ON DEFORMATIONS OF $\mathbb{Q}$-FANO THREEFOLDS II

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Abstract. We investigate some coboundary map associated to a 3-fold terminal singularity which is important in the study of deformations of singular 3-folds. We prove that this map vanishes only for quotient singularities and a $A_{1,2}/4$-singularity, that is, a terminal singularity analytically isomorphic to a $\mathbb{Z}_4$-quotient of the singularity $(x^2 + y^2 + z^3 + u^2 = 0)$.

As an application, we prove that a $\mathbb{Q}$-Fano 3-fold with terminal singularities can be deformed to one with only quotient singularities and $A_{1,2}/4$-singularities. We also treat the $\mathbb{Q}$-smoothability problem on $\mathbb{Q}$-Calabi–Yau 3-folds.

Contents

1. Introduction

We consider algebraic varieties over the complex number field $\mathbb{C}$.

This paper is a continuation of [12]. We study the $\mathbb{Q}$-smoothability of a $\mathbb{Q}$-Fano 3-fold X via certain coboundary maps of local cohomology groups associated to the singularities on X.

1.1. **$\mathbb{Q}$-smoothing of $\mathbb{Q}$-Fano 3-folds.** In this paper, *a $\mathbb{Q}$-Fano 3-fold* means a projective 3-fold with only terminal singularities whose anticanonical divisor is ample. A $\mathbb{Q}$-Fano 3-fold is an important object in the classification theory of algebraic 3-folds. It is one of the end products of the Minimal Model Program. Toward the classification of $\mathbb{Q}$-Fano 3-folds, it is fundamental to study their deformations.

Locally, a 3-fold terminal singularity has *a $\mathbb{Q}$-smoothing*, that is, it can be deformed to a variety with only quotient singularities. In general, local deformations of singularities may not lift to a global deformation of a projective 3-fold as shown for Calabi–Yau 3-folds (cf. [8, Example 5.8]). Nevertheless, Altınok–Brown–Reid

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([1, 4.8.3]) conjectured that a $\mathbb{Q}$-Fano 3-fold has a $\mathbb{Q}$-smoothing. (See Example 3.2 for an example of a $\mathbb{Q}$-smoothing.) This conjecture aims to reduce the classification of $\mathbb{Q}$-Fano 3-folds to those with only quotient singularities. For example, there are several papers (cf. [2], [15]) on the classification of certain $\mathbb{Q}$-Fano 3-folds with only quotient singularities.

Previously, deformations of $\mathbb{Q}$-Fano 3-folds are treated in several papers (cf. [9], [6], [16], [12]). In [12, Theorem 1.5], the author proved that a $\mathbb{Q}$-Fano 3-fold with only "ordinary" terminal singularities has a $\mathbb{Q}$-smoothing. (See Definition 2.1 for the ordinariness of the singularity.) In this article, we treat the remaining case, that is, a $\mathbb{Q}$-Fano 3-fold with non-ordinary terminal singularities. We can deform the non-ordinary terminal singularities except one special singularity as follows.

Theorem 1.1. *A $\mathbb{Q}$-Fano 3-fold can be deformed to one with only quotient singularities and $A_{1,2}/4$-singularities.*

Here, an $A_{1,2}/4$-*singularity* means a singularity analytically isomorphic to

$$0 \in (x^2 + y^2 + z^3 + u^2 = 0)/\mathbb{Z}_4 \subset \mathbb{C}^4/\mathbb{Z}_4(1, 3, 2, 1),$$

where x, y, z, u are coordinates on $\mathbb{C}^4$ and $\mathbb{C}^4/\mathbb{Z}_4(1, 3, 2, 1)$ is the quotient of $\mathbb{C}^4$ by an action of $\mathbb{Z}_4 = \langle \sigma \rangle$ as follows:

$$\sigma \cdot (x, y, z, u) = (\sqrt{-1}x, -\sqrt{-1}y, -z, \sqrt{-1}u).$$

Although we do not know how to deal with $A_{1,2}/4$-singularities, we believe that Theorem 1.1 is useful for the classification.

Remark 1.2. The author studied a deformation of a $\mathbb{Q}$-Fano 3-fold with its anticanonical element in [12] and [13]. In [13, Theorem 1.3], it is proved that, if a $\mathbb{Q}$-Fano 3-fold X has a member $D \in |-K_X|$ with only isolated singularities, then X has a $\mathbb{Q}$-smoothing. In the proof, it is necessary to use [12, Theorem 1.9] and Theorem 1.1 in this paper.

The existence of an elephant with mild singularities is discussed in [13, Section 4] by showing several examples of $\mathbb{Q}$-Fano 3-folds.

