SMALL REPRESENTATIONS OF FINITE CLASSICAL GROUPS

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ABSTRACT. Finite group theorists have established many formulas that express interesting properties of a finite group in terms of sums of characters of the group. An obstacle to applying these formulas is lack of control over the dimensions of representations of the group. In particular, the representations of small dimensions tend to contribute the largest terms to these sums, so a systematic knowledge of these small representations could lead to proofs of important conjectures which are currently out of reach. Despite the classification by Lusztig of the irreducible representations of finite groups of Lie type, it seems that this aspect remains obscure. In this note we develop a language which seems to be adequate for the description of the "small" representations of finite classical groups and puts in the forefront the notion of **rank** of a representation. We describe a method, the "**eta correspondence**", to construct small representations, and we conjecture that our construction is exhaustive. We also give a strong estimate on the dimension of small representations in terms of their rank. For the sake of clarity, in this note we describe in detail only the case of the finite symplectic groups.

0. Introduction

Finite group theorists have established formulas that enable expression of interesting properties of a group G in terms of quantitative statements on sums of values of its characters. There are many examples [9, 10, 34, 33, 39, 48, 52]. We describe a representative one. Consider the commutator map

$$[,]: G \times G \to G; \quad [x, y] = xyx^{-1}y^{-1}, \tag{0.1}$$

and for $g \in G$ denote by $[,]_g$ the set $[,]_g = \{(x, y) \in G \times G; [x, y] = g\}$. In [44] Ore conjectured that for a finite non-commutative simple group G the map (0.1) is onto, i.e., $\#[,]_g \neq 0$, for every $g \in G$. The quantity $\#[,]_g$ is a class function on G and Frobenius developed the formula for its expansion as a linear combination of irreducible characters. Frobenius' formula is

$$\frac{\#[,]_g}{\#G} = 1 + \sum_{\substack{1 \neq \rho \in Irr(G)}} \frac{\chi_{\rho}(g)}{\dim(\rho)},\tag{0.2}$$

where for ρ in the set Irr(G) of isomorphism classes of irreducible representations—aka irreps—of G, we use the symbol χ_{ρ} to denote its character. Estimating the sum in the right-hand side of (0.2) for certain classes of elements in several important finite classical groups was a major technical step in the recent proof [33, 39] of the Ore conjecture. Given the Ore conjecture thus, the following question naturally arises:

Question. What is the distribution of the commutator map (0.1)?

In [52] Shalev conjectured that for a finite non-commutative simple group G the distribution of (0.1) is approximately uniform, i.e.,

$$\sum_{1 \neq \rho \in Irr(G)} \frac{\chi_{\rho}(g)}{\dim(\rho)} = o(1), \quad g \neq 1,$$

$$(0.3)$$

in a well defined quantitative sense (e.g., as $q \to \infty$ for a finite non-commutative simple group of Lie type $G = G(\mathbb{F}_q)$). This conjecture is wide open [52]. It can be proven for the finite symplectic group $Sp_2(\mathbb{F}_q)^1$ invoking its explicit character table, and probably also for $Sp_4(\mathbb{F}_q)$ [55]. As was noted by

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¹For the rest of this note, q is a power of an odd prime number p.

q	3	5	7	11	13
$\sum_{\substack{1\neq \rho\in Irr(Sp_{\delta}(\mathbb{F}_q))}}\frac{\chi_{\rho}(T)}{\dim(\rho)}$	0.269	-0.054	-0.023	-0.008	-0.006

FIGURE 1. Ore sum for the transvection T (0.4) in $G = Sp_6(\mathbb{F}_q)$ for various q's.

Shalev in [52], one can verify the uniformity conjecture for elements of G with small centralizers, using Schur's orthogonality relations for characters [33, 39, 52]. However, relatively little seems to be known about Shalev's conjecture in the case of elements with relatively large centralizers—see Figure 1 for numerical² illustration in the case of $G = Sp_{2n}(\mathbb{F}_q)$ and the transvection element T in G which is given by

$$T = \begin{pmatrix} I & E \\ 0 & I \end{pmatrix}, \quad E_{i,j} = \begin{cases} 1, & i = j = 1; \\ 0, & \text{other } 1 \le i, j \le n. \end{cases}$$
(0.4)

To suggest a possible approach for the resolution of the uniformity conjecture, let us reinterpret (0.3) as a statement about extensive cancellation between the terms

$$\frac{\chi_{\rho}(g)}{\dim(\rho)}, \ \rho \in Irr(G), \tag{0.5}$$

which are called *character ratios*. The dimensions of the irreducible representations of a finite group



FIGURE 2. Partition of $Irr(Sp_6(\mathbb{F}_5))$ according to nearest integer to $\log_5(\dim(\rho))$.

G tend to come in certain layers according to order of magnitude. For example—see Figure 2 for

 $^{^{2}}$ The numerical data in this note was generated with J. Cannon (Sydney) and S. Goldstein (Madison) using Magma.

illustration—it is known [7, 35] that the dimensions of the irreducible representations of $G = Sp_{2n}(\mathbb{F}_q)$ are given by some "universal" set of polynomials in q. In this case the degrees of these polynomials give a natural partition of $Irr(Sp_{2n}(\mathbb{F}_q))$ according to order of magnitude of dimensions. Since the dimension of the representation of a group G is what appears in the denominator of (0.5), it seems reasonable to expect that in (0.3)

- (A) Character ratios of lower dimensional representations tend to contribute larger terms.
- (B) The partial sums over low dimensional representations of "similar" size exhibit cancellations.

A significant amount of numerical data collected recently with Cannon and Goldstein supports assertions (A) and (B). For example, in Figure 3 we plot the numerical values of the character ratios



FIGURE 3. Character ratios at T (0.4) vs. nearest integer to $\log_5(\dim(\rho))$ for $Irr(Sp_6(\mathbb{F}_5))$.

of the irreducible representations of $G = Sp_6(\mathbb{F}_5)$, evaluated at the transvection T (0.4). More precisely, for each $\rho \in Irr(Sp_6(\mathbb{F}_5))$ we marked by a circle the point³ ($\lfloor \log_5(\dim(\rho) \rceil, \chi_{\rho}(T) / \dim(\rho)$) and find that the overall picture is in agreement with (A) and (B). Moreover—see Figures 2 and 3 for illustration—the numerics shows that, although the majority of representations are "large", their character ratios tend to be so small that adding all of them contributes very little to the entire Ore sum (0.3).

The above example illustrates that a possible obstacle to getting group theoretical properties by summing over characters, as in Formula (0.2), is lack of control over the representations with

³We denote by $\lfloor x \rfloor$ the nearest integer value to a real number x.

relatively small dimensions. In particular, it seems that a systematic knowledge on the "small" representations of finite classical groups could lead to proofs of some important open conjectures, which are currently out of reach. However, relatively little seems to be known about these small representations [34, 39, 52, 59].

In this note we develop a language suggesting that the small representations of the finite classical groups can be systematically described by studying their restrictions to unipotent subgroups, and especially, using the notion of **rank** of a representation [23, 31, 47]. In addition, we develop a new method, called the "**eta correspondence**", to construct small representations. We conjecture that our construction is exhaustive. Finally, we use our construction to give a strong estimate on the dimension of the small representations in terms of their rank. For the sake of clarity of exposition we treat in this note only the case of the finite symplectic groups $Sp_{2n}(\mathbb{F}_q)$.

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1. Notion of Rank of Representation

Let us start with the numerical example of the dimensions of the irreducible representations of the group $Sp_6(\mathbb{F}_5)$. The beginning of the list appears in Figure 4. These numbers—see also Figure

1 62 62 63 63 1240 1302 1302 1365 1890 1953 1953 2015 2604 2604 3906 6510 ...

FIGURE 4. Dimensions of Irreps of $Sp_6(\mathbb{F}_5)$.

