ON THE YANG-BAXTER EQUATION AND LEFT NILPOTENT LEFT BRACES

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ABSTRACT. We study non-degenerate involutive set-theoretic solutions (X, r) of the Yang-Baxter equation, we call them simply *solutions*. We show that the structure group $G(X, r)$ of a finite non-trivial solution (X, r) cannot be an Engel group. It is known that the structure group $G(X, r)$ of a finite multipermutation solution (X, r) is a poly-$\mathbb{Z}$ group, thus our result gives a rich source of examples of braided groups and left braces $G(X, r)$ which are poly-$\mathbb{Z}$ groups but not Engel groups.

We also show that a finite solution of the Yang-Baxter equation can be embedded in a convenient way into a finite brace and into a finite braided group.

For a left brace A, we explore the close relation between the multipermutation level of the solution associated with it and the radical chain $A^{(n+1)} = A^{(n)} * A$ introduced by Rump.

1. Introduction

Braces were introduced by Rump [15] to study non-degenerate involutive set-theoretic solutions of the Yang-Baxter equation.

Recall that a left brace is a set B with two operations, $+$ and $\cdot$, such that $(B, +)$ is an abelian group, $(B, \cdot)$ is a group and for every $a, b, c \in B$,

$$a \cdot (b + c) + a = a \cdot b + a \cdot c. \tag{1}$$

Right braces are defined similarly, changing the property (1) by $(a + b) \cdot c + c = a \cdot c + b \cdot c$. A two-sided brace is a left brace which is also a right brace. In any left brace $(B, +, \cdot)$ one defines another operation $*$ by the rule

$$a * b = a \cdot b - a - b,$$

for $a, b \in B$. It is known that $(B, +, \cdot)$ is a two-sided brace if and only if $(B, +, *)$ is a Jacobson radical ring. Conversely, if R is a Jacobson radical ring, then one

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defines a new operation $\circ$ on R by $a\circ b = ab+a+b$ and $(R,\circ)$ is called the adjoint group of the radical ring R. Then $(R,+,\circ)$ is a two-sided brace. Hence the study of two-sided braces is equivalent to the study of Jacobson radical rings.

In general the operation $*$ in a left brace B is not associative, but it is left distributive with respect to the sum, that is

$$a*(b+c) = a*b + a*c,$$

for $a,b,c \in B$.

Let B be a left brace. For $a \in B$, let $\mathcal{L}_a\colon B \longrightarrow B$ be the map defined by $\mathcal{L}_a(b) = ab - a$ for all $b \in B$. It is known that $\mathcal{L}_a$ is an automorphism of the additive group of the left brace B and the map $\mathcal{L}\colon (B,\cdot) \longrightarrow \operatorname{Aut}(B,+)$, defined by $a \mapsto \mathcal{L}_a$, is a morphism of groups. The kernel of this morphism is called the socle of B,

$$\operatorname{Soc}(B) := \{a \in B \mid \mathcal{L}_a = \mathrm{id}\} = \{a \in B \mid ab = a+b, \ \text{ for all } b \in B\}.$$

In fact the socle of a left brace B is an ideal of B, that is, a normal subgroup of its multiplicative group invariant by the maps $\mathcal{L}_a$ for all $a \in B$. In particular, $\operatorname{Soc}(B)$ is also a subgroup of the additive group of B. Note that if $a,b \in \operatorname{Soc}(B)$, then $a - b = \mathcal{L}_b(b^{-1}a) \in \operatorname{Soc}(B)$. Therefore the quotient of the multiplicative group $B/\operatorname{Soc}(B)$ is also the quotient of the additive group and $(B/\operatorname{Soc}(B),+,\cdot)$ is a left brace, the left brace quotient of B modulo its ideal $\operatorname{Soc}(B)$.

Let X be a non-empty set. Recall that a map $r\colon X\times X \longrightarrow X \times X$ is a set-theoretic solution of the Yang-Baxter equation if

$$r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23},$$

where $r_{12}, r_{23}\colon X\times X\times X \longrightarrow X\times X\times X$ are the maps $r_{12} = r\times \mathrm{id}_X$ and $r_{23} = \mathrm{id}_X \times r$. We will write $r(x,y) = ({}^xy, x^y)$. The map r is *non-degenerate* if for every $x \in X$ the maps $y \mapsto {}^xy$ and $y \mapsto y^x$ are bijective, r is *involutive* if $r^2 = \mathrm{id}_{X^2}$.

