COUNTING INTEGRAL POINTS IN CERTAIN HOMOGENEOUS SPACES

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Abstract

The asymptotic formula of the number of integral points in non-compact symmetric homogeneous spaces of semi-simple simply connected algebraic groups is given by the average of the product of the number of local solutions twisted by the Brauer-Manin obstruction. The similar result is also true for homogeneous spaces of reductive groups with some restriction. As application, we will give the explicit asymptotic formulae of the number of integral points of certain norm equations and explain that the asymptotic formula of the number of integral points in Theorem 1.1 of [8] is equal to the product of local integral solutions over all primes and answer a question raised by Borovoi related to the example 6.3 in [3].

1. INTRODUCTION

The Hardy-Littlewood circle method is the classical method for counting integral points. Once this method can be applied, the asymptotic formula of the number of integral points will be given by the product of the number of local solutions and the local-global principle will be true (see [25]). However, the local-global principle can not be held in general. Recently, the existence of the integral points with Brauer-Manin obstruction on various homogeneous spaces has been studied extensively in [4], [9], [5], [34], [35] and [2]. It is natural to ask how the asymptotic formula of the number of integral solutions will be related to Brauer-Manin obstruction for such homogeneous spaces. Indeed, such problem has already been studied by Borovoi and Rudnick in [3] for homogeneous spaces of semi-simple groups, where they measured the difference between the number of integral solutions and that of Hardy-Littlewood expectation by introducing so called the density functions. These density functions were described by using Kottwitz invariant in [3]. This paper can be regarded as the continuation of [3]. More precisely, we show that the asymptotic formula of the integral solutions will be given by the average of the product of local solutions twisted by the Brauer-Manin obstruction.

Notation and terminology are standard if not explained. Let F be a number field, o_F be the ring of integers of F, Ω_F be the set of all primes in F and ∞_F be the set all infinite primes in F. We use $v < \infty_F$ to denote the finite primes of F. Let F_v be the completion of F at v and o_{F_v} be the local completion of o_F at v for each $v \in \Omega_F$. Write $o_{F_v} = F_v$ for $v \in \infty_F$. For any finite subset S of Ω_F containing ∞_F, the

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S-integers are defined to be elements in F which are integral outside S and denoted by o_S. We also denote the adeles of F by $\mathbb{A}_F$ and ideles of F by $\mathbb{I}_F = \mathbb{G}_m(\mathbb{A}_F)$ and $F_\infty = \prod_{v \in \infty_F} F_v$.

Let G be a connected reductive linear algebraic group and $\mathbf{X}$ be a separated scheme of finite type over o_F whose generic fiber

$$X = \mathbf{X} \times_{o_F} F \cong H \backslash G$$

is a right homogeneous space of G, where H is the stabilizer of the fixed point $P \in X(F)$. The map induced by the fixed point P is denoted by

$$p : \; G \longrightarrow X \tag{1.1}$$

The obvious necessary condition for $\mathbf{X}(o_F) \neq \emptyset$ is

$$\prod_{v \in \Omega_F} \mathbf{X}(o_{F_v}) \neq \emptyset \tag{1.2}$$

which is assumed throughout this paper. By the separatedness, one can naturally regard $\mathbf{X}(o_{F_v})$ as an open and compact subset of $X(F_v)$ with v-adic topology for $v < \infty_F$. Since we will count the integral points by using the coordinates, we also assume that X is affine and fix the closed immersion

$$X \hookrightarrow Spec(F[x_1, \cdots, x_n]). \tag{1.3}$$

By Matsushima's criterion (see Theorem 4.17 in §4 of [23]), the assumption (1.3) is equivalent to say that H is reductive as well. For simplicity, we will also assume that H is connected. Therefore any point in $X(E)$ can be regarded as a point in E^n under the closed immersion (1.3) for any F-algebra E. A point $x \in X(F)$ can be written as $x = (z_1, \cdots, z_n) \in F^n$. Then one can define

$$|x|_{\infty_F} = \max_{v \in \infty_F} \max_{1 \leq i \leq n} \{ |z_i|_v \}.$$

For $T > 0$, one can set

$$N(\mathbf{X}, T) = \sharp \{ x \in \mathbf{X}(o_F) : \; |x|_{\infty_F} \leq T \}$$

