = 0$ we observe that f_{14} must contain the monomial xz^3t for X_{14} to be quasi-smooth (see right before Theorem 4.3.1). We may assume that $a_2 = 1$ and that the coefficient of xz^3t in f_{14} is 1. Then Theorem 4.3.1 untwists the singular point O_z.

For the singular points of type $\frac{1}{2}(1,1,1)$ we may assume that $\alpha_3 = 0$ and we have only to consider the singular point O_y. The other singular points can be treated in the same way after suitable coordinate changes.

- For the singular point O_y with $b \neq 0$ consider the linear system generated by xy and z on X_{14}. Its base curves are defined by $x = z = 0$ and $y = z = 0$. The curve defined by $y = z = 0$ does not pass through the singular point O_y. The curve defined by $x = z = 0$ is irreducible. Indeed, the curve is defined by

$$x = z = (t + by^2)w^2 + yt(t - \alpha_1 y^2)(t - \alpha_2 y^2) = 0.$$

Moreover, its proper transform is equivalent to the 1-cycle defined by $(3B+E)\cdot B$. Therefore, the divisor T is nef since $(3B+E)^2 \cdot B > 0$.

- For the singular point O_y with $b = 0$ we take a general member H in the linear system generated by x^3, xy and z. Note that the defining equation of X_{14} must contain either y^3zw or xy^4w. The surface H is a normal surface of degree 14 in $\mathbb{P}(1,2,4,5)$ that is smooth at the point $x = t = w^2 + \alpha_1\alpha_2 y^5 = 0$. Let T be the proper transform of the surface H. The intersection of T with the surface S defines a divisor consisting of two irreducible curves on the normal surface T. One is the proper transform of the curve L_{yw} and the other is the proper transform of the curve C defined by

$$x = z = w^2 + y(t - \alpha_1 y^2)(t - \alpha_2 y^2) = 0.$$

From the intersection numbers

$$(\tilde{L}_{yw} + \tilde{C}) \cdot \tilde{L}_{yw} = -K_Y \cdot \tilde{L}_{yw} = -\frac{2}{5}, \quad (\tilde{L}_{yw} + \tilde{C})^2 = B^2 \cdot (3B + E) = -\frac{3}{20}$$

on the surface T, we obtain

$$\tilde{L}_{yw}^2 = -\frac{2}{5} - \tilde{L}_{yw} \cdot \tilde{C}, \quad \tilde{C}^2 = \frac{1}{4} - \tilde{L}_{yw} \cdot \tilde{C}$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{yw}^2 & \tilde{L}_{yw} \cdot \tilde{C} \\ \tilde{L}_{yw} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{2}{5} - \tilde{L}_{yw} \cdot \tilde{C} & \tilde{L}_{yw} \cdot \tilde{C} \\ \tilde{L}_{yw} \cdot \tilde{C} & \frac{1}{4} - \tilde{L}_{yw} \cdot \tilde{C} \end{pmatrix}$$

is negative-definite since the curves L_{yw} and the curve C intersect at the smooth point of H defined by $x = z = t = w^2 + \alpha_1\alpha_2 y^5 = 0$.

No. 24: $X_{15} \subset \mathbb{P}(1,1,2,5,7)$ $\quad A^3 = 3/14$
$yw^2 + t^3 + z^4(a_1 w + a_2 zt + a_3 xz^3 + a_4 yz^3) + wf_8(x,y,z,t) + f_{15}(x,y,z,t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{7}(1,2,5)$ $\boxed{\mathcal{T}}$	yw^2				
$O_z = \frac{1}{2}(1_x, 1_y, 1_t)$ ⓑ	$-$	B	y	y	$a_1 \neq 0$
$O_z = \frac{1}{2}(1,1,1)$ ⓑ	$-$	B	y	y	$a_1 = 0,\ a_2 = 0$
$O_z = \frac{1}{2}(1_x, 1_y, 1_w)$ ⓢ	$-$	B	$x,\ y$	x, y	$a_1 = 0,\ a_2 \neq 0$

Since X_{15} is quasi-smooth, one of the constants a_1, a_2, a_3 must be non-zero.

• The 1-cycle Γ for the singular point O_z with $a_1 \neq 0$ is irreducible since we have t^3 and z^4w.

• The 1-cycle Γ for the singular point O_z with $a_1 = a_2 = 0$ is also irreducible even though it is not reduced.

• For the singular point O_z with $a_1 = 0$ and $a_2 \neq 0$ we may assume that $a_2 = 1$ and $a_3 = a_4 = 0$. Choose a general member H in the linear system $|-K_{15}|$ and then take the intersection of its proper transform T with S. This gives us a divisor consisting of two irreducible curves on the normal surface T. One is the proper transform of the curve L_{zw}. The other is the proper transform of the curve C defined by

$$x = y = t^2 + z^5 = 0.$$

The curves L_{zw} and C intersect at the point O_w. From the intersection numbers

$$(\tilde{L}_{zw} + \tilde{C}) \cdot \tilde{L}_{zw} = -K_Y \cdot \tilde{L}_{zw} = -\frac{3}{7}, \quad (\tilde{L}_{zw} + \tilde{C})^2 = B^3 = -\frac{2}{7}$$

on the surface T, we obtain

$$\tilde{L}_{zw}^2 = -\frac{3}{7} - \tilde{L}_{zw} \cdot \tilde{C}, \quad \tilde{C}^2 = \frac{1}{7} - \tilde{L}_{zw} \cdot \tilde{C}$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{zw}^2 & \tilde{L}_{zw} \cdot \tilde{C} \\ \tilde{L}_{zw} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{7} - \tilde{L}_{zw} \cdot \tilde{C} & \tilde{L}_{zw} \cdot \tilde{C} \\ \tilde{L}_{zw} \cdot \tilde{C} & \frac{1}{7} - \tilde{L}_{zw} \cdot \tilde{C} \end{pmatrix}$$

is negative-definite since $\tilde{L}_{zw} \cdot \tilde{C} = \frac{5}{7}$.

No. 25: $X_{15} \subset \mathbb{P}(1, 1, 3, 4, 7)$ $A^3 = 5/28$

$yw^2 + t^2(a_1w + a_2zt) + z^5 + wf_8(x, y, z, t) + f_{15}(x, y, z, t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{7}(1, 3, 4)$ $\boxed{\tau}$	yw^2				
$O_t = \frac{1}{4}(1, 1, 3)$ $\boxed{\tau_1}$	wt^2				

No. 26: $X_{15} \subset \mathbb{P}(1, 1, 3, 5, 6)$ $A^3 = 1/6$

$zw^2 + t^3 + z^5 + wf_9(x, y, z, t) + f_{15}(x, y, z, t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{6}(1, 1, 5)$ $\boxed{\tau}$	zw^2				
$O_zO_w = 2\times\frac{1}{3}(1_x, 1_y, 2_t)$ ⓑ	0	B	y	y	

- The 1-cycle Γ for each singular point of type $\frac{1}{3}(1,1,2)$ is irreducible since we have the monomials zw^2 and t^3.

No. 27: $X_{15} \subset \mathbb{P}(1,2,3,5,5)$ $A^3 = 1/10$
$(w-\alpha_1 t)(w-\alpha_2 t)(w-\alpha_3 t) + y^5(a_1 w + a_2 yz + a_3 xy^2) + w^2 f_5(x,y,z,t) + wf_{10}(x,y,z,t) + f_{15}(x,y,z,t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_tO_w = 3 \times \frac{1}{5}(1,2,3)$ $\boxed{\mathcal{T}}$	wt^2				
$O_y = \frac{1}{2}(1,1,1)$ ⓝ	$-$	$5B+2E$	t	t	

We may assume that $\alpha_3 = 0$, i.e., the hypersurface X_{15} has a singular point of type $\frac{1}{5}(1,2,3)$ at the point O_t.

- To see how to treat the singular points of type $\frac{1}{5}(1,2,3)$ we have only to consider the singular point O_t. The others can be dealt with in the same way.

- For the singular point O_y we consider the linear system $|-5K_{X_{15}}|$. Every member in the linear system passes through the point O_y. It has no base curve. Since the proper transform of a general member in $|-5K_{X_{15}}|$ belongs to the linear system $|5B+2E|$, the divisor T is nef.

<u>**No. 28**</u>: $X_{15} \subset \mathbb{P}(1,3,3,4,5)$ $A^3 = 1/12$
$w^3 + zt^3 + h_{15}(y,z) + w^2 f_5(x,y,z,t) + wf_{10}(x,y,z,t) + t^2 g_7(x,y,z) + tg_{11}(x,y,z) + g_{15}(x,y,z)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t = 1 \times \frac{1}{4}(1_x, 3_y, 1_w)$ ⓑ	0	$3B$	y	y	
$O_yO_z = 5 \times \frac{1}{3}(1_x, 1_t, 2_w)$ ⓑ	$-$	$3B$	$y-\alpha_i z$	$t^3 z$	

Note that the homogenous polynomial h_{15} cannot be divisible by z since the hypersurface X_{15} is quasi-smooth. Therefore, we may write

$$h_{15}(y,z) = \prod_{i=1}^{5}(y-\alpha_i z).$$

- The curve defined by $x = y = 0$ is irreducible because we have the monomials w^3 and zt^3.

- The curves defined by $x = y - \alpha_i z = 0$ are also irreducible for the same reason. Therefore, the 1-cycle Γ for each singular point is irreducible.

<u>**No. 29**</u>: $X_{16} \subset \mathbb{P}(1,1,2,5,8)$ $A^3 = 1/5$
$(w-\alpha_1 z^4)(w-\alpha_2 z^4) + yt^3 + az^3t^2 + wf_8(x,y,z,t) + t^2 g_6(x,y,z) + tg_{11}(x,y,z) + g_{16}(x,y,z)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t = \frac{1}{5}(1_x, 2_z, 3_w)$ ⓟ	$+$	$B-E$	y	w^2	
$O_zO_w = 2\times\frac{1}{2}(1_x, 1_y, 1_t)$ ⓑ	$-$	B	y	y	$a\neq 0$
$O_zO_w = 2\times\frac{1}{2}(1_x, 1_y, 1_t)$ ⓢ	$-$	B	x, y	x, y	$a=0$

• If the constant a is non-zero, then the 1-cycle Γ for each singular point of type $\frac{1}{2}(1,1,1)$ is irreducible.

• Suppose that $a=0$. We have only to consider one of the singular points of type $\frac{1}{2}(1,1,1)$. The other singular point can be excluded in the same way. Moreover, we may assume that the singular point is located at the point O_z, i.e., $\alpha_1=0$, by a suitable coordinate change.

We take a general surface H from the pencil $|-K_{X_{16}}|$. It is a K3 surface only with du Val singularities. Let T be the proper transform of the surface. The intersection of T with the surface S gives us a divisor consisting of two irreducible curves on the normal surface T. One is the proper transform $\tilde{L}_{zt}$. The other is the proper transform $\tilde{C}$ of the curve C defined by

$$x=y=w-\alpha_2z^4=0.$$

From the intersection numbers

$$(\tilde{L}_{zt}+\tilde{C})\cdot\tilde{L}_{zt}=-K_Y\cdot\tilde{L}_{zt}=-\frac{2}{5},\quad (\tilde{L}_{zt}+\tilde{C})^2=B^3=-\frac{3}{10}$$

on the surface T, we obtain

$$\tilde{L}_{zt}^2=-\frac{2}{5}-\tilde{L}_{zt}\cdot\tilde{C},\quad \tilde{C}^2=\frac{1}{10}-\tilde{L}_{zt}\cdot\tilde{C}$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{zt}^2 & \tilde{L}_{zt}\cdot\tilde{C} \\ \tilde{L}_{zt}\cdot\tilde{C} & \tilde{C}^2 \end{pmatrix}=\begin{pmatrix} -\frac{2}{5}-\tilde{L}_{zt}\cdot\tilde{C} & \tilde{L}_{zt}\cdot\tilde{C} \\ \tilde{L}_{zt}\cdot\tilde{C} & \frac{1}{10}-\tilde{L}_{zt}\cdot\tilde{C} \end{pmatrix}$$

is negative-definite since $\tilde{L}_{yw}\cdot\tilde{C}=\frac{4}{5}$.

No. 30: $X_{16}\subset\mathbb{P}(1,1,3,4,8)$	$A^3=1/6$
$(w-\alpha_1t^2)(w-\alpha_2t^2)+z^4(a_1t+a_2yz)+wf_8(x,y,z,t)+f_{16}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_tO_w = 2\times\frac{1}{4}(1,1,3)$ $\boxed{\tau}$	wt^2				
$O_z = \frac{1}{3}(1_x, 1_y, 2_w)$ ⓑ	0	B	y	y	$a_1\neq 0$
$O_z = \frac{1}{3}(1_x, 1_t, 2_w)$ ⓑ	0	$B-E$	y	w^2	$a_1=0$

• We may assume that $\alpha_1=0$. To see how to treat the singular points of type $\frac{1}{4}(1,1,3)$, we have only to consider the singular point O_t. The other point can be treated in the same way.

• The 1-cycle Γ for the singular point O_z with $a_1 \neq 0$ is irreducible due to w^2 and z^4t.

• The 1-cycle Γ for the singular point O_z with $a_1 = 0$ consists of the proper transforms of the curves defined by

$$x = y = w - \alpha_1 t^2 = 0$$

and

$$x = y = w - \alpha_2 t^2 = 0.$$

These two irreducible components are symmetric with respect to the biregular involution of X_{16}. Consequently, the components of Γ are numerically equivalent to each other.

No. 31: $X_{16} \subset \mathbb{P}(1,1,4,5,6)$					$A^3 = 2/15$
$zw^2 + t^2(a_1w + a_2yt) + z^4 + wf_{10}(x,y,z,t) + f_{16}(x,y,z,t)$					
Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{6}(1,1,5)$ $\boxed{\tau}$	zw^2				
$O_t = \frac{1}{5}(1,1,4)$ $\boxed{\tau_1}$	wt^2				
$O_zO_w = 1\times\frac{1}{2}(1_x,1_y,1_t)$ ⓑ	$-$	B	y	y	$a_1 \neq 0$
$O_zO_w = 1\times\frac{1}{2}(1_x,1_y,1_t)$ ⓢ	$-$	B	x, y	x, y	$a_1 = 0$

• If $a_1 \neq 0$, the 1-cycle Γ for the singular point of type $\frac{1}{2}(1,1,1)$ is irreducible due to the monomials z^4 and t^2w.

• Suppose $a_1 = 0$. Choose a general member H in the linear system $|-K_X|$. Then it is a normal K3 surface of degree 16 in $\mathbb{P}(1,4,5,6)$. Let T be the proper transform of the surface H. The intersection of T with the surface S defines a divisor consisting of two irreducible curves $\tilde{L}_{tw}$ and $\tilde{C}$ on the normal surface T. The curve $\tilde{C}$ is the proper transform of the curve C defined by

$$x = y = w^2 + z^3 = 0.$$

On the surface T, we have

$$\tilde{L}_{tw} \cdot \tilde{C} = L_{tw} \cdot C = \frac{2}{5}.$$

From the intersections

$$(\tilde{L}_{tw} + \tilde{C}) \cdot \tilde{L}_{tw} = -K_Y \cdot \tilde{L}_{tw} = \frac{1}{30}, \quad (\tilde{L}_{tw} + \tilde{C})^2 = B^3 = -\frac{11}{30}$$

on the surface T, we obtain

$$\tilde{L}_{tw}^2 = -\frac{11}{30}, \quad \tilde{C}^2 = -\frac{4}{5}.$$

The intersection matrix

$$\begin{pmatrix} \tilde{L}_{tw}^2 & \tilde{L}_{tw} \cdot \tilde{C} \\ \tilde{L}_{tw} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{11}{30} & \frac{2}{5} \\ \frac{2}{5} & -\frac{4}{5} \end{pmatrix}$$

is negative-definite.

No. 32: $X_{16} \subset \mathbb{P}(1,2,3,4,7)$	$A^3 = 2/21$
$yw^2 + \prod_{i=1}^{4}(t - \alpha_i y^2) + z^3(a_1 w + a_2 tz + a_3 xz^2) + wf_9(x,y,z,t) + f_{16}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{7}(1,3,4)$ $\boxed{\tau}$	yw^2				
$O_z = \frac{1}{3}(1_x, 2_y, 1_t)$ ⓑ	$-$	$2B$	y	y	$a_1 \neq 0$
$O_z = \frac{1}{3}(1_x, 2_y, 1_w)$ ⓕ	$-$	$2B$	x^2, y	x^2, y	$a_1 = 0$, $a_2 \neq 0$
$O_z = \frac{1}{3}(2_y, 1_t, 1_w)$ ⓑ	$-$	$2B$	y	y	$a_1 = a_2 = 0$
$O_y O_t = 4 \times \frac{1}{2}(1_x, 1_z, 1_w)$ ⓝ	$-$	$3B + E$	xy, z	xy, z	

- The 1-cycle Γ for the singular point O_z with $a_1 \neq 0$ is irreducible due to t^4 and $z^3 w$.

- For the singular point O_z with $a_1 = 0$ and $a_2 \neq 0$ we may assume that $a_3 = 0$. The curve L_{zw} is contained in X_{16} because $a_1 = 0$. Let $Z_{\lambda,\mu}$ be the curve on X_{16} cut out by

$$\begin{cases} y = \lambda x^2 \\ t = \mu x^4, \end{cases}$$

for some sufficiently general complex numbers λ and μ. Then $Z_{\lambda,\mu} = 2L_{zw} + C_{\lambda,\mu}$, where $C_{\lambda,\mu}$ is an irreducible and reduced curve. We have

$$\begin{cases} -K_Y \cdot (2\tilde{L}_{zw} + \tilde{C}_{\lambda,\mu}) = 8B^3 = -\dfrac{4}{7}, \\ -K_Y \cdot \tilde{L}_{zw} = -K_X \cdot L_{zw} - \dfrac{1}{3}E \cdot \tilde{L}_{zw} = -\dfrac{2}{7}, \end{cases}$$

and hence $-K_Y \cdot \tilde{C}_{\lambda,\mu} = 0$.

- The 1-cycle Γ for the singular point O_z with $a_1 = a_2 = 0$ is irreducible even though it is non-reduced.

- For the singular points of type $\frac{1}{2}(1,1,1)$, consider the linear system generated by xy and z. Its base curves are defined by $x = z = 0$ and $y = z = 0$. The curve defined by $y = z = 0$ does not pass through any singular point of type $\frac{1}{2}(1,1,1)$. The curve defined by $x = z = 0$ is irreducible because of the monomial yw^2 and t^4. Since its proper transform is the 1-cycle defined by $(3B + E) \cdot B$ and $(3B + E)^2 \cdot B > 0$, the divisor T is nef.