1.2. **Methods of the proof.** We use a method which is used in [12, Theorem 3.5]. Let (U, p) be a germ of a 3-fold terminal singularity. The key tool of our method is the coboundary map ϕ_U associated to some local cohomology group on a birational modification $\tilde{U} \to U$. (See (2) for the definition of ϕ_U.) If this map is nonzero, it is useful for finding a smoothing or a $\mathbb{Q}$-smoothing of a projective 3-fold. (cf. [10], [6], [12]) The following purely local statement is the main result of Section 2.

Theorem 1.3. *Let (U, p) be a germ of a 3-fold terminal singularity which is not a quotient singularity.*

Then $\phi_U = 0$ if and only if (U, p) is an $A_{1,2}/4$-singularity.

The map ϕ_U is known to be nonzero when (U, p) is Gorenstein ([10, Theorem 1.1]) or (U, p) is an ordinary singularity ([6], [12]). We calculate the coboundary map for a non-ordinary singularity.

Let us mention about the proof of Theorem 1.3. Since a terminal singularity (U, p) of index r is a $\mathbb{Z}_r$-quotient of a hypersurface singularity (V, q), the set $T^1_{(U,p)}$ of first order deformations of (U, p) is the $\mathbb{Z}_r$-invariant part of $T^1_{(V,q)}$. The set $T^1_{(V,q)}$ can be written as $\mathcal{O}_{V,q}/J_{V,q}$ for the Jacobian ideal of (V, q). We calculate the map ϕ_U by using this structure and the inequality (4) proved in [10].

By Theorem 1.3 (ii), the map ϕ_U vanishes for a neighborhood U of an $A_{1,2}/4$-singularity. It seems that we need a new method to treat a $\mathbb{Q}$-Fano 3-fold with $A_{1,2}/4$-singularities. (See Remark 3.3)

1.3. **$\mathbb{Q}$-smoothing of $\mathbb{Q}$-Calabi–Yau 3-folds.** As another corollary of Theorem 1.3, we obtain a similar result for $\mathbb{Q}$-Calabi–Yau 3-folds. Here, a *$\mathbb{Q}$-Calabi–Yau 3-fold* is a normal projective 3-fold with only terminal singularities whose canonical divisor is a torsion class. Let r be the Gorenstein index of X, that is, the minimal positive integer such that $\mathcal{O}_X(rK_X) \simeq \mathcal{O}_X$. The isomorphism $\mathcal{O}_X(rK_X) \simeq \mathcal{O}_X$ determines the global index one cover $\pi\colon Y := \operatorname{Spec} \oplus_{j=0}^{r-1} \mathcal{O}_X(jK_X) \to X$.

As a consequence of Theorem 1.3 and the proof of [6, Main Theorem 1], we obtain the following.

Theorem 1.4. *Let X be a $\mathbb{Q}$-Calabi–Yau 3-fold. Assume that the global index one cover $Y \to X$ is $\mathbb{Q}$-factorial.*

Then a $\mathbb{Q}$-Calabi–Yau 3-fold X can be deformed to one with only quotient singularities and $A_{1,2}/4$-singularities.

Remark 1.5. Namikawa studied another invariant for terminal singularities and $\mathbb{Q}$-smoothability of $\mathbb{Q}$-Calabi–Yau 3-folds in his unpublished note. The invariant is $\mu(X,x)$ defined in [10, Section 2]. It seems that this invariant also vanishes for a $A_{1,2}/4$-singularity (X,x). So we do not know the $\mathbb{Q}$-smoothability of a $\mathbb{Q}$-Calabi-Yau 3-fold with $A_{1,2}/4$-singularities.

2. Calculation of coboundary maps

First, we introduce the coboundary map of local cohomology which is used in [12, 3.2] to find a $\mathbb{Q}$-smoothing of a $\mathbb{Q}$-Fano 3-fold. (See also [10, Section 1], [6, Section 4].)

Let (U,p) be a germ of a 3-fold terminal singularity. Let $\pi_U\colon (V,q) \to (U,p)$ be the index one cover. By the classification ([7], [11]), we see that (V,q) is a hypersurface singularity and π_U is étale outside p. Moreover, we have

$$(V,q) \simeq ((f=0),0) \subset (\mathbb{C}^4,0)$$

for some $f \in \mathbb{C}[x,y,z,u]$, where x,y,z,u are coordinate functions on $\mathbb{C}^4$ and f satisfies $\sigma \cdot f = \zeta_U f$ for the generator $\sigma \in G := \operatorname{Gal}(V/U) \simeq \mathbb{Z}_r$ and $\zeta_U = \pm 1$.

We define the ordinariness of a terminal singularity as follows.

Definition 2.1. Let (U,p) be a germ of a 3-fold terminal singularity. The germ (U,p) is called *ordinary* (resp. *non-ordinary*) if $\zeta_U = 1$ (resp. $\zeta_U = -1$).