and

$$X(F_\infty, T) = \{ x \in X(F_\infty) : \; |x|_{\infty_F} \leq T \}.$$

Since X admits a unique G-invariant gauge form ω_X up to a scalar and ω_X matches with the gauge forms ω_H of H and ω_G of G algebraically in sense of §2.4 (P.24) in [33], the associated Tamagawa measures on $X(\mathbb{A}_F)$, $H(\mathbb{A}_F)$ and $G(\mathbb{A}_F)$ (see Chapter II in [33]) are denoted by

$$d_X = \prod_{v \in \Omega_F} d_v, \quad \nu_H = \prod_{v \in \Omega_F} \nu_v \quad \text{and} \quad \lambda_G = \prod_{v \in \Omega_F} \lambda_v$$

respectively.

Let $Br(X) = H^2_{et}(X, \mathbb{G}_m)$. We would like to write the Brauer-Manin paring (see [29]) in multiplicative way

$$X(\mathbb{A}_F) \times Br(X) \longrightarrow \mu_\infty = \varinjlim_n \mu_n \subset \mathbb{C}^\times$$

$$((x_v)_{v \in \Omega_F}, \alpha) \mapsto \prod_{v \in \Omega_F} \alpha(x_v)$$

where $\alpha(x_v)$'s are all roots of unity and $\alpha(x_v) = 1$ for almost all $v \in \Omega_F$ by fixing the isomorphism

$$\mathbb{Q}/\mathbb{Z} \xrightarrow{\cong} \mu_\infty.$$

Then one can view any element in $Br(X)$ as a locally constant $\mathbb{C}$-value function on $X(\mathbb{A}_F)$. For any subset B of $Br(X)$, one can define

$$X(\mathbb{A}_F)^B = \{(x_v) \in X(\mathbb{A}_F) : \prod_{v \in \Omega_F} \alpha(x_v) = 1 \text{ for all } \alpha \in B\}.$$

Definition 1.4. *For any $\xi \in Br(X)$, one can define*

$$N_v(\mathbf{X}, \xi) = \int_{\mathbf{X}(o_{F_v})} \xi d_v$$

for any $v < \infty_F$ and

$$N_{\infty_F}(X, T, \xi) = \int_{X(F_\infty, T)} \xi d_{\infty_F}$$

for $T > 0$.

The main result of this paper is the following the asymptotic formula.

Theorem 1.5. *If G is simply connected and almost F-simple such that $G(F_\infty)$ is not compact and $X = H \backslash G$ such that H is the group of fixed points of an involution of G, then*

$$N(\mathbf{X}, T) \sim \sum_{\xi \in (Br(X)/Br(F))} (\prod_{v < \infty_F} N_v(\mathbf{X}, \xi)) \cdot N_{\infty_F}(X, T, \xi)$$

as $T \to \infty$ for any separated scheme $\mathbf{X}$ of finite type over o_F with $X = \mathbf{X} \times_{o_F} F$.

For general reductive groups, one can have the similar result with certain restriction. One of the extreme situation is that G is an algebraic torus and X is the trivial torsor of G. Consider any character

$$\chi: \quad G(\mathbb{A}_F)/G(F)St(\mathbf{X}) \longrightarrow \mathbb{C}^\times \quad \text{and} \quad \chi = \prod_{v \in \Omega_F} \chi_v \qquad (1.6)$$

where

$$St(\mathbf{X}) = \{g_\mathbb{A} \in G(\mathbb{A}_F) : (\prod_{v \in \Omega_F} \mathbf{X}(o_{F_v})) \cdot g_\mathbb{A} = \prod_{v \in \Omega_F} \mathbf{X}(o_{F_v})\}$$

and χ_v is induced by

$$\chi_v : G(F_v) \to G(\mathbb{A}_F)/G(F)St(\mathbf{X}) \xrightarrow{\chi} \mathbb{C}^\times$$

for any $v \in \Omega_F$ and $\chi_v = 1$ for $v \in \infty_F$. Each χ_v also induces the locally constant function $\overline{\chi}_v$ on $X(F_v)$ by setting

$$\overline{\chi}_v(x_v) = \chi_v(p^{-1}(x_v))$$

for any $x_v \in X(F_v)$, where p is the map induced by (1.1). Set

$$N_v(\mathbf{X}, \overline{\chi}_v) = \int_{\mathbf{X}(o_{F_v})} \overline{\chi}_v d_v$$

for $v < \infty_F$.