No. 33: $X_{17} \subset \mathbb{P}(1,2,3,5,7)$	$A^3 = 17/210$
$(dx^3 + exy + z)w^2 + t^2(a_1 w + a_2 yt) + z^4(b_1 t + b_2 yz) + y^5(c_1 w + c_2 yt + c_3 y^2 z + c_4 y^3 x) + wf_{10}(x,y,z,t) + f_{17}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{7}(1,2,5)$ $\boxed{\tau}$	zw^2				
$O_t = \frac{1}{5}(1,2,3)$ $\boxed{\tau_1}$	wt^2				
$O_z = \frac{1}{3}(1_x, 2_y, 1_w)$ ⓑ	$-$	$2B$	y	y	$a_1 \neq 0$, $b_1 \neq 0$

$O_z = \frac{1}{3}(1_x, 2_y, 1_w)$ ⓢ	–	$2B$	x^2, y	x^2, y	$a_1 = 0$, $b_1 \neq 0$
$O_z = \frac{1}{3}(1_x, 2_t, 1_w)$ ⓢ	–	$2B$	x^2, y	x^2, zw^2	$b_1 = 0$
$O_y = \frac{1}{2}(1_x, 1_z, 1_t)$ ⓝ	–	$5B + 2E$	t	t	$c_1 \neq 0$
$O_y = \frac{1}{2}(1_x, 1_z, 1_w)$ ⓢ	–	$5B + E$	x^5, t	x^5, zw^2	$c_1 = 0$, $c_2 \neq 0$
$O_y = \frac{1}{2}(1_x, 1_t, 1_w)$ ⓢ	–	$3B$	x^3, z	x^3, yt^3, t^2w	$c_1 = c_2 = 0$ $c_3 \neq 0$
$O_y = \frac{1}{2}(1_z, 1_t, 1_w)$ ⓝ	–	$7B + 3E$	w	w	$c_1 = c_2 = c_3 = 0$

- The 1-cycle Γ for the singular point O_z with $a_1 \neq 0$ and $b_1 \neq 0$ is irreducible since we have the monomials t^2w, z^4t and zw^2.

For the singular point O_z with $a_1b_1 = 0$ choose a general member H in the linear system $|-2K_{X_{17}}|$. Then it is a normal surface of degree 17 in $\mathbb{P}(1, 3, 5, 7)$. Let T be the proper transform of the divisor H. The curve $\tilde{D}$ on T cut out by the surface S is the proper transform of the curve cut by the equations $x = y = 0$.

- Suppose that $b_1 \neq 0$ and $a_1 = 0$. Then $a_2 \neq 0$. The curve $\tilde{D}$ then consists of two irreducible curves $\tilde{L}_{tw}$ and $\tilde{C}_1$. The curve $\tilde{C}_1$ is the proper transform of the curve C_1 defined by

$$x = y = w^2 + b_1z^3t = 0.$$

Note that the curve L_{tw} and C_1 intersect at the point O_t. The surface H is not quasi-smooth at the point O_t. We also consider the divisor D_z on H cut by the equation $z = 0$. We easily see that $D_z = 2L_{tw} + R$, where R is a curve whose support does not contain L_{tw}. The curves R and L_{tw} intersect at the point O_w. The surface H is quasi-smooth at the point O_w. Then we have $\tilde{L}_{tw} \cdot \tilde{R} = \frac{3}{7}$. From the intersection

$$(2\tilde{L}_{tw} + \tilde{R}) \cdot \tilde{L}_{tw} = 3A \cdot \tilde{L}_{tw} = \frac{3}{35}$$

we obtain $\tilde{L}_{tw}^2 = -\frac{6}{35}$. From the intersections

$$(\tilde{L}_{tw} + \tilde{C}_1) \cdot \tilde{L}_{tw} = -K_Y \cdot \tilde{L}_{tw} = \frac{1}{35}, \quad (\tilde{L}_{tw} + \tilde{C}_1)^2 = 2B^3 = -\frac{6}{35}$$

on the surface T, we obtain

$$\tilde{L}_{tw}^2 = -\frac{6}{35}, \quad \tilde{L}_{tw} \cdot \tilde{C}_1 = \frac{1}{5}, \quad \tilde{C}_1^2 = -\frac{2}{5}.$$

The intersection matrix

$$\begin{pmatrix} \tilde{L}_{tw}^2 & \tilde{L}_{tw} \cdot \tilde{C}_1 \\ \tilde{L}_{tw} \cdot \tilde{C}_1 & \tilde{C}_1^2 \end{pmatrix} = \begin{pmatrix} -\frac{6}{35} & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{pmatrix}$$

is negative-definite.

- Suppose that $b_1 = 0$ and $a_1 \neq 0$. Then $b_2 \neq 0$. The curve $\tilde{D}$ consists of two irreducible curves $\tilde{L}_{zt}$ and $\tilde{C}_2$. The curve $\tilde{C}_2$ is the proper transform of the curve C_2 defined by $x = y = zw + a_1t^2 = 0$. From the intersections

$$(\tilde{L}_{zt} + \tilde{C}_2) \cdot \tilde{L}_{zt} = -K_Y \cdot \tilde{L}_{zt} = -\frac{1}{10}, \quad (\tilde{L}_{zt} + \tilde{C}_2)^2 = 2B^3 = -\frac{6}{35}$$

on the surface T, we obtain

$$\tilde{L}_{zt}^2 = -\frac{1}{10} - \tilde{L}_{zt} \cdot \tilde{C}_2, \quad \tilde{C}_2^2 = -\frac{1}{14} - \tilde{L}_{zt} \cdot \tilde{C}_2.$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{zt}^2 & \tilde{L}_{zt} \cdot \tilde{C}_2 \\ \tilde{L}_{zt} \cdot \tilde{C}_2 & \tilde{C}_2^2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{10} - \tilde{L}_{zt} \cdot \tilde{C}_2 & \tilde{L}_{zt} \cdot \tilde{C}_2 \\ \tilde{L}_{zt} \cdot \tilde{C}_2 & -\frac{1}{14} - \tilde{L}_{zt} \cdot \tilde{C}_2 \end{pmatrix}$$

is negative-definite since $\tilde{L}_{zt} \cdot \tilde{C}_2$ is non-negative.

Suppose that $b_1 = 0$ and $a_1 = 0$. We then have $b_2 \neq 0$ and $a_2 \neq 0$. Furthermore, the defining equation of X_{17} must contain xz^2t; otherwise X_{17} would not be quasi-smooth at the point $x = y = w = a_2t^3 + b_2z^5 = 0$. Note that the presence of xz^2t implies the normality of the surfaces H and T. The curve $\tilde{D}$ consists of two irreducible curves $\tilde{L}_{tw}$ and $\tilde{L}_{zt}$. Indeed, $\tilde{D} = \tilde{L}_{tw} + 2\tilde{L}_{zt}$. The curves L_{tw} and L_{zt} intersect at the point O_t. The surface H is not quasi-smooth at the point O_t. We consider the divisor D_z on H cut by the equation $z = 0$. We easily see that $D_z = 2L_{tw} + R$, where R is a curve whose support does not contain L_{tw}. The curves R and L_{tw} intersect at the point O_w. The surface H is quasi-smooth at the point O_w. Then we have $\tilde{L}_{tw} \cdot \tilde{R} = \frac{3}{7}$. From the intersection

$$(2\tilde{L}_{tw} + \tilde{R}) \cdot \tilde{L}_{tw} = 3A \cdot \tilde{L}_{tw} = \frac{3}{35}$$

we obtain $\tilde{L}_{tw}^2 = -\frac{6}{35}$. From the intersections

$$(\tilde{L}_{tw} + 2\tilde{L}_{zt}) \cdot \tilde{L}_{tw} = -K_Y \cdot \tilde{L}_{tw} = \frac{1}{35}, \quad (\tilde{L}_{tw} + 2\tilde{L}_{zt})^2 = 2B^3 = -\frac{6}{35}$$

on the surface T, we obtain

$$\tilde{L}_{tw}^2 = -\frac{6}{35}, \quad \tilde{L}_{tw} \cdot \tilde{L}_{zt} = \frac{1}{10}, \quad \tilde{L}_{zt}^2 = -\frac{1}{10}.$$

Therefore, the curves $\tilde{L}_{tw}$ and $\tilde{L}_{zt}$ form a negative-definite divisor on T.

• For the singular point O_y with $c_1 \neq 0$ we consider the linear system $|-5K_{X_{17}}|$. Every member in the linear system passes through the point O_y. The base locus of $|-5K_{X_{17}}|$ is the union of the loci defined by $x = t = y = 0$ and $x = t = z = 0$. It is a 0-dimensional locus. Since the proper transform of a general member in $|-5K_{X_{15}}|$ belongs to the linear system $|5B + 2E|$, the divisor T is nef.

• For the singular point O_y with $c_1 = 0$ and $c_2 \neq 0$ we may assume that $c_2 = 1$ and $c_3 = c_4 = 0$ by a coordinate change. Choose a general member H in the linear system generated by x^5 and t. Then it is a normal surface of degree 17 in $\mathbb{P}(1, 2, 3, 7)$. Let T be the proper transform of the surface H. The intersection of T with the surface S gives us a divisor consisting of two curves $\tilde{L}_{yw}$ and $\tilde{C}$. The curve $\tilde{C}$ is the proper transform of the curve C defined by

$$x = t = w^2 + b_2yz^4 + awy^2z + by^4z^2 = 0,$$

where a and b are constants.

Suppose that $b_2 \neq 0$. Then the curve $\tilde{C}$ is irreducible. From the intersection numbers

$$(\tilde{L}_{yw} + \tilde{C}) \cdot \tilde{L}_{yw} = -K_Y \cdot \tilde{L}_{yw} = -\frac{3}{7}$$

$$(\tilde{L}_{yw} + \tilde{C})^2 = B^2 \cdot (5B + E) = -\frac{23}{21}$$

on the surface T, we obtain

$$\tilde{L}_{yw}^2 = -\frac{3}{7} - \tilde{L}_{yw} \cdot \tilde{C}, \quad \tilde{C}^2 = -\frac{2}{3} - \tilde{L}_{yw} \cdot \tilde{C}$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{yw}^2 & \tilde{L}_{yw} \cdot \tilde{C} \\ \tilde{L}_{yw} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{7} - \tilde{L}_{yw} \cdot \tilde{C} & \tilde{L}_{yw} \cdot \tilde{C} \\ \tilde{L}_{yw} \cdot \tilde{C} & -\frac{2}{3} - \tilde{L}_{yw} \cdot \tilde{C} \end{pmatrix}$$

is negative-definite since $\tilde{L}_{yw} \cdot \tilde{C}$ is non-negative.

Suppose that $b_2 = 0$. The curve C then consists of two irreducible curves C_1 and C_2 defined by

$$x = t = w - \alpha_1 y^2 z = 0$$

and

$$x = t = w - \alpha_2 y^2 z = 0,$$

respectively. Therefore, the curve $\tilde{C}$ consists of their proper transforms $\tilde{C}_1$ and $\tilde{C}_2$. From the intersections

$$(\tilde{L}_{yw} + \tilde{C}_1 + \tilde{C}_2) \cdot \tilde{L}_{yw} = -K_Y \cdot \tilde{L}_{yw} = -\frac{3}{7},$$

$$(\tilde{L}_{yw} + \tilde{C}_1 + \tilde{C}_2) \cdot \tilde{C}_1 = -K_Y \cdot \tilde{C}_1 = -\frac{1}{3}, \quad (\tilde{L}_{yw} + \tilde{C}_1 + \tilde{C}_2) \cdot \tilde{C}_2 = -K_Y \cdot \tilde{C}_2 = -\frac{1}{3}$$

on the surface T, we obtain the intersection matrix of the curves $\tilde{L}_{yw}$, $\tilde{C}_1$ and $\tilde{C}_2$

$$\begin{pmatrix} -\frac{3}{7} - \tilde{L}_{yw} \cdot \tilde{C}_1 - \tilde{L}_{yw} \cdot \tilde{C}_2 & \tilde{L}_{yw} \cdot \tilde{C}_1 & \tilde{L}_{yw} \cdot \tilde{C}_2 \\ \tilde{L}_{yw} \cdot \tilde{C}_1 & -\frac{1}{3} - \tilde{L}_{yw} \cdot \tilde{C}_1 - \tilde{C}_1 \cdot \tilde{C}_2 & \tilde{C}_1 \cdot \tilde{C}_2 \\ \tilde{L}_{yw} \cdot \tilde{C}_2 & \tilde{C}_1 \cdot \tilde{C}_2 & -\frac{1}{3} - \tilde{L}_{zw} \cdot \tilde{C}_2 - \tilde{C}_1 \cdot \tilde{C}_2 \end{pmatrix}.$$

It is easy to check that it is negative-definite since $\tilde{L}_{yw} \cdot \tilde{C}_1$, $\tilde{L}_{yw} \cdot \tilde{C}_2$ and $\tilde{C}_1 \cdot \tilde{C}_2$ are non-negative.

• For the singular point O_y with $c_1 = c_2 = 0$ and $c_3 \neq 0$ we may assume that $c_3 = 1$ and $c_4 = 0$ by a coordinate change. Note that in such a case, we must have the monomial xyw^2, i.e., $e \neq 0$: otherwise the hypersurface X_{17} is not quasi-smooth at the point defined by $x = z = t = w^2 + y^7 = 0$.

Choose a general member H in the linear system generated by x^3 and z. Then it is a normal surface of degree 17 in $\mathbb{P}(1, 2, 5, 7)$. Let D be the curve on H cut out by the equation $x = 0$. Let T be the proper transform of the surface H. Then T is normal and the curve $\tilde{D}$ is cut out by the surface S.

Suppose that $a_1 \neq 0$. We may then assume that $a_1 = 1$ and $a_2 = 0$ by a coordinate change. The curve $\tilde{D}$ then consists of two irreducible curves $\tilde{L}_{yw}$ and $\tilde{L}_{yt}$. From the intersection numbers

$$(2\tilde{L}_{yw} + \tilde{L}_{yt}) \cdot \tilde{L}_{yw} = -K_Y \cdot \tilde{L}_{yw} = -\frac{3}{7}, \quad (2\tilde{L}_{yw} + \tilde{L}_{yt})^2 = 3B^3 = -\frac{44}{35}$$

on the surface T, we obtain

$$\tilde{L}_{yw}^2 = -\frac{3}{14} - \frac{1}{2}\tilde{L}_{yw} \cdot \tilde{L}_{yt}, \quad \tilde{L}_{yt}^2 = -\frac{2}{5} - 2\tilde{L}_{yw} \cdot \tilde{L}_{yt}$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{yw}^2 & \tilde{L}_{yw} \cdot \tilde{L}_{yt} \\ \tilde{L}_{yw} \cdot \tilde{L}_{yt} & \tilde{L}_{yt}^2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{14} - \frac{1}{2}\tilde{L}_{yw} \cdot \tilde{L}_{yt} & \tilde{L}_{yw} \cdot \tilde{L}_{yt} \\ \tilde{L}_{yw} \cdot \tilde{L}_{yt} & -\frac{2}{5} - 2\tilde{L}_{yw} \cdot \tilde{L}_{yt} \end{pmatrix}$$

is negative-definite since $\tilde{L}_{yw} \cdot \tilde{L}_{yt}$ is non-negative.

Suppose that $a_1 = 0$. By changing the coordinate y, we may assume that the defining equation of X_{17} does not contain the monomial x^2t^3. The curve D consists of two irreducible curves L_{yw} and L_{tw}. In fact, we have $\tilde{D} = 3\tilde{L}_{yw} + \tilde{L}_{tw}$. Since the curve L_{yw} passes through the point O_y but the curve L_{tw} does not, we have

$$L_{yw} \cdot L_{tw} = \tilde{L}_{yw} \cdot \tilde{L}_{tw}, \quad L_{tw}^2 = \tilde{L}_{tw}^2.$$

We also have

$$(3\tilde{L}_{yw} + \tilde{L}_{tw}) \cdot \tilde{L}_{yw} = -K_Y \cdot \tilde{L}_{yw} = -\frac{3}{7}, \quad (3L_{yw} + L_{tw}) \cdot L_{tw} = -K_{X_{17}} \cdot L_{tw} = \frac{1}{35}.$$

To compute $L_{yw} \cdot L_{tw}$, we consider the divisor D_y on H given by the equation $y = 0$. Since the defining equation of X_{17} does not contain the monomial x^2t^3, we have $D_y = 3L_{tw} + R$, where R is a curve whose support does not contain the curve L_{tw}. Note that R meets L_{tw} only at the point O_t. Moreover, we can easily see that $L_{tw} \cdot R = \frac{2}{5}$ since H is quasi-smooth at the point O_t. Then the intersection

$$(3L_{tw} + R) \cdot L_{tw} = -2K_{X_{17}} \cdot L_{tw} = \frac{2}{35}$$

implies that $L_{tw}^2 = -\frac{4}{35}$. This gives a negative-definite matrix

$$\begin{pmatrix} \tilde{L}_{yw}^2 & \tilde{L}_{yw} \cdot \tilde{L}_{tw} \\ \tilde{L}_{yw} \cdot \tilde{L}_{tw} & \tilde{L}_{tw}^2 \end{pmatrix} = \begin{pmatrix} -\frac{10}{63} & \frac{1}{21} \\ \frac{1}{21} & -\frac{4}{35} \end{pmatrix}.$$

• For the singular point O_y with $c_1 = c_2 = c_3 = 0$, we consider linear system $|-7K_{X_{17}}|$. Every member in the linear system passes through the point O_y. The proper transform of a general member in $|-7K_{X_{17}}|$ belongs to the linear system $|7B + 3E|$. The base locus of the linear system $|-7K_{X_{17}}|$ possibly contains only the curve L_{yz} and the curve L_{zt}. If they are contained in X_{17}, we see

$$(7B + 3E) \cdot \tilde{L}_{yz} = -7K_{X_{17}} \cdot L_{yz} - \frac{1}{2}E \cdot \tilde{L}_{yz} = \frac{2}{3}, \quad (7B + 3E) \cdot \tilde{L}_{zt} = -7K_{X_{17}} \cdot L_{zt} = \frac{7}{15}.$$

Therefore, T is nef.

No. 34: $X_{18} \subset \mathbb{P}(1,1,2,6,9)$					$A^3 = 1/6$
$w^2 + t^3 + wf_9(x,y,z,t) + f_{18}(x,y,z,t)$					
Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_tO_w = 1\times\frac{1}{3}(1_x, 1_y, 2_z)$ ⓑ	0	B	y	y	
$O_zO_t = 3\times\frac{1}{2}(1_x, 1_y, 1_w)$ ⓑ	$-$	B	y	y	

• The curve defined by $x = y = 0$ is always irreducible since we have the monomials w^2 and t^3. Therefore, the 1-cycle Γ for each singular point is irreducible.