Convention. By *a solution of the YBE* (or shortly, *a solution*) we mean a non-degenerate involutive set-theoretic solution (X,r) of the Yang-Baxter equation.

Let (X,r) be a solution of the YBE. The structure group of (X,r) is the group $G(X,r)$ with presentation

$$G(X,r) = \langle X \mid xy = ({}^xy)(x^y),\ x,y \in X\rangle.$$

(Some authors call $G(X,r)$ the YB group of (X,r)).

It follows from the results in [8] that X is naturally embedded in $G(X,r)$. One can define a sum on $G(X,r)$ such that $(G(X,r),+)$ is a free abelian group with basis X. Moreover, $(G(X,r),+,\cdot)$ is a left brace such that ${}^{x}y = \mathcal{L}_x(y) \in X$ for all $x, y \in X$, see [5, 9].

We say that this is *the canonical left brace structure on* $G(X,r)$. It is known that the group G acts on the set X from the left (and from the right). Moreover, the assignments $x \mapsto \mathcal{L}_x$ extend to a group homomorphism $\mathcal{L} : G \longrightarrow \mathrm{Sym}_X$. The image $\mathcal{L}(G)$ of this homomorphism is a subgroup of Sym_X called *the permutation group of* (X,r) and denoted by $\mathcal{G}(X,r)$. It is known that $\mathcal{G}(X,r) := \langle \mathcal{L}_x \mid x \in X \rangle$, where $\mathcal{L}_x(y) = {}^{x}y$, for all $x, y \in X$. The group epimorphism $\mathcal{L} : G(X,r) \longrightarrow \mathcal{G}(X,r), x \mapsto \mathcal{L}_x$ has kernel $\mathrm{Ker}\, \mathcal{L} = \mathrm{Soc}(G(X,r))$ (as sets). Thus $\mathcal{G}(X,r)$ inherits a structure of a left brace via this natural isomorphism of groups, we say that this is *the canonical structure of a left brace on* $\mathcal{G}(X,r)$. Moreover, $G(X,r)/\,\mathrm{Soc}(G(X,r)) \cong \mathcal{G}(X,r)$ as symmetric groups (i.e. involutive braided groups) and as left braces, [9, 5].

In this paper, we prove some general results about braces and apply these to study the close relations between the properties of solutions (X,r) and their associated left braces $G(X,r)$. This is in the spirit of [9] and [5].

2. Some results on $G(X,r)$

The results of this section were motivated by a result from [9], which assures that the structure group $G(X,r)$ of a non-trivial solution (X,r) of the Yang Baxter cannot be a two-sided brace.

Let $(B,+,\cdot)$ be a left brace. As usual, for any $a, b \in B$ and positive integer m, ab will denote $a \cdot b$ and a^m will denote $a \cdot a \cdots a$ (where a appears m times).

Lemma 1. *Let B be a left brace whose additive group $(B,+)$ is torsion-free. Assume that $a, b \in B$ and that there is an integer $n(a,b)$ such that $a * (a * (\ldots a * (a * b) \ldots)) = 0$ (where a occurs $n(a,b)$ times and b once in this equation). Assume moreover that $\mathcal{L}_{a^n} = id$ for some integer n. Then $a * b = 0$ or equivalently, $a \cdot b = a + b$.*

Proof. Note that $\mathcal{L}_a(b) = a * b + b$, for $a, b \in B$. Let m be a positive integer. Let $e_1(a,b) = a * b$ and $e_{m+1}(a,b) = a * e_m(a,b)$, for $a, b \in B$. It can be proved by induction on m that

$$\mathcal{L}_{a^m}(b) = b + \sum_{i=1}^{m} \binom{m}{i} e_i(a,b).$$

Since $\mathcal{L}_{a^n} = \mathrm{id}$, we have

$$b = \mathcal{L}_{a^n}(b) = b + \sum_{i=1}^{n} \binom{n}{i} e_i(a,b),$$

and thus

$$ne_1(a,b) = -\sum_{i=2}^{n} \binom{n}{i} e_i(a,b).$$

Hence

$$ne_k(a,b) = -\sum_{i=2}^{n} \binom{n}{i} e_{i+k-1}(a,b), \tag{2}$$

for all positive integer k.