Remark 2.2. Let (U,p) be a germ of a non-ordinary terminal singularity. By the classification ([7], [11]), we have

$$(U,p) \simeq ((x^2+y^2+g(z,u)=0),0)/\mathbb{Z}_4 \subset (\mathbb{C}^4/\mathbb{Z}_4,0), \tag{1}$$

where $g(z,u) \in \mathfrak{m}^2_{\mathbb{C}^4,0}$ is some $\mathbb{Z}_4$-semi-invariant polynomial in z,u and $\sigma \in \mathbb{Z}_4$ acts on $\mathbb{C}^4$ by $\sigma \cdot (x,y,z,u) \mapsto (\sqrt{-1}x, -\sqrt{-1}y, -z, \sqrt{-1}u)$.

Let (U,p) be a germ of a 3-fold terminal singularity and V its index one cover with the $\mathbb{Z}_r$-action as above. Let $\nu\colon \tilde{V} \to V$ be a $\mathbb{Z}_r$-equivariant resolution such that its exceptional divisor $F \subset \tilde{V}$ has SNC support and $\tilde{V} \setminus F \simeq V \setminus \{q\}$. Let $V' := V \setminus \{q\}$ and

$$\tau_V\colon H^1(V', \Omega^2_{V'}(-K_{V'})) \to H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F-\nu^* K_V))$$

the coboundary map of the local cohomology. Note that the sheaf $\mathcal{O}_V(-K_V)$ and $\mathcal{O}_V$ are isomorphic as sheaves, but not isomorphic as $\mathbb{Z}_r$-equivariant sheaves. Let $\tilde{\pi}\colon \tilde{V} \to \tilde{U} := \tilde{V}/\mathbb{Z}_r$ be the finite morphism induced by π and $E \subset \tilde{U}$ the exceptional locus of the birational morphism $\mu\colon \tilde{U} \to U$ induced by ν. Let $U' := U \setminus \{p\}$ and $\mathcal{F}_U^{(0)}$ the $\mathbb{Z}_r$-invariant part of $\tilde{\pi}_* \Omega^2_{\tilde{V}}(\log F)(-F - \nu^* K_V)$. Then we have the coboundary map

$$\phi_U\colon H^1(U', \Omega^2_{U'}(-K_{U'})) \to H^2_E(\tilde{U}, \mathcal{F}_U^{(0)}) \tag{2}$$

which is the $\mathbb{Z}_r$-invariant part of τ_V. We shall study these coboundary maps τ_V and ϕ_U in this section.

For an ordinary terminal singularity, we can calculate the map ϕ_U as follows.

Theorem 2.3. *(cf.* [12, Lemma 3.4]*) Let (U, p) be a germ of a 3-fold ordinary terminal singularity which is not a quotient singularity. Then we have $\phi_U \neq 0$.*

In the following, we prepare ingredients for calculating ϕ_U for a germ (U, p) of a non-ordinary terminal singularities.

We have $H^1_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F)) = 0$ by the proof of [14, Theorem 4]. We also have $H^2(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F)) = 0$ by the Guillén–Navarro Aznar–Puerta–Steenbrink vanishing theorem. Thus we have an exact sequence

$$\begin{aligned} 0 \to H^1(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F - \nu^* K_V)) &\to H^1(V', \Omega^2_{V'}(-K_{V'})) \\ &\overset{\tau_V}{\to} H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F - \nu^* K_V)) \to 0 \end{aligned} \tag{3}$$

The following inequality proved in [10] is useful for the calculation of the coboundary maps.

Proposition 2.4. *We have*

$$\dim \operatorname{Ker} \tau_V \leq \dim \operatorname{Im} \tau_V. \tag{4}$$

Proof. This is proved in Remark after [10, Theorem (1.1)]. Let us recall the proof for the convenience of the reader.

By the exact sequence (3), it is enough to show that

$$h^1(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F)) \leq h^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F)).$$

We have a surjection

$$H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F)) \to H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F))$$

since we have $H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F) \otimes \mathcal{O}_F) = \operatorname{Gr}^2_F H^5_{\{q\}}(V, \mathbb{C}) = 0$. By the local duality, we have

$$H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F))^* \simeq H^1(\tilde{V}, \Omega^1_{\tilde{V}}(\log F)(-F)).$$