No. 35: $X_{18} \subset \mathbb{P}(1,1,3,5,9)$					$A^3 = 2/15$
$w^2 + zt^3 + wf_9(x,y,z,t) + f_{18}(x,y,z,t)$					
Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t = \frac{1}{5}(1_x, 1_y, 4_w)$ ⓟ	$+$	$3B - E$	z	w^2	
$O_zO_w = 2\times\frac{1}{2}(1_x, 1_y, 1_t)$ ⓑ	$-$	B	y	y	

• The 1-cycle Γ for the singular points of type $\frac{1}{2}(1,1,1)$ is irreducible since we have w^2 and zt^3.

No. 36: $X_{18} \subset \mathbb{P}(1,1,4,6,7)$					$A^3 = 3/28$
$zw^2 + t^3 - z^3t + wf_{11}(x,y,z,t) + f_{18}(x,y,z,t)$					
Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{7}(1,1,6)$ $\boxed{\tau}$	zw^2				
$O_z = \frac{1}{4}(1,1,3)$ $\boxed{\epsilon_1}$	$zw^2 - z^3t$				
$O_zO_t = 1\times\frac{1}{2}(1_x, 1_y, 1_w)$ ⓑ	$-$	B	y	y	

• The 1-cycle Γ for the singular point of type $\frac{1}{2}(1,1,1)$ is irreducible because of the monomials zw^2 and t^3.

No. 37: $X_{18} \subset \mathbb{P}(1,2,3,4,9)$					$A^3 = 1/12$
$(w - \beta_1 z^3)(w - \beta_2 z^3) + y\prod_{i=1}^{4}(t - \alpha_i y^2) + at^3z^2 + wf_9(x,y,z,t) + t^3g_6(x,y,z) + t^2g_{10}(x,y,z) + tg_{14}(x,y,z) + g_{18}(x,y,z)$					
Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t = \frac{1}{4}(1_x, 3_z, 1_w)$ ⓑ	0	$2B$	y	w^2	
$O_zO_w = 2\times\frac{1}{3}(1_x, 2_y, 1_t)$ ⓑ	$-$	$2B$	y	y	$a \neq 0$
$O_zO_w = 2\times\frac{1}{3}(1_x, 2_y, 1_t)$ ⓢ	$-$	$2B$	x^2, y	x^2, y	$a = 0$
$O_yO_t = 4\times\frac{1}{2}(1_x, 1_z, 1_w)$ ⓑ	$-$	$4B + E$	$t - \alpha_i y^2$	w^2	

• For the singular point O_t, the 1-cycle Γ can be reducible. In case, we see that Γ consists of the proper transforms of the curves defined by $x = y = w - \beta_1 z^3 = 0$ and $x = y = w - \beta_2 z^3 = 0$. These two irreducible components are symmetric with respect to the biregular involution of X_{18}. In addition, the point O_t is the intersection point of these two curves. Consequently, the components of Γ are numerically equivalent to each other.

• For each singular point of type $\frac{1}{3}(1,2,1)$, the 1-cycle Γ is irreducible if the constant a is not zero.

• Suppose that the constant a is zero. We have only to consider one of the singular points of type $\frac{1}{3}(1,2,1)$. The other singular point can be excluded in the same way. We put $\beta_1 = 0$ and consider the singular point O_z. We may also assume that the defining equation of X_{18} contains neither xz^3t^2 nor x^2z^4t by changing the coordinate w.

We take a general surface H from the pencil $|-2K_{X_{18}}|$ and then let T be the proper transform of the surface. Note that the surface H is normal. However, it is not quasi-smooth at the point O_t. The intersection of T with the surface S gives us a divisor consisting of two irreducible curves on the normal surface T. They are the proper transforms $\tilde{L}_{zt}$ and $\tilde{C}$ of the curve L_{zt} and the curve C defined by

$$x = y = w - \beta_2 z^3 = 0,$$

respectively. From the intersection numbers

$$(\tilde{L}_{zt} + \tilde{C}) \cdot \tilde{L}_{zt} = -K_Y \cdot \tilde{L}_{zt} = -\frac{1}{4}, \quad (\tilde{L}_{zt} + \tilde{C})^2 = 2B^3 = -\frac{1}{6}$$

on the surface T, we obtain

$$\tilde{L}_{zt}^2 = -\frac{1}{4} - \tilde{L}_{zt} \cdot \tilde{C}, \quad \tilde{C}^2 = \frac{1}{12} - \tilde{L}_{zt} \cdot \tilde{C}.$$

To compute the intersection number $\tilde{L}_{zt} \cdot \tilde{C}$, we consider the divisor D_w on H cut by the equation $w = 0$. We easily see that $D_w = 2L_{zt} + R$, where R is a curve whose support does not contain L_{zt}. The curve R and L_{zt} intersects at the point O_z. Let $\tilde{R}$ be the proper transform of R. Then we have $\tilde{L}_{zt} \cdot \tilde{R} = 0$ since they are disconnected on T. From the intersection

$$(2\tilde{L}_{zt} + \tilde{R}) \cdot \tilde{L}_{zt} = (9B + E) \cdot \tilde{L}_{zt} = -\frac{5}{4}$$

we obtain $\tilde{L}_{zt}^2 = -\frac{5}{8}$. Therefore, $\tilde{L}_{zt} \cdot \tilde{C} = \frac{3}{8}$ and $\tilde{C}^2 = -\frac{7}{24}$. With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{zt}^2 & \tilde{L}_{zt} \cdot \tilde{C} \\ \tilde{L}_{zt} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{5}{8} & \frac{3}{8} \\ \frac{3}{8} & -\frac{7}{24} \end{pmatrix}$$

is negative-definite.

• For each singular point of type $\frac{1}{2}(1,1,1)$, the 1-cycle Γ may be reducible. In case, it consists of the proper transforms of the curves defined by

$$x = t - \alpha_i y^2 = w + by^3 z + cz^3 = 0$$

and

$$x = t - \alpha_i y^2 = w + dy^3 z + ez^3 = 0,$$

where b, c, d, e are constants. These two irreducible components are also symmetric with respect to the biregular involution of X_{18}. In addition, the singular point is the intersection point of these two curves. Therefore, the components of Γ are numerically equivalent.

No. 38: $X_{18} \subset \mathbb{P}(1,2,3,5,8)$ $yw^2 + t^2(a_1 w + a_2 zt) + z^6 + y^9 + wf_{10}(x,y,z,t) + f_{18}(x,y,z,t)$					$A^3 = 3/40$
Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{8}(1,3,5)$ $\boxed{\mathcal{T}}$	yw^2				
$O_t = \frac{1}{5}(1,2,3)$ $\boxed{\mathcal{T}_1}$	wt^2				
$O_yO_w = 2\times\frac{1}{2}(1_x,1_z,1_t)$ ⓝ	–	$5B+2E$	t	t	

• For the singular points of type $\frac{1}{2}(1,1,1)$ we consider the linear system $|-5K_{X_{18}}|$. Every member of the linear system passes through the singular points of type $\frac{1}{2}(1,1,1)$ and the base locus of the linear system contains no curves. Since the proper transform of a general member in $|-5K_{X_{18}}|$ belongs to the linear system $|5B+2E|$, the divisor T is nef.

No. 39: $X_{18} \subset \mathbb{P}(1,3,4,5,6)$ $(w-\alpha_1 y^2)(w-\alpha_2 y^2)(w-\alpha_3 y^2) + yt^3 + z^3 w + at^2z^2 + by^2z^3 + w^2 f_6(x,y,z,t) + wf_{12}(x,y,z,t) + t^2 g_8(x,y,z) + tg_{13}(x,y,z) + g_{18}(x,y,z)$					$A^3 = 1/20$
Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t = \frac{1}{5}(1_x,4_z,1_w)$ ⓑ	0	$3B$	y	w^3	
$O_z = \frac{1}{4}(1_x,3_y,1_t)$ ⓑ	–	$3B$	y	y	$a \neq 0$
$O_z = \frac{1}{4}(1_x,3_y,1_t)$ ⓢ	–	$3B$	x^3, y	x^3, y	$a = 0$
$O_zO_w = 1\times\frac{1}{2}(1_x,1_y,1_t)$ ⓝ	–	$5B+2E$	t	t	
$O_yO_w = 3\times\frac{1}{3}(1_x,1_z,2_t)$ ⓝ	–	$5B+E$	t	t	$b \neq 0$
$O_yO_w = 3\times\frac{1}{3}(1_x,1_z,2_t)$ ⓢ	–	$5B+E$	x^5, xz, t	t	$b = 0$

• For the singular point O_t, the 1-cycle Γ may be reducible. However, in case, it consists of two irreducible components. One is the proper transform $\tilde{L}_{zt}$ of the curve L_{zt} and the other is the proper transform $\tilde{C}$ of the curve defined by

$$x = y = w^2 + z^3 = 0.$$

We can easily check that

$$E \cdot \tilde{C} = 2E \cdot \tilde{L}_{zt} = \frac{1}{2}, \quad B \cdot \tilde{C} = 2B \cdot \tilde{L}_{zt} = 0.$$

Therefore, the irreducible curves $\tilde{L}_{zt}$ and $\tilde{C}$ are numerically proportional on Y.

• The 1-cycle Γ for the singular point O_z with $a \neq 0$ is irreducible due to w^3 and t^2z^2.

• For the singular point O_z with $a = 0$ we may assume that the defining equation of X_{18} contains neither xz^3t nor x^2z^4 by changing the coordinate w.

We take a general surface H from the pencil $|-3K_{X_{18}}|$ and then let T be the proper transform of the surface. Note that the surface H is normal. However, it is not quasi-smooth at the point O_t. The intersection of T with the surface S gives us a divisor consisting of two irreducible curves $\tilde{L}_{zt}$ and $\tilde{C}$ on the normal surface T. The curve $\tilde{C}$ is the proper transform of the curve C defined by

$$x = y = w^2 + z^3 = 0.$$

From the intersection numbers

$$(\tilde{L}_{zt} + \tilde{C}) \cdot \tilde{L}_{zt} = -K_Y \cdot \tilde{L}_{zt} = -\frac{1}{5}, \quad (\tilde{L}_{zt} + \tilde{C})^2 = 3B^3 = -\frac{1}{10}$$

on the surface T, we obtain

$$\tilde{L}_{zt}^2 = -\frac{1}{5} - \tilde{L}_{zt} \cdot \tilde{C}, \quad \tilde{C}^2 = \frac{1}{10} - \tilde{L}_{zt} \cdot \tilde{C}.$$

Therefore,

$$(3\tilde{L}_{zt} + \tilde{R}) \cdot \tilde{L}_{zt} = -6K_Y \cdot \tilde{L}_{zt} = -\frac{6}{5}$$

we obtain $\tilde{L}_{zt}^2 = -\frac{2}{5}$. With these intersection numbers we see

$$\begin{pmatrix} \tilde{L}_{zt}^2 & \tilde{L}_{zt} \cdot \tilde{C} \\ \tilde{L}_{zt} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} - \tilde{L}_{zt} \cdot \tilde{C} & \tilde{L}_{zt} \cdot \tilde{C} \\ \tilde{L}_{zt} \cdot \tilde{C} & \frac{1}{10} - \tilde{L}_{zt} \cdot \tilde{C} \end{pmatrix}.$$

To compute the intersection number $\tilde{L}_{zt} \cdot \tilde{C}$, we take the divisor D_w on H cut by the equation $w = 0$. This divisor can be written as $D_w = 3L_{zt} + R$, where R is a curve whose support does not contain L_{zt}. The curve R and L_{zt} intersects at the point O_z. Let $\tilde{R}$ be the proper transform of R. We have $\tilde{L}_{zt} \cdot \tilde{R} = 0$ since they are disconnected on T. From the intersection

$$(3\tilde{L}_{zt} + \tilde{R}) \cdot \tilde{L}_{zt} = -6K_Y \cdot \tilde{L}_{zt} = -\frac{6}{5}$$

we obtain $\tilde{L}_{zt}^2 = -\frac{2}{5}$. Therefore, $\tilde{L}_{zt} \cdot \tilde{C} = \frac{1}{5}$. This shows that the matrix is negative-definite.

• For the singular point of type $\frac{1}{2}(1,1,1)$ we consider the linear system generated by x^{15}, y^5 and t^3 on the hypersurface X_{18}. Its base locus is cut out by $x = y = t = 0$. Since we have the monomial z^3w, the base locus does not contain curves. Therefore the proper transform of a general member in the linear system is nef by Lemma 3.2.6 and it belongs to $|15B + 6E|$. Consequently, the surface T is nef since $3T \sim_{\mathbb{Q}} 15B + 6E$.

For the singular points of type $\frac{1}{3}(1,1,2)$ we may assume that $\alpha_1 = 0$ and consider the singular point O_y. The other points can be dealt with in the same way. Since $\alpha_1 = 0$, the defining equation of X_{18} does not contain the monomial y^6. We may also assume that it does not contain the monomials x^6y^4, x^3y^5, x^2y^4z and xy^4t by changing the coordinate w.

• For the singular point O_y with $b \neq 0$ we consider the linear system generated by x^{30}, t^5 and w^6 on the hypersurface X_{18}. Its base locus is cut out by $x = t = w = 0$. Since $b \neq 0$, its base locus does not contain curves, and hence the proper transform of a general member in the linear system is nef by Lemma 3.2.6. It belongs to $|30B + 6E|$. Consequently, the surface T is nef since $6T \sim_{\mathbb{Q}} 30B + 6E$.

• For the singular point O_y with $b = 0$ we take a general surface H from the linear system generated by x^5, xz and t. Then H is normal. Moreover, the surface H is smooth at the point $x = t = w = z^3 + \alpha_1\alpha_2 y^4 = 0$. Indeed, the defining equation of X_{18} must contain at least one of the monomials xz^2y^3, ty^3z; otherwise X_{18} would be singular at the point $x = t = w = z^3 + \alpha_1\alpha_2 y^4 = 0$. Plugging in $t = \lambda xz + \mu x^5$ with general complex numbers λ and μ into the defining equation of X_{18}, we obtain the defining equation of H in $\mathbb{P}(1, 3, 4, 6)$. It must contain the monomial xz^2y^3. Therefore, the surface H is smooth at the point $x = t = w = z^3 + \alpha_1\alpha_2 y^4 = 0$

Let T be the proper transform of the surface H. The intersection of T with the surface S defines a divisor consisting of two irreducible curves $\tilde{L}_{yz}$ and $\tilde{C}$ on the normal surface T. The curve $\tilde{C}$ is the proper transform of the curve C defined by

$$x = t = z^3 + (w - \alpha_2 y^2)(w - \alpha_3 y^2) = 0.$$

The curves L_{yz} and C intersect at the point defined by $x = t = w = z^3 + \alpha_1\alpha_2 y^4 = 0$. From the intersection numbers

$$(\tilde{L}_{yz} + \tilde{C}) \cdot \tilde{L}_{yz} = -K_Y \cdot \tilde{L}_{yz} = -\frac{1}{4}, \quad (\tilde{L}_{yz} + \tilde{C})^2 = B^2 \cdot (5B + E) = -\frac{1}{12}$$

on the surface T, we obtain

$$\tilde{L}_{yz}^2 = -\frac{1}{4} - \tilde{L}_{yz} \cdot \tilde{C}, \quad \tilde{C}^2 = \frac{1}{6} - \tilde{L}_{yz} \cdot \tilde{C}$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{yz}^2 & \tilde{L}_{yz} \cdot \tilde{C} \\ \tilde{L}_{yz} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} - \tilde{L}_{yz} \cdot \tilde{C} & \tilde{L}_{yz} \cdot \tilde{C} \\ \tilde{L}_{yz} \cdot \tilde{C} & \frac{1}{6} - \tilde{L}_{yz} \cdot \tilde{C} \end{pmatrix}$$

is negative-definite since $\tilde{L}_{yz}$ and $\tilde{C}$ intersect at a smooth point of the surface T.

No. 40: $X_{19} \subset \mathbb{P}(1, 3, 4, 5, 7)$	$A^3 = 19/420$
$tw^2 - zt^3 + z^3(a_1w + a_2yz) + y^4(b_1w + b_2yz + b_3y^2x) + ay^2z^2t + by^3t^2 + wf_{12}(x, y, z, t) + f_{19}(x, y, z, t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{7}(1, 3, 4)$ $\boxed{\tau}$	tw^2				
$O_t = \frac{1}{5}(1, 3, 2)$ $\boxed{\epsilon}$	$tw^2 - zt^3$				
$O_z = \frac{1}{4}(1_x, 3_y, 1_t)$ ⓑ	$-$	$3B$	y	zt^3	$a_1 \neq 0$
$O_z = \frac{1}{4}(1_x, 1_t, 3_w)$ ⓢ	$-$	$3B$	x^3, z	x^3, zt^3	$a_1 = 0$

$O_y = \frac{1}{3}(1_x, 1_z, 2_t)$ ⓑ	–	$7B+E$	w	y^3t^2	$b_1 \neq 0$, $b \neq 0$ $a_2 \neq 0$
$O_y = \frac{1}{3}(1_x, 1_z, 2_t)$ ⓢ	–	$7B+E$	x^7, w	y^3t^2	$b_1 \neq 0$, $b \neq 0$ $a_2 = 0$
$O_y = \frac{1}{3}(1_x, 1_z, 2_t)$ ⓢ	–	$7B+E$	x^7, w	z^4y	$b_1 \neq 0$, $b = 0$ $a_2 \neq 0$
$O_y = \frac{1}{3}(1_x, 1_z, 2_t)$ ⓢ	–	$7B$	x^7, w	x^7, zt^3	$b_1 \neq 0$, $b = 0$ $a_2 = 0$
$O_y = \frac{1}{3}(1_x, 2_t, 1_w)$ ⓢ	–	$4B$	x^4, z	x^4, tw^2	$b_1 = 0$, $b_2 \neq 0$
$O_y = \frac{1}{3}(1_z, 2_t, 1_w)$ ⓝ	–	$7B+2E$	w	w	$b_1 = 0$, $b_2 = 0$

• The 1-cycle Γ for the singular point O_z with $a_1 \neq 0$ is irreducible due to the monomials tw^2, zt^3 and z^3w.

