VINBERG'S θ-GROUPS AND RIGID CONNECTIONS

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Abstract. Let G be a simple complex group of adjoint type. In his unpublished work, Z. Yun associated to each θ-group $(G_0, \mathfrak{g}_1)$ and a vector $X \in \mathfrak{g}_1$ a flat G-connection ∇^X on the trivial G-bundle on $\mathbb{P}^1 - \{0, \infty\}$, generalizing the construction of Frenkel and Gross in [FG]. In this paper we study the local monodromy of those flat G-connections and compute the de Rham cohomology of ∇^X with values in the adjoint representations of G. In particular, we show that in many cases the connection ∇^X is cohomologically rigid.

1. Introduction

1.1. **The goal.** Let G be a simple complex algebraic group of adjoint type. Motivated by Langlands correspondence, Frenkel and Gross [FG] constructed a flat G-connection ∇ on the trivial G-bundle on $\mathbb{P}^1 - \{0, \infty\}$ with following remarkable properties:

(1) ∇ has a regular singularity at 0, and the residue is a regular nilpotent element in the Lie algebra $\mathfrak{g}$ of G.
(2) ∇ has an irregular singularity at ∞ with slope $1/h$, where h is the Coxeter number of G (see [FG, §5] or [CK, §2.2]) for the definition of slope).
(3) ∇ is *cohomologically rigid*, i.e., we have $H^*(\mathbb{P}^1, j_{!*}\nabla^{\mathrm{Ad}}) = 0$, here ∇^{Ad} is the D-module defined by the connection ∇ with values in the adjoint representation of G and $j_{!*}\nabla^{\mathrm{Ad}}$ is the intermediate extension of the D-module ∇^{Ad} to $\mathbb{P}^1$ along $j : \mathbb{P}^1 - \{0, \infty\} \to \mathbb{P}^1$.

The construction used the θ-group $(G_0, \mathfrak{g}_1)$ studied by Vinberg and his school, which comes from a $\mathbb{Z}/h\mathbb{Z}$-grading on $\mathfrak{g}$.

In his unpublished work, Z.Yun generalized the Frenkel-Gross's construction to *all* θ-groups. More precisely, starting with a θ-group $(G_0, \mathfrak{g}_0)$, he constructed a family of flat G-connections ∇^X on $\mathbb{P}^1 - \{0, \infty\}$ parametrizing by vectors $X \in \mathfrak{g}_0$. We called ∇^X the θ-connection associated to $X \in \mathfrak{g}_1$.

The goal of this paper is to study properties of θ-connections ∇^X. In more detail, recall that each θ-group $(G_0, \mathfrak{g}_1)$ corresponds to a torsion automorphism of $\mathfrak{g} = \mathrm{Lie}G$, which we also denote by $\theta \in \mathrm{Aut}(\mathfrak{g})$. Let σ be the image of θ in $\mathrm{Out}(\mathfrak{g})$, the group of outer automorphism of $\mathfrak{g}$. We establish the following properties of ∇^X, parallel to the properties (1), (2) and (3) above:

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(1) ∇^X has a regular singularity at 0, and the residue $\mathrm{Res}(\nabla^X)$ is a nilpotent element in the Lie algebra $\mathfrak{g}^\sigma$. Moreover, for generic vectors $X \in \mathfrak{g}_1$ the residues $\mathrm{Res}(\nabla^X)$ lie in a single nilpotent orbit in $\mathfrak{g}^\sigma$.
(2) ∇^X has an irregular singularity at ∞ with slope e/m for any semi-simple $X \in \mathfrak{g}_1$. Here m (resp. e) is the order of $\theta \in \mathrm{Aut}(\mathfrak{g})$ (resp. $\sigma \in \mathrm{Out}(\mathfrak{g})$).
(3) Assume θ is *stable* with normalized *Kac coordinates* $s_0 = 1$ (see §2.2 for the definition of stable automorphism and normalized Kac coordinates). Then for any stable element $X \in \mathfrak{g}_1$ the connection ∇^X is *cohomologically rigid*.

1.2. Let me explain briefly how these properties are obtained in the untwisted case $\sigma = id$. Properties (1) and (2) follow basically from the construction. To prove property (3), we need to show that the cohomology groups $H^*(\mathbb{P}^1, j_{!*}\nabla^{X,\mathrm{Ad}})$ vanishes (see Definition 4.1). Here we follow an argument in [FG]. First, it follows from a general result in [FG, §7] that those cohomology groups are isomorphic to kernels of certain $\mathbb{C}$-linear maps on the loop algebra $\tilde{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}]$. So we reduce to show that kernels of those maps vanish. In the case of [FG], the authors observe that if we consider the *principal grading* on the loop algebra $\tilde{\mathfrak{g}}$ (see Example 3.1), then in terms of a homogeneous basis of $\tilde{\mathfrak{g}}$ (with respect to the principal grading), the relevant maps become more tractable and the desired cohomology vanishing follows from Kac's Theorem on *principal* Heisenberg subalgebras of affine Kac-Moody algebras. Now our observation is that for general θ-connections ∇^X, if we consider the *Kac grading* on $\tilde{\mathfrak{g}}$ corresponding to the torsion automorphism θ (see §3), then the relevant maps again become more tractable and the desired cohomology vanishing follows from the results in [Kac, V, RLYG, RY] about gradings on Lie algebras (known as Vinberg's theory of θ-groups), and a generalization of Kac's Theorem to *general* Heisenberg subalgebras of affine Kac-Moody algebras (see Proposition 3.5 and Remark 4.4).

1.3. **Relation with** [Yun] **and ramified geometric Langlands.** In [Yun], starting with a stable torsion automorphism $\theta \in \mathrm{Aut}(\check{\mathfrak{g}})$ for the Langlands dual of $\mathfrak{g}$ and a stable functional $\phi \in \check{\mathfrak{g}}_1^{*,s}$, the author constructed a remarkable ℓ-adic G-local system $\mathrm{KL}_G(\phi)$ on $\mathbb{P}^1 - \{0, \infty\}$, which generalized his early work in [HNY] with Heinloth and Ngô. This ℓ-adic local system is tamely ramified at 0 and ramified at ∞. He furthermore described the monodromy of $\mathrm{KL}_G(\phi)$ at 0 and conditionally deduced the cohomologically rigid of $\mathrm{KL}_G(\phi)$ (see [Yun, Theorem 4.7 and Proposition 5.2]). The construction can carry out over the complex number with ℓ-adic sheaves replaced by D-modules. Thus, starting with a stable automorphism θ of $\check{\mathfrak{g}}$ and a stable function $\phi \in \check{\mathfrak{g}}_1^{*,s}$, we get a flat G-connection $\mathrm{KL}_G(\phi)_{dR}$ on $\mathbb{P}^1 - \{0, \infty\}$. The result of this paper gives strong evidence of the following conjecture:

Conjecture 1.1 (Z.Yun and [HNY] Conjecture 2.14)**.** *There is a bijection between the set of stable linear functions $\phi \in \mathfrak{g}_1^{*,s}$ and the set of stable vectors $X \in \check{\mathfrak{g}}_1^s$, such that whenever ϕ corresponds to X under this bijection, there is a natural isomorphism between σ-twisted flat $\check{G}$-connection on $\mathbb{G}_m$*

$$\mathrm{KL}_{\check{G}}(\phi) \simeq \nabla^X.$$

The solution of the conjecture above will provide many interesting examples of geometric Langlands correspondences with wild ramifications. When $m = h$ is the Coxeter number, i.e., in the Frenkel-Gross case (see §4.3.1), the conjecture above was proved in [Zhu] using a ramified version of Beilinson-Drinfeld's work on quantization of Hitchin's integrable systems. We plan to extend the methods in [Zhu] to more general stable automorphisms.

1.4. The paper is organized as follows. In §2 we give a review of Vinberg's theory of θ-groups. In §3 we recall Kac gradings for loop algebras and Kac's theories on automorphism of loop algebras and Heisenberg subalgebras of affine Kac-Moody algebras. In §4 we recall Yun's construction of θ-connections ∇^X. We compute the residue of ∇^X at 0 and the slope and irregularity at ∞. In §5 we prove the main results of this paper: In Theorem 5.1, assume the θ-group $(G_0, \mathfrak{g}_1)$ is regular, i.e., $\mathfrak{g}_1$ contains regular semi-simple elements, we compute the de Rham cohomology of the θ-connections with values in the adjoint representation. In Theorem 5.2, assume the θ-group is stable and with normalized Kac coordinates $s_0 = 1$, we establish the cohomological rigidity of ∇^X. Finally, in §7 we give several examples of θ-connections.

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2. Gradings on simple Lie algebras

2.1. **Notation.** Let $\mathfrak{g}$ be the Lie algebra of a simple complex algebraic group G of adjoint type. Let ℓ be the rank of $\mathfrak{g}$. Let B be a Borel subgroup of G and let $T \subset B$ be maximal torus. We denote by $\mathfrak{g} = \mathrm{Lie}G$, $\mathfrak{t} = \mathrm{Lie}T$ and $\mathfrak{b} = \mathrm{Lie}B$. We let $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ (resp. $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$) denote the adjoint representation of G (resp. $\mathfrak{g}$). For any element $x \in \mathfrak{g}$ we denote by $\mathfrak{g}^x$ the kernel of $\mathrm{ad}(x)$.

Let $\mathbb{X}$ (resp. $\check{\mathbb{X}}$) be the weight lattices (resp. coweight lattices) of T, and R (resp. $\check{R}$) be the set of roots (resp. co-roots) of T in G. We fixed a pinning $(\mathbb{X}, R, \check{\mathbb{X}}, \check{R}, \{E_i\})$, where $E_i \in \mathfrak{g}$ is a root vector for the simple roots $\alpha_i \in \Delta$.