Theorem 1.7. *If G is an anisotropic torus and X is the trivial torsor of G, then*

$$N(\mathbf{X}, T) \sim \frac{1}{\tau(G)} (\sum_\chi \prod_{v < \infty_F} N_v(\mathbf{X}, \overline{\chi}_v)) \cdot d_{\infty_F}(X(F_\infty, T))$$

as $T \to \infty$ for any separated scheme $\mathbf{X}$ of finite type over o_F with $X = \mathbf{X} \times_{o_F} F$, where $\tau(G)$ is the Tamagawa number of G and χ runs over all characters in (1.6).

The basic idea for proving Theorem 1.5 is to partition the orbits of the arithmetic group action in $\mathbf{X}(o_F)$ in proper way and apply the equi-distribution property developed in [6], [7] and [1]. The paper is organized as follows. We first study various orbits of $\mathbf{X}(o_F)$ in Section 2. In order to obtain the similar result for general reductive groups under the assumption that the map on the F-points induced by (1.1) is surjective, we establish the mass formulae associated the Brauer-Manin obstruction following from [36] in Section 3. Such mass formulae with characters are initial from [12], [31] and [26]. In Section 4, we prove our main theorems based on the results on previous sections and the equi-distribution property. As application, we first study the asymptotic formulae of the integral solutions for norm equations in Section 5. In Section 6, we will explain that the asymptotic formula in Theorem 1.1 of [8] can be given by the product of all local solutions and answer a question raised by Borovoi related to the example 6.3 in [3] .

2. ORBITS

Fix a finite subset S containing ∞_F satisfying that there are the group schemes $\mathbf{G}$ and $\mathbf{H}$ of finite type over o_S whose generic fibers are G and H respectively such that

$$\mathbf{X}_S = \mathbf{X} \times_{o_F} o_S \cong \mathbf{H} \backslash \mathbf{G}$$

and $P \in \mathbf{X}(o_{F_v})$ for all $v \notin S$. For each $v \in S \setminus \infty_F$, one can fix a group scheme $\mathbf{G}_v$ of finite type over o_{F_v} such that $\mathbf{G}_v \times_{o_{F_v}} F_v = G \times_F F_v$.

Definition 2.1. *For each $v \in \Omega_F$, one defines*

$$St(\mathbf{X}(o_{F_v})) = \begin{cases} \mathbf{G}(o_{F_v}) & v \notin S \\ \{g \in \mathbf{G}_v(o_{F_v}): \quad \mathbf{X}(o_{F_v}) = \mathbf{X}(o_{F_v}) \cdot g\} & v \in S \setminus \infty_F \\ G(F_v) & v \in \infty_F \end{cases}$$

Then $St(\mathbf{X}(o_{F_v}))$ is an open subgroup of $G(F_v)$ for each $v \in \Omega_F$ and it is compact for $v < \infty_F$.

Lemma 2.2. *The number of orbits*

$$[\mathbf{X}(o_{F_v})/St(\mathbf{X}(o_{F_v}))]$$

is finite.

Proof. If $v \in \infty_F$, then

$$\sharp[\mathbf{X}(o_{F_v})/St(\mathbf{X}(o_{F_v}))] \leq \sharp H^1(F_v, H)$$

which is finite by Theorem 6.14 and Corollary 2 of Chapter 6 in [22].

For any $x \in \mathbf{X}(o_{F_v})$ with $v < \infty_F$, the morphism induced by the point x

$$f_x: \quad G \cong \{x\} \times_F G \longrightarrow X \times_F G \xrightarrow{m} X$$

is dominant and smooth. By Proposition 3.3 of Chapter 3 in [22], we have that f_x is an open map over F_v points. This implies that $xSt(\mathbf{X}(o_{F_v}))$ is open in $\mathbf{X}(o_{F_v})$. By compactness of $\mathbf{X}(o_{F_v})$, one concludes $[\mathbf{X}(o_{F_v})/St(\mathbf{X}(o_{F_v}))]$ is finite. $\square$