Suppose that $e_1(a,b) = a * b \neq 0$. Let $n(a,b)$ be the smallest positive integer such that $e_{n(a,b)}(a,b) = 0$. Then, by (2),

$$ne_{n(a,b)-1}(a,b) = -\sum_{i=2}^{n} \binom{n}{i} e_{i+n(a,b)-2}(a,b) = 0.$$

Since $(B,+)$ is torsion-free, we have that $e_{n(a,b)-1}(a,b) = 0$, in contradiction with the definition of $n(a,b)$. □

Let G be a group. Following the notation of [14], for $g,h \in G$, we denote by $[g,h]$ the element $[g,h] = g^{-1}h^{-1}gh$. Recall that the group G is an Engel group if and only if for each $g,h \in G$ there exists a positive integer $n(g,h)$ such that $[[\dots[[g,h],h]\dots],h] = 1$, where h occurs $n(g,h)$ times.

Theorem 2. *Let B be a left brace such that its additive group $(B,+)$ is torsion-free and $[B : \mathrm{Soc}(B)] = n < \infty$. If the multiplicative group $(B,\cdot)$ of the left brace B is an Engel group, then B is a trivial brace, that is $a \cdot b = a + b$, for all $a,b \in B$.*

Proof. Let $a \in B$ and $c \in \mathrm{Soc}(B)$. Note that

$$\mathcal{L}_a(c) = ac - a = aca^{-1}a - a = aca^{-1} + a - a = aca^{-1}.$$

Hence

$$\begin{aligned} [a,c] &= a^{-1}c^{-1}ac = a^{-1}c^{-1}a + c \\ &= \mathcal{L}_{a^{-1}}(c^{-1}) + c = \mathcal{L}_{a^{-1}}(-c) + c \\ &= -\mathcal{L}_{a^{-1}}(c) + c = -a^{-1} * c - c + c \\ &= -a^{-1} * c = (a^{-1} * c)^{-1}. \end{aligned}$$

Hence $[c,a] = a^{-1}*c$. Therefore $[[\dots[[c,a],a]\dots],a] = a^{-1}*(\dots a^{-1}*(a^{-1}*c)\dots)$. Let $b \in B$. Since $[B : \operatorname{Soc}(B)] = n < \infty$ and $\operatorname{Soc}(B)$ is an ideal of B, $nb \in \operatorname{Soc}(B)$. Since $(B,\cdot)$ is an Engel group, there exists a positive integer m such that

$$[[\dots[[(nb),a],a]\dots],a] = 1,$$

(where a occurs m times). Hence

$$0 = 1 = a^{-1}*(\dots a^{-1}*(a^{-1}*(nb))\dots) = n(a^{-1}*(\dots a^{-1}*(a^{-1}*b)\dots)),$$

(where a^{-1} appears m times). But $(B,+)$ is torsion free, hence

$$a^{-1}*(\dots a^{-1}*(a^{-1}*b)\dots) = 0,$$

where a^{-1} occurs m times.

By Lemma 1, $a^{-1}*b = 0$. Therefore $a*b = 0$, for all $a,b \in B$, or equivalently, B is a trivial left brace. □

We call a left brace B left nilpotent if $B^n = 0$ for some n, where $B^{n+1} = B*B^n$ is the chain introduced by Rump in [15]. As a consequence of Lemma 1 and Theorem 2, we have the following two results.

Theorem 3. *Let (X,r) be a finite solution of the YBE. Assume that for each $a,b \in X$ there is a positive integer $n = n(a,b)$ such that the equality $a*(a*(\dots a*(a*b))) = 0$ holds in $G(X,r)$,(a occurs n times and b occurs once in this equality). Then (X,r) is the trivial solution. In particular, if $G(X,r)$ is a left nilpotent left brace, then (X,r) is the trivial solution.*

Proof. Since $G(X,r)/\operatorname{Soc}(G(X,r)) \cong \mathcal{G}(X,r)$ is a subgroup of the symmetric group Sym_X of the finite set X, we have that $[G(X,r) : \operatorname{Soc}(G(X,r))] < \infty$. Hence, by Lemma 1, ${}^{a}b = \mathcal{L}_a(b) = ab - a = a + b - a = b$, for all $a,b \in X$. In particular, (X,r) is the trivial solution. □

It is known that any ordered abelian-by-finite group is abelian, see for example [12, Section 4]. It is also known that any torsion-free nilpotent group is ordered (see [13, Lemma 13.1.6]). Recall that if (X,r) is a finite solution of the YBE, then $G(X,r)$ is a torsion-free, solvable and abelian-by-finite group (see [8] and [10]). Therefore, if $G(X,r)$ is nilpotent, then it is abelian. In this case the canonical left brace structure on $G(X,r)$ is trivial and (X,r) is the trivial solution. We have the following related result.