2—reveal the story of the hierarchy in the world of representations of finite classical groups. A lot of useful information is available on the "minimal" representations of these groups, i.e., the ones of lowest dimensions [34, 39, 52, 59]. In the case of $Sp_{2n}(\mathbb{F}_q)$ these are the 4 components of the oscillator (aka Weil) representations [14, 18, 21, 22, 61], 2 of dimension $(q^n - 1)/2$ and 2 of dimension $(q^n + 1)/2$, which in Figure 4 are the ones of dimensions 62, 62, 63, 63. In addition, a lot is known about the "big" representations of the finite classical groups, i.e., those of considerably large dimension (See [6, 7, 8, 32, 34, 35, 36, 37, 52, 57] and references therein). We will not attempt to define the "big" representations at this stage, but in Figure 4 the ones of dimension 6510 and above fall in that category. However, relatively little seems to be known about the representations of the classical groups which are in the range between "minimal" and "big" [34, 39, 52, 59]. In Figure 4 those form the layer of 11 representations of dimensions between 1240 and 3906.

In this section we introduce a language that will enable us to define the "small" representations of finite classical groups. This language will extend well beyond the notion of minimal representations and will induce a partition of the set of isomorphism classes of irreducible representations which is closely related to the hierarchy afforded by dimension. In particular, this language gives an explicit organization of all the representations in Figure 4, and explains why this list is, in a suitable sense, complete. The key idea we will use is that of the *rank* of a representation. This notion was developed in the 1980s by Howe, in the context of unitary representations of classical groups over local fields [25], but it has not been applied to finite groups. For the sake of clarity of exposition, in this note we give the definition of rank only in the symplectic case, leaving the more general treatment to future publication. We start by discussing necessary ingredients from the structure theory of $Sp_{2n}(\mathbb{F}_q)$.

1.1. The Siegel Unipotent Radical. Let (V, \langle, \rangle) be a 2*n*-dimensional symplectic vector space over the finite field \mathbb{F}_q . In order to simplify certain formulas, let us assume that

$$V = X \oplus Y, \tag{1.1}$$

where X and Y are vector spaces dual to each other with pairing \bullet , and that the symplectic form \langle, \rangle is the natural one which is defined by that pairing, i.e.,

$$\left\langle \begin{pmatrix} x_1\\ y_1 \end{pmatrix}, \begin{pmatrix} x_2\\ y_2 \end{pmatrix} \right\rangle = x_1 \bullet y_2 - x_2 \bullet y_1.$$
(1.2)

Note that X and Y are maximal isotropic—aka Lagrangian—subspaces of V. Consider the symplectic group Sp = Sp(V) of elements of GL(V) which preserve the form \langle, \rangle . Denote by $P = P_X$ the subgroup of all elements in Sp that preserve X. The group P is called the *Siegel parabolic* [54] and can be described explicitly in terms of the decomposition (1.1)

$$P = \left\{ \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} C & 0 \\ 0 & {}^tC^{-1} \end{pmatrix}; \ A: Y \to X \text{ symmetric, } C \in GL(X) \right\},$$

where ${}^{t}C^{-1} \in GL(Y)$ is the inverse of the transpose of C. In particular, P has the form of a semi-direct product, known also as its Levi decomposition [8],

$$P \simeq N \rtimes GL(X),\tag{1.3}$$

where $N = N_X$, called the *unipotent radical* of P, is the normal subgroup

$$N = \left\{ \begin{pmatrix} I & A \\ 0 & I \end{pmatrix}; A : Y \to X \text{ symmetric} \right\}.$$

The group N is abelian and we have a tautological GL(X)-equivariant isomorphism

$$N \xrightarrow{\sim} Sym^2(X),$$
 (1.4)

where $Sym^2(X)$ denotes the space of symmetric bilinear forms on $Y = X^*$, and the GL(X) action on $Sym^2(X)$ is the standard one. In addition, if we fix a non-trivial additive character ψ of \mathbb{F}_q we obtain a GL(X)-equivariant isomorphism

$$\begin{cases} Sym^2(Y) \widehat{\longrightarrow} \widehat{N}, \\ B \mapsto \psi_B, \end{cases}$$
(1.5)

where $Sym^2(Y)$ denotes the space of symmetric bilinear forms on $X = Y^*$, the GL(X) action on $Sym^2(Y)$ is the standard one, the symbol \hat{N} stands for the Pontryagin dual (group of characters) of N, and

$$\psi_B(A) = \psi(Tr(BA)),\tag{1.6}$$

for every $A \in Sym^2(X)$, where Tr(BA) indicates the trace of the composite operator $Y \xrightarrow{A} X \xrightarrow{B} Y$.

1.2. The *N*-spectrum of a Representation. Now, take a representation ρ of Sp and look at the restriction to *N*. It decomposes [58] as a sum of characters with certain multiplicities

$$\rho_{|N} = \sum_{B \in Sym^2(Y)} m_B \psi_B. \tag{1.7}$$

The function m and its support will be called, respectively, the <u>N</u>-spectrum of ρ , and the <u>N</u>-support of ρ , and will be denoted by $Spec_N(\rho)$, and $Supp_N(\rho)$, respectively.

We would like to organize the decomposition (1.7) in a more meaningful way. Note that the restriction to N of a representation ρ of Sp can be thought of as the restriction to N of the restriction of ρ to P. Using (1.3), this implies [41]:

Proposition 1.2.1. The N-spectrum of a representation ρ of Sp is GL(X) invariant. That is, $m_B = m_{B'}$ if B and B' define equivalent symmetric bilinear forms on X.

The first major invariant of a symmetric bilinear form is its rank. It is well known [30] that, over finite fields of odd characteristics, there are just two isomorphism classes of symmetric bilinear forms of a given rank r. They are classified by their discriminant [30], which is an element in $\mathbb{F}_q^*/\mathbb{F}_q^{*2}$. We denote by \mathcal{O}_{r+} and \mathcal{O}_{r-} , the two classes of symmetric bilinear forms, these whose discriminant is the coset of squares, and these whose coset consists of non-squares, respectively; or we will denote the pair of them, or whichever one is relevant in a given context as $\mathcal{O}_{r\pm}$. If B is a form of rank r, we will also say that the associated character ψ_B has rank r. We may also refer to the character as being of type + or type -, according to the type of B. With this notation, we can reorganize the expansion (1.7) of $\rho_{|N}$. Namely, we split the sum according to the ranks of characters, and within each rank we split the sum into two partial sub-sums according to the two isomorphism classes of the associated forms:

$$\rho_{|N} = \sum_{r} \sum_{\pm} m_{r\pm} \sum_{B \in \mathcal{O}_{r\pm}} \psi_B.$$
(1.8)

Note that, Formula (1.8) implies, by evaluation at the identity of N, that the dimension of ρ must be

$$\dim(\rho) = \sum_{r} \sum_{\pm} m_{r\pm} \cdot \#\mathcal{O}_{r\pm}, \qquad (1.9)$$

i.e., a weighted sum of the cardinalities $\#\mathcal{O}_{r\pm}$ of the isomorphism classes of symmetric bilinear forms. It is easy to write formulas for these cardinalities [1]. We have

$$#\mathcal{O}_{r\pm} = #Gr_{n,r} \cdot \frac{\#GL_r}{\#O_{r\pm}},\tag{1.10}$$

where $Gr_{n,r}$ denotes the Grassmannian of r-dimensional subspaces of \mathbb{F}_q^n , the symbol GL_r stands for the group of automorphisms of \mathbb{F}_q^r , and $O_{r\pm}$ is the isometry group of a non-degenerate form of type \pm on \mathbb{F}_q^r , i.e., it is O_{r+} in case of a form from \mathcal{O}_{r+} and likewise with + and - switched. In particular, using standard formulas [1] for $\#Gr_{n,r}, \#GL_r$, and $\#O_{r\pm}$, we obtain

$$\#\mathcal{O}_{r\pm} \approx \frac{1}{2} q^{r(n-\frac{r-1}{2})} .$$
 (1.11)

1.3. Smallest Possible Irreducible Representation. From (1.11) we get, in particular, that the smallest non-trivial orbits are those of rank one forms. Using (1.10) we see that these have size $\#\mathcal{O}_{1\pm} = (q^n - 1)/2$. It follows from this that the smallest possible dimension of a non-trivial irreducible representation ρ of Sp should satisfy

$$\dim(\rho) \ge \frac{q^n - 1}{2}.\tag{1.12}$$

Indeed, we have the following lemma:

Lemma 1.3.1. The only irreducible representation of Sp with N-spectrum concentrated at zero is the trivial one.