Moreover we see that the differential homomorphism

$$d\colon H^1(\tilde{V}, \Omega^1_{\tilde{V}}(\log F)(-F)) \to H^1(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F))$$

is surjective by studying the spectral sequence

$$H^q(\tilde{V}, \Omega^p_{\tilde{V}}(\log F)(-F)) \Rightarrow \mathbb{H}^{p+q}(\tilde{V}, \Omega^\bullet_{\tilde{V}}(\log F)(-F)) = 0$$

as in the proof of [10, Theorem (1.1)]. Thus we obtain relations

$$\begin{aligned} h^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F)) \geq h^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)) &= h^1(\tilde{V}, \Omega^1_{\tilde{V}}(\log F)(-F)) \\ &\geq h^1(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F)) \end{aligned} \tag{5}$$

and this implies (4). □

Let $T^1_{(V,q)}$, $T^1_{(U,p)}$ be the sets of first order deformations of the germs (V,q) and (U,p) respectively. Recall that we have an isomorphism $T^1_{(V,q)} \simeq \mathcal{O}_{V,q}/J_{V,q}$ of $\mathcal{O}_{V,q}$-modules for the Jacobian ideal $J_{V,q} \subset \mathcal{O}_{V,q}$. Hence we have a surjective $\mathcal{O}_{V,q}$-module homomorphism $\varepsilon \colon \mathcal{O}_{V,q} \to T^1_{(V,q)}$ which sends $h \in \mathcal{O}_{V,q}$ to the corresponding deformation $\varepsilon_h \in T^1_{(V,q)}$. Also we have a commutative diagram

$$\begin{array}{ccc} T^1_{(U,p)} & \xrightarrow{\simeq} & H^1(U', \Omega^2_{U'}(-K_{U'})) \\ \big\downarrow & & \big\downarrow \\ T^1_{(V,q)} & \xrightarrow{\simeq} & H^1(V', \Omega^2_{V'}(-K_{V'})), \end{array}$$

where the horizontal isomorphisms are restrictions by open immersions and the upper terms inject into the lower terms as the $\mathbb{Z}_r$-invariant parts. Note that we have the horizontal isomorphisms since $\{p\} \hookrightarrow U$ and $\{q\} \hookrightarrow V$ have codimensions 3, and the spaces U and V are Cohen-Macaulay. Thus we identify $T^1_{(V,q)}, T^1_{(U,p)}$ and $H^1(V', \Omega^2_{V'}(-K_{V'})), H^1(U', \Omega^2_{U'}(-K_{U'}))$ respectively via these isomorphisms.

We use the following notion of right equivalence ([4, Definition 2.9]).

Definition 2.5. Let $\mathbb{C}\{x_1, \ldots, x_n\}$ be the convergent power series ring of n variables. Let $f, g \in \mathbb{C}\{x_1, \ldots, x_n\}$.

We say that f is *right equivalent* to g if there exists an automorphism φ of $\mathbb{C}\{x_1, \ldots, x_n\}$ such that $\varphi(f) = g$. We write this as $f \overset{r}{\sim} g$.

By using these ingredients, we calculate the coboundary map for a non-ordinary singularity. The following theorem and Theorem 2.3 imply Theorem 1.3.

Theorem 2.6. *Let (U,p) be a germ of a non-ordinary 3-fold terminal singularity which is not a quotient singularity.*

(i) *Assume that the index one cover $(V,q) \not\simeq ((x^2+y^2+z^3+u^2=0),0)$. Then we have $\phi_U \neq 0$.*

(ii) *Assume that $(V,q) \simeq ((x^2+y^2+z^3+u^2=0),0)$. Then $\phi_U = 0$.*

Proof. (i) Suppose that $\phi_U = 0$. We show the claim by contradiction. We can write $g(z,u) = \sum a_{i,j} z^i u^j \in \mathbb{C}[z,u]$ for some $a_{i,j} \in \mathbb{C}$ for $i,j \geq 0$. Since the generator $\sigma \in \mathbb{Z}_4$ acts on g by $\sigma \cdot g = -g$ and on $z^i u^j$ by $\sigma \cdot z^i u^j = \sqrt{-1}^{2i+j} z^i u^j$, we see that $a_{i,j} \neq 0$ only if

$$2i + j \equiv 2 \mod 4. \tag{6}$$

Let $J_g := (\frac{\partial g}{\partial z}, \frac{\partial g}{\partial u}) \subset \mathbb{C}[z,u]$ be the Jacobian ideal of the polynomial g. Note that we have $T^1_{(V,q)} \simeq \mathbb{C}[z,u]/(g, J_g)$ since $\varepsilon_x = \varepsilon_y = 0 \in T^1_{(V,q)}$.