Theorem 4. *Let (X,r) be a finite solution of the YBE. If the structure group $G(X,r)$ is an Engel group, then (X,r) is the trivial solution.*

Proof. This is a consequence of Theorem 2. □

3. Right nilpotent left braces

Etingof, Schedler and Soloviev in [8] introduced the retract solution of a given solution of the YBE. Let (X, r) be a solution of the YBE. The retract relation $\sim$ on the set X with respect to r is defined by $x \sim y$ if $\sigma_x = \sigma_y$, where $\sigma_x(z) = {}^x z$. Then the retraction of (X, r) is $\mathrm{Ret}(X, r) = ([X], r_{[X]})$, where $[X] = X/\sim$ and

$$r_{[X]}([x], [y]) = ([{}^x y], [x^y]),$$

where $[x]$ denotes the $\sim$-class of $x \in X$. We define $\mathrm{Ret}^1(X, r) = \mathrm{Ret}(X, r)$ and $\mathrm{Ret}^k(X, r) = \mathrm{Ret}(\mathrm{Ret}^{k-1}(X, r))$ for $k > 1$. A solution (X, r) of the YBE is called a multipermutation solution of level m if m is the smallest nonnegative integer such that the solution $\mathrm{Ret}^m(X, r)$ has cardinality 1; in this case we write $\mathrm{mpl}(X, r) = m$.

Let B be a left brace. By $B^{(m)}$ we mean the chain of ideals introduced by Rump in [15], so $B^{(1)} = B$ and $B^{(n+1)} = B^{(n)} * B$. We say that B is right nilpotent if there exists a positive integer n such that $B^{(n)} = 0$.

Recall that if B is a left brace, then the map $r\colon B \times B \longrightarrow B \times B$ defined by

$$r(a, b) = (\mathcal{L}_a(b), \mathcal{L}^{-1}_{\mathcal{L}_a(b)}(a)),$$

is a solution of the YBE. This is the solution of the YBE associated with the left brace B (see [6]).

Proposition 5. *Let B be a nonzero left brace and let (B, r) be its associated solution of the YBE. Then the multipermutation level of $(B, r) = m < \infty$ if and only if $B^{(m+1)} = 0$ and $B^{(m)} \neq 0$.*

Proof. Note that $\mathrm{Soc}(B) = \{b \in B \mid b * a = 0 \text{ for every } a \in B\}$.

First we shall prove the implication $(\mathrm{mpl}(B, r) = m) \Rightarrow (B^{(m+1)} = 0$ and $B^{(m)} \neq 0)$. We use induction on $m = \mathrm{mpl}(B, r)$. Suppose $\mathrm{mpl}(B, r) = 1$. Therefore, $\mathcal{L}_a(b) = a * b + b = b$ which is equivalent to $a * b = 0$ for all $a, b \in B$. It follows that $B * B = 0$, so $B^{(2)} = 0$. But B is a nonzero left brace, hence $B^{(1)} = B \neq 0$. This gives the base for induction.

Suppose now that for all k, $1 \leq k \leq m-1$, the condition $\mathrm{mpl}(B, r) = k \leq m-1$ implies $B^{(k+1)} = 0$ and $B^{(k)} \neq 0$. Assume that $\mathrm{mpl}(B, r) = m$, then the retraction $\mathrm{Ret}(B, r) = ([B], r_{[B]})$ has multipermutation level $m - 1$.

Moreover, there is an isomorphism of left braces (or equivalently an isomorphism of braided groups) $B/\mathrm{Soc}(B) \cong [B]$ and $\mathrm{Ret}(B, r)$ is isomorphic to the solution of the YBE associated with $B/\mathrm{Soc}(B)$ ([15], [6], [9]). Hence by the inductive assumption $(B/\operatorname{Soc}(B))^{(m)} = 0$ and $(B/\operatorname{Soc}(B))^{(m-1)} \neq 0$. This implies $B^{(m)} \subseteq \operatorname{Soc}(B)$ and that $B^{(m-1)}$ is not a subset of $\operatorname{Soc}(B)$. Therefore $B^{(m+1)} = 0$ and $B^{(m)} \neq 0$.

Now we prove the inverse implication: $(B^{(m+1)} = 0$ and $B^{(m)} \neq 0) \Rightarrow (\operatorname{mpl}(B, r) = m)$.