Let $\mathrm{Aut}(R)$ be the subgroup of $\mathrm{Aut}(\mathbb{X})$ preserving R and let $\mathrm{Aut}(R, \Delta)$ be the subgroup of $\mathrm{Aut}(R)$ preserving Δ. The choice of pinning induced an isomorphism $\mathrm{Aut}(\mathfrak{g}) = G \rtimes \mathrm{Aut}(R, \Delta)$. Let $\exp : V := \check{\mathbb{X}} \otimes \mathbb{R} \to T$ be the exponential map given by $\alpha(\exp(x)) = e^{2\pi i \alpha(x)}$, for all $\alpha \in \mathbb{X}$.

For any $m \in \mathbb{Z}^\times$, we let $\xi_m = e^{\frac{2\pi i}{m}}$. For any $\mathbb{C}$-vector space V, we denote by $V^* = \mathrm{Hom}(V, \mathbb{C})$ the dual of V.

2.2. **Affine simple roots.** In this subsection we collect some basic definitions and properties of twisted affine Kac-Moody algebra and affine simple roots. For details, see [Kac, §8].

Let $\sigma \in \mathrm{Aut}(R, \Delta)$ and let e be the order of σ. We have $e = 1$ or $e = 2$ (type A, D, E_6) or $e = 3$ (type D_4). Consider the affine Kac-Moody algebra $\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathbb{C}K \oplus \mathbb{C}\mathrm{d}$. Here $\tilde{\mathfrak{g}} \oplus \mathbb{C}K$ is the universal central extension of the loop algebra $\tilde{\mathfrak{g}} := \mathfrak{g}[t, t^{-1}]$ and $[\mathrm{d}, t^i \otimes x + K] = it^i \otimes x$. The automorphism σ extends to an automorphism of $\hat{\mathfrak{g}}$ by

$$\sigma(t^i \otimes x) = \xi_e^{-i} t^i \otimes \sigma(x), \quad \sigma(K) = K, \quad \sigma(\mathrm{d}) = \mathrm{d}.$$

The fixed point subalgebra

$$^{\sigma}\hat{\mathfrak{g}} := {}^{\sigma}\tilde{\mathfrak{g}} \oplus \mathbb{C}K \oplus \mathbb{C}\mathrm{d},$$

where ${}^{\sigma}\tilde{\mathfrak{g}} := \mathfrak{g}[t, t^{-1}]^{\sigma}$, is the twisted affine Kac-Moody algebra associated to $(\mathfrak{g}, \sigma)$.

Set ${}^{\sigma}\hat{\mathfrak{t}} := \mathfrak{t}^{\sigma} \oplus \mathbb{C}K \oplus \mathbb{C}\mathrm{d}$ (here $\mathfrak{t}^{\sigma}$ is the σ-fixed vectors in $\mathfrak{t}$) and define $\delta \in ({}^{\sigma}\hat{\mathfrak{t}})^*$ by $\delta|_{\mathfrak{t}^{\sigma}\oplus\mathbb{C}K} = 0$, $\delta(\mathrm{d}) = 1$. Then there is an affine root spaces decomposition of ${}^{\sigma}\hat{\mathfrak{g}}$ with respect to ${}^{\sigma}\hat{\mathfrak{t}}$

$$(1) \qquad {}^{\sigma}\hat{\mathfrak{g}} = {}^{\sigma}\hat{\mathfrak{t}} \oplus \bigoplus_{\alpha\in\Phi_{\mathrm{aff}}} {}^{\sigma}\hat{\mathfrak{g}}_{\alpha}$$

here $\Phi^{\sigma}_{\mathrm{aff}} \subset (\mathfrak{t}^{\sigma})^* \oplus \mathbb{C}\delta$ is the set of affine roots. We identify affine roots $\Phi^{\sigma}_{\mathrm{aff}}$ with affine functions on $\mathfrak{t}^{\sigma}$ by sending δ to the constant function 1.

We now recall the construction of affine simple roots $\Delta^{\sigma}_{\mathrm{aff}} \subset \Phi^{\sigma}_{\mathrm{aff}}$. Let R/σ (resp. Δ/σ) be the set of orbits in R (resp, Δ) under σ. For any orbit $a \in R/\sigma$, let β_a denote the restriction to $\mathfrak{t}^{\sigma}$ of any $\alpha \in a$. Then the collection $R^{\sigma} := \{\beta_a | a \in R/\sigma\}$ is a root system (possibly non-reduced) with basis $\Delta^{\sigma} = \{\beta_a | a \in \Delta/\sigma\}$. Let us choose a numbering

$$\Delta^{\sigma} = \{\beta_1, ..., \beta_{\ell_\sigma}\},$$

where ℓ_σ is the number of σ-orbits on the set of simple roots Δ. We define a certain positive root η as follows. If $\sigma = id$, then η is the highest root. If $\sigma \neq id$, unless (R, σ) is of type ${}^2A_{2n}$, η is the highest short root of R^{σ}; when (R, σ) is of type ${}^2A_{2n}$, η is twice the highest short root of R^{σ}. There are unique positive integers $\{b_0 = 1, b_1, ..., b_{\ell_\sigma}\}$ such that

$$(2) \qquad \eta = \sum_{i=1}^{\ell_\sigma} b_i \beta_i.$$

The sum $h_\sigma := e \sum_{i=0}^{\ell_\sigma} b_i$ is the twisted Coxeter number of (R, σ). We define

$$\beta_0 = 1/e - \eta.$$

Then $\Delta^{\sigma}_{\mathrm{aff}} := \{\beta_0, ..., \beta_{\ell_\sigma}\}$ is the set of affine simple roots associated to $(\mathfrak{g}, \sigma)$. The set $C^{\sigma} = \{x \in V^{\sigma} | \beta_i(x) > 0, \ i = 0, ..., \ell_\sigma\}$ is called the fundamental alcove. We denote by $\bar{C}^{\sigma}$ the closure of C^{σ} in V^{σ}.

2.3. **Normalized Kac coordinates.** Let $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_i$ be a grading of $\mathfrak{g}$. The grading on $\mathfrak{g}$ corresponds to a torsion automorphism $\theta = \theta' \rtimes \sigma \in \mathrm{Aut}(\mathfrak{g})$ such that $\theta(v) = \xi^i_m v$ for $v \in \mathfrak{g}_i$. The automorphism θ is G-conjugate to one of the form $t \rtimes \sigma$ with $t \in T^{\sigma}$. Thus without loss of generality, we can assume $\theta = t \rtimes \sigma$, $t \in T^{\sigma}$.

According to [Kac, Proposition 8.1] (see also [OV, §3]), there exists $x \in \bar{C}^{\sigma}$ such that $\theta = \exp(x) \rtimes \sigma$. Since θ has order m, we have

$$(3) \qquad \beta_i(x) = \frac{s_i}{m}.$$

The integers $(s_i)_{i=0,\ldots,\ell_\sigma}$ are the normalized *Kac coordinates* of θ (see [RLYG, §2.2]). These coordinates satisfy

$$(4) \qquad e\sum_{i=0}^{\ell_\sigma} b_i s_i = m,$$

here e is the order of σ and the b_i are integers mentioned earlier in §2.2.

Let $\check{\lambda} = mx \in \check{\mathbb{X}}^\sigma$. The action of $\mathbb{G}_m$ on $\mathfrak{g}$ via $\check{\lambda}$ give a grading $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}(k)$ and each $\mathfrak{g}_i$ decomposes as

$$\mathfrak{g}_i = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_i(k). \tag{5}$$

Lemma 2.1. *In above decomposition, we have 1) $\mathfrak{g}_i(k) = 0$ unless $k \equiv i \mod \frac{m}{e}$ and 2) $-m + es_0 \leq k \leq m - es_0$.*

Proof. Let $0 \neq v \in \mathfrak{g}_i(k)$. Then we have $\theta(v) = \xi_m^i v$. On the other hand, since $\theta = \check{\lambda}(\xi_m) \rtimes \sigma$, we have $\theta(v) = \sigma(\xi_m^k v) = \xi_e^l \xi_m^k v = (\xi_m)^{\frac{ml}{e}+k} v$ for some $l \in \mathbb{Z}$. This implies $k \equiv i \mod \frac{m}{e}$. For part 2), it is enough to show that $\beta(\check{\lambda}) \leq m - es_0$, where β is the highest root of $\mathfrak{g}$ (here we regard β as an element in $(\mathfrak{t}^\sigma)^*$). For this, observe that we have $e\eta - \beta \in \sum_{i=1}^{\ell_\sigma} \mathbb{Z}_{\geq 0}\beta_i$[1], here η is the root introduced in (2). Hence

$$\beta(\check{\lambda}) \leq e\eta(\check{\lambda}) = e\sum_{i=1}^{\ell_\sigma} b_i\beta_i(\check{\lambda}) \overset{(3)}{=} e\sum_{i=1}^{\ell_\sigma} b_i s_i \overset{(4)}{=} m - es_0.$$

We are done.

□

2.4. **The θ-group.** Let G_0 be the reductive subgroup of G with Lie algebra $\mathfrak{g}_0$. There are natural actions of $G_0 \rtimes \sigma$ on $\mathfrak{g}_i$. The pair $(G_0, \mathfrak{g}_1)$ is called θ-group in the terminology of the Vinberg school.