Define

$$St(\mathbf{X}) = \prod_{v \in \Omega_F} St(\mathbf{X}(o_{F_v}))$$

Then $St(\mathbf{X})$ is an open subgroup of $G(\mathbb{A}_F)$. We set

$$\mathbf{X} \cdot \sigma_{\mathbb{A}} := (\prod_{v \in \Omega_F} \mathbf{X}(o_{F_v}) \cdot \sigma_v) \subseteq X(\mathbb{A}_F)$$

for any $\sigma_{\mathbb{A}} = (\sigma_v)_{v \in \Omega_F} \in G(\mathbb{A}_F)$. In particular,

$$\mathbf{X} \cdot 1_{\mathbb{A}} = \prod_{v \in \Omega_F} \mathbf{X}(o_{F_v}).$$

It is clear that $\sigma_{\mathbb{A}}^{-1} St(\mathbf{X}) \sigma_{\mathbb{A}}$ acts on $\mathbf{X} \cdot \sigma_{\mathbb{A}}$.

Corollary 2.3. *The number of orbits*

$$\mathbf{X} \cdot \sigma_{\mathbb{A}} / \sigma_{\mathbb{A}}^{-1} St(\mathbf{X})) \sigma_{\mathbb{A}}$$

is finite.

Proof. Write $\sigma_{\mathbb{A}} = (\sigma_v)_{v \in \Omega_F}$. There is a finite subset $S_1 \supseteq S$ such that $\sigma_v \in \mathbf{G}(o_{F_v})$ for all $v \notin S_1$. Then one has the component-wise bijection

$$\mathbf{X} \cdot \sigma_{\mathbb{A}} / \sigma_{\mathbb{A}}^{-1} St(\mathbf{X}) \sigma_{\mathbb{A}} \cong [\prod_{v \in S_1} \mathbf{X}(o_{F_v}) \cdot \sigma_v / \sigma_v^{-1} St(\mathbf{X}(o_{F_v})) \sigma_v] \times [\prod_{v \notin S_1} \mathbf{X}(o_{F_v}) / \mathbf{G}(o_{F_v})].$$

For each $v \notin S_1$, one has the short exact sequence as pointed sets

$$1 \to \mathbf{H}(o_{F_v}) \to \mathbf{G}(o_{F_v}) \to \mathbf{X}(o_{F_v}) \to H^1_{et}(o_{F_v}, \mathbf{H}).$$

Since $\mathbf{H}$ is connected, one has $H^1_{et}(o_{F_v}, \mathbf{H}) = 1$ by Hensel's Lemma and Lang's Theorem and $|\mathbf{X}(o_{F_v}) / \mathbf{G}(o_{F_v})| = 1$ for $v \notin S_1$.

For each $v \in S_1$, the map

$$\mathbf{X}(o_{F_v}) \cdot \sigma_v / \sigma_v^{-1} St(\mathbf{X}(o_{F_v})) \sigma_v \to \mathbf{X}(o_{F_v}) / St(\mathbf{X}(o_{F_v})); \qquad \bar{x} \mapsto \overline{x \cdot \sigma_v^{-1}}$$

is also bijective. The result follows from Lemma 2.2. $\qquad \square$

Definition 2.4. *Let* $\Gamma = G(F) \cap St(\mathbf{X})$.

It is clear that Γ acts on $\mathbf{X}(o_F)$.

Corollary 2.5. *The number of orbits* $\mathbf{X}(o_F) / \Gamma$ *is finite.*

Proof. Since the natural map

$$\mathbf{X}(o_F) / \Gamma \longrightarrow (\prod_{v \in S} \mathbf{X}(o_{F_v}) / St(\mathbf{X}(o_{F_v}))) \times (\prod_{v \notin S} \mathbf{X}(o_{F_v}) / \mathbf{G}(o_{F_v}))$$

is injective, the result follows from Corollary 2.3.

$\qquad \square$

Lemma 2.6. *For any* $u_{\mathbb{A}} \in G(\mathbb{A}_F)$, *the natural map*

$$H(F) \backslash H(\mathbb{A}_F) u_{\mathbb{A}} (\sigma_{\mathbb{A}}^{-1} St(\mathbf{X}) \sigma_{\mathbb{A}}) / (\sigma_{\mathbb{A}}^{-1} St(\mathbf{X}) \sigma_{\mathbb{A}})$$

$$\xrightarrow{\cong} H(F) \backslash H(\mathbb{A}_F) / [H(\mathbb{A}_F) \cap ((u_{\mathbb{A}} \sigma_{\mathbb{A}}^{-1}) St(\mathbf{X}) (u_{\mathbb{A}} \sigma_{\mathbb{A}}^{-1})^{-1})]$$

by sending $h_{\mathbb{A}} u_{\mathbb{A}}$ *to* $h_{\mathbb{A}}$ *is bijective.*

Proof. It follows from the direct verification.