• For the singular point O_z with $a_1 = 0$ we choose a general member H in the linear system $|-3K_{X_{19}}|$. Then it is a normal surface of degree 19 in $\mathbb{P}(1,4,5,7)$. Let T be the proper transform of the divisor H. The intersection of T with the surface S defines a divisor consisting of two irreducible curves $\tilde{L}_{zw}$ and $\tilde{C}$. The curve $\tilde{C}$ is the proper transform of the curve C defined by

$$x = y = w^2 - zt^2 = 0.$$

From the intersection

$$(\tilde{L}_{zw} + \tilde{C}) \cdot \tilde{L}_{zw} = -K_Y \cdot \tilde{L}_{zw} = -\frac{1}{21}, \quad (\tilde{L}_{zw} + \tilde{C})^2 = 3B^3 = -\frac{4}{35}$$

on the surface T, we obtain

$$\tilde{L}_{zw}^2 = -\frac{1}{21} - \tilde{L}_{zw} \cdot \tilde{C}, \quad \tilde{C}^2 = -\frac{1}{15} - \tilde{L}_{zw} \cdot \tilde{C}.$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{zw}^2 & \tilde{L}_{zw} \cdot \tilde{C} \\ \tilde{L}_{zw} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{21} - \tilde{L}_{zw} \cdot \tilde{C} & \tilde{L}_{zw} \cdot \tilde{C} \\ \tilde{L}_{zw} \cdot \tilde{C} & -\frac{1}{15} - \tilde{L}_{zw} \cdot \tilde{C} \end{pmatrix}$$

is negative-definite since $\tilde{L}_{zw} \cdot \tilde{C}$ is non-negative number.

Consider the singular point O_y with $b_1 \neq 0$. In this case, we may assume that $b_1 = 1$ and $b_2 = b_3 = 0$ by a suitable coordinate change.

• For the singular point O_y with $b_1 \neq 0$, $b \neq 0$ and $a_2 \neq 0$ the 1-cycle Γ is irreducible because of the monomials zt^3, yz^4 and y^3t^2.

• For the singular point O_y with $b_1 \neq 0$, $b \neq 0$ and $a_2 = 0$ we may assume that the monomial xy^2z^3 does not appear in f_{19} by changing the coordinate w in a suitable way. We must then have $a \neq 0$; otherwise the hypersurface would not be quasi-smooth at the point defined by

$$x = t = w = a_1z^3 + y^4 = 0.$$

Take a general member H in the linear system generated by x^7 and w. Then it is a normal surface of degree 19 in $\mathbb{P}(1,3,4,5)$. Let T be the proper transform of the divisor H. The intersection of T with the surface S gives us a divisor consisting of two irreducible curves $\tilde{L}_{yz}$ and $\tilde{C}_1$. The curve $\tilde{C}_1$ is the proper transform of the curve C_1 defined by $x = w = -zt^2 + ay^2z^2 + by^3t = 0$. From the intersection

$$(\tilde{L}_{yz} + \tilde{C}_1) \cdot \tilde{L}_{yz} = -K_Y \cdot \tilde{L}_{yz} = -\frac{1}{4}, \quad (\tilde{L}_{yz} + \tilde{C}_1)^2 = B^2 \cdot (7B + E) = -\frac{7}{20}$$

on the surface T, we obtain

$$\tilde{L}_{yz}^2 = -\frac{1}{4} - \tilde{L}_{yz} \cdot \tilde{C}_1, \quad \tilde{C}_1^2 = -\frac{1}{10} - \tilde{L}_{yz} \cdot \tilde{C}_1.$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{yz}^2 & \tilde{L}_{yz} \cdot \tilde{C}_1 \\ \tilde{L}_{yz} \cdot \tilde{C}_1 & \tilde{C}_1^2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} - \tilde{L}_{yz} \cdot \tilde{C}_1 & \tilde{L}_{yz} \cdot \tilde{C}_1 \\ \tilde{L}_{yz} \cdot \tilde{C}_1 & -\frac{1}{10} - \tilde{L}_{yz} \cdot \tilde{C}_1 \end{pmatrix}$$

is negative-definite since $\tilde{L}_{yz} \cdot \tilde{C}_1$ is non-negative number.

• For the singular point O_y with $b_1 \neq 0$, $b = 0$ and $a_2 \neq 0$, we do the same as in the case where $b_1 \neq 0$, $b \neq 0$ and $a_2 = 0$. The difference is that the intersection of T with the surface S gives us a divisor consisting of two irreducible curves $\tilde{L}_{yt}$ and $\tilde{C}_2$. The curve $\tilde{C}_2$ is the proper transform of the curve C_2 defined by $x = w = -t^3 + a_2yz^3 + ay^2zt = 0$. From the intersections

$$(\tilde{L}_{yt} + \tilde{C}_2) \cdot \tilde{L}_{yt} = -K_Y \cdot \tilde{L}_{yt} = -\frac{1}{10}, \quad (\tilde{L}_{yt} + \tilde{C}_2)^2 = B^2 \cdot (7B + E) = -\frac{7}{20}$$

on the surface T, we obtain

$$\tilde{L}_{yt}^2 = -\frac{1}{10} - \tilde{L}_{yt} \cdot \tilde{C}_2, \quad \tilde{C}_2^2 = -\frac{1}{4} - \tilde{L}_{yt} \cdot \tilde{C}_2.$$

This shows $\tilde{L}_{yt}$ and $\tilde{C}_2$ forms a negative-definite divisor on T.

• For the singular point O_y with $b_1 \neq 0$, $b = 0$ and $a_2 = 0$ we may assume that the monomial xy^2z^3 does not appear in f_{19} by changing the coordinate w in a suitable way. We must then have $a \neq 0$; otherwise the hypersurface would not be quasi-smooth at the point defined by $x = t = w = a_1z^3 + y^4 = 0$.

We do the same as the previous case. In this case, we obtain a divisor consisting of three irreducible curves $\tilde{L}_{yz}$, $\tilde{L}_{yt}$ and $\tilde{C}_3$. The curve $\tilde{C}_3$ is the proper transform of the curve C_3 defined by $x = w = -t^2 + ay^2z = 0$. From the intersections

$$(\tilde{L}_{yz} + \tilde{L}_{yt} + \tilde{C}_3) \cdot \tilde{L}_{yz} = -K_Y \cdot \tilde{L}_{yz} = -\frac{1}{4},$$
$$(\tilde{L}_{yz} + \tilde{L}_{yt} + \tilde{C}_3) \cdot \tilde{L}_{yt} = -K_Y \cdot \tilde{L}_{yt} = -\frac{1}{10},$$
$$(\tilde{L}_{yz} + \tilde{L}_{yt} + \tilde{C}_3) \cdot \tilde{C}_3 = -K_Y \cdot \tilde{C}_3 = -\frac{1}{6}.$$

on the surface T, we obtain the intersection matrix of the curves $\tilde{L}_{yz}$, $\tilde{L}_{yt}$ and $\tilde{C}_3$

$$\begin{pmatrix} -\frac{1}{4} - \tilde{L}_{yz} \cdot \tilde{L}_{yt} - \tilde{L}_{yz} \cdot \tilde{C}_3 & \tilde{L}_{yz} \cdot \tilde{L}_{yt} & \tilde{L}_{yz} \cdot \tilde{C}_3 \\ \tilde{L}_{yz} \cdot \tilde{L}_{yt} & -\frac{1}{10} - \tilde{L}_{yz} \cdot \tilde{L}_{yt} - \tilde{L}_{yt} \cdot \tilde{C}_3 & \tilde{L}_{yt} \cdot \tilde{C}_3 \\ \tilde{L}_{yz} \cdot \tilde{C}_3 & \tilde{L}_{yt} \cdot \tilde{C}_3 & -\frac{1}{6} - \tilde{L}_{yz} \cdot \tilde{C}_3 - \tilde{L}_{yt} \cdot \tilde{C}_3 \end{pmatrix}.$$

It is easy to check that it is negative-definite since $\tilde{L}_{yz}\cdot\tilde{L}_{yt}$, $\tilde{L}_{yz}\cdot\tilde{C}_3$ and $\tilde{L}_{yt}\cdot\tilde{C}_3$ are non-negative numbers.

• For the singular point O_y with $b_1=0$ and $b_2\neq 0$ we may put $b_2=1$ by a coordinate change. Choose a general member H in the pencil on X_{19} generated by x^4 and z. Then it is a normal surface of degree 19 in $\mathbb{P}(1,3,5,7)$. Let T be the proper transform of the divisor H. The surface S cuts out the surface T into a divisor $\tilde{D}$.

We suppose that $b\neq 0$. The divisor $\tilde{D}$ then consists of two irreducible curves $\tilde{L}_{yw}$ and $\tilde{C}$. The curve $\tilde{C}$ is the proper transform of the curve C defined by

$$x=z=w^2+by^3t=0.$$

From the intersections

$$(\tilde{L}_{yw}+\tilde{C})\cdot\tilde{L}_{yw}=-K_Y\cdot\tilde{L}_{yw}=-\frac{2}{7},\quad (\tilde{L}_{yw}+\tilde{C})^2=4B^3=-\frac{17}{35}$$

on the surface T, we obtain

$$\tilde{L}_{yw}^2=-\frac{2}{7}-\tilde{L}_{yw}\cdot\tilde{C},\quad \tilde{C}^2=-\frac{1}{5}-\tilde{L}_{yw}\cdot\tilde{C}.$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix}\tilde{L}_{yw}^2 & \tilde{L}_{yw}\cdot\tilde{C}\\ \tilde{L}_{yw}\cdot\tilde{C} & \tilde{C}^2\end{pmatrix}=\begin{pmatrix}-\frac{2}{7}-\tilde{L}_{yw}\cdot\tilde{C} & \tilde{L}_{yw}\cdot\tilde{C}\\ \tilde{L}_{yw}\cdot\tilde{C} & -\frac{1}{5}-\tilde{L}_{yw}\cdot\tilde{C}\end{pmatrix}$$

is negative-definite since $\tilde{L}_{yw}\cdot\tilde{C}$ is non-negative number.

We now suppose that $b=0$. The divisor $\tilde{D}$ then consists of two irreducible curves $\tilde{L}_{yw}$ and $\tilde{L}_{yt}$. From the intersection

$$(\tilde{L}_{yw}+2\tilde{L}_{yt})\cdot\tilde{L}_{yw}=-K_Y\cdot\tilde{L}_{yw}=-\frac{2}{7},\quad (\tilde{L}_{yw}+2\tilde{L}_{yt})^2=4B^3=-\frac{17}{35}$$

on the surface T, we obtain

$$\tilde{L}_{yw}^2=-\frac{2}{7}-2\tilde{L}_{yw}\cdot\tilde{L}_{yt},\quad \tilde{L}_{yt}^2=-\frac{1}{20}-\frac{1}{2}\tilde{L}_{yw}\cdot\tilde{L}_{yt}.$$

This again shows that $\tilde{L}_{yw}$ and $\tilde{L}_{yt}$ form a negative-definite divisor on T.

• For the singular point O_y with $b_1=b_2=0$ we consider the linear system generated by z^{35}, t^{28} and w^{20} on the hypersurface X_{19}. Its base locus is cut out by $z=t=w=0$. Since we have the monomial xy^6, the base locus does not contain curves. Therefore the proper transform of a general member in the linear system is nef by Lemma 3.2.6. It belongs to $|140B+40E|$. Consequently, the surface T is nef since $20T\sim_{\mathbb{Q}}140B+40E$.

No. 41: $X_{20}\subset\mathbb{P}(1,1,4,5,10)$	$A^3=1/10$
$(w-\alpha_1t^2)(w-\alpha_2t^2)+z^5+wf_{10}(x,y,z,t)+f_{20}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_tO_w=2\times\frac{1}{5}(1,1,4)$ $\boxed{T}$	wt^2				
$O_zO_w=1\times\frac{1}{2}(1_x,1_y,1_t)$ ⓑ	$-$	B	y	y	

• We may assume that $\alpha_1 = 0$. To see how to treat the singular points of type $\frac{1}{5}(1,1,4)$, we have only to consider the singular point O_t. The other point can be untwisted or excluded in the same way.

• The 1-cycle Γ for the singular point of type $\frac{1}{2}(1,1,1)$ is irreducible because of the monomial w^2 and z^5.

No. 42: $X_{20} \subset \mathbb{P}(1,2,3,5,10)$					$A^3 = 1/15$
$(w-\alpha_1 y^5)(w-\alpha_2 y^5) + wt^2 + z^5(a_1 t + a_2 yz) + wf_{10}(x,y,z,t) + f_{20}(x,y,z,t)$					
Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_z = \frac{1}{3}(1_x, 2_y, 1_w)$ ⓑ	$-$	$2B$	y	y	$a_1 \neq 0$
$O_z = \frac{1}{3}(1_x, 2_t, 1_w)$ ⓑ	$-$	$2B$	y	w^2	$a_1 = 0$
$O_t O_w = 2 \times \frac{1}{5}(1,2,3)$ $\boxed{\tau}$	wt^2				
$O_y O_w = 2 \times \frac{1}{2}(1,1,1)$ ⓝ	$-$	$5B+2E$	t	t	

• The 1-cycle Γ for the singular point O_z with $a_1 \neq 0$ is irreducible due to w^2 and $z^5 t$.

• The 1-cycle Γ for the singular point O_z with $a_1 = 0$ consists of the proper transforms of the curve L_{zt} and the curve defined by

$$x = y = w + t^2 = 0.$$

These two irreducible components are symmetric with respect to the biregular involution of X_{20}. Consequently the components of Γ are numerically equivalent to each other.

• By changing the coordinate w we may assume that t^4 is not in the polynomial f_{20}. To see how to untwist or exclude the singular points of type $\frac{1}{5}(1,2,3)$ we have only to consider the singular point O_t. The other point can be treated in the same way.

• For the singular points of type $\frac{1}{2}(1,1,1)$, we consider the linear system $|-5K_{X_{20}}|$. Every member of the linear system passes through the singular points of type $\frac{1}{2}(1,1,1)$ and the base locus of the linear system contains no curves. Since the proper transform of a general member in $|-5K_{X_{20}}|$ belongs to the linear system $|5B+2E|$, the divisor T is nef.

No. 43: $X_{20} \subset \mathbb{P}(1,2,4,5,9)$					$A^3 = 1/18$
$yw^2 + t^4 + \prod_{i=1}^{5}(z - \alpha_i y^2) + wf_{11}(x,y,z,t) + f_{20}(x,y,z,t)$					
Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{9}(1,4,5)$ $\boxed{\tau}$	yw^2				
$O_y O_z = 5 \times \frac{1}{2}(1_x, 1_t, 1_w)$ ⓑ	$-$	$4B+E$	$z - \alpha_i y^2$	yw^2	

• The 1-cycles Γ for the singular points of type $\frac{1}{2}(1,1,1)$ are irreducible due to the monomials yw^2 and t^4.

No. 44: $X_{20} \subset \mathbb{P}(1,2,5,6,7)$ $A^3 = 1/21$

$tw^2 + y(t-\alpha_1 y^3)(t-\alpha_2 y^3)(t-\alpha_3 y^3) + z^4 + wf_{14}(x,y,z,t) + f_{20}(x,y,z,t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{7}(1,2,5)$ $\boxed{\tau}$	tw^2				
$O_t = \frac{1}{6}(1,5,1)$ $\boxed{\epsilon}$	$tw^2 - yt^3$				
$O_yO_t = 3\times\frac{1}{2}(1_x,1_z,1_w)$ ⓝ	$-$	$7B+3E$	w	w	

• Consider the linear system generated by x^{35}, z^7 and w^5. Since the defining equation of X_{20} contains yt^3, the base locus of the linear system contains no curve. Therefore, the proper transform of a general member in this linear system is nef. Since it belongs to $|35B+15E|$, the surface T is nef.

No. 45: $X_{20} \subset \mathbb{P}(1,3,4,5,8)$ $A^3 = 1/24$

$z(w-\alpha_1 z^2)(w-\alpha_2 z^2) + t^4 + y^4(a_1 w + a_2 yt) + wf_{12}(x,y,z,t) + f_{20}(x,y,z,t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{8}(1,3,5)$ $\boxed{\tau}$	zw^2				
$O_y = \frac{1}{3}(1_x,1_z,2_t)$ ⓝ	$-$	$4B+E$	z	z	
$O_zO_w = 2\times\frac{1}{4}(1_x,3_y,1_t)$ ⓑ	$-$	$3B$	y	y	

• For the singular point O_y we consider the linear system generated by x^{40}, z^{10}, t^8 and w^5 on the hypersurface X_{20}. Its base locus does not contain curves. Therefore the proper transform of a general member in the linear system is nef by Lemma 3.2.6. The proper transform belongs to $|40B+10E|$. Consequently, the surface T is nef since $10T \sim_{\mathbb{Q}} 40B+10E$.

• The 1-cycles Γ for the singular points of type $\frac{1}{4}(1,3,1)$ are irreducible due to the monomials zw^2 and t^4.

No. 46: $X_{21} \subset \mathbb{P}(1,1,3,7,10)$ $A^3 = 1/10$

$yw^2 + t^3 + z^7 + wf_{11}(x,y,z,t) + f_{21}(x,y,z,t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{10}(1,3,7)$ $\boxed{\tau}$	yw^2				

No. 47: $X_{21} \subset \mathbb{P}(1,1,5,7,8)$ $A^3 = 3/40$

$zw^2 + t^3 + yz^4 + wf_{13}(x,y,z,t) + f_{21}(x,y,z,t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{8}(1,1,7)$ $\boxed{\tau}$	zw^2				
$O_z = \frac{1}{5}(1_x,2_t,3_w)$ ⓕ	$+$	$B-E$	y	t^3	

• For the singular point O_z, let C_λ be the curve on the surface S_y defined by

$$\begin{cases} y = 0, \\ w = \lambda x^8 \end{cases}$$

for a sufficiently general complex number λ. We then have

$$-K_Y \cdot \tilde{C}_\lambda = (B - E) \cdot (8B + E) \cdot B = 0.$$

Consider the linear system generated by x^{72}, y^9t^9 and y^8w^8. Its base curve is defined by $x = y = 0$. It is an irreducible curve because we have the monomials zw^2 and t^3. The proper transform of a general member of the linear system is equivalent to $72B$. The only curve that intersects the divisor B negatively is the proper transform of the irreducible curve defined by $x = y = 0$. It is not on the surface T. Therefore, if the curve $\tilde{C}_\lambda$ is reducible, each component of the curve $\tilde{C}_\lambda$ intersects B trivially.

No. 48: $X_{21} \subset \mathbb{P}(1, 2, 3, 7, 9)$	$A^3 = 1/18$
$zw^2 + t^3 + z^7 + y^6(a_1w + a_2yt + a_3y^3z + a_4xy^4) + wf_{12}(x, y, z, t) + f_{21}(x, y, z, t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{9}(1, 2, 7)$ $\boxed{T}$	zw^2				
$O_y = \frac{1}{2}(1, 1, 1)$ ⓝ	$-$	$9B + 4E$	$w + yt$	w or yt	
$O_zO_w = 2\times\frac{1}{3}(1_x, 2_y, 1_t)$ ⓑ	$-$	$2B$	y	y	

• For the singular point O_y we consider the linear system $|-9K_{X_{24}}|$. Every member of the linear system passes through the singular point O_y and its base locus contains no curves. Since the proper transform of a general member in $|-9K_{X_{24}}|$ belongs to the linear system $|9B + 4E|$, the divisor T is nef.