A grading $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_i$ (resp. a torsion automorphism $\theta \in \mathrm{Aut}(\mathfrak{g})$, resp. a θ-group $(G_0, \mathfrak{g}_1)$) is called *regular*, if $\mathfrak{g}_1$ contains a regular semi-simple element; *stable* if $\mathfrak{g}_1$ contains a stable element (recall that an element v is called stable if G_0-orbit of v is closed and the stabilizer in G_0 is finite). According to [RLYG, §5.3], a vector $v \in \mathfrak{g}_1$ is stable if only if v is a regular semi-simple elements of $\mathfrak{g}$ and the action of θ on the Cartan sub-algebra centralizing v is *elliptic*, i.e. $Z_{\mathfrak{g}_0}(v) = 0$. We denote by $\mathfrak{g}_1^r$ (resp. $\mathfrak{g}_1^s$) the open set of regular semi-simple (resp. stable) elements.

For future reference, we include a lemma about structure of $\mathfrak{g}_0$:

Lemma 2.2.

(1) $\mathfrak{g}_0$ *is the reductive subalgebra with Cartan subalgebra $\mathfrak{t}^\sigma$ and the system of simple roots $\Delta_0^\sigma = \{\bar{\beta}_i | \beta_i \in \Delta_{\mathrm{aff}}^\sigma, s_i = 0\}$. Here $\bar{\beta}_i$ denotes the linear part of β_i.*
(2) *If $s_0 \neq 0$, we have $\mathfrak{g}_0 = \mathfrak{g}_0 \cap \mathfrak{g}(0) = \mathfrak{g}(0)^\sigma$.*

Proof. Part (1) is proved in [Kac, Proposition 8.6] (see also [OV, §3.11]). For part (2) we first note that $\mathfrak{g}_0 \supset \mathfrak{g}_0 \cap \mathfrak{g}(0) = \mathfrak{g}(0)^\sigma$. Since $\mathfrak{g}(0)^\sigma$ is the Levi subalgebra of $\mathfrak{g}^\sigma$ with system of simple roots $\{\bar{\beta}_i | \beta_i \in \Delta_{\mathrm{aff}}^\sigma, i \neq 0, s_i = 0\}$, part (1) implies $\dim \mathfrak{g}_0 = \dim \mathfrak{g}(0)^\sigma$. The claim follows.

□

[1]This is obvious when $\sigma = id$, for $\sigma \neq id$ one can use the table 1 in [RLYG] to check it.

3. Kac gradings on loop algebras

3.1. Let ${}^{\sigma}\tilde{\mathfrak{g}}=\mathfrak{g}[t,t^{-1}]^{\sigma}$ be the twisted loop algebra. The decomposition in (1) induces a roots space decomposition of the twisted loop algebra ${}^{\sigma}\tilde{\mathfrak{g}}=\bigoplus_{\alpha\in\Phi_{\mathrm{aff}}^{\sigma}\cup\{0\}}{}^{\sigma}\tilde{\mathfrak{g}}_{\alpha}$ (here we set ${}^{\sigma}\tilde{\mathfrak{g}}_0=\mathfrak{t}^{\sigma}$). For any $x\in V^{\sigma}$, Kac has introduced a $\mathbb{Z}$-grading

$$
{}^{\sigma}\tilde{\mathfrak{g}}=\bigoplus_{i\in\mathbb{Z}}{}^{\sigma}\tilde{\mathfrak{g}}_{x,i}. \tag{6}
$$

Explicitly, we have

$$
{}^{\sigma}\tilde{\mathfrak{g}}_{x,i}=\bigoplus_{\alpha\in\Phi_{\mathrm{aff}}^{\sigma}\cup\{0\},\ \alpha(x)=\frac{i}{m}}{}^{\sigma}\tilde{\mathfrak{g}}_{\alpha}.
$$

We called the $\mathbb{Z}$-grading in (6) the *Kac grading* associated to x.

Example 3.1 (Principal grading)**.** Consider the case $\theta=\exp(x)$ where $x=\frac{\check{\rho}}{h}$, $\check{\rho}(\alpha_i)=1$ for $i=1,...,\ell$. In this case the corresponding Kac grading can be described as follows. Let E_i, H_i, F_i, $i=1,...,\ell$ be a Chevalley basis of $\mathfrak{g}$ such that $E_i\in\mathfrak{n}$, $H_i\in\mathfrak{t}$, $F_i\in\bar{\mathfrak{n}}$. Then $\tilde{\mathfrak{g}}=\mathfrak{g}[t,t^{-1}]$ has Kac-Moody generators $e_i=E_i\otimes 1$, $h_i=H_i\otimes 1$, $f_i=F_i\otimes 1$, $i=1,...,\ell$; $e_0=F_{-\beta}\otimes t$, $f_0=E_{\beta}\otimes t^{-1}$, here β is the highest root and E_{β} (resp. $F_{-\beta}$) is a generator of the root space $\mathfrak{g}_{\beta}$ (resp. $\mathfrak{g}_{-\beta}$). Now the grading on $\tilde{\mathfrak{g}}$ is given by $\deg(e_i)=-\deg(f_i)=1$, $\deg(h_i)=0$. We call this grading the *principal grading* (see [Kac, §14]).

Remark 3.2. In [Kac, §1.3], Kac called the $\mathbb{Z}$-grading in (6) the gradation of type $(s_0,...,s_{\ell_{\sigma}})$.

3.2. We preserve the setup in §2.3. Let $\theta=\exp(x)\rtimes\sigma\in\mathrm{Aut}(\mathfrak{g})$, $x\in\bar{C}^{\sigma}$ and let $\mathfrak{g}=\bigoplus_{i\in\mathbb{Z}/m\mathbb{Z}}\mathfrak{g}_i$ be the grading on $\mathfrak{g}$ defined by the automorphism θ. Let $u=t^{\frac{e}{m}}$, and consider the following $\mathbb{Z}$-graded Lie algebra

$$
\mathfrak{g}(\theta,m):=\bigoplus_{i\in\mathbb{Z}}u^i\mathfrak{g}_i\subset\mathfrak{g}[u,u^{-1}],
$$

with $\mathfrak{g}(\theta,m)_i:=u^i\mathfrak{g}_i$. In his book, Kac proved the following.

Theorem 3.3 (See [Kac], Theorem 8.5)**.**

(1) *Let $\check{\lambda}=mx\in\check{\mathbb{X}}^{\sigma}$. The automorphism $\mathrm{Ad}(\lambda(u^{-1})):\mathfrak{g}[u,u^{-1}]\simeq\mathfrak{g}[u,u^{-1}]$ induces an isomorphism*
$$
\Phi:\mathfrak{g}(\theta,m)\simeq{}^{\sigma}\tilde{\mathfrak{g}}.
$$
(2) *Under the isomorphism Φ, the $\mathbb{Z}$-grading of $\mathfrak{g}(\theta,m)$ becomes to the Kac-Moy-Prasad grading of ${}^{\sigma}\tilde{\mathfrak{g}}$ associated to x.*
(3) *Under the isomorphism Φ, the derivation $u\partial_u$ on $\mathfrak{g}(\theta,m)$ becomes the derivation $D=\frac{m}{e}t\partial_t+\mathrm{ad}\check{\lambda}$ on ${}^{\sigma}\tilde{\mathfrak{g}}$.*
(4) *The invariant form $\langle u^ix,u^jy\rangle=\delta_{i,-j}(x,y)_{Kill}$ on $\mathfrak{g}[u,u^{-1}]$ induced an invariant from $\langle,\rangle_{\theta}$ (resp. $\langle,\rangle_{\sigma}$) on $\mathfrak{g}(\theta,m)$ (resp. ${}^{\sigma}\tilde{\mathfrak{g}}$) which is compatible with the grading, i.e., we have $\langle v,w\rangle_{\theta}=0$ (resp. $\langle v,w\rangle_{\sigma}=0$) for $v\in\mathfrak{g}(\theta,m)_i$, $w\in\mathfrak{g}(\theta,m)_j$, $i+j\neq 0$ (resp $v\in{}^{\sigma}\tilde{\mathfrak{g}}_{x,i}$, $w\in{}^{\sigma}\tilde{\mathfrak{g}}_{x,j}$, $i+j\neq 0$).*

We have the following corollary:

Corollary 3.4 ([Kac], [RY])**.**

(1) *For each* $i = 0, 1, .., m-1$, *there is a canonical isomorphism*

$$^{\sigma}\tilde{\mathfrak{g}}_{x,i} = \bigoplus_k t^{\frac{e(i-k)}{m}} \mathfrak{g}_i(k) \simeq \mathfrak{g}_i.$$

where the sum is over $-m + es_0 \leq k \leq m - es_0,\ k \equiv i \bmod \frac{m}{e}$.

(2) *If* $i > 0$, *all powers* $t^{\frac{e(i-k)}{m}}$ *appearing in the above sum are positive, i.e., for* $0 < i < m$, $\mathfrak{g}_i(k) = 0$ *unless* $-m + es_0 \leq k \leq i,\ k \equiv i \bmod \frac{m}{e}$.

Proof. Since $\mathfrak{g}(\theta, m)_i = u^i \mathfrak{g}_i$ for $i = 0, 1, .., m-1$, result in §2.2 and above Theorem implies

$$^{\sigma}\tilde{\mathfrak{g}}_{x,i} = \Phi(u^i \mathfrak{g}_i) = \mathrm{Ad}(\lambda(u^{-1}))(u^i \mathfrak{g}_i) = \bigoplus_k t^{\frac{e(i-k)}{m}} \mathfrak{g}_i(k),$$

here the sum is over $-m + es_0 \leq k \leq m - es_0,\ k \equiv i \bmod \frac{m}{e}$. Part (1) follows. Since x is in the fundamental alcove $\bar{C}^{\sigma}$, direct calculation shows that, for $i > 0$, $^{\sigma}\tilde{\mathfrak{g}}_{x,i}$ is contained in $^{\sigma}\tilde{\mathfrak{g}} \cap \mathfrak{g}[t]$. Part (2) follows.