The proof of Lemma 1.3.1 is easy, but to avoid interrupting this discussion, we defer it to Appendix A.1.

A representation whose dimension attaining the lower bound (1.12) would contain each rank one character of one type, and nothing else. Since N is such a small subgroup of Sp, it is unclear whether to expect such a representation to exist. In particular, it would be irreducible already on the Siegel parabolic P, and it would be the smallest possible faithful representation of P. It turns out, however, that it does exist; in fact, there are two [14, 21, 22, 29, 61].

Proposition 1.3.2. There are two irreducible representations of Sp of dimension $\frac{q^n-1}{2}$, one containing either one of the two rank one GL(X) orbits in \hat{N} .

What is the next largest possible dimension? Well, one more - the N-spectrum could include a rank one orbit, and a trivial representation. It turns out that these also exist [14, 21, 22, 29, 61].

Proposition 1.3.3. There are two irreducible representations of Sp of dimension $\frac{q^n-1}{2} + 1 = \frac{q^n+1}{2}$ one whose N-spectrum contains one of the rank one orbits in \hat{N} .

For a proof of Propositions 1.3.2 and 1.3.3, see Section 2.4.

1.4. Definition of Rank of Representation. The existence of the above smallest possible representations, plus considerations of tensor products, tell us that, for any orbit $\mathcal{O}_{k\pm}$ in \widehat{N} there will be representations of Sp whose N-spectrum contains the given orbit, together with orbits of smaller rank. Since the size—see Formulas (1.10) and (1.11)—of the orbits $\mathcal{O}_{k\pm}$ is increasing rapidly with k, representations whose N-spectrum is concentrated on orbits of smaller rank can be expected to have smaller dimensions. This motivates us to introduce the following key notion in our approach for small representations.

Definition 1.4.1 (Rank). Let ρ be a representation of Sp.

- (1) We say that ρ is of <u>rank</u> k, denoted $rk(\rho) = k$, iff the restriction $\rho_{|N|}$ contains characters of rank k, but of no higher rank.
- (2) If ρ is of rank k and contains characters of type \mathcal{O}_{k+} , but not of type \mathcal{O}_{k-} , then we say that ρ is of type \mathcal{O}_{k+} ; and likewise with + and switched.

Let us convey some intuition for this notion using numerical data obtained for the irreducible representations of the group $Sp_6(\mathbb{F}_5)$ —see Figure 5. The computations of the multiplicities and rank in this case reveal a striking compatibility with the families of representations appearing in the list of Figure 4. For example, it shows that the trivial representation is the one with rank k = 0; the

$\mathbf{m} \setminus \dim(\rho)$	1	62	62	63	63	1240	1302	1302	1365	1890	1953	1953	2015	2604	2604	3906	6510	
\mathbf{m}_0	1			1	1				1	30	31	31	31			62		
m ₁₋		1		1			1		1		1		1	1	1	1		
m ₁₊			1		1			1	1			1	1	1	1	1	5	
m ₂₋						1	1	1	1					2	2			
m ₂₊										1	1	1	1			2		
m ₃₋																	1	
m ₃₊																		
$\mathbf{rk}(\rho)$	0	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	3	

FIGURE 5. Multiplicities and rank for irreps of $Sp_6(\mathbb{F}_5)$.

4 components of the two oscillator representations are those of rank k = 1 and they split into 2 of type \mathcal{O}_{1+} and 2 of type \mathcal{O}_{1-} ; the 11 representations of dimensions between 1240 and 3906 are the ones of rank k = 2 and they split into 5 of type \mathcal{O}_{2+} and 6 of type \mathcal{O}_{2-} ; and above that the "big" representations are those with rank k = 3.

The main quest now is for a systematic construction of the "low rank" irreducible representations. In the next section we take the first step toward that goal by treating the smallest non-trivial representations of Sp which are of rank k = 1—see Propositions 1.3.2 and 1.3.3.

2. The Heisenberg and Oscillator Representations

Where do the smallest representations of Sp come from? A conceptual answer to this question was given by Weil in [61]. They can be found by considering the Heisenberg group.

2.1. The Heisenberg Group. The *Heisenberg* group attached to (V, \langle, \rangle) is a two-step nilpotent group that can be realized by the set

$$H = V \times \mathbb{F}_q,$$

with the group law

$$(v, z) \cdot (v', z') = (v + v', z + z' + \frac{1}{2} \langle v, v' \rangle)$$

In particular, the center Z of the Heisenberg group

 $Z = \{(0, z); z \in \mathbb{F}_q\},\$

is equal to its commutator subgroup. Moreover, the commutator operation in H induces a skew-symmetric bilinear form on $H/Z \simeq V$ that coincides with the original symplectic form.

The group H is the analog over a finite field of the Lie group associated with the Canonical Commutation Relations (CCR) of Werner Heisenberg, of Uncertainty Principle fame.

2.2. Representations of the Heisenberg Group. We would like to describe the representation theory, i.e., the irreducible representations, of the Heisenberg group. This theory is simultaneously simple and deep, with fundamental connections to a wide range of areas in mathematics and its applications. Take an irreducible representation π of H. Then, by Schur's lemma, the center Z will act by scalars

$$\pi(0,z) = \psi_{\pi}(z)I, \quad z \in Z,$$

where I is the the identity operator on the representation space of π , and $\psi_{\pi} \in \widehat{Z}$ is a character of Z, called the *central character* of π . If $\psi_{\pi} = 1$, then π factors through $H/Z \simeq V$, which is abelian, so π is itself a character of V. The case of non-trivial central character is described by the following celebrated theorem [40]:

Theorem 2.2.1 (Stone–von Neumann–Mackey). Up to equivalence, there is a unique irreducible representation π_{ψ} with given non-trivial central character ψ in $\widehat{Z} \smallsetminus \{1\}$.

We will call the (isomorphism class of the) representation π_{ψ} the *Heisenberg representation* associated to the central character ψ .

Remark 2.2.2 (Realization). There are many ways to realize (i.e., to write explicit formulas for) π_{ψ} [14, 18, 19, 21, 22, 29, 61]. In particular, it can be constructed as induced representation from any character extending ψ to any maximal abelian subgroup of H [24, 41]. To have a concrete one, note that the inverse image in H of any Lagrangian subspace of V will be a maximal abelian subgroup for which we can naturally extend the character ψ . For example, consider the Lagrangian $X \subset V$ and the associated maximal abelian subgroup \widetilde{X} with character $\widetilde{\psi}$ on it, given by

$$\widetilde{X} = X \times \mathbb{F}_q, \quad \widetilde{\psi}(x,z) = \psi(z).$$

Then we have the explicit realization of π_{ψ} , given by the action of H, by right translations, on the space

$$Ind_{\widetilde{X}}^{H}(\widetilde{\psi}) = \{ f : H \to \mathbb{C}; \quad f(\widetilde{x}h) = \widetilde{\psi}(\widetilde{x})f(h), \quad \widetilde{x} \in \widetilde{X}, h \in H \}.$$

$$(2.1)$$

In particular, we have $\dim(\pi_{\psi}) = q^n$.