(**Case 1**) Assume that $a_{0,2} \neq 0$. We can write

$$g(z,u) = u^2(1 + h_1(z,u)) + h_2(z)$$

for some polynomials $h_1(z,u) \in (z,u) \subset \mathbb{C}[z,u]$ and $h_2(z) \in (z) \subset \mathbb{C}[z]$. Thus $g(z,u) \in \mathcal{O}_{\mathbb{C}^2,0}$ is right equivalent to $u^2 + h_2(z)$. We see that $h_2(z) \in \mathcal{O}_{\mathbb{C},0}$ is right equivalent to z^{2i_0+1} for some positive integer i_0 since $(g=0)$ has an isolated singularity and by the condition (6). Thus we have

$$(V,q) \simeq ((x^2 + y^2 + z^{2i_0+1} + u^2 = 0), 0).$$

If $i_0 = 1$, it contradicts the assumption $(V, q) \not\simeq ((x^2 + y^2 + z^3 + u^2 = 0), 0)$. Hence we have $i_0 \geq 2$. By calculating the partial derivatives of $x^2 + y^2 + z^{2i_0+1} + u^2$, we see that $\varepsilon_1, \varepsilon_z, \varepsilon_{z^2} \in T^1_{(V,q)}$ are linearly independent and

$$\dim T^1_{(V,q)} \geq 3.$$

On the other hand, we see that $\tau_V(\varepsilon_z) = 0$ since we assumed $\phi_U = 0$ and $\varepsilon_z \in T^1_{(U,p)}$. By this and the fact that τ_V is an $\mathcal{O}_{V,q}$-module homomorphism, we obtain a surjection $\mathbb{C}[z, u]/(z, u) \to \operatorname{Im} \tau_V$ since $\varepsilon_u = 0$. By this surjection and $\mathbb{C}[z, u]/(z, u) \simeq \mathbb{C}$, we obtain $\dim \operatorname{Im} \tau_V \leq 1$. By this and the inequality (4), we obtain an inequality

$$\dim T^1_{(V,q)} = \dim \operatorname{Im} \tau_V + \dim \operatorname{Ker} \tau_V \leq 1 + 1 = 2$$

and it is a contradiction.

(**Case 2**) Assume that $a_{0,2} = 0$. Then we see that $a_{i,j} \neq 0$ only if $2i + j \geq 6$ by (6). Note that a monomial $z^i u^j$ with $2i + j \geq 6$ is some multiple of either z^3, z^2u^2, zu^4 or u^6. By computing partial derivatives of these monomials, we see that $(g, J_g) \subset (z^2, zu^2, u^4)$. Thus we see that $\varepsilon_1, \varepsilon_z, \varepsilon_{zu}, \varepsilon_u, \varepsilon_{u^2}, \varepsilon_{u^3} \in T^1_{(V,q)}$ are linearly independent and we obtain

$$\dim T^1_{(V,q)} \geq 6. \tag{7}$$

On the other hand, by the assumption $\phi_U = 0$, we have $\tau_V(\varepsilon_z) = 0, \tau_V(\varepsilon_{u^2}) = 0$ since $\varepsilon_z, \varepsilon_{u^2} \in T^1_{(U,p)}$. Thus we have a relation $(z, u^2) \subset \operatorname{Ker} \tau_V \circ \varepsilon \subset \mathcal{O}_{V,q}$ and obtain a surjection $\mathbb{C}[z, u]/(z, u^2) \to \operatorname{Im} \tau_V$. This implies an inequality $\dim \operatorname{Im} \tau_V \leq \dim \mathbb{C}[z, u]/(z, u^2) = 2$. By this inequality and the inequality (4), we have an inequality

$$\dim T^1_{(V,q)} = \dim \operatorname{Ker} \tau_V + \dim \operatorname{Im} \tau_V \leq 2 + 2 = 4.$$

This contradicts (7).

Hence we obtain $\phi_U \neq 0$ and finish the proof of (i).

(ii) For non-negative integers i, j, we set

$$b^{i,j} := \dim H^j(\tilde{V}, \Omega^i_{\tilde{V}}(\log F)(-F)),$$

$$l^{i,j} := \dim H^j(F, \Omega^i_{\tilde{V}}(\log F) \otimes \mathcal{O}_F).$$

Let $s_k(V, q)$ for $k = 0, 1, 2, 3$ be the Hodge number of the Milnor fiber of (V, q) as in [14, Section 4]. By [14, Theorem 6], we have $s_0 = 0, s_1 = b^{1,1}, s_2 = b^{1,1} + l^{1,1}$ and $s_3 = l^{0,2}$. We see that $l^{0,2} = 0$ by [14, Lemma 2]. Since the sum $\sum_{k=0}^3 s_k(V, q)$ is the Milnor number of (V, q), we obtain $2b^{1,1} + l^{1,1} = 2$. Since $b^{1,1} \neq 0$ by [10, Theorem 2.2], we obtain

$$b^{1,1} = 1, \quad l^{1,1} = 0. \tag{8}$$

There exists an exact sequence

$$\begin{aligned} H^0(F, \Omega^1_{\tilde{V}}(\log F) \otimes \mathcal{O}_F) \to H^1(\tilde{V}, \Omega^1_{\tilde{V}}(\log F)(-F)) \to H^1(\tilde{V}, \Omega^1_{\tilde{V}}(\log F)) \\ \to H^1(F, \Omega^1_{\tilde{V}}(\log F) \otimes \mathcal{O}_F). \end{aligned} \tag{9}$$