□

Let us assume θ is regular and let $X \in \mathfrak{g}_1^r$ be a regular semi-simple element. Consider

$$p_1 = \Phi(uX) \in {}^{\sigma}\tilde{\mathfrak{g}}_{x,1}.$$

We have the following generalization of [Kac1, Proposition 3.8]

Proposition 3.5. *Let* $\mathfrak{a} = \mathrm{Ker}(\mathrm{ad}p_1)$ *and* $\mathfrak{c} = \mathrm{Im}(\mathrm{ad}p_1)$. *We have*

(1) *The twisted loop algebra* $^{\sigma}\tilde{\mathfrak{g}}$ *has an orthogonal decomposition* $^{\sigma}\tilde{\mathfrak{g}} = \mathfrak{a} \oplus \mathfrak{c}$ *with respect to the invariant form* $\langle , \rangle_{\sigma}$ *in Theorem 3.3.*

(2) *The Lie subalgebra* $\mathfrak{a}$ *is commutative. With respect to the Kac grading,* $\mathfrak{a} = \bigoplus \mathfrak{a}_i$, *the subspaces* $\mathfrak{a}_i$ *and* $\mathfrak{a}_j$ *are orthogonal (*resp. *non degenerately paired) with respect to the invariant form* $\langle , \rangle_{\sigma}$ *on* $^{\sigma}\tilde{\mathfrak{g}}$ *if* $i + j \neq 0$ *(*resp. $i + j = 0$*).*

(3) *Consider the Kac-Moody central extension* $^{\sigma}\tilde{\mathfrak{g}} \oplus \mathbb{C}K$ *of* $^{\sigma}\tilde{\mathfrak{g}}$ *(cf. §2.2). The pre-image* $\hat{\mathfrak{a}} = \mathfrak{a} \oplus \mathbb{C}K$ *of* $\mathfrak{a}$ *in* $^{\sigma}\tilde{\mathfrak{g}} \oplus \mathbb{C}K$ *is a non-split central extension of* $\mathfrak{a}$.

Proof. We first prove part (1) and (2). Since the isomorphism $\Phi : \mathfrak{g}(\theta, m) \simeq {}^{\sigma}\tilde{\mathfrak{g}}$ is compatible with the invariant forms on both side, it is enough to prove the corresponding statement for $\mathfrak{g}(\theta, m)$. Let $\mathfrak{s} = \bigoplus_{i \in \mathbb{Z}/m} \mathfrak{s}_i$ be the centralizer of X in $\mathfrak{g}$ and $\mathfrak{b} = \bigoplus_{i \in \mathbb{Z}/m} \mathfrak{b}_i$ be its orthogonal complement with respect to the Killing form $(,)_{Kill}$. Consider $\mathfrak{a}' = \mathrm{Ker}(\mathrm{ad}(uX))$ and its orthogonal complement $\mathfrak{c}'$ in $\mathfrak{g}(\theta, m)$ with respect to the invariant form $\langle , \rangle_{\theta}$. We have

$$\mathfrak{a}' = \bigoplus_{i \in \mathbb{Z}} \mathfrak{a}'_i, \quad \mathfrak{c}' = \bigoplus_{i \in \mathbb{Z}} \mathfrak{c}'_i,$$

where $\mathfrak{a}'_i = u^i \mathfrak{s}_i$, $\mathfrak{c}'_i = u^i \mathfrak{b}_i$. Now since the restriction of $(,)_{Kill}$ to any Cartan subalgebra is non-degenerate and $\mathrm{ad}(X)$ in invertible on $\mathfrak{b}$, we have i) $\mathfrak{b}_{i+1} = [X, \mathfrak{b}_i]$ and ii) $\mathfrak{s}_i$ and $\mathfrak{s}_j$ are orthogonal (resp. non degenerately paired) with respect to $(,)_{Kill}$ if $i + j \neq 0$ (resp. $i + j = 0$). Part (1) and (2) follow.

For part (3), we first notice that a cocycle corresponding to the central extension is given by ${}^{\sigma}\tilde{\mathfrak{g}} \times {}^{\sigma}\tilde{\mathfrak{g}} \to \mathbb{C}$, $(v, w) \to \langle t\partial_t(v), w\rangle_\sigma$. Thus it is enough to show that for any $z \neq 0 \in \mathfrak{a}_n$ there exists $z' \in \mathfrak{a}_{-n}$ such that $\langle t\partial_t(z), z'\rangle_\sigma \neq 0$. Now observe that

$$\langle t\partial_t(z), z'\rangle_\sigma = \langle \frac{e}{m}(D - \mathrm{ad}\check{\lambda})(z), z'\rangle_\sigma = \langle \frac{ne}{m} z, z'\rangle_\sigma - \langle \frac{e}{m}[\check{\lambda}, z], z'\rangle_\sigma,$$

here D is the derivation of ${}^{\sigma}\tilde{\mathfrak{g}}$ in Theorem 3.3. Since $\mathfrak{a}$ is commutative by part (2) we have $\langle \frac{e}{m}[\check{\lambda}, z], z'\rangle_\sigma = \langle \frac{e}{m}\check{\lambda}, [z, z']\rangle_\sigma = 0$, hence $\langle t\partial_t(z), z'\rangle_\sigma = \langle \frac{ne}{m} z, z'\rangle_\sigma$ and the desired claim follows again from part (2). □

4. Yun's θ-connections

In his unpublished work, Z. Yun associated to each torsion automorphism $\theta \in \mathrm{Aut}(\mathfrak{g})$ and a nonzero vector $X \in \mathfrak{g}_1$, a *twisted* flat G-connection ∇^X on the trivial G-bundle on $\mathbb{G}_m = \mathbb{P}^1 - \{0, \infty\}$, called the θ-connection associated to X. In this section we shall recall his construction of ∇^X and compute its residue at 0 and slope and irregularity at ∞.

4.1. Twisted flat G-connection and cohomological rigidity.

In this subsection we recall the definition of twisted flat G-connection and cohomological rigidity (see [Yun1] in the setting of ℓ-adic sheaves).

Let C be a smooth curve and let $\mathcal{F}$ be a G-bundle on C. Denote by $\pi_{\mathcal{F}} : \mathcal{F} \to C$ the natural projection. A G-connection ∇ on $\mathcal{F}$ is a G-equivaraint map $\nabla : \Omega^1_{\mathcal{F}} \to \pi^*_{\mathcal{F}}\Omega^1_C$ such that the composition $\pi^*_{\mathcal{F}}\Omega^1_C \xrightarrow{d\pi_{\mathcal{F}}} \Omega^1_{\mathcal{F}} \xrightarrow{\nabla} \pi^*_{\mathcal{F}}\Omega^1_C$ is equal to the identity map. Now let $\widetilde{C} \to C$ be a finite étale Galois cover with Galois group Γ. Let $\sigma : \Gamma \to \mathrm{Aut}(G)$ be a homomorphism. We define a σ*-twisted* flat G-connection on C to be a triple $(\mathcal{F}, \nabla, \delta)$ where $\mathcal{F}$ is G-bundle on $\widetilde{C}$, ∇ is a flat G-connection on it, and δ is a collection of isomorphisms $\delta_\gamma : (\mathcal{F}, \nabla) \simeq \gamma^*(\mathcal{F}, \nabla)$, $\gamma \in \Gamma$ satisfying the usual cocycle relations with respect to the multiplication on Γ.

When $\mathcal{F}$ is the trivial G-bundle, then a σ-twisted flat connection on it may be described as an operator

$$\nabla = d + A(t)dt,$$

where d is the exterior derivative and $A(t)dt$ is a Γ-invariant $\mathfrak{g}$-valued one-form on $\widetilde{C}$, with Γ acting by deck transformation and by the map $\Gamma \xrightarrow{\sigma} \mathrm{Aut}(G) \to \mathrm{Aut}(\mathfrak{g})$ on $\mathfrak{g}$.

Let $(\mathcal{F}, \nabla, \delta)$ be a σ-twisted flat G-connection on a smooth curve C. Then the corresponding flat vector bundle ∇^{Ad} on $\widetilde{C}$ associated to the adjoint representation descends to C by Γ-equivariance. Let $\bar{\nabla}^{\mathrm{Ad}}$ be the flat vector bundle on C after descent.

Definition 4.1. A σ-twisted flat G-connection $(\mathcal{F}, \nabla, \delta)$ over an open subset C of a complete smooth curve $\bar{C}$ is called *cohomologically rigid* if

$$H^*(\bar{C}, j_{!*}\bar{\nabla}^{\mathrm{Ad}}) = 0,$$

where $j : C \hookrightarrow \bar{C}$ is the inclusion, and $j_{!*}\bar{\nabla}^{\mathrm{Ad}}$ is the *non-derived* push forward of the D-module $\bar{\nabla}^{\mathrm{Ad}}$ along j.[2]

[2] We use the notation $j_{!*}$ because in this case $j_{!*}\bar{\nabla}^{\mathrm{Ad}}$ is also the intermediate extension of the D-module $\bar{\nabla}^{\mathrm{Ad}}$ to $\bar{C}$ (see [BBD, §5.2.2]).

Remark 4.2. Our definition of cohomological rigidity here is stronger than the usual definition. Usually one only requires $H^1(\bar{C}, j_{!*}\bar{\nabla}^{\mathrm{Ad}}) = 0$.