The base for the induction is clear. Assume that for all $k \leq m$ the implication is true. Suppose that B is a left brace such that $B^{(m+2)} = 0$ and $B^{(m+1)} \neq 0$. Recall that $B^{(m+2)} = B^{(m+1)} * B$, therefore $(B/\operatorname{Soc}(B))^{(m+1)} = 0$. On the other hand $B^{(m+1)} \neq 0$ and $B^{(m+1)} = B^{(m)} * B$ imply $(B/\operatorname{Soc}(B))^{(m)} \neq 0$. By the inductive assumption $\operatorname{mpl}(\operatorname{Ret}(B, r)) = m$, and therefore, $\operatorname{mpl}(B, r) = m + 1$. This proves the proposition. □

4. Embedding solutions and groups into finite braces and finite rings

In this section we will show that a finite solution of the YBE can be embedded (in an explicit way) into a finite left brace. Recall that it was shown in [9] that there is a canonical one-to-one correspondence between left braces and symmetric groups (in the sense of Takeuchi [17]). Therefore Proposition 6 also shows explicitly how to embed a finite solution of the YBE into a finite symmetric group.

Proposition 6. *Let (X, r) be a finite solution of the YBE. Then there is a finite left brace $(B, +, \cdot)$ such that $X \subseteq B$ generates the additive group of B. Moreover, (X, r) is a subsolution of the solution (B, σ) associated canonically with the left brace B, that is, X is σ-invariant and $r = \sigma_{|X\times X}$ is the restriction of σ on $X \times X$.*

Proof. Let $G = G(X, r)$ be the structure group of the finite solution (X, r). We know that the additive group of the left brace $(G, +, .)$ associated with G is free abelian with basis X and ${}^a b = \mathcal{L}_a(b) = ab - a$ and $a^b = \mathcal{L}^{-1}_{\mathcal{L}_a(b)}(a)$, for all $a, b \in X$. Since X is finite, $[G : \operatorname{Soc}(G)] < \infty$, say $[G : \operatorname{Soc}(G)] = n$. Consider the set $I = \{ng \mid g \in G(X, r)\}$. We claim that I is an ideal of the brace G, that is, I is a normal subgroup of the multiplicative group $(G, .)$ which is invariant with respect to the left actions by elements of G, [5]. It is clear that I is an additive subgroup of $(G, +)$ and $I \subseteq \operatorname{Soc}(G(X, r))$. Then for $u, v \in I$ one has $uv = u + {}^u v = u + v \in I$, $u^{-1} = -u \in I$, so I is a subgroup of G. Let $g, h \in G$. Then

$$h(ng)h^{-1} = ({}^h(ng))(h^{ng})(h^{-1}) = ({}^h(ng))(hh^{-1}) = {}^h(ng) = n(hg) \in I.$$

Thus, I is a normal subgroup of $(G, .)$ which is also invariant under the left action by elements of G. Therefore I is an ideal of the left brace $(G, +, .)$. It is not difficult to show that brace quotient $B = G/I$ is a finite left brace of order n^m, where m is the cardinality of X. Observe also that for any two elements $x, y \in X, x \neq y$, one has $x - y \notin I$, since the additive group of $(G, +)$ is free abelian with a basis X. Now the restriction of the natural map $G \to G/I = B$ on the set X is injective. The proposition has been proved. □

At a conference in Porto Cesareo, B. Amberg mentioned that he and his collaborators first became interested in Jacobson radical rings because they gave them a way to construct examples of triply-factorizable groups. Later, they found more ways of constructing such examples. Triply factorized groups can be also used to define braces; see [16, Theorem 18]. Interesting results on triply factorized groups can be found in [1, 2, 3, 4, 16, 11]. Triply factorized groups are for example useful for investigating the structure of normal subgroups of a group $G = AB$ which is a product of two subgroups. Several authors investigated connections between triply factorized groups and nearrings [16], [11]. It might be interesting to investigate the connections between nearrings and braces. We would like to pose a related open question:

Question 1. *Investigate whether there is any relation between nearrings and solutions of the YBE?*

The multiplicative group of a brace A is also called an adjoint group of brace A. Observe that [7, Corollary 3.6] asserts that every finite solvable group is a subgroup of an adjoint group of some left brace. We also make the following simple remark which follows from [5, Lemma 8.1] and [7, Corollary 3.8].

Remark. (Related to [7, Corollary 3.8] and [6, Lemma 8.1]) Every finite nilpotent group is a subgroup of the adjoint group of a finite nilpotent ring.