2.3. The Oscillator Representation. A compelling property of the Heisenberg group is that it has a large automorphism group. In particular, the action of Sp on V lifts to an action on H by automorphisms leaving the center point-wise fixed. The precise formula is $g(v, z) = (gv, z), g \in Sp$. It follows from the *Stone-von Neumann-Mackey* theorem, that the induced action of Sp on the set Irr(H) will leave fixed each isomorphism class $\pi_{\psi}, \psi \in \widehat{Z} \setminus \{1\}$. This means that, if we fix a vector space \mathcal{H}_{ψ} realizing π_{ψ} , then for each g in Sp there is an operator $\omega_{\psi}(g)$ which acts on space \mathcal{H}_{ψ} and satisfies the equation

$$\omega_{\psi}(g)\pi_{\psi}(h)\omega_{\psi}(g)^{-1} = \pi_{\psi}(g(h)), \qquad (2.2)$$

which is also known as the exact Egorov identity [11] in the mathematical physics literature. Note that, by Schur's lemma, the operator $\omega_{\psi}(g)$ is defined by (2.2) up to scalar multiples. This implies that for any $g, g' \in Sp$ we have $\omega_{\psi}(g)\omega_{\psi}(g') = c(g,g')\omega_{\psi}(gg')$, where c(g,g') is an appropriate complex number of absolute value 1. It is well known (see [14, 18, 19] for explicit formulas) that over finite fields of odd characteristic this mapping can be lifted to a genuine representation.

Theorem 2.3.1 (Oscillator Representation). There $exists^4$ a representation

$$\omega_{\psi}: Sp \longrightarrow GL(\mathcal{H}),$$

that satisfies the Egorov identity (2.2).

We will call ω_{ψ} the *oscillator* representation. This is a name that was given to this representation in [22] due to its origin in physics [50, 51]. Another popular name for ω_{ψ} is the *Weil* representation, following the influential paper [61].

Remark 2.3.2 (Schrödinger Model). We would like to have some useful formulas for the representation ω_{ψ} . Note that the space (2.1) is naturally identified with

$$L^2(Y)$$
 - functions on Y. (2.3)

On the space (2.3) we realize the representation ω_{ψ} . This realization is sometime called the Schrödinger model. In particular, in that model for every $f \in L^2(Y)$ we have [14, 61]

(A)
$$\begin{bmatrix} \omega_{\psi} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} f \end{bmatrix} (y) = \psi(\frac{1}{2}A(y,y))f(y),$$

where $A: Y \to X$ is symmetric;

$$(B) \begin{bmatrix} \omega_{\psi} \begin{pmatrix} 0 & B \\ -B^{-1} & 0 \end{pmatrix} f \end{bmatrix} (y) = \frac{1}{\gamma(B,\psi)} \sum_{y' \in Y} \psi(B(y,y')) f(y'),$$

where $B: Y \xrightarrow{\longrightarrow} X$ is symmetric, and $\gamma(B, \psi) = \sum_{y \in Y} \psi(-\frac{1}{2}B(y, y))$ the quadratic Gauss sum;

 $(C) \begin{bmatrix} \omega_{\psi} \begin{pmatrix} {}^{t}C^{-1} & 0 \\ 0 & C \end{pmatrix} f \end{bmatrix} (y) = \begin{pmatrix} \frac{\det(C)}{q} \end{pmatrix} f(C^{-1}y),$

where $C \in GL(Y)$, ${}^{t}C^{-1} \in GL(X)$ its transpose inverse, and $\left(\frac{\cdot}{a}\right)$ is the Legendre symbol⁵.

It turns out that the isomorphism class of ω_{ψ} does change when varying the central character ψ in $\widehat{Z} \setminus \{1\}$. However, this dependence is weak. The following result indicates that there are only two possible oscillator representations. For a character ψ in $\widehat{Z} \setminus \{1\}$ denote by $\psi_a, a \in \mathbb{F}_q^*$, the character $\psi_a(0, z) = \psi(0, az)$.

Proposition 2.3.3. We have $\omega_{\psi} \simeq \omega_{\psi'}$ iff $\psi' = \psi_{s^2}$ for some $s \in \mathbb{F}_q^*$.

For a proof of Proposition 2.3.3, see Appendix A.2.

⁴The lift is unique except the case n = 2 and q = 3, where still there is a canonical lift [18, 19].

⁵For $x \in \mathbb{F}_q^*$ the Legendre symbol $\left(\frac{x}{q}\right) = +1$ or -1, according to x being a square or not, respectively.

2.4. The Smallest Possible Representations. Using Formula (A) given in Remark 2.3.2, it is easy to determine the rank of the oscillator representation.

Proposition 2.4.1. Each representation ω_{ψ} is of rank 1. One isomorphism class is of type \mathcal{O}_{1+} and the other is of type \mathcal{O}_{1-} .

In addition, the oscillator representations are slightly reducible. The center $Z(Sp) = \{\pm I\}$ acts on the representation ω_{ψ} —see Remark 2.3.2 for the explicit action of -I. We have the direct sum decomposition

$$\omega_{\psi} = \omega_{\psi,1} \oplus \omega_{\psi,sgn},\tag{2.4}$$

with

$$\dim(\omega_{\psi,1}) = \begin{cases} \frac{q^n+1}{2} & \text{if } q \equiv 1 \mod 4; \\ \frac{q^n-1}{2} & \text{if } q \equiv 3 \mod 4; \end{cases} \text{ and } \dim(\omega_{\psi,sgn}) = \begin{cases} \frac{q^n-1}{2} & \text{if } q \equiv 1 \mod 4; \\ \frac{q^n+1}{2} & \text{if } q \equiv 3 \mod 4, \end{cases}$$

where $\omega_{\psi,1}$ is the subspace of "even vectors", i.e., vectors on which Z(Sp) acts trivially, and $\omega_{\psi,sgn}$ is the subspace of "odd vectors", i.e., vectors on which Z(Sp) acts via the sign character. The above discussion also implies the following:

Theorem 2.4.2. The decomposition (2.4) is the decomposition of ω_{ψ} into irreducible representations.

To conclude, our study of the oscillator representation has established Propositions 1.3.2 and 1.3.3. More precisely, the representations (2.4) have rank one, they are of type $\mathcal{O}_{1\pm}$, and have the required dimensions.

3. Construction of Rank k Representations

Where do higher rank representations of Sp come from? This section will include an answer to this question in the regime of "small" representations. More, precisely we give here a systematic construction of rank k irreducible representations of Sp in the so-called "stable range"

$$k < n = \frac{\dim(V)}{2}.$$

We will also refer to such representations as "small" or "low rank".

3.1. The Symplectic-Orthogonal Dual Pair. Let U be a k-dimensional vector space over \mathbb{F}_q , and let β be an inner product (i.e., a non-degenerate symmetric bilinear form) on U. The pair (U, β) is called a *quadratic space*. We denote by O_β the isometry group of the form β . Consider the vector space $V \otimes U$ —the tensor product of V and U [4]. It has a natural structure of a symplectic space, with the symplectic form given by $\langle, \rangle \otimes \beta$. The groups Sp = Sp(V) and O_β act on $V \otimes U$ via their actions on the first and second factors, respectively,

$$Sp \curvearrowright V \otimes U \curvearrowleft O_{\beta}.$$

Both actions preserve the form $\langle , \rangle \otimes \beta$, and moreover the action of Sp commutes with that of O_{β} , and vice versa. In particular, we have a map

$$Sp \times O_{\beta} \longrightarrow Sp(V \otimes U),$$
 (3.1)

which embeds the two factors Sp and O_{β} in $Sp(V \otimes U)$, and they form a pair of commuting subgroups. In fact, each is the full centralizer of the other inside $Sp(V \otimes U)$. Thus, the pair (Sp, O_{β}) forms what has been called in [22] a *dual pair* of subgroups of $Sp(V \otimes U)$. 3.2. The Schrödinger Model. We write down a specific model for the oscillator representation⁶ $\omega_{V\otimes U}$ of $Sp(V \otimes U)$ which is convenient for us when we consider the restriction of $\omega_{V\otimes U}$ to the subgroups Sp and O_{β} . Let us identify the Lagrangian subspace $Y \otimes U$ of $V \otimes U$ with Hom(X, U). This enables to realize (see Remark 2.3.2) the representation $\omega_{V\otimes U}$ on the space of functions

$$\mathcal{H} = L^2(Hom(X, U)).$$

In this realization, the action of an element $A: Y \to X$ of the Siegel unipotent N of Sp is given by

$$\left(\omega_{V\otimes U}(A)f\right)(T) = \psi(\frac{1}{2}Tr(\beta_T \circ A))f(T), \qquad (3.2)$$

where for $T: X \to U$ we denote by $\beta_T: X \to X^* = Y$ the quadratic form

$$\beta_T(x, x') = \beta(T(x), T(x')), \qquad (3.3)$$

and we denote by $Tr(\beta_T \circ A)$ the trace of the composite operator $Y \xrightarrow{A} X \xrightarrow{\beta_T} Y$. In addition, in this model the action of an element $r \in O_\beta$ is given by

$$\left(\omega_{V\otimes U}(r)f\right)(T) = \left(\frac{\det(r)^n}{q}\right)f(r^{-1}\circ T).$$
(3.4)