Remark 4.3. For a general connected reductive group G, one should modify the definition above by replacing $\bar{\nabla}^{\mathrm{Ad}}$ by $\bar{\nabla}^{\mathrm{Ad,der}}$ where $\bar{\nabla}^{\mathrm{Ad,der}}$ is the flat vector bundle associated to the representation $\mathfrak{g}^{\mathrm{der}} = \mathrm{Lie}\ G^{\mathrm{der}}$. For example, consider the case σ is trivial, $G = GL_n$ and $C \subset \bar{C} = \mathbb{P}^1$. Then for an irreducible GL_n-connection $(\mathcal{F}, \nabla)$ we have $\nabla^{\mathrm{Ad,der}} = \mathrm{End}^0(\mathcal{E})$, the flat vector bundle of traceless endomorphisms of $\mathcal{E} := \mathcal{F} \times^{GL_n} \mathbb{C}^n$. The connection ∇ is cohomologically rigid if and only if $H^1(\mathbb{P}^1, j_{!*}\nabla^{\mathrm{Ad}}) = 0$, $H^0(\mathbb{P}^1, j_{!*}\nabla^{\mathrm{Ad}}) = H^2(\mathbb{P}^1, j_{!*}\nabla^{\mathrm{Ad}}) = \mathbb{C}$, which is equivalent to the condition that the Euler characteristic $\chi(\mathbb{P}^1, j_{!*}\nabla^{\mathrm{Ad}}) = 2$. In particular, we see that our definition is compatible with the one in [Katz1, §5].

4.2. **Construction of ∇^X.** We preserve the setup in §2.2. Let $\theta = \exp(x) \rtimes \sigma \in \mathrm{Aut}(\mathfrak{g}) = G \rtimes \mathrm{Aut}(R, \Delta)$ be a torsion automorphism of $\mathfrak{g}$. Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/m} \mathfrak{g}_i$ be the corresponding grading. Let $X \in \mathfrak{g}_1$ and let us write $X = \sum X_k$, $X_k \in \mathfrak{g}_1(k)$ according to (5). By Corollary 3.4, we have $X_k = 0$ unless $-m + es_0 \leq k \leq 1$ and $k \equiv 1 \bmod \frac{m}{e}$. Define

$$p_1 = \Phi(uX) = \sum X_k t^{\frac{e(1-k)}{m}} \in {}^{\sigma}\tilde{\mathfrak{g}},$$

here Φ is the isomorphism in Theorem 3.3. Then the θ-connection associated to X is the following flat G-connection on the trivial G-bundle on $\mathbb{G}_m = \mathrm{Spec}\mathbb{C}[t, t^{-1}]$

$$\nabla^X = d + p_1 \frac{dt}{t} = d + \sum_{-m+es_0 \leq k \leq 1,\ k \equiv 1 \bmod \frac{m}{e}} X_k t^{\frac{e(1-k)}{m}} \frac{dt}{t}. \tag{7}$$

Note that $\frac{e(i-k)}{m} \in \mathbb{Z}$.

The $\mathfrak{g}$-valued one form $p_1 \frac{dt}{t}$ is σ-invariant, where σ acts on $\mathbb{G}_m$ by the formula $t \to \xi_e^{-1} t$ and by the pinned automorphism on $\mathfrak{g}$. Therefore, by the discussion in §4.1, we can regard ∇^X as a σ-*twisted* flat G-connection on the trivial G-bundle on $\mathbb{G}_m$, where we regard σ as a map $\sigma : \mu_e = \langle \xi_e \rangle \to \mathrm{Aut}(G)$ sending $\xi_e \to \sigma$, and $\Gamma = \mu_e$ is the Galois group of the finite étale Galois cover $[e] : \widetilde{\mathbb{G}}_m = \mathbb{G}_m \to \mathbb{G}_m$ given by the e-th power map.

4.3. **Residue at** 0. Notice that $\frac{e(1-k)}{m} > 0$ for $k < 1$, thus the connection ∇^X has regular singularity at 0 with residue $\mathrm{Res}(\nabla^X) = X_1 \in \mathfrak{g}_1(1)^{\sigma}$. Since $\mathfrak{g}(1)^{\sigma}$ consists of nilpotent elements of $\mathfrak{g}^{\sigma}$, the residue is nilpotent.

Moreover, since there are only finitely many G^{σ} orbits on $\mathfrak{g}(1)^{\sigma}$ (see [V]), there is a dense open subset of $\mathfrak{g}(1)^{\sigma}$ which lies in a single nilpotent G^{σ}-orbit of $\mathfrak{g}^{\sigma}$. We denote this orbit by $\mathcal{O}_{\theta}$. Thus, for generic $X \in \mathfrak{g}_1$ the residue $\mathrm{Res}(\nabla^X)$ lies in $\mathcal{O}_{\theta}$.

The assignment $\theta \to \mathcal{O}_{\theta}$ gives a well defined map [3]

$$\{\textit{torsion automorphism of } \mathfrak{g} \textit{ whose image in } \mathrm{Aut}(R, \Delta) \textit{ is } \sigma\} \to \{\textit{nilpotent orbits in } \mathfrak{g}^{\sigma}\}.$$

We now assume θ is stable. Consider the normalized Kac coordinates $\{s_0, s_1, ..., s_{\ell_\sigma}\}$ of θ. If we omit s_0 and double the remaining Kac coordinates we obtain the weighted Dynkin

[3] This map and the map in (8) are due to Z. Yun.

diagram for the nilpotent orbit $\mathcal{O}_\theta$. Thus for Y in $\mathcal{O}_\theta$, we have $\dim \mathfrak{g}^{\sigma,Y} = \dim \mathfrak{g}(0)^\sigma$. The nilpotent class $\mathcal{O}_\theta$ is distinguished if and only if $\dim \mathfrak{g}(0)^\sigma = \dim \mathfrak{g}(1)^\sigma$.

Recall that stable torsion automorphisms θ are classified by regular elliptic W-conjugacy classes in the coset $W\sigma$ (see [RLYG, Corollary 15]). We therefore get a map

$$\{\textit{regular elliptic classes in } W\sigma\} \to \{\textit{nilpotent orbits in } \mathfrak{g}^\sigma\}. \tag{8}$$

In the case $\sigma = \mathrm{id}$ and the normalized Kac coordinates satisfies $s_0 = 1$, this map is studied in [S] and [RLYG, §7.3] (see §7.1 for more details). The relation between this map and Kazhdan-Luszitg map [KL] is discussed in [RLYG, §8.3, Remark 2].

We expect that for any stable vector $X \in \mathfrak{g}_1^s$ we have $X_1 \in \mathcal{O}_\theta$. In other worlds, we expect the conjugacy classes of the residue $\mathrm{Res}(\nabla^X)$, $X \in \mathfrak{g}_1^s$ depends only on θ and is given by the map (8). We will verify this expectation in some examples in §7.

4.3.1. *An example.* We preserve the setup in example 3.1. Consider $\theta = \exp(\check{\rho}/h)$, where $\check{\rho}$ is the half-sum of positive co-roots and h is the Coxeter number. We have $\mathfrak{g}_0 = \mathfrak{t}$ and $\mathfrak{g}_1 = \mathfrak{g}(1) \bigoplus \mathfrak{g}(-h+1)$, $\mathfrak{g}(1) = \bigoplus_{i=1}^{\ell} \mathfrak{g}_{\alpha_i}$, $\mathfrak{g}(-h+1) = \mathfrak{g}_{-\beta}$. Here β is the highest root. Choosing a generator E_i for each $\mathfrak{g}_{\alpha_i}$, a generator E_0 for $\mathfrak{g}_{-\beta}$, and identifying $\mathfrak{g}_1$ with $\bigoplus_{i=0}^{\ell} \mathbb{C}E_i$, the open subset $\mathfrak{g}_1^s$ of stable vectors can be identified with $\mathfrak{g}_1^s = \{\sum c_i E_i | c_i \neq 0 \textit{ for } i = 0, ..., \ell\}$. For any $X = \sum_{i=0}^{\ell} c_i E_i \in \mathfrak{g}_1^s$, the corresponding θ-connection takes the form

$$\nabla^X = d + \frac{\sum_{i=1}^{\ell} c_i E_i}{t} dt + c_0 E_0 dt.$$

This is the rigid connections constructed in [FG]. The residue of ∇^X at 0 is $N' = \sum_{i=1}^{\ell} c_i X_i$, which is regular nilpotent.

Remark 4.4. Recall that Heisenberg algebras of the Kac-Moody central extension ${}^\sigma\tilde{\mathfrak{g}} \oplus \mathbb{C}K$ are parametrized, up to conjugacy, by W-conjugacy classes of the coset Wσ (see, e.g.,[KP] for the case $\sigma = \mathrm{id}$). Given $w \in \mathrm{W}\sigma$, let $\hat{\mathfrak{a}}_w$ denote the associated Heisenberg subalgebra. One can show that, when θ is stable torsion automorphism, the algebra $\hat{\mathfrak{a}}$ in Proposition 3.5 is conjugate to the Heisenberg sub-algebra $\hat{\mathfrak{a}}_w$ where w is an element in the regular elliptic conjugacy class of Wσ corresponding to θ.

4.4. **Slope and Irregularity at ∞.** In this section we compute the slope and irregularity of ∇^X at ∞. We adapt the definition of the slope of a connection on a principal G-bundle from [D, FG] (see [BS, CK] for other equivalent definitions): a connection on a principal G-bundle with irregular singularity at a point x on a curve X has slope $a/b > 0$ at this point if the following holds. Let s be a uniformizing parameter at x, and pass to the extension given by adjoining the b-th root of s: $u^b = s$. Then the connection, written using the parameter u in the extension and a particular trivialization of the bundle on the punctured disc at x should have a pole of order $a+1$ at x, and its polar part at x should not be nilpotent.