Since $l^{1,0} = 0$ by [14, Lemma 1], the both outer terms are zero and the homomorphism in the middle is an isomorphism. By this and (8), we have

$$\mathbb{C} \simeq H^1(\tilde{V}, \Omega^1_{\tilde{V}}(\log F)) \simeq H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F))^*. \tag{10}$$

Suppose that $\tau_V(\varepsilon_z) \neq 0$. Then $\varepsilon_z \notin \operatorname{Ker} \tau_V$. This implies that $\operatorname{Ker} \tau_V = 0$ since $T^1_{(V,q)} \simeq \mathbb{C}[z]/(z^2)$ as $\mathbb{C}[z]$-modules. Thus $\mathbb{C}^2 \simeq \operatorname{Im} \tau_V \simeq H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F))$. This contradicts (10).

Thus we obtain $\tau_V(\varepsilon_z) = 0$. Since $T^1_{(U,p)} \simeq \mathbb{C}$ is generated by ε_z, we see that $\phi_U = 0$. Thus we finish the proof of (ii). □

Now we prepare another coboundary map to study $\mathbb{Q}$-smoothability of a $\mathbb{Q}$-Calabi–Yau 3-fold.

Let (U, p) be a germ of a 3-fold terminal singularity and $V, \tilde{V}, F, \tilde{U}$ as before. We have the coboundary map

$$\bar{\tau}_V \colon H^1(V', \Omega^2_{V'}(-K_{V'})) \to H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(-\nu^* K_V))$$

and this fits in the commutative diagram

$$\begin{array}{ccc} H^1(V', \Omega^2_{V'}(-K_{V'})) & \xrightarrow{\bar{\tau}_V} & H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(-\nu^* K_V)) \\ \downarrow{\scriptstyle \tau_V} & \nearrow{\scriptstyle \tau'_V} & \\ H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F - \nu^* K_V)), & & \end{array} \tag{11}$$

where the injectivity of τ'_V is proved in the proof of [10, Theorem 1.1].

Let $\bar{\mathcal{F}}^{(0)}_U := (\tilde{\pi}_* \Omega^2_{\tilde{V}}(-\nu^* K_V))^{\mathbb{Z}_r}$ be the $\mathbb{Z}_r$-invariant part. Let

$$\bar{\phi}_U \colon H^1(U', \Omega^2_{U'}(-K_{U'})) \to H^2_E(\tilde{U}, \bar{\mathcal{F}}^{(0)}_U)$$

be the coboundary map. It is the $\mathbb{Z}_r$-invariant part of $\bar{\tau}_V$. As the $\mathbb{Z}_r$-invariant part of the diagram (11), we obtain the following diagram;

$$\begin{array}{ccc} H^1(U', \Omega^2_{U'}(-K_{U'})) & \xrightarrow{\bar{\phi}_U} & H^2_E(\tilde{U}, \bar{\mathcal{F}}^{(0)}_U) \\ \downarrow{\scriptstyle \phi_U} & \nearrow{\scriptstyle \phi'_U} & \\ H^2_E(\tilde{U}, \mathcal{F}^{(0)}_U). & & \end{array}$$

By these arguments, we obtain the following result as a corollary of Theorem 2.3 and Theorem 2.6.

Corollary 2.7. *Let (U, p) be a germ of a 3-fold terminal singularity which is not a quotient singularity.*

Then $\bar{\phi}_U = 0$ if and only if the germ (U, p) is an $A_{1,2}/4$-singularity.

We use the blow-down morphism of deformations by a resolution $\tilde{V} \to V$ to find a $\mathbb{Q}$-smoothing. It is already used in several papers on deformations of singular 3-folds. (cf. [10], [9], [12])

Let

$$\nu_* \colon H^1(\tilde{V}, \Omega^2_{\tilde{V}}(-K_{\tilde{V}})) \to H^1(V', \Omega^2_{V'}(-K_{V'}))$$

be the restriction homomorphism by the open immersion $V' \hookrightarrow \tilde{V}$. We use this notation since there is a commutative diagram

$$\begin{array}{ccc} H^1(\tilde{V}, \Omega^2_{\tilde{V}}(-K_{\tilde{V}})) & \xrightarrow{\nu_*} & H^1(V', \Omega^2_{V'}(-K_{V'})) \\ \downarrow \simeq & & \downarrow \simeq \\ T^1_{\tilde{V}} & \longrightarrow & T^1_V, \end{array}$$

where the lower horizontal homomorphism is the blow-down homomorphism of deformations ([17]). We can prove the relation

$$\operatorname{Im} \nu_* \subset \operatorname{Ker} \tau_V = \operatorname{Ker} \bar{\tau}_V \tag{12}$$

by the same argument as in [12, Claim 3.7].