Let p be a prime. By [6, Lemma 8.1], every finite p-group is isomorphic to a subgroup of the adjoint group of a finite nilpotent ring R such that R has cardinality a power of p. Let G be a finite nilpotent group. Let $p_1, \ldots p_m$ be the the distinct prime divisors of the order of G. Let P_i be the Sylow p_i-subgroup of G. Then P_i is isomorphic to a subgroup of the adjoint group of a finite nilpotent ring R_i. Since $G \cong P_1 \times \cdots \times P_m$, it is clear that G is isomorphic to a subgroup of the adjoint group of the finite nilpotent ring $R_1 \times \cdots \times R_m$. If R is a ring, then the adjoint semigroup of R is defined by $a \circ b = a + b + a \cdot b$.

By following the technique of the proof of [6, Lemma 8.1] we get the following result.

Proposition 7. *Let G be a group and let R be a ring (with unit). Then G is isomorphic to a subgroup of the adjoint semigroup of the group ring $R[G]$.*

Proof. Let $f\colon G \longrightarrow R[G]$ be the map defined by $f(g) = g - 1$, for $g \in G$. Clearly f is injective. Let $g, h \in G$. We have that

$$f(gh) = gh - 1 = (g-1)(h-1) + g - 1 + h - 1 = (g-1)\circ(h-1) = f(g)\circ f(h).$$

Therefore f is an injective homomorphism of semigroups from G into the adjoint semigroup of $R[G]$. □

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References

[1] B. Amberg, *Triply factorized groups* in the Groups St Andrews 1989, Vol 1, London Mathematical Society Lecture Note Series, Cambridge University Press 1991, 1–13.

[2] B.Amberg, S. Franciosi, F. de Giovanni, *Triply factorized groups*, Communications in Algebra Volume 18, Issue 3, 1990, 789–809.

[3] B. Amberg, L. Kazarin, *Nilpotent p-algebras and factorized p-groups*, Proceedings of "Groups St. Andrews 2005" , London Math. Society Lecture Note Series 339, Vol 1, Cambridge Univ. Press (2007), 130–147.

[4] B. Amberg, Ya.P. Sysak, *Radical rings and products of groups*, Groups St Andrews 1997 in Bath: Edited by C. M. Campbell, 1–19.

[5] Ferran Cedó, Eric Jespers, Jan Okniński, *Braces and the Yang-Baxter Equation*, arXiv:1205.3587v1 [math.RA].

[6] Ferran Cedó, Eric Jespers, Jan Okniński, *Braces and the Yang-Baxter Equation*, Communications in Mathematical Physics April 2014, Volume 327, Issue 1, pp 101–116.

[7] Ferran Cedó, Eric Jespers, Ángel de Río, *Involutive Yang-Baxter groups*, Trans. Amer. Math. Soc. 362 (2010), 2541–2558.

[8] P. Etingof, T. Schedler and A. Soloviev, Set-theoretical solutions to the quantum Yang-Baxter equation, Duke Math. J. 100 (1999), 169–209.

[9] Tatiana Gateva-Ivanova, *Set-theoretic solutions of the Yang-Baxter equation, Braces, and Symmetric groups*, arXiv:1507.02602 [math.QA].

[10] T. Gateva-Ivanova and M. Van den Bergh, Semigroups of I-type, J. Algebra 206 (1998), 97–112.

[11] Peter Hubert, *Nearrings and a Construction of Triply Factorized Groups*, doctoral dissertation, Johannes Gutenberg-Universitat in Mainz, 2005.

[12] Eric Jespers and Jan Okninski *Binomial semigroups*, J. Algebra 202 (1998), 250–275.

[13] D.S. Passman, *The Algebraic Structure of Group Rings*, Wiley, New York, 1977.

[14] D. J. S. Robinson, *A Course in the Theory of Groups*, Springer-Verlag GTM Vol. 80 (1982).

[15] Wolfgang Rump, *Braces, radical rings, and the quantum Yang-Baxter equation*, Journal of Algebra Volume 307, Issue 1, 1 January 2007, Pages 153–170.

[16] Ya. P. Sysak, *Products of groups and local nearrings*, Note. Mat. 28 (2008), suppl. n. 2. 177-214.

[17] M. Takeuchi, *Survey on mached pairs of groups. An elementary approach to the ESS-LYZ theory*, Banach Center Publ. 61 (2003), 305–331.

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