3.3. The Eta Correspondence. Consider the oscillator representation $\omega_{V\otimes U}$ of $Sp(V\otimes U)$.

Remark 3.3.1. For the rest of this section we make the following choice. If ψ is the central character we use to define $\omega_{V\otimes U}$, then the character $\psi_{\frac{1}{2}}$, $\psi_{\frac{1}{2}}(z) = \psi(\frac{1}{2}z)$, of \mathbb{F}_q is the one we use in (1.5) to identify $Sym^2(Y)$ and \hat{N} .

With this choice of parameters we can make the following precise statement:

Proposition 3.3.2. Assume that $\dim(U) = k < n$. As a representation of Sp, $\omega_{V\otimes U}$ is of rank k and type⁷ \mathcal{O}_{β} .

For a proof of Proposition 3.3.2, see Appendix A.3.

Now, consider the restriction, via the map (3.1), of $\omega_{V\otimes U}$ to the product $Sp \times O_{\beta}$. We decompose this restriction into isotypic components for O_{β} :

$$\omega_{V \otimes U|Sp \times O_{\beta}} \simeq \sum_{\tau \in Irr(O_{\beta})} \Theta(\tau) \otimes \tau, \qquad (3.5)$$

where $\Theta(\tau)$ is a representation of Sp. Although the factors $\Theta(\tau)$ in (3.5) will in general not be irreducible, we can say something about how they decompose. Let us denote by

$$Irr(Sp)_k \supset Irr(Sp)_{k\beta},$$

the sets of (equivalence classes of) irreducible representations of Sp of rank k, and of rank k and type \mathcal{O}_{β} , respectively. The next theorem—the main result of this note—announces that each $\Theta(\tau)$ has a certain largest "chunk", which is in fact what we are searching for.

Theorem 3.3.3 (Eta Correspondence). Assume that $\dim(U) = k < n$. The following hold true:

- (1) **Rank** k piece. For each τ in $Irr(O_{\beta})$ the representation $\Theta(\tau)$ contains a unique irreducible constituent $\eta(\tau)$ of rank k; all other constituents have rank less than k.
- (2) **Injection.** The mapping $\tau \mapsto \eta(\tau)$ gives an embedding

$$\eta: Irr(O_{\beta}) \longrightarrow Irr(Sp)_{k\beta}.$$
(3.6)

(3) **Spectrum.** The multiplicity of the orbit \mathcal{O}_{β} in $\eta(\tau)_{|N|}$ is dim (τ) .

⁶We suppress the dependence of $\omega_{V\otimes U}$ on the central character, but we record which symplectic group it belongs to. ⁷A rank k form B on Y is of type \mathcal{O}_{β} if Y/rad(B) is isometric to (U,β) .

For a proof of Theorem 3.3.3, see Appendix B.

It also seems that this construction should produce all of the rank k representations. We formulate this as a conjecture.

Conjecture 3.3.4 (Exhaustion). Assume that $\dim(U) = k < n$. We have

$$Irr(Sp)_k = \eta(Irr(O_{\beta+})) \bigsqcup \eta(Irr(O_{\beta-})), \qquad (3.7)$$

where β + and β - represent the two isomorphism classes of inner products of rank k.

Remark 3.3.5. Note that by (3.6) the union in (3.7) is indeed disjoint.

Conjecture 3.3.4 is backed up by theoretical observations and numerical computations—see Subsection 3.4 for illustration.

Remark 3.3.6 (The case dim(U) = n). Proposition 3.3.2 and Theorem 3.3.3, and their proofs, hold also in the case dim(U) = n. However, due to Conjecture 3.3.4 we decided to formulate them with k < n.

We give now several additional remarks that, in particular, will clarify the novelty of our main result, and will also explain why we decided to call (3.6) the eta correspondence.

Remark 3.3.7. We would like to comment that

(a) Eta vs. Theta correspondence over local fields: Considering the groups Sp and O_β over a local field, one can associate, in a similar fashion as above, to every irreducible representation τ of O_β , a representation $\Theta(\tau)$ of Sp. It will in general not be irreducible and the question is, what component to select from it? One option is to take the "minimal" piece of $\Theta(\tau)$. Indeed, it turns out that $\Theta(\tau)$ has a unique irreducible quotient $\theta(\tau)$. The assignment

$$au\mapsto heta(au)$$
 - "minimal" piece,

is the famous theta correspondence, which has been studied by many authors [13, 17, 23, 26, 28, 42, 45, 46, 60] for its usefulness in the theory of automorphic forms. A second option is to take the "maximal" piece of $\Theta(\tau)$. Indeed, repeating in the local field case, verbatim, the scheme we proposed above, we find that $\Theta(\tau)$ has a largest chunk in the form of a unique irreducible sub-representation $\eta(\tau)$ of rank k, which will equal $\theta(\tau)$ exactly when $\Theta(\tau)$ is irreducible. The assignment

$$\tau \mapsto \eta(\tau)$$
 - "maximal" piece,

is our eta correspondence (3.6). Application of the new correspondence to representation theory of classical groups over local fields, will be a subject for future publications.

(b) Eta Correspondence over finite fields: As noted by several authors (see [2, 3, 5, 22, 56], and in particular [2] where the case of unipotent representations was considered) the theta correspondence is not defined over finite fields⁸. The eta correspondence comes as the appropriate construction in this case. This is also the reason we use a related, although different, notation for the correspondence (3.6).

Finally, we would like to make the following remark on the generality of our work.

Remark 3.3.8 (Generalized Eta Correspondence). The notion of rank for the group Sp_{2n} over local fields was described in [25]. The theory for general classical groups over local fields was developed by Li in [31], and it was extended to all semi-simple algebraic groups over local fields by Salmasian in [47]. The development of the eta correspondence (3.6) for all finite classical groups will be discussed in future publications. For expositional purposes, in this note we describe only the case of the finite symplectic groups.

⁸In fact, the attempt [22] to develop a duality theory over finite fields preceded the one over the local fields [23].

3.4. Numerical Justification for the Exhaustion Conjecture . Conjecture 3.3.4 is backed up by numerical data collected for the groups $Sp_6(\mathbb{F}_q)$, q = 3, 5, 7, 9, 11, 13; $Sp_8(\mathbb{F}_q)$, q = 3, 5, and $Sp_{10}(\mathbb{F}_3)$. Indeed, the Magma computations done with Cannon and Goldstein, for the various sizes of symplectic

$m \setminus \dim(\rho)$	1	40	40	41	41	780	820	820	861	1599	1640	1640	1640	1680
m 0	1			1	1				1	39		40	40	40
<i>m</i> ₁₊		1		1			1		1		1	1		1
<i>m</i> ₁₋			1		1			1	1		1		1	1
<i>m</i> ₂₊						1	1	1	1		2			
<i>m</i> ₂₋										1		1	1	1
$rk(\rho)$	0	1	1	1	1	2	2	2	2	2	2	2	2	2

FIGURE 6. Multiplicities and rank for irreps of $Sp_8(\mathbb{F}_3)$: Ranks k = 0, 1, 2.