Let us compute the slopes of ∇^X at ∞ using the definition above. Consider the covering given by $t = a^{-\frac{m}{e}}$. Then the connection ∇^X becomes

$$d - \frac{m}{e} \sum_k X_k a^{k-1} \frac{da}{a}.$$

Taking the gauge transform with $\check{\lambda}^{-1}(a)$, then $\mathrm{Ad}(\check{\lambda}^{-1}(a))X_i = a^{-k}X_k$, hence the connection becomes

$$d - \frac{m}{e}X\frac{da}{a^2} + \check{\lambda}\frac{da}{a}. \tag{9}$$

Assume X is semi-simple, then according to the above definition of slopes, we see that the slope of ∇^X at ∞ are either 0 or e/m.

Recall that any representation V of G gives rise to a flat vector bundle $\nabla^{X,V}$ on $\mathbb{G}_m$. We compute the irregularity of the connection $\mathrm{Irr}_\infty(\nabla^{X,V})$ at infinity when X is semi-simple following [FG, §13]. Since X is semi-simple the leading term of the connection in (9) is diagonalizable in any representation V of G. It implies the slopes of the connection $\nabla^{X,V}$ is either 0 or e/m, the former occurring at the zero eigenspaces of X on V and the later occurring at the non-zero eigenspaces. According to [Katz, §1 and §2.3], the irregularity $\mathrm{Irr}_\infty(\nabla^{X,V})$ is equal to the sum of the slopes of the connection at ∞. This implies

$$\mathrm{Irr}_\infty(\nabla^{X,V}) = \frac{e}{m}(\dim V - \dim V^X).$$

Assume $\nabla^{X,V}$ descends to a flat vector bundle $\bar{\nabla}^{X,V}$ via the e-th power map $[e] : \widetilde{\mathbb{G}}_m \to \mathbb{G}_m$. Then again by [Katz, §2.3], we have

$$\mathrm{Irr}_\infty(\bar{\nabla}^{X,V}) = \mathrm{Irr}_\infty(\nabla^{X,V})/e = \frac{1}{m}(\dim V - \dim V^X). \tag{10}$$

Remark 4.5. Let me mention that the θ-connections constructed by Z. Yun (in the untwisted case) and their slopes can be also constructed and computed using the theory of regular strata developed by C. Bremer and D. Sage [BS].

5. Main results

We preserve the setup of §4.1. Let $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_i$ be a grading of $\mathfrak{g}$ and let $\theta = \theta' \rtimes \sigma \in \mathrm{Aut}(\mathfrak{g})$ be the corresponding automorphism. Let $X \in \mathfrak{g}_1$ be a nonzero vector and ∇^X be the corresponding θ-connection, which is a σ-twisted flat G-connection on $\mathbb{G}_m$.

Consider the adjoint representation Ad of G on its Lie algebra $\mathfrak{g}$. The corresponding flat vector bundle $\nabla^{X,\mathrm{Ad}}$ descends via the e-th power map $[e] : \widetilde{\mathbb{G}}_m \to \mathbb{G}_m$ by σ-equivariance. Let $\bar{\nabla}^{X,\mathrm{Ad}}$ be the connection after descent.

Here are the main results of this note, generalizing [FG, Theorem 1 and Proposition 11] to general θ-groups:

Theorem 5.1. *Assume θ is regular. Then for any regular semi-simple vector $X \in \mathfrak{g}_1^r$, we have*

$$H^0(\mathbb{P}^1, j_{!*}\bar{\nabla}^{X,\mathrm{Ad}}) = H^2(\mathbb{P}^1, j_{!*}\bar{\nabla}^{X,\mathrm{Ad}}) = 0$$

and

$$\dim H^1(\mathbb{P}^1, j_{!*}\bar{\nabla}^{X,\mathrm{Ad}}) = \frac{\#R}{m} - \dim \mathfrak{g}^{\sigma,X_1}. \tag{11}$$

Here $j : \mathbb{G}_m \hookrightarrow \mathbb{P}^1$ is the canonical embedding and $X_1 \in \mathfrak{g}_1(1)^\sigma$ is the residue of the connection ∇^X at 0 (see §4.3).

Theorem 5.2. *Assume θ is stable and its normalized Kac coordinates satisfies $s_0 = 1$. Then for any stable vector $X \in \mathfrak{g}_1^s$, we have*

$$H^i(\mathbb{P}^1, j_{!*}\bar{\nabla}^{X,\mathrm{Ad}}) = 0$$

for all i, that is, ∇^X is cohomologically rigid (see Definition 4.1).

6. Proofs

6.1. The key step leading the proofs of Theorem 5.1 and Theorem 5.2 is the computation of the cohomology groups $H^0(D_0^\times, \bar{\nabla}^{X,\mathrm{Ad}})$ and $H^0(D_\infty^\times, \bar{\nabla}^{X,\mathrm{Ad}})$. Here $D_0^\times = \mathrm{Spec}\mathbb{C}((t))$ (resp. $D_\infty^\times = \mathrm{Spec}\mathbb{C}((t^{-1}))$) is the formal punctured disc around 0 (resp. ∞).

We first introduce some auxiliary notations that will be used in the rest of the section. Let $\nabla^X = d + p_1\frac{dt}{t}$ be the θ-connection associated to $X \in \mathfrak{g}_1^s$ (see §4.2). Recall $p_1 = \sum X_k t^{\frac{e(1-k)}{m}} \in {}^\sigma\tilde{\mathfrak{g}}$. The connection ∇^X gives a $\mathbb{C}$-linear map

$$\nabla^{X,\mathrm{Ad}} : \mathfrak{g}[[t,t^{-1}]] \to \mathfrak{g}[[t,t^{-1}]]\frac{dt}{t}.$$

Let $f = \sum v_n t^n \in \mathfrak{g}[[t,t^{-1}]]$ be a solution to $\nabla^{X,\mathrm{Ad}}(f) = 0$. The components v_n satisfy

$$nv_n + [X_1, v_n] + \sum_{-m+es_0\le i\le 0,\ i\equiv 1 \bmod \frac{m}{e}} [X_i, v_{a_i}] = 0, \tag{12}$$

where $a_i = n - \frac{e(1-i)}{m} \in \mathbb{Z}$. Notice that $a_i < n$ for all i.

6.1.1. We compute $H^0(D_0^\times, \bar{\nabla}^{X,\mathrm{Ad}})$. Recall $\bar{\nabla}^{X,\mathrm{Ad}}$ is the descent of $\nabla^{X,\mathrm{Ad}}$ along the e-th power map $[e] : \mathbb{G}_m \to \mathbb{G}_m$. Thus $H^0(D_0^\times, \bar{\nabla}^{X,\mathrm{Ad}}) = H^0(D_0^\times, [e]^*\bar{\nabla}^{X,\mathrm{Ad}})^\sigma = H^0(D_0^\times, \nabla^{X,\mathrm{Ad}})^\sigma = \mathrm{Ker}(\nabla^{X,\mathrm{Ad}} : \mathfrak{g}((t))^\sigma \to \mathfrak{g}((t))^\sigma\frac{dt}{t})$. Let $f = \sum v_n t^n \in \mathfrak{g}((t))^\sigma$ be a solution to $\nabla^{X,\mathrm{Ad}}(f) = 0$. If $f \neq 0$, then there exists $b \in \mathbb{Z}$ such that $v_b \neq 0$ and $v_s = 0$ for $s < b$. We claim that $b \geq 0$. Indeed, if $b < 0$, then equation (12) implies

$$bv_b + [X_1, v_b] = 0,$$

which is impossible since the operator $b\cdot\mathrm{Id}+X_1$ is invertible (recall $X_1 \in \mathfrak{g}(1)^\sigma$ is nilpotent). Thus $f = \sum v_n t^n \in \mathfrak{g}[[t]]^\sigma$ and $v_0 \in \mathfrak{g}^\sigma$ lies in the kernel of $\mathrm{ad}(X_1)$. The equation (12) also implies there is a unique solution $f = \sum v_n t^n$ in $\mathfrak{g}[[t]]^\sigma$ for each $v_0 \in \mathfrak{g}^{\sigma,X_1}$. Above discussion shows that

$$H^0(D_0^\times, \bar{\nabla}^{X,\mathrm{Ad}}) = \mathrm{Ker}(\nabla^{X,\mathrm{Ad}} : \mathfrak{g}((t))^\sigma \to \mathfrak{g}((t))^\sigma\frac{dt}{t}) = \mathfrak{g}^{\sigma,X_1}.$$

6.1.2. We show that $H^0(D_\infty^\times, \bar{\nabla}^{X,\mathrm{Ad}})$ is zero. For this, we need some preliminary results about solutions $f \in \mathfrak{g}[[t,t]]^\sigma$ to $\nabla^{X,\mathrm{Ad}}(f) = 0$. Let f be such a solution. If we write f in its components for the *Kac grading*: $f = \sum y_n$ where $y_n \in {}^\sigma\tilde{\mathfrak{g}}_{x,n}$, then we have

$$(\frac{m}{e}t\nabla^{X,\mathrm{Ad}})(f) = \sum_n((\frac{m}{e}t\partial_t + \mathrm{ad}\check{\lambda})y_n + \frac{m}{e}[p_1, y_n] - [\check{\lambda}, y_n])dt = 0.$$

Recall $p_1 = \sum X_k t^{\frac{e(1-k)}{m}} \in {}^\sigma\tilde{\mathfrak{g}}$.