$\qquad \square$

Proposition 2.7. *With the map (1.1), the double coset decomposition*

$$H(F) \backslash p^{-1}(\mathbf{X} \cdot \sigma_{\mathbb{A}}) / (\sigma_{\mathbb{A}}^{-1} St(\mathbf{X}) \sigma_{\mathbb{A}})$$

is finite for any given $\sigma_{\mathbb{A}} \in G(\mathbb{A}_F)$.

Proof. Since the map p induces the bijection

$$H(\mathbb{A}_F)\backslash p^{-1}(\mathbf{X}\cdot\sigma_\mathbb{A})/(\sigma_\mathbb{A}^{-1}St(\mathbf{X})\sigma_\mathbb{A}) \cong \mathbf{X}\cdot\sigma_\mathbb{A}/(\sigma_\mathbb{A}^{-1}St(\mathbf{X})\sigma_\mathbb{A}),$$

one only needs to show that the further double coset decomposition for each piece

$$H(F)\backslash H(\mathbb{A}_F)u_\mathbb{A}(\sigma_\mathbb{A}^{-1}St(\mathbf{X})\sigma_\mathbb{A})/(\sigma_\mathbb{A}^{-1}St(\mathbf{X})\sigma_\mathbb{A})$$

is finite by Corollary 2.3.

By Lemma 2.6, one only needs to show the finiteness of the following double coset decomposition

$$H(F)\backslash H(\mathbb{A}_F)/[H(\mathbb{A}_F)\cap((u_\mathbb{A}\sigma_\mathbb{A}^{-1})St(\mathbf{X})(u_\mathbb{A}\sigma_\mathbb{A}^{-1})^{-1})].$$

Indeed, the v-component of

$$H(\mathbb{A}_F)\cap((u_\mathbb{A}\sigma_\mathbb{A}^{-1})St(\mathbf{X})(u_\mathbb{A}\sigma_\mathbb{A}^{-1})^{-1})$$

is an open subgroup of $H(F_v)$. Moreover, it is equal to $H(F_v)$ for $v\in\infty_F$ and is equal to $\mathbf{H}(\mathfrak{o}_{F_v})$ for almost all $v\in\Omega_F$. The result follows from Theorem 5.1 of Chapter 5 in [22]. $\qquad\square$

Definition 2.8. *For any $x,y\in\mathbf{X}(o_F)$, we define the equivalent relation over $\mathbf{X}(o_F)$*

$$x\sim y \quad\Leftrightarrow\quad x=y\cdot s_\mathbb{A}$$

for some $s_\mathbb{A}\in St(\mathbf{X})$. The set of the equivalent classes is denoted by $\mathbf{X}(o_F)/\sim$.

It is clear that

$$\sharp(\mathbf{X}(o_F)/\sim)\leq\sharp(\mathbf{X}(o_F)/\Gamma)<\infty$$

by Corollary 2.5.

Proposition 2.9.
1). If G is semi-simple and simply connected such that $G'(F_\infty)$ is not compact for any simple factor of G, then the diagonal map

$$\mathbf{X}(o_F)/\sim \ \xrightarrow{\cong}\ (\mathbf{X}\cdot 1_\mathbb{A})^{Br(X)}/St(\mathbf{X})$$

is bijective.

2). If the map $G(F)\xrightarrow{p}X(F)$ in (1.1) is surjective, there is a bijection

$$\mathbf{X}(o_F)/\sim \ \xrightarrow{\cong}\ H(\mathbb{A}_F)\backslash(p^{-1}(\mathbf{X}\cdot 1_\mathbb{A}))\cap(H(\mathbb{A}_F)G(F)St(\mathbf{X}))/St(\mathbf{X})$$

Proof.
1). Since $St(\mathbf{X})$ acts on $(\mathbf{X}\cdot 1_\mathbb{A})^{Br(X)}$ by Corollary 3.6 in [2], it is clear that the diagonal map is injective. Since any orbit of $St(\mathbf{X})$ is open by the proof of Lemma 2.2, there is an integral point in $\mathbf{X}(o_F)$ for any orbit of $St(\mathbf{X})$ inside $(\mathbf{X}\cdot 1_\mathbb{A})^{Br(X)}$ by the proof of Theorem 3.7 (b) in [4]. This implies that the diagonal map is surjective.