$m \setminus \dim(\rho)$	9840	9840	10660	10660	11480	11480	12300	12300	19188	21320	21320	22960	22960
\mathbf{m}_0					40	40	40	40					
m ₁₊		12		13		13	1	13			26	1	27
m ₁₋	12		13		13		13	1		26		27	1
m ₂₊			1	1			1	1	3			2	2
m ₂₋					1	1	1	1		1	1	1	1
m ₃₊	1		1		1		1			2		2	
m ₃₋		1		1		1		1			2		2
m ₄₊													
m ₄₋									1				
rk(p)	3	3	3	3	3	3	3	3	4	3	3	3	3

FIGURE 7. Multiplicities and rank for irreps of $Sp_8(\mathbb{F}_3)$: Ranks k = 3, 4.

$m \setminus \dim{(\rho)}$	25830	29848	29848	31980	31980	32760	32760	33579	33579	34440	34440	37310	
\mathbf{m}_0	90							39	39	40	40	130	
m ₁₊			13		39		39		39	1	40		
m ₁₋		13		39		39		39		40	1	13	
m ₂₊		4	4	1	1	2	2	1	1	2	2		
m ₂₋	3			1	1	1	1	2	2	2	2	4	
m ₃₊		1		3		3		3		3		1	
m ₃₋			1		3		3		3		3		
m ₄₊	1											1	
m ₄₋		1	1										
rk(p)	4	4	4	3	3	3	3	3	3	3	3	4	

FIGURE 8. Multiplicities and rank for irreps of $Sp_8(\mathbb{F}_3)$: Ranks k = 3, 4.

groups, repeatedly confirm the assertion made in the exhaustion conjecture, i.e., Identity (3.7). For

example, the number of conjugacy classes in the orthogonal groups $O_{1+}(\mathbb{F}_q)$ and $O_{1-}(\mathbb{F}_q)$ together is 4, each contributes 2 classes; the number of conjugacy classes in the groups $O_{2+}(\mathbb{F}_q)$ and $O_{2-}(\mathbb{F}_q)$ together is q + 6, one contributes $\frac{q+5}{2}$ classes and the other $\frac{q+7}{2}$ classes. In addition, the number of conjugacy classes in the groups $O_{3+}(\mathbb{F}_q)$ and $O_{3-}(\mathbb{F}_q)$ together is 4(q+2), each contributes 2(q+2)classes. Hence, the computations of the multiplicities and rank for the groups $Sp_6(\mathbb{F}_5)$ and $Sp_8(\mathbb{F}_3)$ presented in Figures 5 and 6–7–8, respectively, give the required numerical confirmation of (3.7) in these cases.

4. Dimension of Rank k Representations

We would like to clarify the strong relationship between the dimension of a representation of Sp and its rank.

4.1. **Dimension.** Fix k < n and consider a rank k irreducible representation $\rho \in Irr(Sp)_k$. Let us assume that ρ appears in the image of the eta correspondence (3.6). Namely, there exist $\tau \in Irr(O_{k\pm})$ such that $\rho = \eta(\tau)$ —see Section 3.3. Using Part 3 of Theorem 3.3.3, and the dimension formula (1.9), we have

$$\dim(\eta(\tau)) = \dim(\tau) \cdot \#\mathcal{O}_{k\pm} + \sum_{r < k} \sum_{\pm} m_{r\pm} \cdot \#\mathcal{O}_{r\pm}.$$
(4.1)

The point now is—see Figure 9 for illustration—that the term $\dim(\tau) \cdot \#\mathcal{O}_{k\pm}$ dominates the right hand side of (4.1). Indeed, we have

Theorem 4.1.1 (Dimension Estimate). Let $\eta(\tau)$ be a rank k < n irreducible representation of Sp associated to an irreducible representation τ of $O_{k\pm}$. Then

$$1 \le \frac{\dim(\eta(\tau))}{\dim(\tau) \# \mathcal{O}_{k\pm}} \le 1 + \frac{2 + \varepsilon(q)}{q^{n-k+1}}, \quad with \quad \varepsilon(q) = O(1/q).$$

$$(4.2)$$

A proof of Theorem 4.1.1 will be given in a sequel paper.

Remark 4.1.2. The term $\varepsilon(q)$ can be estimated explicitly. For example we have $\varepsilon(q) < 2/q + 4/q^2$.

Theorem 4.1.1 seems to substantially extend the current knowledge [34, 43, 52, 59] on the dimensions of representations of the finite symplectic groups (See Lemma 2.3. in [39]).

$m \smallsetminus \dim\left(\eta(\tau)\right)$	1	62	62	63	63	1240	1302	1302	1365	1890	1953	1953	2015	2604	2604	3906
m ₀	1			1	1				1	30	31	31	31			62
m ₁₋		1		1			1		1		1		1	1	1	1
m ₁₊			1		1			1	1			1	1	1	1	1
m ₂₋						1	1	1	1					2	2	
m ₂₊										1	1	1	1			2
$\frac{\dim(\eta(\tau))}{\dim(\tau) \# \mathcal{O}_{\underline{k} \pm}}$	1	1	1	1.01	1.01	1	1.05	1.05	1.10	1.01	1.05	1.05	1.08	1.05	1.05	1.05
$rk(\eta(\tau))$	0	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2

FIGURE 9. The relation between dim $(\tau) \cdot \#\mathcal{O}_{k\pm}$ and dim $(\eta(\tau))$ for rank k < 3 irreps of $Sp_6(\mathbb{F}_5)$.

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4.2. Compatibility of Dimension and Rank. Although dimension tends to increase with rank, because of the factor dim(τ) in (4.1), it may happen, see Figures 7–8, that a representation of rank k has larger dimension than one of rank k + 1. However, for a given k, if n is large enough then the representations of rank k will have smaller dimension than those of rank k + 1. For example, it seems that if q is sufficiently large, then for k = 1, one can take n = 2, and for k = 2, one can take n = 3—see Figures 9 and 6 for illustration. In general, using Theorem 4.1.1 and the known estimates on the dimensions of the largest irreducible representations of the orthogonal groups, we have the following result:

Proposition 4.2.1 (Compatibility of Dimension and Rank). For sufficiently large q, in the regime

$$k < 2\sqrt{n} - 1$$

the rank k representations appearing in the image of the eta correspondence (3.6) always have smaller dimension then those of rank k + 1.

The exact computation leading to a verification of Proposition 4.2.1 will be given in a sequel paper.

APPENDIX A. Proofs

A.1. Proof of Lemma 1.3.1.

Proof. If $Supp_N(\rho) = 0$ then $\rho|_N$ is trivial. The Lemma now follows from the well known fact that the N conjugates generate the group Sp [1].

A.2. Proof of Proposition 2.3.3.

Proof. Consider the automorphism $\alpha_s : H \to H$ given by $\alpha_s(v, z) = (v, s^2 z)$. It can be extended to an automorphism of the semi-direct product of Sp with H, by letting it act trivially on Sp. The equivalence of oscillator representations follows. The fact that for a non-square $\varepsilon \in \mathbb{F}_q^*$, the representations ω_{ψ} and $\omega_{\psi_{\varepsilon}}$ are not isomorphic, can be verified using the realization given in Remark 2.3.2. This completes the proof of the proposition.

A.3. Proof of Proposition 3.3.2.

Proof. The proposition follows immediately from Equation (3.2) in Section 3.2.

APPENDIX B. Proof of the Eta Correspondence Theorem

We give a proof of Theorem 3.3.3 that is an elementary application of the *double commutant* theorem [62].