• The 1-cycles Γ for the singular points of type $\frac{1}{3}(1, 2, 1)$ are irreducible because of the monomials zw^2 and t^3.

<u>**No. 49**</u>: $X_{21} \subset \mathbb{P}(1, 3, 5, 6, 7)$	$A^3 = 1/30$
$w^3 + yt^3 + z^3(a_1t + a_2xz) + w^2f_7(x, y, z, t) + wf_{14}(x, y, z, t) + f_{21}(x, y, z, t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t = \frac{1}{6}(1_x, 5_z, 1_w)$ ⓑ	0	$3B$	y	w^3	
$O_z = \frac{1}{5}(1_x, 3_y, 2_w)$ ⓑ	0	$3B$	y	y	$a_1 \neq 0$
$O_z = \frac{1}{5}(3_y, 1_t, 2_w)$ ⓑ	0	$3B$	y	y	$a_1 = 0$
$O_yO_t = 3\times\frac{1}{3}(1_x, 2_z, 1_w)$ ⓝ	$-$	$5B + E$	x^2y, z	x^2y, z	

• For the singular points of types $\frac{1}{6}(1, 5, 1)$ and $\frac{1}{5}(1, 3, 2)$, the 1-cycle Γ is the proper transform of the curve defined by

$$x = y = w^3 + a_1z^3t = 0.$$

It is irreducible even though it can be non-reduced.

- For the singular points of type $\frac{1}{3}(1,2,1)$, consider the linear system on X_{21} generated by x^2y and z. Its base curves are defined by $x=z=0$ and $y=z=0$. The curve defined by $x=z=0$ is irreducible because of the monomials w^3 and yt^3. For its proper transform $\tilde{C}$, we have $T\cdot\tilde{C}=\frac{1}{6}$. Therefore, the divisor T is nef since the curve defined by $y=z=0$ does not pass through any singular point of type $\frac{1}{3}(1,2,1)$.

No. 50: $X_{22}\subset\mathbb{P}(1,1,3,7,11)$	$A^3=2/21$
$w^2+yt^3+z^5(a_1t+a_2xz^2+a_3yz^2)+wf_{11}(x,y,z,t)+f_{22}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t=\frac{1}{7}(1_x,3_z,4_w)$ ⓟ	$+$	$B-E$	y	w^2	
$O_z=\frac{1}{3}(1_x,1_y,2_w)$ ⓑ	$-$	B	y	y	$a_1\neq 0$
$O_z=\frac{1}{3}(1_y,1_t,2_w)$ ⓑ	$-$	B	y	y	$a_1=0$

- If $a_1=0$, then $a_2\neq 0$: otherwise the hypersurface X_{22} would be singular at the point defined by $x=y=w=0$ and $t^3+a_3z^7=0$.

- If $a_1\neq 0$, the 1-cycle Γ for the singular point O_z is irreducible because of the monomials w^2 and z^5t. If $a_1=0$, the 1-cycle Γ for the singular point O_z is still irreducible even though it is not reduced.

No. 51: $X_{22}\subset\mathbb{P}(1,1,4,6,11)$	$A^3=1/12$
$w^2+zt^3+z^4t+wf_{11}(x,y,z,t)+f_{22}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t=\frac{1}{6}(1_x,1_y,5_w)$ ⓟ	$+$	$4B-E$	z	w^2	
$O_z=\frac{1}{4}(1_x,1_y,3_w)$ ⓝ	0	B	x, y	x, y	
$O_zO_t=1\times\frac{1}{2}(1_x,1_y,1_w)$ ⓑ	$-$	B	y	y	

- For the singular point O_z, we can easily see that the surface T is nef since the base locus of the linear system $|-K_{X_{22}}|$ is the irreducible curve cut by $x=y=0$ and $B^3=0$.

- For the singular point of type $\frac{1}{2}(1,1,1)$, the intersection Γ is irreducible since we have the monomials w^2, z^4t, and zt^3.

No. 52: $X_{22}\subset\mathbb{P}(1,2,4,5,11)$	$A^3=1/20$
$w^2+yt^4+y\prod_{i=1}^{5}(z-\alpha_iy^2)+wf_{11}(x,y,z,t)+f_{22}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t=\frac{1}{5}(1_x,4_z,1_w)$ ⓑ	0	$4B$	z	z	
$O_z=\frac{1}{4}(1_x,1_t,3_w)$ ⓝ	$-$	$5B+E$	xz, t	xz, t	

$O_yO_z = 5\times\frac{1}{2}(1_x, 1_t, 1_w)$ ⓑ	$-$	$4B + E$	$z - \alpha_i y^2$	w^2	

• For the singular points of types $\frac{1}{5}(1, 4, 1)$ and $\frac{1}{2}(1, 1, 1)$, the 1-cycle Γ is always irreducible because the defining polynomial of X_{22} contains the monomials w^2 and yt^4.

• For the singular point O_z, consider the linear system on X_{22} generated by xz and t. Its base curves are defined by $x = t = 0$ and $z = t = 0$. The curve defined by $x = t = 0$ is irreducible because of the monomials w^2 and yz^5. Its proper transform intersects the divisor T positively. Since the curve defined by $z = t = 0$ does not pass through the singular point O_z, its proper transform also intersects T positively. Therefore, the divisor T is nef.

No. 53: $X_{24} \subset \mathbb{P}(1, 1, 3, 8, 12)$ $A^3 = 1/12$

$w^2 + t^3 + z^8 + wf_{12}(x, y, z, t) + f_{24}(x, y, z, t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_tO_w = 1\times\frac{1}{4}(1_x, 1_y, 3_z)$ ⓑ	0	B	y	y	
$O_zO_w = 2\times\frac{1}{3}(1_x, 1_y, 2_t)$ ⓑ	$-$	B	y	y	

• The 1-cycle Γ for each singular point is irreducible because of the monomials w^2 and t^3.

No. 54: $X_{24} \subset \mathbb{P}(1, 1, 6, 8, 9)$ $A^3 = 1/18$

$zw^2 + t^3 + z^4 + wf_{15}(x, y, z, t) + f_{24}(x, y, z, t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{9}(1, 1, 8)$ $\boxed{T}$	zw^2				
$O_zO_w = 1\times\frac{1}{3}(1_x, 1_y, 2_t)$ ⓑ	$-$	B	y	y	
$O_zO_t = 1\times\frac{1}{2}(1_x, 1_y, 1_w)$ ⓑ	$-$	B	y	y	

• The 1-cycles Γ for the singular points of types $\frac{1}{3}(1, 1, 2)$ and $\frac{1}{2}(1, 1, 1)$ are irreducible since we have the monomials z^4 and t^3.

No. 55: $X_{24} \subset \mathbb{P}(1, 2, 3, 7, 12)$ $A^3 = 1/21$

$(w - \alpha_1 y^6)(w - \alpha_2 y^6) + zt^3 + wf_{12}(x, y, z, t) + f_{24}(x, y, z, t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t = \frac{1}{7}(1_x, 2_y, 5_w)$ ⓟ	$+$	$3B - E$	z	w^2	
$O_zO_w = 2\times\frac{1}{3}(1_x, 2_y, 1_t)$ ⓑ	$-$	$2B$	y	y	
$O_yO_w = 2\times\frac{1}{2}(1_x, 1_z, 1_t)$ ⓝ	$-$	$7B + 3E$	xy^3, y^2z, t	xy^3, y^2z, t	

• The 1-cycle Γ for each singular point of type $\frac{1}{3}(1, 2, 1)$ is irreducible because of the monomials w^2 and zt^3.

• For each singular point of type $\frac{1}{2}(1,1,1)$, the divisor T is nef. Indeed, the base curve of the linear system on X_{24} generated by xy^3, y^2z and t is cut out by $y=t=0$. It does not passes through any singular point of type $\frac{1}{2}(1,1,1)$. Therefore, the surface T must be nef.

No. 56: $X_{24} \subset \mathbb{P}(1,2,3,8,11)$	$A^3 = 1/22$
$yw^2+t^3+z^8+y^{12}+wf_{13}(x,y,z,t)+f_{24}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{11}(1,3,8)$ $\boxed{\tau}$	yw^2				
$O_yO_t = 3\times\frac{1}{2}(1_x,1_z,1_w)$ ⓑ	$-$	$3B+E$	z	z	

• The 1-cycle Γ for each singular point of type $\frac{1}{2}(1,1,1)$ is irreducible due to the monomials yw^2 and t^3.

No. 57: $X_{24} \subset \mathbb{P}(1,3,4,5,12)$	$A^3 = 1/30$
$(w-\alpha_1y^4)(w-\alpha_2y^4)+zt^4+z^6+wf_{12}(x,y,z,t)+f_{24}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t = \frac{1}{5}(1_x,3_y,2_w)$ ⓑ	0	$3B$	y	y	
$O_zO_w = 2\times\frac{1}{4}(1_x,3_y,1_t)$ ⓑ	$-$	$3B$	y	y	
$O_yO_w = 2\times\frac{1}{3}(1_x,1_z,2_t)$ ⓝ	$-$	$4B+E$	z	z	

• The cycles Γ for the singular points of types $\frac{1}{5}(1,3,2)$ and $\frac{1}{4}(1,3,1)$ are irreducible because of the monomials w^2 and zt^4.

• For each singular point of type $\frac{1}{3}(1,1,2)$ we consider the linear system generated by x^{20}, z^5 and t^4 on the hypersurface X_{24}. Its base locus is cut out by $x=z=t=0$. Since the defining equation of X_{24} contains the monomial wy^4, its base locus does not contain any curves. Therefore the proper transform of a general member in the linear system is nef by Lemma 3.2.6. The proper transform belongs to $|20B+5E|$. Consequently, the surface T is nef since $5T \sim_{\mathbb{Q}} 20B+5E$.

No. 58: $X_{24} \subset \mathbb{P}(1,3,4,7,10)$	$A^3 = 1/35$
$zw^2+t^2(a_1w+a_2yt)+z^6+y^8+wf_{14}(x,y,z,t)+f_{24}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{10}(1,3,7)$ $\boxed{\tau}$	zw^2				
$O_t = \frac{1}{7}(1,3,4)$ $\boxed{\tau_1}$	t^2w				
$O_zO_w = 1\times\frac{1}{2}(1_x,1_y,1_t)$ ⓝ	$-$	$7B+3E$	t	t	

• For the singular point of type $\frac{1}{2}(1,1,1)$, we consider the linear system $|-7K_{X_{24}}|$. Every member of the linear system passes through the singular point of type $\frac{1}{2}(1,1,1)$ and its base locus contains no curves. Since the proper transform of a general member in $|-7K_{X_{24}}|$ belongs to the linear system $|7B+3E|$, the divisor T is nef.

No. 59: $X_{24}\subset\mathbb{P}(1,3,6,7,8)$	$A^3=1/42$
$w^3+yt^3+\prod_{i=1}^4(z-\alpha_i y^2)+w^2f_8(x,y,z,t)+wf_{16}(x,y,z,t)+f_{24}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t=\frac{1}{7}(1_x,6_z,1_w)$ ⓑ	0	$3B$	y	w^3	
$O_zO_w=1\times\frac{1}{2}(1_x,1_y,1_t)$ ⓑ	$-$	$3B+E$	y	y	
$O_yO_z=4\times\frac{1}{3}(1_x,1_t,2_w)$ ⓑ	$-$	$6B+E$	$z-\alpha_i y^2$	yt^3	

- The 1-cycle Γ for each singular point is irreducible due to the monomials w^3, z^4 and yt^3.

No. 60: $X_{24}\subset\mathbb{P}(1,4,5,6,9)$	$A^3=1/45$
$tw^2+(t^2-\alpha_1y^3)(t^2-\alpha_2y^3)+z^3(a_1w+a_2yz)+wf_{15}(x,y,z,t)+f_{24}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w=\frac{1}{9}(1,4,5)$ $\boxed{\tau}$	tw^2				
$O_z=\frac{1}{5}(1_x,4_y,1_t)$ ⓑ	$-$	$4B$	y	y	$a_1\neq 0$
$O_z=\frac{1}{5}(1_x,1_t,4_w)$ ⓑ	$-$	$4B$	y	t^4	$a_1=0$
$O_tO_w=1\times\frac{1}{3}(1_x,1_y,2_z)$ ⓑ	$-$	$5B+E$	z	z	
$O_yO_t=2\times\frac{1}{2}(1_x,1_z,1_w)$ ⓝ	$-$	$5B+2E$	xy, z	xy, z	

- The 1-cycle Γ for the singular point O_z with $a_1\neq 0$ is irreducible due to the monomials t^4 and z^3w.

- The 1-cycle Γ for the singular point O_z with $a_1=0$ has two irreducible components. One is $\tilde{L}_{zw}$ and the other is the proper transform $\tilde{C}$ of the curve defined by

$$x=y=w^2+t^3=0.$$

Then we see that

$$E\cdot\tilde{C}=3E\cdot\tilde{L}_{zw},\quad B\cdot\tilde{C}=3B\cdot\tilde{L}_{zw}.$$

Therefore these two components are numerically proportional on Y.

- The 1-cycle Γ for the singular point of type $\frac{1}{3}(1,1,2)$ is irreducible since we have terms tw^2 and $(t^2-\alpha_1y^3)(t^2-\alpha_2y^3)$. Note that the constants α_i's cannot be zero.

- For the singular points of type $\frac{1}{2}(1,1,1)$, we consider the linear system generated by xy and z on X_{24}. Its base curves are defined by $x=z=0$ and $y=z=0$. The curve defined by $y=z=0$ passes though no singular point of type $\frac{1}{2}(1,1,1)$. The curve defined by $x=z=0$ is irreducible. Moreover, its proper transform is the 1-cycle defined by $(5B+2E)\cdot B$. Consequently, the divisor T is nef since $(5B+2E)^2\cdot B>0$.

No. 61: $X_{25} \subset \mathbb{P}(1,4,5,7,9)$	$A^3 = 5/252$
$tw^2 - yt^3 + z^5 + y^4(a_1w + a_2yz + a_3xy^2) + wf_{16}(x,y,z,t) + f_{25}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{9}(1,4,5)$ $\boxed{\tau}$	tw^2				
$O_t = \frac{1}{7}(1,5,2)$ $\boxed{\epsilon}$	$tw^2 - yt^3$				
$O_y = \frac{1}{4}(1_x,1_z,3_t)$ ⓑ	$-$	$9B+E$	w	z^5	$a_1 \neq 0$
$O_y = \frac{1}{4}(1_x,3_t,1_w)$ ⓢ	$-$	$5B$	x^5, z	x^5, tw^2	$a_1 = 0$, $a_2 \neq 0$
$O_y = \frac{1}{4}(1_z,3_t,1_w)$ ⓑ	$-$	$7B+E$	t	t	$a_1 = a_2 = 0$

- If $a_1 \neq 0$, the 1-cycle Γ for the singular point O_y is irreducible due to the monomials yt^3 and z^5.

- If $a_1 = a_2 = 0$, the 1-cycle Γ for the singular point O_y is irreducible even though it is not reduced.

- Now we suppose that $a_1 = 0$ and $a_2 \neq 0$. Then we may assume that $a_2 = 1$ and $a_3 = 0$. We take a surface H cut by an equation $z = \lambda x^5$ with a general complex number λ and then let T be the proper transform of the surface. The surface H is normal but it is not quasi-smooth at the point O_y.

The intersection of T with the surface S gives us a divisor consisting of two irreducible curves on the normal surface T. One is the proper transform of the curve L_{yw}. The other is the proper transform of the curve C defined by

$$x = z = w^2 - yt^2 = 0.$$

From the intersection numbers

$$(\tilde{L}_{yw} + \tilde{C}) \cdot \tilde{L}_{yw} = -K_Y \cdot \tilde{L}_{yw} = -\frac{2}{9}, \quad (\tilde{L}_{yw} + \tilde{C})^2 = 5B^3 = -\frac{20}{63}$$

on the surface T, we obtain

$$\tilde{L}_{yw}^2 = -\frac{2}{9} - \tilde{L}_{yw} \cdot \tilde{C}, \quad \tilde{C}^2 = -\frac{2}{21} - \tilde{L}_{yw} \cdot \tilde{C}.$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{yw}^2 & \tilde{L}_{yw} \cdot \tilde{C} \\ \tilde{L}_{yw} \cdot \tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{2}{9} - \tilde{L}_{yw} \cdot \tilde{C} & \tilde{L}_{yw} \cdot \tilde{C} \\ \tilde{L}_{yw} \cdot \tilde{C} & -\frac{2}{21} - \tilde{L}_{yw} \cdot \tilde{C} \end{pmatrix}$$

is negative-definite since $\tilde{L}_{zw} \cdot \tilde{C}$ is positive.

No. 62: $X_{26} \subset \mathbb{P}(1,1,5,7,13)$	$A^3 = 2/35$
$w^2 + zt^3 + yz^5 + wf_{13}(x,y,z,t) + f_{26}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t = \frac{1}{7}(1_x,1_y,6_w)$ ⓟ	$+$	$5B-E$	z	w^2	
$O_z = \frac{1}{5}(1_x,2_t,3_w)$ ⓕ	$+$	$B-E$	y	w^2	

- For the singular point O_z, let C_λ be the curve on the surface S_y defined by

$$\begin{cases} y = 0, \\ t = \lambda x^7 \end{cases}$$

for a sufficiently general complex number λ. Then

$$-K_Y \cdot \tilde{C}_\lambda = (B - E)(7B + E)B = 0.$$

If the curve $\tilde{C}_\lambda$ is reducible, it consists of two irreducible components that are numerically equivalent to each other since the two components of the curve C_λ are symmetric with respect to the biregular quadratic involution of X_{26}. Then each component of $\tilde{C}_\lambda$ intersects $-K_Y$ trivially.

No. 63: $X_{26} \subset \mathbb{P}(1, 2, 3, 8, 13)$ $A^3 = 1/24$

$w^2+y(t-\alpha_1 y^4)(t-\alpha_2 y^4)(t-\alpha_3 y^4)+z^6(a_1 t+a_2 yz^2)+wf_{13}(x, y, z, t)+f_{26}(x, y, z, t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t = \frac{1}{8}(1_x, 3_z, 5_w)$ ⓟ	$+$	$2B - E$	y	w^2	
$O_z = \frac{1}{3}(1_x, 2_y, 1_w)$ ⓑ	$-$	$2B$	y	y	$a_1 \neq 0$
$O_z = \frac{1}{3}(1_x, 2_t, 1_w)$ ⓑ	$-$	$2B$	y	w^2	$a_1 = 0$
$O_y O_t = 3 \times \frac{1}{2}(1_x, 1_z, 1_w)$ ⓑ	$-$	$3B + E$	z	z	

- For each of the singular points of types $\frac{1}{3}(1, 2, 1)$ and $\frac{1}{2}(1, 1, 1)$, the 1-cycle Γ is always irreducible because of the monomials w^2, yt^3 and $z^6 t$ even though it is possibly non-reduced.