2). For any element $H(\mathbb{A}_F)u_\mathbb{A}St(\mathbf{X})$ in the right side, there are $g\in G(F)$, $h_\mathbb{A}\in H(\mathbb{A}_F)$ and $s_\mathbb{A}\in St(\mathbf{X})$ such that $u_\mathbb{A}=h_\mathbb{A}\cdot g\cdot s_\mathbb{A}$. Then $P\cdot g\cdot s_\mathbb{A}\in\mathbf{X}\cdot 1_\mathbb{A}$. Therefore $P\cdot g\in\mathbf{X}(o_F)$. One can define the map

$$\phi:\ H(\mathbb{A}_F)u_\mathbb{A}St(\mathbf{X})\mapsto[P\cdot g]$$

For the different choices of $g'\in G(F)$, $h'_\mathbb{A}\in H(\mathbb{A}_F)$ and $s'_\mathbb{A}\in St(\mathbf{X})$, one has

$$P\cdot g\sim P\cdot g'.$$

Therefore ϕ is well-defined and injective. Since the map $G(F)\xrightarrow{p}X(F)$ in (1.1) is surjective, one can verify directly that ϕ is bijective. $\qquad\square$

Write

$$\mathbf{X}(o_F) \;=\; \bigcup_i (\mathbf{X}(o_F) \cap x_i St(\mathbf{X})) \tag{2.10}$$

with $x_i \in \mathbf{X}(o_F)$. For

$$y, z \in \mathbf{X}(o_F) \cap x_i St(\mathbf{X}),$$

we can define the further equivalent relation $\sim_G$ on $\mathbf{X}(o_F) \cap x_i St(\mathbf{X})$ as follows

$$y \sim_G z \quad \Leftrightarrow \quad y = z \cdot g$$

if there is $g \in G(F)$.

The following result is proved implicitly in the proof of Theorem 4.2 in [3]. For convenience, we will provide the proof as well.

Proposition 2.11.

$$\sharp((\mathbf{X}(o_F) \cap x_i St(\mathbf{X})) / \sim_G) \le \sharp(ker(H^1(F, H) \to \prod_{v \in \Omega_F} H^1(F_v, H))).$$

If G is semi-simple and simply connected and $G'(F_\infty)$ is not compact for any simple factor G' of G, then the equality holds.

Proof. Since the exact sequence

$$1 \to H(F) \to G(F) \to X(F) \xrightarrow{\delta} H^1(F, H) \to H^1(F, G)$$

by Galois cohomology (see Chapter I, §5.4 Proposition 36 in [27]), one has that $\delta(x_i)$ is an H-torsor over F. Since the twisting of H by $\delta(x_i)$ in sense of [27] is the stabilizer H_{x_i} of x_i in G by the direct computation, one has the following commutative diagram

$$
\begin{array}{ccc}
H^1(F, H_{x_i}) & \xrightarrow[\circ[x_i]]{\cong} & H^1(F, H) \\
\downarrow & & \downarrow \\
\prod_{v \in \Omega_F} H^1(F_v, H_{x_i}) & \xrightarrow[\circ[x_i]]{\cong} & \prod_{v \in \Omega_F} H^1(F_v, H)
\end{array}
$$

where the neutral element in $H^1(F, H_{x_i})$ will be sent to $\delta(x_i)$ in $H^1(F, H)$ by Chapter I, §5.3 Proposition 35 in [27]. Therefore

$$\sharp((\mathbf{X}(o_F) \cap x_i St(\mathbf{X})) / \sim_G) \le \sharp(ker(H^1(F, H_{x_i}) \to \prod_{v \in \Omega_F} H^1(F_v, H_{x_i})))$$

and

$$\sharp(ker(H^1(F, H) \to \prod_{v \in \Omega_F} H^1(F_v, H))) = \sharp(ker(H^1(F, H_{x_i}) \to \prod_{v \in \Omega_F} H^1(F_v, H_{x_i})))$$

by the functoriality of Galois cohomology and the above diagram.