3. Application to $\mathbb{Q}$-smoothing problems

In [12, Theorem 3.2], we proved the following.

Theorem 3.1. *Let X be a $\mathbb{Q}$-Fano 3-fold.*

Then there exists a deformation $\mathcal{X} \to \Delta^1$ of X over a unit disc Δ^1 such that the general fiber $\mathcal{X}_t$ for $t \in \Delta^1 \setminus \{0\}$ satisfies the following; For each singular point $p \in \mathcal{X}_t$ and its Stein neighborhood U_p, the coboundary map ϕ_{U_p} vanishes.

As an application of this result and Theorem 2.6, we obtain a proof of Theorem 1.1 as follows.

Proof of Theorem 1.1. By Theorem 3.1, we can deform a $\mathbb{Q}$-Fano 3-fold X to one with only singularities $p_1, \ldots, p_l$ such that $\phi_{U_i} = 0$, where U_i is a Stein neighborhood of p_i for $i = 1, \ldots, l$. By Theorem 1.3, such a terminal singularity is either a quotient singularity or an $A_{1,2}/4$-singularity. Thus we finish the proof. □

Example 3.2. There exists an example of a $\mathbb{Q}$-Fano 3-fold with an $A_{1,2}/4$-singularity. This example has a $\mathbb{Q}$-smoothing.

Let $X := X_{10} \subset \mathbb{P}(1,1,2,3,4)$ be a weighted hypersurface of degree 10 defined by the polynomial

$$f_{X_{10}} := w^2(x_1^2 + x_2^2) + w(y^3 + z^2) + x_1^{10} + x_2^{10} + y^5 + z^3 x_1,$$

where x_1, x_2, y, z, w are coodinates of weights $1,1,2,3,4$, respectively. By perturbing the coefficients of the polynomial, we obtain that

$$\operatorname{Sing} X = \{[0:0:0:1:0], [0:0:0:0:1]\},$$

$p_z := [0:0:0:1:0]$ is a $1/3(1,1,2)$-singularity and $p_w := [0:0:0:0:1]$ is an $A_{1,2}/4$-singularity. Let

$$\mathcal{X} := (f_{X_{10}} + t \cdot yw^2 = 0) \subset \mathbb{P}(1,1,2,3,4) \times \mathbb{A}^1 \to \mathbb{A}^1$$

be a deformation of X, where t is a coordinate of $\mathbb{A}^1$. Then we see that $\mathcal{X}$ is a $\mathbb{Q}$-smoothing of X. The general fiber $\mathcal{X}_t$ has two $1/2(1,1,1)$-singularities, a $1/3(1,1,2)$-singularity and a $1/4(1,3,1)$-singularity.

Remark 3.3. We give a comment on a $\mathbb{Q}$-Fano 3-fold with $A_{1,2}/4$-singularities.

Let X be a $\mathbb{Q}$-Fano 3-fold. The local-to-global spectral sequence of Ext groups induces an exact sequence

$$\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) \to H^0(X, \underline{\mathrm{Ext}}^1(\Omega^1_X, \mathcal{O}_X)) \to H^2(X, \Theta_X),$$

where $\underline{\mathrm{Ext}}^1$ is a sheaf of Ext groups. Recall that $\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X)$ and $H^0(X, \underline{\mathrm{Ext}}^1(\Omega^1_X, \mathcal{O}_X))$ are the sets of first order deformations of X and the singularities on X, respectively. Thus, if we have $H^2(X, \Theta_X) = 0$, we see that X is $\mathbb{Q}$-smoothable.

However, this approach does not work in general. Namikawa constructed an example of a Fano 3-fold X with $A_{1,2}$-singularities such that $H^2(X, \Theta_X) \neq 0$ ([9, Example 5]). Here an $A_{1,2}$-singularity is a hypersurface singularity locally isomorphic to $(x^2 + y^2 + z^3 + u^2 = 0) \subset \mathbb{C}^4$. This X has a smoothing. The author expects that there also exists a $\mathbb{Q}$-Fano 3-fold X with $A_{1,2}/4$-singularities such that $H^2(X, \Theta_X) \neq 0$.

Thus we do not know $\mathbb{Q}$-smoothability of a $\mathbb{Q}$-Fano 3-fold with $A_{1,2}/4$-singularities.

As another application of Theorem 2.6, we obtain a proof of Theorem 1.4 as follows.

Proof of Theorem 1.4. The proof is a modification of the proof of [6, Main Theorem 1]. We sketch the proof for the convenience of the reader.

First we prepare notations to define the diagram (13).