No. 64: $X_{26} \subset \mathbb{P}(1, 2, 5, 6, 13)$ $A^3 = 1/30$

$w^2 + y \prod_{i=1}^{4}(t - \alpha_i y^3) + z^4(a_1 t + a_2 x) + wf_{13}(x, y, z, t) + f_{26}(x, y, z, t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t = \frac{1}{6}(1_x, 5_z, 1_w)$ ⓑ	0	$2B$	y	w^2	
$O_z = \frac{1}{5}(1_x, 2_y, 3_w)$ ⓑ	0	$2B$	y	y	$a_1 \neq 0$
$O_z = \frac{1}{5}(2_y, 1_t, 3_w)$ ⓑ	0	$2B$	y	y	$a_1 = 0$
$O_y O_t = 4 \times \frac{1}{2}(1_x, 1_z, 1_w)$ ⓑ	$-$	$6B + 2E$	$t - \alpha_i y^3$	w^2	

- For each of the singular points the 1-cycle Γ is irreducible due to the monomials w^2 and $z^4 t$. In particular, if $a_1 = 0$, then the 1-cycle Γ is $2L_{zt}$.

No. 65: $X_{27} \subset \mathbb{P}(1, 2, 5, 9, 11)$ $A^3 = 3/110$

$zw^2 + t^3 + yz^5 + y^8(a_1 w + a_2 yt + a_3 y^3 z + a_4 xy^5) + wf_{16}(x, y, z, t) + f_{27}(x, y, z, t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition

$O_w = \frac{1}{11}(1,2,9)$ $\boxed{\tau}$	zw^2				
$O_z = \frac{1}{5}(1_x,4_t,1_w)$ ⓑ	$-$	$2B$	y	zw^2	
$O_y = \frac{1}{2}(1,1,1)$ ⓝ	$-$	$11B+5E$	$w+xy^5$	xy^5 or w	

• The 1-cycle Γ for the singular point O_z is irreducible due to the monomials zw^2 and t^3

• For the singular point O_y, we consider the linear system $|-11K_{X_{27}}|$. Note that every member of the linear system passes through the point O_y and the base locus of the linear system contains no curves. Since the proper transform of a general member in $|-11K_{X_{27}}|$ belongs to the linear system $|11B+5E|$, the divisor T is nef.

No. 66: $X_{27}\subset\mathbb{P}(1,5,6,7,9)$	$A^3=1/70$
$w^3+zt^3+z^3w+y^4t+ay^3z^2+w^2f_9(x,y,z,t)+wf_{18}(x,y,z,t)+f_{27}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t = \frac{1}{7}(1_x,5_y,2_w)$ ⓑ	0	$5B$	y	y	
$O_z = \frac{1}{6}(1_x,5_y,1_t)$ ⓑ	$-$	$5B$	y	y	
$O_y = \frac{1}{5}(1_x,1_z,4_w)$ ⓝ	$-$	$7B+E$	t	y^3z^2	$a\neq 0$
$O_y = \frac{1}{5}(1_x,1_z,4_w)$ ⓢ	$-$	$7B$	x^7, t	x^7, wz^3	$a=0$
$O_zO_w = 1\times\frac{1}{3}(1_x,2_y,1_t)$ ⓑ	$-$	$5B+E$	y	y	

We may assume that the polynomial f_{27} contains neither xy^4z nor x^2y^5 by changing the coordinate t in an appropriate way.

• For the singular points except the point O_y, the 1-cycles Γ are always irreducible because of the monomials w^3, zt^3 and z^3w.

• For the singular point O_y with $a\neq 0$, we consider the linear system generated by x^2y, xz and t. Its base curve C is cut out by $x=t=0$. It is irreducible because of the monomials w^3 and y^3z^2. Since we have $T\cdot\tilde{C}=(7B+E)^2\cdot B=\frac{1}{2}$, the divisor T is nef.

• For the singular point O_y with $a=0$, we take a general member H in the linear system generated by x^7 and t. Then it is a normal surface of degree 27 in $\mathbb{P}(1,5,6,9)$. Let T be the proper transform of the surface H. The intersection of T with the surface S gives us a divisor consisting of two irreducible curves $\tilde{L}_{yz}$ and $\tilde{C}$ on the normal surface T. The curve $\tilde{C}$ is the proper transform of the curve C defined by

$$x=t=w^2+z^3=0.$$

From the intersection numbers

$$(\tilde{L}_{yz}+\tilde{C})\cdot\tilde{L}_{yz}=-K_Y\cdot\tilde{L}_{yz}=-\frac{1}{6},\quad (\tilde{L}_{yz}+\tilde{C})^2=7B^3=-\frac{1}{4}$$

on the surface T, we obtain

$$\tilde{L}_{yz}^2=-\frac{1}{6}-\tilde{L}_{yz}\cdot\tilde{C},\quad \tilde{C}^2=-\frac{1}{12}-\tilde{L}_{yz}\cdot\tilde{C}.$$

With these intersection numbers we see that the matrix

$$\begin{pmatrix} \tilde{L}_{yz}^2 & \tilde{L}_{yz}\cdot\tilde{C} \\ \tilde{L}_{yz}\cdot\tilde{C} & \tilde{C}^2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{6}-\tilde{L}_{yz}\cdot\tilde{C} & \tilde{L}_{yz}\cdot\tilde{C} \\ \tilde{L}_{yz}\cdot\tilde{C} & -\frac{1}{12}-\tilde{L}_{yz}\cdot\tilde{C} \end{pmatrix}$$

is negative-definite since $\tilde{L}_{yz}\cdot\tilde{C}$ is non-negative.

No. 67: $X_{28}\subset\mathbb{P}(1,1,4,9,14)$ $A^3=1/18$

$w^2+yt^3+z^7+wf_{14}(x,y,z,t)+f_{28}(x,y,z,t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t=\frac{1}{9}(1_x,4_z,5_w)$ ⓟ	$+$	$B-E$	y	w^2	
$O_zO_w=1\times\frac{1}{2}(1_x,1_y,1_t)$ ⓑ	$-$	B	y	y	

• The 1-cycle Γ for the singular point of type $\frac{1}{2}(1,1,1)$ is irreducible since we have the monomials w^2 and z^7.

No. 68: $X_{28}\subset\mathbb{P}(1,3,4,7,14)$ $A^3=1/42$

$(w-\alpha_1t^2)(w-\alpha_2t^2)+z^7+y^7(a_1t+a_2xy^2)+wf_{14}(x,y,z,t)+f_{28}(x,y,z,t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_y=\frac{1}{3}(1_x,1_z,2_w)$ ⓑ	$-$	$7B+E$	t	w^2	$a_1\neq 0$
$O_y=\frac{1}{3}(1_z,1_t,2_w)$ ⓑ	$-$	$4B+E$	z	z	$a_1=0$
$O_tO_w=2\times\frac{1}{7}(1,3,4)$ $\boxed{\tau}$	wt^2				
$O_zO_w=1\times\frac{1}{2}(1_x,1_y,1_t)$ ⓑ	$-$	$3B+E$	y	y	

• For the singular point O_y with $a_1\neq 0$ the 1-cycle Γ is irreducible because of the monomials w^2 and z^7.

• The 1-cycle Γ for the singular point O_y with $a_1=0$ consists of two irreducible curves. These are the proper transforms of the curves defined by $x=z=w-\alpha_it^2=0$. Since these two curves on X_{28} are interchanged by the automorphism defined by

$$[x,y,z,t,w]\mapsto[x,y,z,t,(\alpha_1+\alpha_2)t^2-f_{14}-w],$$

their proper transforms are numerically equivalent on Y.

• To see how to deal with the singular points of type $\frac{1}{7}(1,3,4)$ we may assume that $\alpha_1=0$ and we have only to consider the singular point O_t. The other point can be treated in the same way.

• The 1-cycle Γ for the singular point of type $\frac{1}{2}(1,1,1)$ is irreducible due to the monomials w^2 and z^7.

No. 69: $X_{28}\subset\mathbb{P}(1,4,6,7,11)$ $A^3=1/66$

$zw^2+t^4+y(z^2-\alpha_1y^3)(z^2-\alpha_2y^3)+wf_{17}(x,y,z,t)+f_{28}(x,y,z,t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{11}(1,4,7)$ $\boxed{\tau}$	zw^2				
$O_z = \frac{1}{6}(1_x, 1_t, 5_w)$ ⓑ	$-$	$4B$	y	t^4	
$O_yO_z = 2\times\frac{1}{2}(1_x, 1_t, 1_w)$ ⓝ	$-$	$11B+5E$	$xyz,\ yt,\ w$	$xyz,\ yt,\ w$	

- The 1-cycle Γ for the singular point O_z is irreducible due to the monomials zw^2 and t^4.
- For the singular points of type $\frac{1}{2}(1,1,1)$ consider the linear system generated by xyz, yt and w. Since the base curves of the linear system pass through no singular points of type $\frac{1}{2}(1,1,1)$ the divisor T is nef.

No. 70: $X_{30} \subset \mathbb{P}(1,1,4,10,15)$ $\quad A^3 = 1/20$
$w^2 + t^3 + z^5t + wf_{15}(x,y,z,t) + f_{30}(x,y,z,t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_z = \frac{1}{4}(1_x, 1_y, 3_w)$ ⓑ	$-$	B	y	y	
$O_tO_w = 1\times\frac{1}{5}(1_x, 1_y, 4_z)$ ⓑ	0	B	y	y	
$O_zO_t = 1\times\frac{1}{2}(1_x, 1_y, 1_w)$ ⓑ	$-$	B	y	y	

- For each singular point the 1-cycle Γ is irreducible due to the monomials w^2 and t^3.

No. 71: $X_{30} \subset \mathbb{P}(1,1,6,8,15)$ $\quad A^3 = 1/24$
$w^2 + zt^3 + wf_{15}(x,y,z,t) + f_{30}(x,y,z,t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t = \frac{1}{8}(1_x, 1_y, 7_w)$ ⓟ	$+$	$6B-E$	z	w^2	
$O_zO_w = 1\times\frac{1}{3}(1_x, 1_y, 2_t)$ ⓑ	$-$	B	y	y	
$O_zO_t = 1\times\frac{1}{2}(1_x, 1_y, 1_w)$ ⓑ	$-$	B	y	y	

- For the singular points of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,1,2)$, the 1-cycles Γ are irreducible because of w^2 and t^3z.

No. 72: $X_{30} \subset \mathbb{P}(1,2,3,10,15)$ $\quad A^3 = 1/30$
$w^2 + t^3 + z^{10} + y^{15} + wf_{15}(x,y,z,t) + f_{30}(x,y,z,t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_tO_w = 1\times\frac{1}{5}(1_x, 2_y, 3_z)$ ⓑ	0	$2B$	y	y	
$O_zO_w = 2\times\frac{1}{3}(1_x, 2_y, 1_t)$ ⓑ	$-$	$2B$	y	y	
$O_yO_t = 3\times\frac{1}{2}(1_x, 1_z, 1_w)$ ⓑ	$-$	$3B+E$	z	z	

• For each singular point the 1-cycle Γ is irreducible due to the monomials w^2 and t^3.

No. 73: $X_{30} \subset \mathbb{P}(1,2,6,7,15)$ $A^3 = 1/42$

$w^2 + yt^4 + \prod_{i=1}^{5}(z - \alpha_i y^3) + wf_{15}(x,y,z,t) + f_{30}(x,y,z,t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t = \frac{1}{7}(1_x, 6_z, 1_w)$ ⓑ	0	$2B$	y	w^2	
$O_zO_w = 1\times\frac{1}{3}(1_x, 2_y, 1_t)$ ⓑ	–	$2B$	y	y	
$O_yO_z = 5\times\frac{1}{2}(1_x, 1_t, 1_w)$ ⓑ	–	$6B + 2E$	$z - \alpha_i y^3$	w^2	

• For each singular point the 1-cycle Γ is irreducible due to the monomials w^2, z^5 and yt^4.

No. 74: $X_{30} \subset \mathbb{P}(1,3,4,10,13)$ $A^3 = 1/52$

$zw^2 + t^3 + z^5t + y^{10} + wf_{17}(x,y,z,t) + f_{30}(x,y,z,t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{13}(1,3,7)$ $\boxed{T}$	zw^2				
$O_z = \frac{1}{4}(1_x, 3_y, 1_w)$ ⓑ	–	$3B$	y	y	
$O_zO_t = 1\times\frac{1}{2}(1_x, 1_y, 1_w)$ ⓑ	–	$3B + E$	y	y	

• The 1-cycles Γ for the singular points of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{4}(1,3,1)$ are irreducible because of the monomials zw^2, t^3 and z^5t.

No. 75: $X_{30} \subset \mathbb{P}(1,4,5,6,15)$ $A^3 = 1/60$

$w^2 + t^5 + z^6 + y^6t + wf_{15}(x,y,z,t) + f_{30}(x,y,z,t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_y = \frac{1}{4}(1_x, 1_z, 3_w)$ ⓝ	–	$5B + E$	xy, z	xy, z	
$O_tO_w = 1\times\frac{1}{3}(1_x, 1_y, 2_z)$ ⓑ	–	$5B + E$	z	z	
$O_zO_w = 2\times\frac{1}{5}(1_x, 4_y, 1_t)$ ⓑ	–	$4B$	y	y	
$O_yO_t = 2\times\frac{1}{2}(1_x, 1_z, 1_w)$ ⓑ	–	$5B + 2E$	z	z	

• For the singular point O_y, consider the linear system generated by xy and z. Its base curves are defined by $x = z = 0$ and $y = z = 0$. The curve defined by $y = z = 0$ does not pass through the point O_y. The curve defined by $x = z = 0$ is irreducible. Moreover, its proper transform is the 1-cycle defined by $(5B + E) \cdot B$. Consequently, the divisor T is nef since $(5B + E)^2 \cdot B > 0$.

• For the other singular points we immediately see that the 1-cycles Γ are irreducible due to the monomials w^2 and t^5.

No. 76: $X_{30} \subset \mathbb{P}(1,5,6,8,11)$	$A^3 = 1/88$
$tw^2 + zt^3 + z^5 + y^5 + wf_{19}(x,y,z,t) + f_{30}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{11}(1,5,6)$ $\boxed{\tau}$	tw^2				
$O_t = \frac{1}{8}(1,5,3)$ $\boxed{\epsilon}$	$tw^2 - zt^3$				
$O_zO_t = 1\times\frac{1}{2}(1_x,1_y,1_w)$ ⓑ	$-$	$5B+2E$	y	y	

• The 1-cycle Γ for the singular point of type $\frac{1}{2}(1,1,1)$ is irreducible because of the monomials tw^2 and z^5.

No. 77: $X_{32} \subset \mathbb{P}(1,2,5,9,16)$	$A^3 = 1/45$
$w^2 + zt^3 + yz^6 + wf_{16}(x,y,z,t) + f_{32}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t = \frac{1}{9}(1_x,2_y,7_w)$ ⓟ	$+$	$5B-E$	z	w^2	
$O_z = \frac{1}{5}(1_x,4_t,1_w)$ ⓑ	$-$	$2B$	y	w^2	
$O_yO_w = 2\times\frac{1}{2}(1_x,1_z,1_t)$ ⓝ	$-$	$9B+4E$	xy^4, y^2z, t	xy^4, y^2z, t	

• For the singular point O_z, the 1-cycle Γ is irreducible due to the monomials w^2 and zt^3.

• For the singular points of type $\frac{1}{2}(1,1,1)$, we consider the linear system generated by xy^4, y^2z and t on X_{32}. Its base curve is defined by $y = t = 0$. The curve defined by $y = t = 0$ passes though no singular point of type $\frac{1}{2}(1,1,1)$. Consequently, the divisor T is nef.

No. 78: $X_{32} \subset \mathbb{P}(1,4,5,7,16)$	$A^3 = 1/70$
$w^2 + yt^4 + z^5t + wf_{16}(x,y,z,t) + f_{32}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t = \frac{1}{7}(1_x,5_z,2_w)$ ⓑ	0	$4B$	y	w^2	
$O_z = \frac{1}{5}(1_x,4_y,1_w)$ ⓑ	$-$	$4B$	y	y	
$O_yO_w = 2\times\frac{1}{4}(1_x,1_z,3_t)$ ⓝ	$-$	$5B+E$	xy, z	xy, z	

• For the singular points other than those of type $\frac{1}{4}(1,1,3)$, the 1-cycles Γ are always irreducible due to the monomials w^2 and z^5t.

• For the singular points of type $\frac{1}{4}(1,1,3)$, we consider the linear system generated by xy and z on X_{32}. Its base curves are defined by $x = z = 0$ and $y = z = 0$. The curve defined by $y = z = 0$ passes though no singular point of type $\frac{1}{4}(1,1,3)$. The curve defined by $x = z = 0$ is irreducible because of the monomials w^2 and yt^4. Its proper transform is the 1-cycle defined by $(5B+E)\cdot B$. Therefore, the divisor T is nef since $(5B+E)^2 \cdot B > 0$.

No. 79: $X_{33} \subset \mathbb{P}(1,3,5,11,14)$ $A^3 = 1/70$					
$zw^2 + t^3 + yz^6 + y^{11} + wf_{19}(x,y,z,t) + f_{33}(x,y,z,t)$					
Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_w = \frac{1}{14}(1,3,11)$ $\boxed{\tau}$	zw^2				
$O_z = \frac{1}{5}(1_x,1_t,4_w)$ ⓑ	$-$	$3B$	y	t^3	

• The 1-cycle Γ for the singular point O_z is irreducible because of the monomials zw^2 and t^3.

No. 80: $X_{34} \subset \mathbb{P}(1,3,4,10,17)$ $A^3 = 1/60$					
$w^2 + zt^3 + z^6t + y^8(a_1t + a_2y^2z + a_3xy^4) + wf_{17}(x,y,z,t) + t^2g_{14}(x,y,z) + tg_{24}(x,y,z) + g_{34}(x,y,z)$					
Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t = \frac{1}{10}(1_x,3_y,7_w)$ ⓟ	$+$	$4B-E$	z	w^2	
$O_z = \frac{1}{4}(1_x,3_y,1_w)$ ⓑ	$-$	$3B$	y	y	
$O_y = \frac{1}{3}(1_x,1_z,2_w)$ ⓝ	$-$	$4B+E$	z	z	$a_1 \neq 0$
$O_y = \frac{1}{3}(1_x,1_t,2_w)$ ⓑ	$-$	$4B$	z	w^2	$a_1 = 0$, $a_2 \neq 0$
$O_y = \frac{1}{3}(1_z,1_t,2_w)$ ⓑ	$-$	$4B+E$	z	z	$a_1 = a_2 = 0$
$O_zO_t = 1 \times \frac{1}{2}(1_x,1_y,1_w)$ ⓑ	$-$	$3B+E$	y	y	

• For each of the singular points to which the method ⓑ is applied, the 1-cycle Γ is always irreducible even though it is possibly non-reduced.