If G is semi-simple and simply connected and $G'(F_\infty)$ is not compact for any simple factor G' of G, then $G(\mathbb{A}_F) = St(\mathbf{X})G(F)$ by the strong approximation for G (see Theorem 7.12 of Chapter 7 in [22]). Let

$$\xi \in ker(H^1(F, H_{x_i}) \to \prod_{v \in \Omega_F} H^1(F_v, H_{x_i})).$$

By the Hasse principle for G, one obtains $x \in X(F)$ such that $\delta(x) = \xi$ and $x = x_i \cdot g_{\mathbb{A}}$ with $g_{\mathbb{A}} \in G(\mathbb{A}_F)$. There are $s_{\mathbb{A}} \in St(\mathbf{X})$ and $\sigma \in G(F)$ such that $g_{\mathbb{A}} = s_{\mathbb{A}} \sigma$. Therefore

$$x \cdot \sigma^{-1} = x_i \cdot s_{\mathbb{A}} \in \mathbf{X}(o_F) \cap x_i St(\mathbf{X})$$

and $\delta(x \cdot \sigma^{-1}) = \delta(x) = \xi$. $\qquad\square$

One can further decompose

$$\mathbf{X}(o_F) \cap x_i St(\mathbf{X}) = \bigcup_j (y_j^{(i)} G(F) \cap x_i St(\mathbf{X})) = \bigcup_j (y_j^{(i)} G(F) \cap y_j^{(i)} St(\mathbf{X}))$$

with $y_j^{(i)} \in \mathbf{X}(o_F)$. The following proposition is essentially the same as Lemma 4.4.1 (i) in [3], which is originally from [31].

Proposition 2.12. *With the above notation, there is a bijection*

$$(y_j^{(i)} G(F) \cap y_j^{(i)} St(\mathbf{X}))/\Gamma \;\xrightarrow{\cong}\; H_{ij}(F)\backslash(H_{ij}(\mathbb{A}_F) \cap G(F)St(\mathbf{X}))/(H_{ij}(\mathbb{A}_F) \cap St(\mathbf{X}))$$

where H_{ij} is the stabilizer of $y_j^{(i)}$.

In particular, if the map $G(F) \xrightarrow{p} X(F)$ in (1.1) is surjective, then there is a bijection

$$\mathbf{X}(o_F) \cap x_i St(\mathbf{X})/\Gamma \;\xrightarrow{\cong}\; H(F)\backslash(H(\mathbb{A}_F) \cap G(F)St(\mathbf{X})\sigma_i^{-1})/(H(\mathbb{A}_F) \cap \sigma_i St(\mathbf{X})\sigma_i^{-1})$$

where $\sigma_i \in G(F)$ such that $P \cdot \sigma_i = x_i$.

Proof. Since

$$x \in y_j^{(i)} G(F) \cap y_j^{(i)} St(\mathbf{X}) \quad \Leftrightarrow \quad x = y_j^{(i)} \cdot g = y_j^{(i)} \cdot s_{\mathbb{A}}$$

with $g \in G(F)$ and $s_{\mathbb{A}} \in St(\mathbf{X})$, one has that

$$g s_{\mathbb{A}}^{-1} \in H_{ij}(\mathbb{A}_F) \cap G(F)St(\mathbf{X}).$$

One can verify directly the map

$$x \mapsto H_{ij}(F) g s_{\mathbb{A}}^{-1}(H_{ij}(\mathbb{A}_F) \cap St(\mathbf{X}))$$

is well-defined and induces the map

$$(y_j^{(i)} G(F) \cap y_j^{(i)} St(\mathbf{X}))/\Gamma \xrightarrow{\phi} H_{ij}(F)\backslash(H_{ij}(\mathbb{A}_F) \cap G(F)St(\mathbf{X}))/(H_{ij}(\mathbb{A}_F) \cap St(\mathbf{X}))$$

which is well-defined as well.
   If

$$H_{ij}(F) g s_{\mathbb{A}}^{-1}(H_{ij}(\mathbb{A}_F) \cap St(\mathbf{X})) = H_{ij}(F) g' s_{\mathbb{A}}'^{-1}(H_{ij}(\mathbb{A}_F) \cap St(\mathbf{X})),$$