B.1. The Double Commutant Theorem. We will use the following version:

Theorem B.1.1 (Double Commutant Theorem). Let W be a finite dimensional vector space. Let $\mathcal{A}, \mathcal{A}' \subset End(W)$ be two sub-algebras, such that

- (1) The algebra \mathcal{A} acts semi-simply on W.
- (2) Each of \mathcal{A} and \mathcal{A}' is the full commutant of the other in End(W).

Then \mathcal{A}' acts semi-simply on W, and as a representation of $\mathcal{A} \otimes \mathcal{A}'$ we have

$$W = \bigoplus_{i \in I} W_i \otimes W'_i,$$

where W_i are all the irreducible representations of \mathcal{A} , and W'_i are all the irreducible representations of \mathcal{A}' . In particular, we have a bijection between irreducible representations of \mathcal{A} and \mathcal{A}' , and moreover, every isotypic component for \mathcal{A} is an irreducible representation of $\mathcal{A} \otimes \mathcal{A}'$.

B.2. **Preliminaries.** Let us start with several preliminary steps. We work with the Schrödinger model of $\omega_{V\otimes U}$ appearing in Section 3.2. It is realized on the space

$$\mathcal{H} = L^2(Hom(X, U)), \tag{B.1}$$

and there, the actions of an element A of the Siegel unipotent radical $N \subset Sp$, and an element $r \in O_{\beta}$, are given by Formulas (3.2) and (3.4), respectively. In particular, we have

Claim B.2.1. Every character appearing in the restriction of $\omega_{V\otimes U}$ to N is of the form ψ_{β_T} for some $T \in Hom(X, U)$. Moreover, we have $rank(\beta_T) = k$ iff T is onto.

For the rest of the section, we fix a transformation $T: X \twoheadrightarrow U$ which is <u>onto</u> and consider the character subspace

$$\mathcal{H}^{\psi_{\beta_T}} = \{ f \in \mathcal{H}; \ \omega_{V \otimes U}(A) f = \psi_{\beta_T}(A) f, \ A \in N \}.$$

We would like to have a better description of the space $\mathcal{H}^{\psi_{\beta_T}}$. The orthogonal group O_{β} acts naturally on Hom(X, U) and we denote by \mathcal{O}_T the orbit of T under this action.

Proposition B.2.2. We have $\mathcal{H}^{\psi_{\beta_T}} = L^2(\mathcal{O}_T)$ the space of functions on \mathcal{O}_T .

For a proof of Proposition B.2.2, see Section B.4.1.

Note that, because T is onto, the action of O_{β} on \mathcal{O}_T is free. In particular, we can identify \mathcal{O}_T with O_{β} , and the Peter–Weyl theorem [50] for the regular representation implies

Corollary B.2.3. Under the action of O_{β} , the space $\mathcal{H}^{\psi_{\beta_T}}$ decomposes as

$$\mathcal{H}^{\psi_{\beta_T}} \simeq \bigoplus_{\tau \in Irr(O_\beta)} \dim(\tau)\tau. \tag{B.2}$$

We would like now to describe the commutant of O_{β} in $End(\mathcal{H}^{\psi_{\beta_T}})$. Considering the group

$$G_{\beta_T} = Stab_{GL(X)}(\beta_T),$$

of automorphisms of X that stabilize the form β_T , we obtain two commuting actions

$$O_{\beta} \curvearrowright \mathcal{H}^{\psi_{\beta_T}} \curvearrowleft G_{\beta_T}.$$

Moreover, we have

Proposition B.2.4. The groups O_{β} and $G_{\beta_{T}}$ generate each other's commutant in $End(\mathcal{H}^{\psi_{\beta_{T}}})$.

For a proof of Proposition B.2.4, see Section B.4.2.

B.3. Proof of Theorem 3.3.3.

Proof. Write

$$\Theta(\tau) \simeq \sum \eta_i(\tau)$$

for various irreducible representations $\eta_i(\tau)$ of Sp. Then

$$\Theta(\tau)^{\psi_{\beta_T}} \simeq \sum \eta_i(\tau)^{\psi_{\beta_T}}.$$
(B.3)

In addition, $\Theta(\tau)^{\psi_{\beta_T}}$ is a G_{β_T} -module, and so is each $\eta_i(\tau)^{\psi_{\beta_T}}$. Hence, Identity (B.3) gives a decomposition of $\Theta(\tau)^{\psi_{\beta_T}}$ into (not necessarily irreducible) submodules for G_{β_T} . But Proposition B.2.4 together with the Double Commutant Theorem says that $\Theta(\tau)^{\psi_{\beta_T}}$ is irreducible as a G_{β_T} -module. Therefore, exactly one of the $\eta_i(\tau)^{\psi_{\beta_T}}$ will be non-zero, and it defines an irreducible representation of G_{β_T} , which has dimension equal to dim (τ) , by equation (B.2). To conclude, there exists a unique irreducible sub-representation $\eta(\tau) < \Theta(\tau)$ of rank k and type \mathcal{O}_{β_T} , the multiplicity of the orbit \mathcal{O}_{β} in $\eta(\tau)_{|N}$ is dim (τ) , and finally, the Double Commutant Theorem implies that for $\tau \ncong \tau'$ in $Irr(O_{\beta})$, we have $\eta(\tau) \ncong \eta(\tau')$. This completes the proof of Theorem 3.3.3.

B.4. Proofs.

B.4.1. Proof of Proposition B.2.2.

Proof. Using the delta basis $\{\delta_T; T \in Hom(X, U)\}$, we can verify Claim B.2.2, by showing that if $\beta_{T'} = \beta_T$ then there exists $r \in O_\beta$ such that $T' = r \circ T$. Indeed, let r be the composition

$$U \xrightarrow{\longrightarrow} X/rad(\beta_T) \xrightarrow{\longrightarrow} U$$

where the first and second isomorphisms are these induced by T', and T, respectively, and $rad(\beta_T)$ is the radical of β_T . This completes the proof of Proposition B.2.2.

B.4.2. Proof of Proposition B.2.4.

Proof. To verify this assertion, note that we have a short exact sequence

$$1 \to N_{k,n-k} \to G_{\beta_T} \to O(X/rad(\beta_T)) \times GL(Z) \to 1, \tag{B.4}$$

where $Z = \ker(T) = rad(\beta_T)$, $O(X/rad(\beta_T))$ is the orthogonal group of $X/rad(\beta_T)$, and $N_{k,n-k}$ is the appropriate unipotent group. The group $O(X/rad(\beta_T))$ acts simply transitively on the orbit \mathcal{O}_T , as does the group O_β , and these two actions commute with each other. If we use the map $r \mapsto r^{-1} \circ T$ to identify O_β with \mathcal{O}_T , then the action of O_β becomes the action of O_β on itself by left translation, and the action of $O(X/rad(\beta_T))$ can be identified with the action of O_β on itself by right translation. By the Peter-Weyl Theorem [50], we conclude that the groups $O(X/rad(\beta_T))$ and O_β generate mutual commutants in the operators on $L^2(\mathcal{O}_T) \simeq \mathcal{H}^{\psi_{\beta_T}}$. A fortiori the groups G_{β_T} and O_β generate mutual commutants on $L^2(\mathcal{O}_T)$. This completes the proof of Proposition B.2.4.