Notice that the operator $\frac{m}{e}t\partial_t + \mathrm{ad}\check{\lambda}$ is exactly the derivation D of ${}^{\sigma}\tilde{\mathfrak{g}}$ in Theorem 3.3 which defines the *Kac grading*. Thus we have $(\frac{m}{e}t\partial_t + \mathrm{ad}\check{\lambda})y_n = ny_n$ and above equation gives rise to the identity

$$ny_n - [\check{\lambda}, y_n] + \frac{m}{e}[p_1, y_{n-1}] = 0 \tag{13}$$

for all $n \in \mathbb{Z}$.

We have the following lemma

Lemma 6.1 ([FG], Lemma 6)**.** *Suppose that y_n satisfying (13) and $y_n \in \mathfrak{a}_n$ for some n. Then $y_m = 0$ for all $m \leq n$.*

Proof. Assume that $y_n \neq 0$. In the course of the proof of Corollary 3.5 (part (3)), we have shown that there exists $z \in \mathfrak{a}_{-n}$ such that $\langle t\partial_t(y_n), z\rangle_\sigma \neq 0$. On the other hand, since y_n satisfies (13), we have

$$t\partial_t(y_n) = \frac{e}{m}(D - \mathrm{ad}\check{\lambda})(y_n) = \frac{e}{m}(ny_n - [\check{\lambda}, y_n]) = -[p_1, y_{n-1}] \in \mathfrak{c}$$

and it implies $\langle t\partial_t(y_n), z\rangle_\sigma = 0$. We get a contradiction. Hence y_n must be zero.

Now, the equation (13) shows that if $y_n = 0$ then $y_{n-1} \in \mathfrak{a}_{n-1}$, hence, by induction that $y_m = 0$ for all $m \leq n$. □

Above lemma implies $H^0(D_\infty^\times, \bar{\nabla}^{X,\mathrm{Ad}}) = 0$. To see this, observe that

$$H^0(D_\infty^\times, \bar{\nabla}^{X,\mathrm{Ad}}) = \mathrm{Ker}(\nabla^{X,\mathrm{Ad}} : \mathfrak{g}((t^{-1}))^\sigma \to \mathfrak{g}((t^{-1}))^\sigma \frac{dt}{t}).$$

Let $f \in \mathrm{Ker}(\nabla^{X,\mathrm{Ad}} : \mathfrak{g}((t^{-1}))^\sigma \to \mathfrak{g}((t^{-1}))^\sigma \frac{dt}{t})$. Then we have $v_n = 0$ for $n \gg 0$. This implies $y_n = 0$ for $n \gg 0$ (recall that y_n are the components of f for the *Kac grading*), hence by above lemma $y_n = 0$ for all n. So we must have $f = 0$.

6.2. **Proof of Theorem 5.1.** According to [FG, §8], we have

$$H^0(\mathbb{P}^1, j_{!*}\bar{\nabla}^{X,\mathrm{Ad}}) = H^0(\mathbb{G}_m, \bar{\nabla}^{X,\mathrm{Ad}}),$$
$$H^2(\mathbb{P}^1, j_{!*}\bar{\nabla}^{X,\mathrm{Ad}}) = H_c^2(\mathbb{G}_m, \bar{\nabla}^{X,\mathrm{Ad}}),$$

and there is an exact sequence

$$0 \to H^0(\mathbb{G}_m, \bar{\nabla}^{X,\mathrm{Ad}}) \to H^0(D_0^\times, \bar{\nabla}^{X,\mathrm{Ad}}) \oplus H^0(D_\infty^\times, \bar{\nabla}^{X,\mathrm{Ad}}) \to$$
$$H_c^1(\mathbb{G}_m, \bar{\nabla}^{X,\mathrm{Ad}}) \to H^1(\mathbb{P}^1, j_{!*}\bar{\nabla}^{X,\mathrm{Ad}}) \to 0.$$

We first prove $H^0(\mathbb{P}^1, j_{!*}\bar{\nabla}^{X,\mathrm{Ad}}) = H^2(\mathbb{P}^1, j_{!*}\bar{\nabla}^{X,\mathrm{Ad}}) = 0$. Since $H^0(D_\infty^\times, \bar{\nabla}^{X,\mathrm{Ad}}) = 0$ by the result in §6.1.2, $\bar{\nabla}^{X,\mathrm{Ad}}$ admits no global sections, i.e., $H^0(\mathbb{P}^1, j_{!*}\bar{\nabla}^{X,\mathrm{Ad}}) = H^0(\mathbb{G}_m, \bar{\nabla}^{X,\mathrm{Ad}}) = 0$. Dually, $H^2(\mathbb{P}^1, j_{!*}\bar{\nabla}^{X,\mathrm{Ad}}) = H_c^2(\mathbb{G}_m, \bar{\nabla}^{X,\mathrm{Ad}}) = H^0(\mathbb{G}_m, \bar{\nabla}^{X,\mathrm{Ad}})^* = 0$. Here we used the fact adjoint representation Ad is self-dual, hence $(\bar{\nabla}^{X,\mathrm{Ad}})^* \simeq \bar{\nabla}^{X,\mathrm{Ad}}$.

Now we prove $\dim H^1(\mathbb{P}^1, j_{!*}\bar{\nabla}^{X,\mathrm{Ad}}) = \#R/m - \dim \mathfrak{g}^{\sigma, X_1}$. Results from §6.1.1, §6.1.2 and above exact sequence imply

$$0 \to \mathfrak{g}^{\sigma, X_1} \to H_c^1(\mathbb{G}_m, \bar{\nabla}^{X,\mathrm{Ad}}) \to H^1(\mathbb{P}^1, j_{!*}\bar{\nabla}^{X,\mathrm{Ad}}) \to 0. \tag{14}$$

Thus it suffices to prove that $\dim H_c^1(\mathbb{G}_m, \bar{\nabla}^{X,\mathrm{Ad}}) = \#R/m$.

Recall the Deligne's formula in [D, §6.21.1] for the Euler characteristic

$$\chi_c(\mathbb{G}_m, \bar{\nabla}^{X,\mathrm{Ad}}) := \sum_i (-1)^i \dim H^i_c(\mathbb{G}_m, \bar{\nabla}^{X,\mathrm{Ad}}) = \chi_c(\mathbb{G}_m)\,\mathrm{rank}(\bar{\nabla}^{X,\mathrm{Ad}}) - \sum_{\alpha=0,\infty} \mathrm{Irr}_\alpha(\bar{\nabla}^{X,\mathrm{Ad}}).$$

Since $\chi_c(\mathbb{G}_m) = 0$ and $\bar{\nabla}^{X,\mathrm{Ad}}$ is regular at 0, it implies

$$\chi_c(\mathbb{G}_m, \bar{\nabla}^{X,\mathrm{Ad}}) = -\,\mathrm{Irr}_\infty(\bar{\nabla}^{X,\mathrm{Ad}}).$$

Using the vanishing of H^0_c, H^2_c and the formula in line (10), we get

$$\dim H^1_c(\mathbb{G}_m, \bar{\nabla}^{X,\mathrm{Ad}}) = \mathrm{Irr}_\infty(\bar{\nabla}^{X,\mathrm{Ad}}) = \frac{1}{m}(\dim \mathfrak{g} - \dim \mathfrak{g}^X).$$

Since X is regular semi-simple, we have $\frac{1}{m}(\dim \mathfrak{g} - \dim \mathfrak{g}^X) = \#R/m$, hence

$$\dim H^1_c(\mathbb{G}_m, \bar{\nabla}^{X,\mathrm{Ad}}) = \#R/m.$$

This finished the proof of Theorem 5.1.

6.3. **Proof of Theorem 5.2.** It is enough to show that $H^1(\mathbb{P}^1, j_{!*}\bar{\nabla}^{X,\mathrm{Ad}}) = 0$. We begin with the following lemma:

Lemma 6.2. *For any solution $f = \sum v_n t^n$ of $\nabla^{X,\mathrm{Ad}}(f) = 0$ in $\mathfrak{g}[[t, t^{-1}]]^\sigma$ we have $v_n = 0$ for all $n < 0$.*

Proof. Write $f = \sum y_n$ in the components for the Kac grading. When $n = 0$, the equation (13) becomes

$$-[\check{\lambda}, y_0] + m[p_1, y_{-1}] = 0.$$

Since $s_0 = 1$ by assumption, Lemma 2.2 implies $y_0 \in \mathfrak{g}_0 = \mathfrak{g}_0 \cap \mathfrak{g}(0) \subset \ker(\mathrm{ad}\check{\lambda})$. Therefore above equation implies $y_{-1} \in \mathfrak{a}_{-1}$, thus by Lemma 6.1, we have $y_n = 0$ for $n < 0$, or equivalently $f = \sum_{n\geq 0} y_n$. On the other hand, Corollary 3.4 and the fact $\mathfrak{g}_0 = \mathfrak{g}_0 \cap \mathfrak{g}(0)$ imply $y_n \in \mathfrak{g}[t]$ for $n \geq 0$. The Lemma follows. □

By [FG, §9], we have $H^1_c(\mathbb{G}_m, \bar{\nabla}^{X,\mathrm{Ad}}) \simeq \mathrm{Ker}(\nabla^{X,\mathrm{Ad}} : \mathfrak{g}[[t, t^{-1}]]^\sigma \to \mathfrak{g}[[t, t^{-1}]]^\sigma \frac{dt}{t})$, which is equal to $\mathrm{Ker}(\nabla^{X,\mathrm{Ad}} : \mathfrak{g}[[t]]^\sigma \to \mathfrak{g}[[t]]^\sigma \frac{dt}{t})$ by above Lemma. The same argument as in §6.1.1 shows that $\mathrm{Ker}(\nabla^{X,\mathrm{Ad}} : \mathfrak{g}[[t]]^\sigma \to \mathfrak{g}[[t]]^\sigma \frac{dt}{t}) = \mathfrak{g}^{\sigma,X_1}$. Therefore the first two terms in the short exact sequence (14) both have dimension $\dim \mathfrak{g}^{\sigma,X_1}$. This proves the vanishing of $H^1(\mathbb{P}^1, j_{!*}\bar{\nabla}^{X,\mathrm{Ad}})$, hence finished the proof of Theorem 5.2

7. Examples

In this section we give several examples of θ-connections ∇^X. In each example we write down the connection explicitly and check its cohomological rigidity using the formula in (11). We also check that, in each case, the residue of the θ-connection at $0 \in \mathbb{P}^1$ (or rather its conjugacy classes) depends only on θ, hence verify our expectation in §4.3. References for this section are [FG, RLYG, RY].