Let $p_1, \ldots, p_l \in X$ be the non-quotient singularities and $U_1, \ldots, U_l$ their Stein neighborhoods. Let $\nu\colon \tilde{Y} \to Y$ be a $\mathbb{Z}_r$-equivariant resolution such that its exceptional divisor F is a SNC divisor and $\tilde{Y} \setminus F \simeq Y \setminus \nu^{-1}(\{p_1, \ldots, p_l\})$. Let $\tilde{\pi}\colon \tilde{Y} \to \tilde{X} := \tilde{Y}/\mathbb{Z}_r$ be the quotient morphism and $\mu\colon \tilde{X} \to X$ the induced birational morphism with the exceptional divisor E.

Let $V_i := \pi^{-1}(U_i)$, $\tilde{V}_i := \nu^{-1}(V_i)$, $F_i := F \cap \tilde{V}_i$ and $\nu_i := \nu|_{\tilde{V}_i}\colon \tilde{V}_i \to V_i$ be the restrictions. Let $\tilde{U}_i := \mu^{-1}(U_i)$, $E_i := E \cap \tilde{U}_i$ and $\tilde{\pi}_i := \tilde{\pi}|_{\tilde{V}_i}\colon \tilde{V}_i \to \tilde{U}_i$ the induced finite morphism. Let $\bar{\mathcal{F}}^{(0)} := \left(\tilde{\pi}_* \Omega^2_{\tilde{Y}}(-\nu^* K_V)\right)^{\mathbb{Z}_r}$ be the $\mathbb{Z}_r$-invariant part and $\bar{\mathcal{F}}^{(0)}_i := \bar{\mathcal{F}}^{(0)}|_{\tilde{U}_i}$ its restriction.

Then we have the diagram

$$\begin{array}{ccccc} H^1(X', \Omega^2_{X'}(-K_{X'})) & \xrightarrow{\oplus \psi_i} & \oplus_{i=1}^l H^2_{E_i}(\tilde{X}, \bar{\mathcal{F}}^{(0)}) & \xrightarrow{\oplus B_i} & H^2(\tilde{X}, \bar{\mathcal{F}}^{(0)}) \\ \downarrow{\scriptstyle \oplus p_{U_i}} & & \oplus\varphi_i \downarrow \simeq & & \\ \oplus_{i=1}^l H^1(U'_i, \Omega^2_{U'_i}(-K_{U'_i})) & \xrightarrow{\bar{\phi}_i} & \oplus_{i=1}^l H^2_{E_i}(\tilde{U}_i, \bar{\mathcal{F}}^{(0)}_i), & & \end{array} \tag{13}$$

where $X' := X \setminus \{p_1, \ldots, p_l\}$ and $U'_i := U_i \cap X'$.

Let $V'_i := \pi^{-1}(U'_i)$. Note that $B_i \circ \varphi_i^{-1} \circ \bar{\phi}_i$ is the $\mathbb{Z}_r$-invariant part of the composition

$$\begin{aligned} H^1(V'_i, \Omega^2_{V'_i}(-K_{V'_i})) \to H^2_{F_i}(\tilde{V}_i, \Omega^2_{\tilde{V}_i}(-\nu_i^* K_{V_i})) \to H^2_{F_i}(\tilde{Y}, \Omega^2_{\tilde{Y}}(-\nu^* K_Y)) \\ \to H^2(\tilde{Y}, \Omega^2_{\tilde{Y}}(-\nu^* K_Y)). \end{aligned} \tag{14}$$

We see that this is zero by [10, Proposition 1.2] since we assumed that Y is $\mathbb{Q}$-factorial. Thus we also see that $B_i \circ \varphi_i^{-1} \circ \bar{\phi}_i = 0$.

There exists an element $\eta_i \in H^1(U_i', \Omega^2_{U_i'}(-K_{U_i'}))$ such that $\bar{\phi}_i(\eta_i) \neq 0$ by Theorem 1.3. Since $B_i \circ \varphi_i^{-1} \circ \bar{\phi}_i(\eta_i) = 0$, there exists $\eta \in H^1(X', \Omega^2_{X'}(-K_{X'}))$ such that $\psi_i(\eta) = \varphi_i^{-1}(\phi_i(\eta_i))$. By the relation (12) and $p_{U_i}(\eta) - \eta_i \in \operatorname{Ker} \bar{\phi}_i$, we see that $p_{U_i}(\eta) \notin \operatorname{Im}(\nu_i)_*$, where we use the inclusion $H^1(U_i', \Omega^2_{U_i'}(-K_{U_i'})) \subset H^1(V_i', \Omega^2_{V_i'}(-K_{U_i'}))$. By arguing as in the proof of [12, Theorem 3.5], we can deform singularity $p_i \in U_i$ as long as $\bar{\phi}_i \neq 0$. By Corollary 2.7, we obtain a required deformation since the deformations of a $\mathbb{Q}$-Calabi–Yau 3-fold are unobstructed ([8, Theorem A]).

□

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