References

- [1] Artin E., Geometric Algebra. Interscience Publisher (1957).
- [2] Aubert A.M., Kraskiewicz W., and Przebinda T., Howe correspondence and Springer correspondence for dual pairs over a finite field. Lie Algebras, Lie Superalgebras, Vertex Algebras and Related Topics, Proceedings of Symposia in Pure Mathematics, vol. 92, AMS (2016).
- [3] Adams J. and Moy A., Unipotent representations and reductive dual pairs over finite fields. Tra. Am. Math. Soc., 340 (1993), 309-321.
- [4] Atiyah M. and Macdonald I., Introduction to Commutative Algebra. Addison-Wesley (1969).
- [5] Aubert A.M., Michel J., and Rouquier R., Correspondance de Howe pour les groupes réductifs sur les corps finis. Duke Math. Journal 83 (1996), 353-397.
- [6] Carter R.W., Finite Groups of Lie Type: Conjugacy Classes and Complex Characters. Wiley Classics Library (1993).
- [7] Deligne P. and Lusztig G., Representations of reductive groups over finite fields. Annals of Math. vol. 103 (1976), 103-161.
- [8] Digne F. and Michel J., Representations of finite groups of Lie type. LMS Student text 21 (1991).
- [9] Diaconis P., Group representations in probability and statistics. IMS Lecture Notes Monogr. Ser. vol. 11 (1988).
- [10] Diaconis P. and Shahshahani M., Generating a random permutation with random transpositions. Z. Wahrsch. 57 (1981), 159-179.
- [11] Egorov Y., The canonical transformations of a pseudo-differential operator. Uspehi. Mat. Nauk. 24, 235-236 (1969).
- [12] Ellers E. W. and Gordeev N. L., On the conjectures of J. Thompson and O. Ore. Trans. Amer. Math. Soc. 350 (1998), 3657-3671.
- [13] Gelbart S., Weil's Representation and the Spectrum of the Metaplectic group. LNM 530, Springer-Verlag (1975).
- [14] Gérardin P., Weil representations associated to finite fields. J. Alg. 46 (1977), 54-101.
- [15] Gluck D., Sharper character value estimates for groups of Lie type. J. Alg. 174 (1995), 229-266.
- [16] Gluck D., Characters and Random Walks on Finite Classical Groups. Adv. in Math. 129 (1997), 46-72.
- [17] Gan W. T. and Takeda S., A proof of the Howe duality conjecture. JAMS 29 (2016), 473-493.
- [18] Gurevich S. and Hadani R., The geometric Weil representation. Selecta Math. 13 (2007), 465-481.
- [19] Gurevich S. and Hadani R., Quantization of Symplectic Vector Spaces over Finite Fields. Journal of Symplectic Geometry, Vol. 7, N. 4 (2009), 475-502.
- [20] Gow R., Commutators in the symplectic group. Arch. Math. (Basel) 50 (1988), 204-209.
- [21] Howe R., On the character of Weil's representation. Trans. Amer. Math. Soc. 177 (1973), 287-298.
- [22] Howe R., Invariant theory and duality for classical groups over finite fields with applications to their singular representation theory. *Preprint, Yale University (1973)*.

- [23] Howe R., θ -series and invariant theory. Proc. Sympos. Pure Math., Corvallis, Ore. (1977), 275-285.
- [24] Howe R., On the role of the Heisenberg group in harmonic analysis. Bull. AMS 3 (1980), N. 2, 821-843.
- [25] Howe R., On a notion of rank for unitary representations of the classical groups. Harmonic analysis and group representations, Liguori, Naples (1982), 223-331.
- [26] Howe R., Transcending classical invariant theory. JAMS vol.2 (1989), 535-552.
- [27] Hausel, T. Letellier, E., Rodriguez-Villegas, F., Arithmetic harmonic analysis on character and quiver varieties, Duke Math. Journal, Vol. 160, N. 2 (2011), 323-400.
- [28] Kudla S.S., On the local theta-correspondence. Invent. math. (1986), Vol. 83, Issue 2, 229-255.
- [29] Isaacs I.M., Characters of solvable and symplectic groups. Amer. J. Math. 95 (1973), 594-635.
- [30] Lam T.Y., Introduction to Quadratic Forms over Fields. American Mathematical Society (2005).
- [31] Li J-S., On the classification of irreducible low rank unitary representations of classical groups. Comp. Math. 71 (1989), 29-48.
- [32] Larsen M., Malle G., and Tiep P.H., The largest irreducible representations of simple groups. Proc. London Math. Soc. 106 (2013), 65-96.
- [33] Liebeck M.W., O'Brien E.A., Shalev A., and P. H. Tiep., The Ore conjecture. JEMS 12 (2010), 939-1008.
- [34] Liebeck M.W. and Shalev A., Character degrees and random walks in finite groups of Lie type. Proc. London Math. Soc. 90 (2005), 61-86.
- [35] Lusztig G., Characters of reductive groups over a finite field. Annals of Math. Studies, 107. Princeton Univ. Press (1984).
- [36] Lusztig G., Algebraic and geometric methods in representation theory. Lecture at the Chinese University of Hong Kong (2014). arXiv:1409.8003.
- [37] Lusztig, G., Intersection cohomology methods in representation theory. Proc. ICM Kyoto (1990), 155-174.
- [38] Lusztig, G., On the character values of finite Chevalley groups at unipotent elements. Journal of Alg. vol. 104 (1986), 146-194.
- [39] Malle G., The proof of Ore's conjecture [after Ellers-Gordeev and Liebeck-O'Brien-Shalev-Tiep]. Sém. Bourbaki, Astérisque 361 (2014), 325-348.
- [40] Mackey G.W., A theorem of Stone and von Neumann. Duke Math. J. 16 (1949), no. 2, 313-326.
- [41] Mackey G.W., Imprimitivity for representations of locally compact groups. Proc. NAS. U.S.A. 35 (1949), 537–545.
- [42] Moeglin C., Vigneras M.F., and Waldspurger J.L., Correspondence de Howe sur un corps p-adique. LNM 1291, Springer-Verlag (1987).
- [43] Nguyen H.N., Low-dimensional complex characters of the symplectic and orthogonal groups. Comm. Algebra 38 (2010), 1157-1197.
- [44] Ore O., Some remarks on commutators. Proc. Amer. Math. Soc. 2 (1951), 307-314.
- [45] Prasad D., On the local Howe duality correspondence. IMRN (1993), No. 11, 279-287.
- [46] Rallis S., L-functions and the oscillator representation. LNM, vol. 1245, Springer-Verlag (1987).
- [47] Salmasian H., A notion of rank for unitary representations of reductive groups based on Kirillov's orbit method. Duke Math. J. 136 (2007), 1-49.
- [48] Saloff-Coste L., Random Walks on Finite Groups. Vol. 110, Encyclopaedia of Math. Sciences (2004), 263-346.
- [49] Segal I., Transforms for operators and symplectic automorphisms over a locally compact group. Math. Scand. 13 (1963), 31-43.
- [50] Serre J.P., Linear Representations of Finite Groups. Springer-Verlag (1977).
- [51] Shale D., Linear symmetries of free boson fields. Trans. AMS 103 (1962), 149-167.
- [52] Shalev A., Commutators, Words, Conjugacy Classes and Character Methods. Turk J Math 31 (2007), Suppl, 131-148.
- [53] Shoji T., On the Green polynomials of classical groups. Invent. Math., 74 (1983), 239-267.
- [54] Siegel C.L., Symplectic Geometry. American Journal of Mathematics Vol. 65, N. 1 (1943), 1-86.
- [55] Srinivasan B., The characters of the finite symplectic group $Sp_4(q)$. Trans. of the AMS 131 (1968), 488-525.
- [56] Srinivasan B., Weil representations of classical groups. Invent. Math. 51 (1979), 143-153.
- [57] Srinivasan B., Representations of finite Chevalley groups. LNM 764, Springer-Verlag (1979).
- [58] Terras A., Fourier Analysis on Finite Groups and Applications. LMS Student Texts 43, Cambridge University Press (1999).
- [59] Tiep P.H. and Zalesskii A.E., Minimal characters of the finite classical groups. Comm. in Alg. vol. 24 (1996).
- [60] Waldspurger J.-L., Demonstration d'une conjecture de dualite de Howe dans le cas p- adique, $p \neq 2$. Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Israel Math. Conf. Proc., vol. 2 (1990), 267–324.
- [61] Weil A., Sur certains groups d'operateurs unitaires. Acta Math. 111 (1964), 143-211.
- [62] Weyl H., The classical groups, their invariants and representations. Princeton Univ. Press (1946).

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