• For the singular point O_y with $a_1 \neq 0$, we consider the linear system generated by xy and z on X_{34}. Its base curves are defined by $x = z = 0$ and $y = z = 0$. The curve defined by $y = z = 0$ does not pass through the point O_y. The curve defined by $x = z = 0$ is irreducible because of the monomials w^2 and y^8t. Its proper transform is the 1-cycle defined by $(4B+E) \cdot B$, and hence it intersects T positively. Consequently, the divisor T is nef.

No. 81: $X_{34} \subset \mathbb{P}(1,4,6,7,17)$ $A^3 = 1/84$					
$w^2 + zt^4 + yz^5 + y^7z + wf_{17}(x,y,z,t) + f_{34}(x,y,z,t)$					
Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t = \frac{1}{7}(1_x,4_y,3_w)$ ⓑ	0	$6B$	z	w^2	
$O_z = \frac{1}{6}(1_x,1_t,5_w)$ ⓑ	$-$	$4B$	y	zt^4	
$O_y = \frac{1}{4}(1_x,3_t,1_w)$ ⓑ	$-$	$7B+E$	t	t	
$O_yO_z = 2 \times \frac{1}{2}(1_x,1_t,1_w)$ ⓝ	$-$	$7B+3E$	xz, t	xz, t	

• The 1-cycle Γ for the singular point O_t is irreducible due to the monomials w^2 and y^5t^2 even though it can be non-reduced.

• For the singular point O_z, the 1-cycle Γ is irreducible due to the monomials w^2 and zt^4.

• For the singular point O_y, the 1-cycle Γ is irreducible due to the monomials w^2, yz^5 and y^7z

• For the singular points of type $\frac{1}{2}(1,1,1)$, we consider the linear system generated by xz and t on X_{34}. Its base curves are defined by $x=t=0$ and $z=t=0$. The curve defined by $z=t=0$ passes though no singular point of type $\frac{1}{2}(1,1,1)$. The curve defined by $x=t=0$ is irreducible due to the monomials w^2, yz^5 and y^7z. Its proper transform is equivalent to the 1-cycle defined by $(7B+3E)\cdot B$ that intersects T positively. Therefore, the divisor T is nef.

No. 82: $X_{36}\subset\mathbb{P}(1,1,5,12,18)$	$A^3=1/30$
$w^2+t^3+yz^7+wf_{18}(x,y,z,t)+f_{36}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_z=\frac{1}{5}(1_x,2_t,3_w)$ ⓕ	0	$B-E$	y	w^2	
$O_tO_w=1\times\frac{1}{6}(1_x,1_y,5_z)$ ⓑ	0	B	y	y	

• For the singular point O_z, let C_λ be the curve on the surface S_y cut by $t=\lambda x^{12}$ for a general complex number λ. Then

$$-K_Y\cdot\tilde{C}_\lambda=(B-E)(12B+2E)B=0.$$

If the curve $\tilde{C}_\lambda$ is reducible, it consists of two irreducible components. Because these two components are symmetric with respect to the biregular quadratic involution of X_{36}, they must be numerically equivalent to each other. Therefore, each component of $\tilde{C}_\lambda$ intersects $-K_Y$ trivially.

• For the singular point of type $\frac{1}{6}(1,1,5)$, the 1-cycle Γ is irreducible due to w^2 and t^3.

No. 83: $X_{36}\subset\mathbb{P}(1,3,4,11,18)$	$A^3=1/66$
$(w-\alpha_1y^6)(w-\alpha_2y^6)+yt^3+z^9+wf_{18}(x,y,z,t)+f_{36}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t=\frac{1}{11}(1_x,4_z,7_w)$ ⓟ	$+$	$3B-E$	y	w^2	
$O_zO_w=1\times\frac{1}{2}(1_x,1_y,1_t)$ ⓑ	$-$	$3B+E$	y	y	
$O_yO_w=2\times\frac{1}{3}(1_x,1_z,2_t)$ ⓑ	$-$	$18B+4E$	$w-\alpha_iy^6$	yt^3	

• For each of the singular points corresponding to the method ⓑ, the 1-cycle Γ is irreducible since we have the monomials w^2, yt^3, and z^9.

No. 84: $X_{36}\subset\mathbb{P}(1,7,8,9,12)$	$A^3=1/168$
$w^3+t^4+z^3w+y^4(a_1z+a_2xy)+w^2f_{12}(x,y,z,t)+wf_{24}(x,y,z,t)+f_{36}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_z=\frac{1}{8}(1_x,7_y,1_t)$ ⓑ	$-$	$7B$	y	y	
$O_y=\frac{1}{7}(1_x,2_t,5_w)$ ⓑ	$-$	$8B$	z	t^4	$a_1\neq 0$
$O_y=\frac{1}{7}(1_z,2_t,5_w)$ ⓑ	$-$	$12B+E$	w	w	$a_1=0$
$O_tO_w=1\times\frac{1}{3}(1_x,1_y,2_z)$ ⓑ	$-$	$8B+2E$	z	z	
$O_zO_w=1\times\frac{1}{4}(1_x,3_y,1_t)$ ⓑ	$-$	$7B+E$	y	y	

• For each singular point the 1-cycle Γ is always irreducible because of the monomials w^3 and t^4. In particular, the intersection Γ for the singular point O_y with $a_1=0$ is irreducible even though it is non-reduced.

No. 85: $X_{38}\subset\mathbb{P}(1,3,5,11,19)$	$A^3=2/165$
$w^2+zt^3+yz^7+y^9(a_1t+a_2y^2z)+wf_{19}(x,y,z,t)+f_{38}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t=\frac{1}{11}(1_x,3_y,8_w)$ ⓟ	$+$	$5B-E$	z	w^2	
$O_z=\frac{1}{5}(1_x,1_t,4_w)$ ⓑ	$-$	$3B$	y	zt^3	
$O_y=\frac{1}{3}(1_x,2_z,1_w)$ ⓑ	$-$	$5B+E$	z	z	$a_1\neq 0$
$O_y=\frac{1}{3}(1_x,2_t,1_w)$ ⓑ	$-$	$5B+E$	z	w^2	$a_1=0$

• For the singular point O_z, the 1-cycle Γ is irreducible because of the monomials w^2 and zt^3.

• For the singular point O_y, the 1-cycle Γ is irreducible because of the monomials w^2 and y^9t. Note that in case when $a_1=0$ the 1-cycle Γ is still irreducible but non-reduced.

No. 86: $X_{38}\subset\mathbb{P}(1,5,6,8,19)$	$A^3=1/120$
$w^2+zt^4+z^5t+y^6t+wf_{19}(x,y,z,t)+f_{38}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t=\frac{1}{8}(1_x,5_y,3_w)$ ⓑ	0	$5B$	y	y	
$O_z=\frac{1}{6}(1_x,5_y,1_w)$ ⓑ	$-$	$5B$	y	y	
$O_y=\frac{1}{5}(1_x,1_z,4_w)$ ⓝ	$-$	$6B+E$	xy, z	xy, z	
$O_zO_t=1\times\frac{1}{2}(1_x,1_y,1_w)$ ⓑ	$-$	$5B+2E$	y	y	

• For the singular points except the point O_y, the 1-cycles Γ are always irreducible because of the monomials w^2, zt^4 and z^5t.

• For the singularity O_y, we consider the linear system generated by xy and z on X_{38}. Its base curves are defined by $x=z=0$ and $y=z=0$. The curve defined by $y=z=0$ does not pass though the singular point O_y. The curve defined by $x=z=0$ is irreducible because of the monomials w^2 and y^6t. The proper transform is equivalent to the 1-cycle defined by

$(6B+E)\cdot B$ and $(6B+E)^2\cdot B>0$. Therefore, the divisor T is nef.

No. 87: $X_{40}\subset\mathbb{P}(1,5,7,8,20)$	$A^3=1/140$
$(w-\alpha_1y^4)(w-\alpha_2y^4)+t^5+yz^5+wf_{20}(x,y,z,t)+f_{40}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_z=\frac{1}{7}(1_x,1_t,6_w)$ ⓑ	$-$	$5B$	y	t^5	
$O_tO_w=1\times\frac{1}{4}(1_x,1_y,3_z)$ ⓑ	$-$	$7B+E$	z	z	
$O_yO_w=2\times\frac{1}{5}(1_x,2_z,3_t)$ ⓝ	$-$	$7B+E$	$x^2y,\ z$	$x^2y,\ z$	

- The irreducibility of the 1-cycle Γ can be immediately checked for each singular point corresponding to the method ⓑ since we have the monomials w^2 and t^5.

- For the singular points of type $\frac{1}{5}(1,2,3)$, we consider the linear system generated by x^2y and z on X_{40}. Its base curves are defined by $x=z=0$ and $y=z=0$. The curve defined by $y=z=0$ passes though no singular points of type $\frac{1}{5}(1,2,3)$. The curve defined by $x=z=0$ is irreducible because of the monomials w^2 and t^5. Its proper transform is equivalent to the 1-cycle defined by $(7B+E)\cdot B$. Consequently, the divisor T is nef since $(7B+E)^2\cdot B>0$.

No. 88: $X_{42}\subset\mathbb{P}(1,1,6,14,21)$	$A^3=1/42$
$w^2+t^3+z^7+wf_{21}(x,y,z,t)+f_{42}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_tO_w=1\times\frac{1}{7}(1_x,1_y,6_z)$ ⓑ	0	B	y	y	
$O_zO_w=1\times\frac{1}{3}(1_x,1_y,2_t)$ ⓑ	$-$	B	y	y	
$O_zO_t=1\times\frac{1}{2}(1_x,1_y,1_w)$ ⓑ	$-$	B	y	y	

- For each singular point the 1-cycle Γ is irreducible due to the monomials w^2, t^3, and z^7.

No. 89: $X_{42}\subset\mathbb{P}(1,2,5,14,21)$	$A^3=1/70$
$w^2+t^3+yz^8+y^{21}+wf_{21}(x,y,z,t)+f_{42}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_z=\frac{1}{5}(1_x,4_t,1_w)$ ⓑ	$-$	$2B$	y	w^2	
$O_tO_w=1\times\frac{1}{7}(1_x,2_y,5_z)$ ⓑ	0	$2B$	y	y	
$O_yO_t=3\times\frac{1}{2}(1_x,1_z,1_w)$ ⓑ	$-$	$5B+2E$	z	z	

- For each singular point the 1-cycle Γ is irreducible because of the monomials w^2 and t^3.

No. 90: $X_{42}\subset\mathbb{P}(1,3,4,14,21)$	$A^3=1/84$
$(w-\alpha_1y^7)(w-\alpha_2y^7)+t^3+z^7t+wf_{21}(x,y,z,t)+f_{42}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_z = \frac{1}{4}(1_x, 3_y, 1_w)$ ⓑ	$-$	$3B$	y	y	
$O_tO_w = 1\times\frac{1}{7}(1_x, 3_y, 4_z)$ ⓑ	0	$3B$	y	y	
$O_yO_w = 2\times\frac{1}{3}(1_x, 1_z, 2_t)$ ⓝ	$-$	$4B+E$	xy, z	xy, z	
$O_zO_t = 1\times\frac{1}{2}(1_x, 1_y, 1_w)$ ⓑ	$-$	$3B+E$	y	y	

- For the singular points other than those of type $\frac{1}{3}(1,1,2)$, the 1-cycle Γ is irreducible since we have monomials w^2, t^3, and z^7t.

- For the singular points of type $\frac{1}{3}(1,1,2)$, consider the linear system generated by xy and z on X_{42}. Its base curves are defined by $x = z = 0$ and $y = z = 0$. The curve defined by $y = z = 0$ passes through no singular points of type $\frac{1}{3}(1,1,2)$. The curve defined by $x = z = 0$ is irreducible because of the monomials w^2 and t^3. Its proper transform is equivalent to the 1-cycle defined by $(4B+E)\cdot B$ and $(4B+E)^2\cdot B > 0$. Therefore, the divisor T is nef.

No. 91: $X_{44} \subset \mathbb{P}(1,4,5,13,22)$	$A^3 = 1/130$
$w^2 + zt^3 + yz^8 + y^{11} + wf_{22}(x,y,z,t) + f_{44}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_t = \frac{1}{13}(1_x, 4_y, 9_w)$ ⓟ	$+$	$5B-E$	z	w^2	
$O_z = \frac{1}{5}(1_x, 3_t, 2_w)$ ⓑ	$-$	$4B$	y	w^2	
$O_yO_w = 1\times\frac{1}{2}(1_x, 1_z, 1_t)$ ⓑ	$-$	$5B+2E$	z	z	

- For each singular point the 1-cycle Γ is irreducible due to the monomials w^2, y^{11}, and zt^3.

No. 92: $X_{48} \subset \mathbb{P}(1,3,5,16,24)$	$A^3 = 1/120$
$w^2 + t^3 + yz^9 + y^{16} + wf_{24}(x,y,z,t) + f_{48}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_z = \frac{1}{5}(1_x, 1_t, 4_w)$ ⓑ	$-$	$3B$	y	t^3	
$O_tO_w = 1\times\frac{1}{8}(1_x, 3_y, 5_z)$ ⓑ	0	$3B$	y	y	
$O_yO_w = 2\times\frac{1}{3}(1_x, 2_z, 1_t)$ ⓑ	$-$	$5B+E$	z	z	

- For each singular point the 1-cycle Γ is irreducible due to the monomials w^2 and t^3.

No. 93: $X_{50} \subset \mathbb{P}(1,7,8,10,25)$	$A^3 = 1/280$
$w^2 + t^5 + z^5t + y^6(a_1z + a_2xy) + wf_{25}(x,y,z,t) + f_{50}(x,y,z,t)$	

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_z = \frac{1}{8}(1_x, 7_y, 1_w)$ ⓑ	$-$	$7B$	y	y	

$O_y = \frac{1}{7}(1_x, 3_t, 4_w)$ ⓑ	$-$	$8B$	z	w^2	$a_1 \neq 0$
$O_y = \frac{1}{7}(1_z, 3_t, 4_w)$ ⓑ	$-$	$10B + E$	t	t	$a_1 = 0$
$O_tO_w = 1 \times \frac{1}{5}(1_x, 2_y, 3_z)$ ⓑ	$-$	$8B + E$	z	z	
$O_zO_t = 1 \times \frac{1}{2}(1_x, 1_y, 1_w)$ ⓑ	$-$	$7B + 3E$	y	y	

- For each singular point the 1-cycle Γ is always irreducible because of the monomials w^2 and t^5. In particular, the 1-cycle Γ for the singular point O_y with $a_1 = 0$ is irreducible even though it is non-reduced.

No. 94: $X_{54} \subset \mathbb{P}(1, 4, 5, 18, 27)$ $A^3 = 1/180$

$w^2 + t^3 + yz^{10} + y^9t + wf_{27}(x, y, z, t) + f_{54}(x, y, z, t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_z = \frac{1}{5}(1_x, 3_t, 2_w)$ ⓑ	$-$	$4B$	y	w^2	
$O_y = \frac{1}{4}(1_x, 1_z, 3_w)$ ⓑ	$-$	$18B + 3E$	t	w^2	
$O_tO_w = 1 \times \frac{1}{9}(1_x, 4_y, 5_z)$ ⓑ	0	$4B$	y	y	
$O_yO_t = 1 \times \frac{1}{2}(1_x, 1_z, 1_w)$ ⓑ	$-$	$5B + 2E$	z	z	

- For each singular point the 1-cycle Γ is irreducible due to the monomials w^2, t^3 and yz^{10}.

No. 95: $X_{66} \subset \mathbb{P}(1, 5, 6, 22, 33)$ $A^3 = 1/330$

$w^2 + t^3 + z^{11} + y^{12}(a_1z + a_2xy) + wf_{33}(x, y, z, t) + f_{66}(x, y, z, t)$

Singularity	B^3	Linear system	Surface T	Vanishing order	Condition
$O_y = \frac{1}{5}(1_x, 2_t, 3_w)$ ⓑ	$-$	$6B$	z	w^2	$a_1 \neq 0$
$O_y = \frac{1}{5}(1_z, 2_t, 3_w)$ ⓑ	$-$	$6B + E$	z	z	$a_1 = 0$
$O_tO_w = 1 \times \frac{1}{11}(1_x, 5_y, 6_z)$ ⓑ	0	$5B$	y	y	
$O_zO_w = 1 \times \frac{1}{3}(1_x, 2_y, 1_t)$ ⓑ	$-$	$5B + E$	y	y	
$O_zO_t = 1 \times \frac{1}{2}(1_x, 1_y, 1_w)$ ⓑ	$-$	$5B + 2E$	y	y	

- The 1-cycle Γ for each singular point is irreducible because of the monomials w^2 and t^3.

□

6 Epilogue

Open problems

Let X be a quasi-smooth hypersurface of degrees d with only terminal singularities in weighted projective space $\mathbb{P}(1, a_1, a_2, a_3, a_4)$, where $d = \sum a_i$. In Main Theorem, we prove that X is birationally rigid. In particular, X is non-rational. Moreover, the proof also explicitly describes the generators of the group of birational automorphisms $\mathrm{Bir}(X)$ modulo subgroup of biregular automorphisms $\mathrm{Aut}(X)$ (see Theorem 3.3.4). Furthermore, Theorem 1.1.10 says that $\mathrm{Bir}(X) = \mathrm{Aut}(X)$ for those families in the list of Fletcher and Reid with entry numbers No. 1, 3, 10, 11, 14, 19, 21, 22, 28, 29, 34, 35, 37, 39, 49, 50, 51, 52, 53, 55, 57, 59, 62, 63, 64, 66, 67, 70, 71, 72, 73, 75, 77, 78, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94 and 95. Of course, some quasi-smooth threefolds in other families may also be birationally super-rigid.