7.1. **S-distinguished nilpotent case.** Recall that a nilpotent element N in $\mathfrak{g}$ is called *distinguished* if $\mathfrak{g}^N$ consists of nilpotent elements. Let $N \in \mathfrak{g}$ be a distinguished nilpotent element. There is a co-character $\check{\lambda}$ such that $\mathrm{Ad}\check{\lambda}(t)N = tN$ for all $t \in \mathbb{C}^\times$. This gives a grading $\mathfrak{g} = \bigoplus_{k=-a}^{a} \mathfrak{g}(k)$ where $\mathfrak{g}(k) = \{x \in \mathfrak{g} | \mathrm{Ad}\check{\lambda}(t)x = t^k x\}$. Set $m = a+1$ and consider the inner automorphism $\theta_N := \check{\lambda}(\xi_m) \in \mathrm{Aut}(\mathfrak{g})$. We have $\mathfrak{g}_0 = \mathfrak{g}(0)$ and $\mathfrak{g}_1 = \mathfrak{g}(1) \oplus \mathfrak{g}(-a)$. Following [RLYG, §7.3], we say that a distinguished nilpotent element $N \in \mathfrak{g}$ is *S-distinghuished* if the automorphism θ_N is stable. According to *loc. cit.*, a nilpotent element N is S-distinguished if and only if there exits $E \in \mathfrak{g}(-a)$ such that $N + E \in \mathfrak{g}_1$ is stable. Moreover, assume $\mathfrak{g}$ is of exceptional type, the map $N \to \theta_N$ defines a bijection between the set of S-distinguished nilpotent orbits in $\mathfrak{g}$ to the set of stable inner automorphism on $\mathfrak{g}$ with $s_0 = 1$.

Let θ_N be the stable automorphism of $\mathfrak{g}$ corresponding to a S-distinguished nilpotent element $N \in \mathfrak{g}$. Let $X = N + E \in \mathfrak{g}_1 = \mathfrak{g}(1) \oplus \mathfrak{g}(-a)$ be a stable vector. Then by the formula in (7), the corresponding θ-connection ∇^X takes the form

$$\nabla^X = d + \frac{N}{t}dt + Edt.$$

Note that N is the residue of ∇^X at zero.

Let us verify that ∇^X is cohomologically rigid, i.e., $\dim H^*(\mathbb{P}^1, j_{!*}\nabla^{X,\mathrm{Ad}}) = 0$. By Theorem 5.1, we have $H^0 = H^2 = 0$. Thus it remains to show $H^1(\mathbb{P}^1, j_{!*}\nabla^{X,\mathrm{Ad}}) = \frac{\#R}{m} - \dim \mathfrak{g}^N = 0$. To see this recall that N is distinguished, thus we have $\dim \mathfrak{g}^N = \dim \mathfrak{g}(0) = \dim \mathfrak{g}_0$. On the other hand, we have $\dim \mathfrak{g}_0 = \frac{\#R}{m}$ (see [P, Theorem 4.2]). Result follows.

7.1.1. *Type G_2.* Let $\mathfrak{g}$ is the simple Lie algebra of type G_2. Let α_1, α_2 be the simple root of $\mathfrak{g}$, where α_2 is the short root. Consider the automorphism $\theta = \check{\lambda}(\xi_3)$, where $\check{\lambda} = \check{\omega}_1$ is the fundamental co-weight dual to α_1 and ξ_3 is a 3-th primitive root of unity. According to [RLYG], θ is a stable inner automorphism of order 3 with normalized Kac coordinates

$$1\ 1 \Rrightarrow 0.$$

Observe that if we omit s_0 and double remaining the Kac coordinates we obtain

$$2 \Rrightarrow 0,$$

which is the weighted Dynkin diagram for the nilpotent orbit $G_2(2)$. This implies θ is equal to θ_N in §7.1 for some $N \in G_2(2)$.

We have $G_0 = \mathrm{GL}_2(\mathbb{C})$ and $\mathfrak{g}_1 = \mathfrak{g}(1) \oplus \mathfrak{g}(-2)$, $\mathfrak{g}(1) = \bigoplus_{k=0}^{3} \mathfrak{g}_{\alpha_1 + k\alpha_2}$, $\mathfrak{g}(-2) = \mathfrak{g}_{-2\alpha_1 - 3\alpha_2}$. As a representation of $G_0 = \mathrm{GL}_2(\mathbb{C})$, we have

$$\mathfrak{g}(1) \simeq \det{}^2 \otimes P_3, \quad \mathfrak{g}(-2) \simeq \det{}^{-1} \otimes P_0, \tag{15}$$

where P_d is the space of homogeneous polynomials of degree d on $\mathbb{C}^2$, with the natural action of $G_0 = \mathrm{GL}_2(\mathbb{C})$. Choosing coordinates, we regard a vector $X \in \mathfrak{g}_1$ as a pair (f, z), where $f = f(x, y)$ is a binary cubic polynomials over $\mathbb{C}$ and $a \in \mathbb{C}$. According to [RY, §7.5], we have $(f, z) \in \mathfrak{g}_1^s$ if and only if $z \neq 0$ and f has three distinct roots in the projective line. For any $X \in \mathfrak{g}_1^s$, let us write $X = X_1 + X_{-2}$ according to the decomposition $\mathfrak{g}_1 = \mathfrak{g}(1) \oplus \mathfrak{g}(-2)$. The corresponding θ-connection takes the form

$$\nabla^X = d + \frac{X_1}{t}dt + X_{-2}dt.$$

We claim that the residue X_1 is in the subregular nilpotent orbit $G_2(2)$. In particular, the conjugacy classes of the residue $\mathrm{Res}(\nabla^X)$ is independent of the choice $X \in \mathfrak{g}_1^s$. To prove the claim, observe that the intersection of $G_2(2)$ with $\mathfrak{g}(1)$ is open dense. Thus to show that X_1 is in $G_2(2)$ it is enough to show that $\dim \mathrm{Ad}G_0(X_1) = \dim \mathfrak{g}(1) = 4$. But it follows from the fact that the centralizer $Z_{G_0}(X_1)$ of X_1 in G_0 is the symmetric group S_3 (permuting the roots of f, where f is the binary cubic polynomial corresponding to X_1 under the isomorphism (15)), hence $\dim \mathrm{Ad}G_0(X_1) = \dim G_0 = 4$.

7.2. **Type $^2A_{2n}$.** Let $\mathfrak{g} = \mathfrak{sl}_{2n+1}(\mathbb{C})$ $(n \geq 1)$. Let σ be a pinned automorphism of $\mathfrak{g}$. We define $\theta = \check{\rho}(-1) \rtimes \sigma \in \mathrm{Aut}(\mathfrak{g})$. According to [RLYG], it is a stable involution with Kac coordinates

$$1 \Rightarrow 0\ 0 \cdots 0\ 0 \Rightarrow 0.$$

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the corresponding grading. We give a description of $\mathfrak{g}_0$ and $\mathfrak{g}_1$. Let V be a vector space over $\mathbb{C}$ of dimension $2n+1$ with basis $\{x_{-n}, ..., x_{-1}, x_0, x_1, ..., x_n\}$. We define an inner product $\langle , \rangle$ on V by the formula $\langle \sum a_i x_i, \sum b_i x_i \rangle = \sum_{i=-n}^{n} a_i b_{-i}$. For any $X \in \mathfrak{gl}(V)$, let X^* be the adjoint of X with respect to this inner product. Then under the canonical isomorphism $\mathfrak{sl}(V) \simeq \mathfrak{g}$, we have $\theta(X) = -X^*$ for any $X \in \mathfrak{g}$. Thus $\mathfrak{g}_0 \simeq \mathfrak{so}(V) = \{X \in \mathfrak{sl}(V) | X = -X^*\}$, $\mathfrak{g}_1 = \{X \in \mathfrak{sl}(V) | X = X^*\}$. Moreover, we have $\mathfrak{g}_1^s = \mathfrak{g}_1 \cap \mathfrak{g}^{rs}$, here $\mathfrak{g}^{rs}$ is the open subset of regular semi-simple elements in $\mathfrak{g}$.

Since $m = e = 2$ (recall m and e are the order of θ and σ), Corollary 3.4 implies $\mathfrak{g}_1 = \mathfrak{g}_1(0)$. Thus for any $X \in \mathfrak{g}_1$, the θ-connection ∇^X has the form

$$\nabla^X = d + X dt.$$

In particular, it is unramified at zero.

Finally, since $\mathfrak{g}^\sigma$ is a simple lie algebra of type B_n we have $\dim \mathfrak{g}^\sigma = n(2n+1)$. Thus for $X \in \mathfrak{g}_1^s$ we have

$$\dim H^1(\mathbb{P}^1, j_{!*}\bar{\nabla}^{X,\mathrm{Ad}}) = \frac{\#R}{2} - \dim \mathfrak{g}^\sigma = \frac{2n(2n+1)}{2} - n(2n+1) = 0.$$

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