Explicit birational involutions play a key role in the proof of Main Theorem. In many cases, they arise from generically 2-to-1 rational maps of X to suitable 3-dimensional weighted projective spaces (*quadratic involutions*). However, in some cases they arise from rational maps of X to suitable 2-dimensional weighted projective spaces whose general fibers are birational to smooth elliptic curves (*elliptic involutions*). Moreover, we often use such elliptic rational fibrations in order to exclude some singular points of X as centers of non-canonical singularities of any log pair $\left(X, \frac{1}{n}\mathcal{M}\right)$, where $\mathcal{M}$ is a mobile linear subsystem in $|-nK_X|$. The latter is done using Corollary 2.2.2 or Lemma 3.2.8. A similar role in the proof of Main Theorem is played by so-called *Halphen pencils* on X, i.e., pencils whose general members are irreducible surfaces of Kodaira dimension zero. Implicitly Halphen pencils appear almost every time when we apply Lemmas 3.2.2 and 3.2.7. This leads us to three problems that are closely related to Main Theorem. They are

(1) to find relations between generators of the birational automorphism group $\mathrm{Bir}(X)$;

(2) to describe birational transformations of X into elliptic fibrations;

(3) to classify Halphen pencils on X.

While proving Main Theorem, we noticed many interesting Halphen pencils on X even though we did not mention them explicitly in the proofs. We also observed that their general members are K3 surfaces. This gives an evidence for

Conjecture 6.1. *Every Halphen pencil on X is a pencil of K3 surfaces.*

We do not know any deep reason why this conjecture should be true. When X is a general threefold in its family, Conjecture 6.1 was proved in [14].

The original proof of Theorem 1.1.3 given by Iskovskikh and Manin in [33] holds in arbitrary characteristic. This also follows from [44]. The short proof of Theorem 1.1.3 given by Corti in [24] holds only in characteristic zero. For some families in the list of Fletcher and Reid, the proof of Main Theorem requires vanishing type results and, thus, is valid only in characteristic zero. This suggests the birational rigidity problem of X and problems (1), (2) and (3) over an algebraically closed field of positive characteristic. For double covers of $\mathbb{P}^3$ ramified along smooth sextic surfaces, this was done in [13] and [15], which revealed special phenomenon of small characteristics (see [13, Example 1.5]).

General vs. special

The first three problems listed in the previous section are solved in the case when X is a general hypersurface in its family. This is done in [6], [7], [12] and [14]. In many cases, the same methods can be applied regardless of the assumption that X is general. For example, we proved in [12] that a general hypersurface in the families No. 3, 60, 75, 83, 87, 93 cannot have a birational transformation to an elliptic fibration. We are able to prove that it is also true for every quasi-smooth hypersurface in the families No. 3, 75, 83, 87, 93, using the methods given in this paper. However, in the family No. 60, it is no longer true for an arbitrary quasi-smooth hypersurface.

Example 6.2. Let X_{24} be a quasi-smooth hypersurface in the family No. 60. Suppose, in addition, that X_{24} contains the curve L_{zw}. We may then assume that it is defined by the equation

$$\begin{aligned}&w^2t + w(at^2x^3 + tg_9(x, y, z) + g_{15}(x, y, z))+\\&t^4 + t^3h_6(x, y, z) + t^2h_{12}(x, y, z) + th_{18}(x, y, z) + h_{24}(x, y, z) = 0\end{aligned}$$

in $\mathbb{P}(1, 4, 5, 6, 9)$. For the hypersurface X_{24} to be quasi-smooth at the point O_z, the polynomial h_{24} must contain the monomial yz^4. For the hypersurface X_{24} to contain the curve L_{zw}, the polynomial g_{15} does not contain the monomial z^3.

Consider the projection $\pi\colon X_{24} \dashrightarrow \mathbb{P}(1, 4, 6)$. Its general fiber is an irreducible curve birational to an elliptic curve. To see this, on the hypersurface X_{24}, consider the surface cut by $y = \lambda x^4$ and the surface cut by $t = \mu x^6$, where λ and μ are sufficiently general complex numbers. Then the intersection of these two surfaces is the 1-cycle $4L_{zw} + C_{\lambda,\mu}$, where $C_{\lambda,\mu}$ is a curve defined by the equation

$$\lambda w^2x^2 + w\left(\mu^2x^{11}g_0 + \mu x^2g_9(x, \lambda x^4, z) + \frac{g_{15}(x, \lambda x^4, z)}{x^4}\right)+$$

$$\mu^4x^{20} + \mu^3x^{14}h_6(x, \lambda x^4, z) + \mu^2x^8h_{12}(x, \lambda x^4, z) + \mu x^2h_{18}(x, \lambda x^4, z) + \frac{h_{24}(x, \lambda x^4, z)}{x^4} = 0$$

in $\mathbb{P}(1, 5, 9)$. Plugging $x = 1$ into the equation, we see that the curve $C_{\lambda,\mu}$ is birational to a double cover of $\mathbb{C}$ ramified at four distinct points.

In some of the 95 families of Reid and Fletcher, special quasi-smooth hypersurfaces may have simpler geometry than their general representatives.

Example 6.3. Let X_5 be a quasi-smooth hypersurface in $\mathbb{P}(1, 1, 1, 1, 2)$ (the family No. 2). The hypersurface X_5 can be given by

$$tw^2 + wf_3(x, y, z, t) + f_5(x, y, z, t) = 0.$$

The natural projection $X_5 \dashrightarrow \mathbb{P}^3$ is a generically double cover. Therefore, it induces a birational involution of X_5, denoted by τ. By Theorem 3.3.4, the birational automorphism group $\mathrm{Bir}(X)$ is generated by the biregular automorphism group $\mathrm{Aut}(X)$ and the involution τ. By Main Theorem, the hypersurface X_5 is birationally rigid. Moreover, if the hypersurface X_5 is general, then it is not birationally super-rigid, i.e., $\mathrm{Bir}(X) \neq \mathrm{Aut}(X)$. However, in a special case, the involution τ can be biregular, and hence the hypersurface X_5 is birationally super-rigid. To be precise, the involution τ is biregular if and only if the coefficient polynomial f_3 of w is a zero polynomial. Thus, the hypersurface X_5 is birationally super-rigid if and only if f_3 is a zero polynomial.

However, this is not always the case, i.e., special quasi-smooth hypersurfaces usually have more complicated geometry than their general representatives. Here we provide three illustrating examples.

Example 6.4. Let X_4 be a smooth quartic threefold in $\mathbb{P}^4$ (the family No. 1). From Theorem 1.1.3 we know that every smooth quartic hypersurface in $\mathbb{P}^4$ admits no non-biregular birational automorphisms. Moreover, it was proved in [6] that every rational map $\rho\colon X_4 \dashrightarrow \mathbb{P}^2$ whose general fiber is birational to a smooth elliptic curve fits a commutative diagram

$$\begin{array}{ccc} & X_4 & \\ {\scriptstyle\rho}\swarrow & & \searrow{\scriptstyle\pi} \\ \mathbb{P}^2 & \overset{}{\underset{\sigma}{\dashrightarrow}} & \mathbb{P}^2, \end{array}$$

where π is a linear projection from a line and σ is a birational map. Furthermore, it was proved in [14] that every Halphen pencil on X_4 is contained in $|-K_{X_4}|$ provided that X_4 satisfies some generality assumptions. Earlier, Iskovskikh pointed out in [32] that this is no longer true for an *arbitrary* smooth quartic hypersurface in $\mathbb{P}^4$. Indeed, a special smooth quartic hypersurface in $\mathbb{P}^4$ may have a Halphen pencil contained in $|-2K_{X_4}|$. The complete classification of Halphen pencils on X_4 was obtained in [11].

Example 6.5 (For details see the proof of Theorem 4.3.1)**.** Let X_{14} be a quasi-smooth hypersurface in $\mathbb{P}(1,2,3,4,5)$ (the family No. 23). If X_{14} is a general such hypersurface, then there exists an exact sequence of groups

$$1 \longrightarrow \Gamma_{X_{14}} \longrightarrow \mathrm{Bir}(X_{14}) \longrightarrow \mathrm{Aut}(X_{14}) \longrightarrow 1,$$

where $\Gamma_{X_{14}}$ is a free product of two birational involutions constructed in Section 4.2. This follows from [12, Lemma 4.2] (cf. Theorem 3.3.4). Moreover, let $\rho\colon X_{14} \dashrightarrow \mathbb{P}^2$ be a rational map whose general fiber is birational to a smooth elliptic curve. If X_{14} is general, then there exists a commutative diagram

$$\begin{array}{ccc} & X_{14} & \\ {\scriptstyle\phi}\swarrow & & \searrow{\scriptstyle\rho} \\ \mathbb{P}(1,2,3) & \underset{\sigma}{\dashrightarrow} & \mathbb{P}^2 \end{array}$$

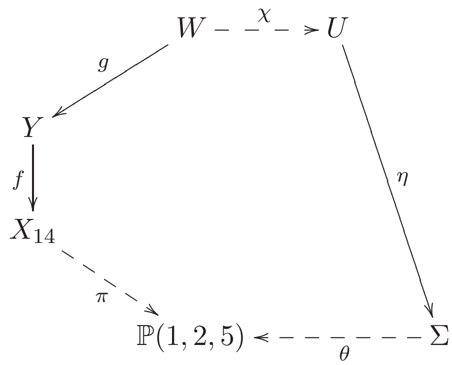
where ϕ is the natural projection and σ is some birational map. Suppose now that X_{14} is defined by the equation

$$(t+by^2)w^2 + yt(t-\alpha_1y^2)(t-\alpha_2y^2) + z^4y + xtz^3 + xf_{13}(x,y,z,t,w) + yg_{12}(y,z,t,w) = 0.$$

Then none of these assertions are true. Indeed, let $\mathcal{H}$ be the linear subsystem of $|-5K_{X_{14}}|$ generated by x^5, xy^2, x^3y and $yz+xt$. Let $\pi\colon X_{14} \dashrightarrow \mathbb{P}(1,2,5)$ be the rational map induced by

$$[x:y:z:t:w] \mapsto [x:y:yz+xt].$$

Then π is dominant and its general fiber is birational to an elliptic curve . Let $f\colon Y\to X_{14}$ be the weighted blow up at the point O_z with weight $(1,1,2)$. Denote by E its exceptional surface. Let $g\colon W\to Y$ be the weighted blow up at the point over O_w with weight $(1,2,3)$. Denote by G be its exceptional divisor. Denote by $\hat{L}_{zw}$, $\hat{L}_{zt}$ and $\hat{L}_{yw}$ the proper transforms of the curves L_{zw}, L_{zt} and L_{yw} by the morphism $f\circ g$. Then the curves $\hat{L}_{zw}$ and $\hat{L}_{zt}$ are the only curves that intersect $-K_W$ negatively. Moreover, there is an anti-flip $\chi\colon W\dashrightarrow U$ along the curves $\hat{L}_{zw}$ and $\hat{L}_{zt}$ (see the proof of Theorem 4.3.1). Let $\check{E}$ and $\check{G}$ be the proper transforms on U of the divisors E and G, respectively. For $m\gg 0$, the linear system $|-mK_U|$ is free and gives an elliptic fibration $\eta\colon U\to\Sigma$, where Σ is a normal surface. Furthermore, there exist a commutative diagram



where θ is a birational map. The divisor $\check{G}$ is a section of the elliptic fibration η and $\check{E}$ is a 2-section of η. Let τ_U be a birational involution of the threefold U that is induced by the reflection of the general fiber of η with respect to the section $\check{G}$. The involution τ_U induces a birational involution of X_{14}. This new involution is not biregular and not contained in the subgroup of the birational automorphism group $\mathrm{Bir}(X_{14})$ generated by two birational involutions constructed in Section 4.2.

Example 6.6. Let X_{17} be a quasi-smooth hypersurface in $\mathbb{P}(1,2,3,5,7)$ (the family No. 33). Then it can be given by the quasi-homogenous polynomial equation

$$(dx^3+exy+z)w^2+t^2(a_1w+a_2yt)+z^4(b_1t+b_2yz)+$$
$$y^5(c_1w+c_2yt+c_3y^2z+c_4y^3x)+wf_{10}(x,y,z,t)+f_{17}(x,y,z,t)=0.$$

The pencil $|-2K_{X_{17}}|$ is a Halphen pencil. Moreover, if the defining equation of X_{17} is sufficiently general, then this is the only Halphen pencil on X_{17} (see [14, Corollary 1.1]). Suppose that $c_1=c_2=0$ and $c_3\neq 0$. Then we may assume that $c_3=1$ and $c_4=0$ by a coordinate change. Here we encounter an extra Halphen pencil. Indeed, the pencil on X_{17} cut out by $\lambda x^3+\mu z=0$, where $[\lambda:\mu]\in\mathbb{P}^1$, is a Halphen pencil contained in $|-3K_{X_{17}}|$ and different from the Halphen pencil $|-2K_{X_{17}}|$.

Calabi problem

In many applications it is useful to *measure* how singular effective $\mathbb{Q}$-divisors D equivalent to $-K_X$ can be. A possible *measurement* is given by the so-called α-invariant of the Fano hypersurface X. It is defined by the number

$$\alpha(X)=\sup\left\{\lambda\in\mathbb{Q}\,\middle|\,\begin{array}{l}\text{the log pair } (X,\lambda D) \text{ is Kawamata log terminal}\\ \text{for every effective } \mathbb{Q}\text{-divisor } D\sim_{\mathbb{Q}}-K_X.\end{array}\right\}.$$

If X is a general hypersurface in its family, then $\alpha(X) = 1$ by [8, Theorem 1.3] and [9, Theorem 1.15] except the case when X belongs to the families No. 1, 2, 3, 4 or 5. If X is a general quartic threefold in $\mathbb{P}^3$ (the family No. 1), we have $\alpha(X) \geq \frac{7}{9}$ by [16, Theorem 1.1.6]. If X is a double cover of $\mathbb{P}^3$ ramified along smooth sextic surface (the family No. 3), then all possible values of $\alpha(X)$ are found in [16, Theorem 1.1.5]. For general threefolds in the families No. 2, 4 and 5, the bound $\alpha(X) > \frac{3}{4}$ proved in [8] and [10]. In particular, we have

Corollary 6.7. *Let X be a quasi-smooth hypersurface of degrees d with only terminal singularities in the weighted projective space $\mathbb{P}(1, a_1, a_2, a_3, a_4)$, where $d = \sum a_i$. Suppose that X is a general hypersurface in this family. Then $\alpha(X) > \frac{3}{4}$.*

Similarly, we can define the α-invariant of any Fano variety with at most Kawamata log terminal singularities. This invariant has been studied intensively by many people who used different notations for it. The notation $\alpha(X)$ is due to Tian who defined the α-invariant in a different way (see [49]). However, his definition coincides with the one we just gave (see [17, Theorem A.3]).

Tian proved in [49] that a smooth Fano variety of dimension n whose α-invariant is greater than $\frac{n}{n+1}$ admits a Kähler–Einstein metric. This result was generalized for Fano varieties with quotient singularities by Demailly and Kollár (see [26, Criterion 6.4]). Thus, Corollary 6.7 implies

Theorem 6.8. *Let X be a quasi-smooth hypersurface of degrees d with only terminal singularities in the weighted projective space $\mathbb{P}(1, a_1, a_2, a_3, a_4)$, where $d = \sum a_i$. Suppose that X is a general hypersurface in this family. Then X is admits an orbifold Kähler–Einstein metric.*

Recently, Chen, Donaldson and Sun and independently Tian proved that a smooth Fano variety admits a Kähler–Einstein metric if and only if it is K-stable (see [19], [20], [21], [22] and [50]). Earlier Odaka and Okada proved that birationally super-rigid smooth Fano varieties with base-point-free anticanonical linear systems must be slope-stable (see [40]). Furthermore, Odaka and Sano proved that Fano varieties of dimension n with at most log terminal singularities whose α-invariants are greater than $\frac{n}{n+1}$ must be K-stable (see [40]). These results suggest that *every* quasi-smooth hypersurface in the 95 families of Fletcher and Reid should admit an orbifold Kähler–Einstein metric.

Using methods we developed in the proof of Main Theorem, it is possible to explicitly describe all quasi-smooth hypersurfaces in the 95 families of Fletcher and Reid whose α-invariants are greater than $\frac{3}{4}$. All of them must admit orbifold Kähler–Einstein metrics by [26, Criterion 6.4].

The α-invariants can be applied to the non-rationality problem on products of Fano varieties. In particular, we can apply [8, Theorem 6.5] to quasi-smooth hypersurfaces in the 95 families of Fletcher and Reid whose α-invariants are 1.

Arithmetics

As it was pointed out by Pukhlikov and Tschinkel, the problem (1) is closely related to the problem of potential density of rational points on X in the case when X is defined over a number field. For example, if $\mathrm{Bir}(X)$ is infinite, then we are able to show that X contains infinitely many rational surfaces. It implies the potential density of rational points on X.

The papers [4], [5], [30] use birational transformations into elliptic fibrations in order to prove the potential density on all smooth Fano threefolds possibly except double covers of $\mathbb{P}^3$ ramified along smooth sextic surfaces (the family No. 3 in the list of Fletcher and Reid).

If X is defined over a number field, it seems likely that the set of rational points on X is potentially dense. For every smooth quartic threefold in $\mathbb{P}^4$ (the family No. 1), this was proved by Harris and Tschinkel in [30]. For general Fano hypersurfaces in the families No. 2, 4, 5, 6, 7, 9, 11, 12, 13, 15, 17, 19, 20, 23, 25, 27, 30, 31, 33, 36, 38, 40, 41, 42, 44, 58, 61, 68 and 76, this was proved in [12] and [15]. Despite many attempts, this problem is still open for double covers of $\mathbb{P}^3$ ramified along smooth sextic surfaces.

The methods we use in the proof of Main Theorem can be applied to prove the potential density of rational points on the quasi-smooth hypersurfaces in some families in the list of Fletcher and Reid. In fact, for some families we can use our methods to prove the density of rational points on X (see [12, Page 84 and Section 5]).

Fano threefold complete intersections

In 2013 and 2014, after the present paper was announced on ArXiv, new results on the birational rigidity of Fano threefold complete intersections were introduced ([1], [41]). Like the 95 families of Fano threefold hypersurfaces, it is well known that there are 85 families of Fano threefold complete intersections of codimension 2 ([29, Table 6]). In addition, it is also known that there is only one family of Fano threefold complete intersections of codimension 3, *i.e.*, complete intersections of three quadrics in $\mathbb{P}^6$. There is no Fano threefold complete intersections of codimensions 4 and higher ([18]). The lists of Fano threefold complete intersections in [29, Tables 5, 6, and 7] are proved to be complete ([18]). In 1996, a general member in the family of complete intersections of quadrics and cubics in $\mathbb{P}^5$ is proved to be birationally rigid ([34]). In 2013, Odaka announced that general members in 19 families out of the 85 families of Fano threefold complete intersections of codimension 2 are birationally rigid and that general members in the other 64 families are not birationally rigid ([41]). After Odaka, a proof of the birational rigidity of quasi-smooth complete intersections in the 19 families (except the family of smooth complete intersections of quadrics and cubics in $\mathbb{P}^5$) is announced by Ahmadinezhad and Zucconi ([1]).

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