there is $h \in H_{ij}(F)$ and $\xi \in H_{ij}(\mathbb{A}_F) \cap St(\mathbf{X})$ such that $g s_{\mathbb{A}}^{-1} = h g' s_{\mathbb{A}}'^{-1}\xi$. Then

$$g'^{-1} h^{-1} g = s_{\mathbb{A}}'^{-1}\xi s_{\mathbb{A}} \in \Gamma.$$

This implies that ϕ is injective. It is clear that ϕ is surjective.
   Suppose the map $G(F) \xrightarrow{p} X(F)$ in (1.1) is surjective. Then

$$y_j^{(i)} G(F) \cap y_j^{(i)} St(\mathbf{X}) = X(F) \cap y_j^{(i)} St(\mathbf{X}) = X(F) \cap x_i St(\mathbf{X}) = \mathbf{X}(o_F) \cap x_i St(\mathbf{X})$$

and the result follows from applying the conjugation isomorphism given by σ_i. □

3. MASS FORMULAE ASSOCIATED TO ELEMENTS OF BRAUER GROUPS

In this section, we will establish the mass formulae associated the Brauer-Manin obstruction following from [36]. Such mass formulae with characters are initial from [12], [31] and [26]. We will keep the same notations as those in Section 1 and Section 2 and further assume that both G and H have no non-trivial F-characters. Then the Tamagawa numbers of G and H will be given by

$$\tau(H) = \int_{H(F)\backslash H(\mathbb{A}_F)} \nu_H \quad \text{and} \quad \tau(G) = \int_{G(F)\backslash G(\mathbb{A}_F)} \lambda_G$$

respectively (see [15]). Define

$$m(\mathbf{X} \cdot \sigma_\mathbb{A}) := \nu_{\infty_F}((H(F) \cap (\sigma_\mathbb{A}^{-1} St(\mathbf{X})\sigma_\mathbb{A}))\backslash H(F_\infty))$$

and

$$M(\mathbf{X} \cdot \sigma_\mathbb{A}) := \lambda_{\infty_F}((G(F) \cap (\sigma_\mathbb{A}^{-1} St(\mathbf{X})\sigma_\mathbb{A}))\backslash G(F_\infty)).$$

Then both $m(\mathbf{X} \cdot \sigma_\mathbb{A})$ and $M(\mathbf{X} \cdot \sigma_\mathbb{A})$ are finite by Theorem 4.17 in [22] since both G and H have no non-trivial F-characters.

Denote

$$[G, \mathbf{X}] := G(F)[G(\mathbb{A}_F), G(\mathbb{A}_F)]St(\mathbf{X}) \quad \text{and} \quad s(\mathbf{X}) := [G(\mathbb{A}_F) : [G, \mathbf{X}]].$$

By Theorem 5.1 of Chapter 5 in [22], the index $s(\mathbf{X})$ is finite.

Definition 3.1. *For any given $\sigma_\mathbb{A} \in G(\mathbb{A}_F)$, we define*

$$R(\mathbf{X} \cdot \sigma_\mathbb{A}) := \sum_{\gamma_\mathbb{A}} M(\mathbf{X} \cdot \sigma_\mathbb{A}\gamma_\mathbb{A}^{-1})$$

where $\gamma_\mathbb{A}$ runs over the double cosets $G(F)\backslash[G, \mathbf{X}]/\sigma_\mathbb{A}^{-1} St(\mathbf{X})\sigma_\mathbb{A}$ and

$$r(\mathbf{X} \cdot \sigma_\mathbb{A}) := \sum_{\gamma_\mathbb{A}} m(\mathbf{X} \cdot \sigma_\mathbb{A}\gamma_\mathbb{A}^{-1})$$

where $\gamma_\mathbb{A}$ runs over the double coset decomposition

$$H(F)\backslash p^{-1}(\mathbf{X} \cdot \sigma_\mathbb{A}) \cap [G, \mathbf{X}]/\sigma_\mathbb{A}^{-1} St(\mathbf{X})\sigma_\mathbb{A}.$$

Lemma 3.2. *For any $\sigma_\mathbb{A} \in G(\mathbb{A}_F)$, one has*

$$R(\mathbf{X} \cdot 1_\mathbb{A}) = R(\mathbf{X} \cdot \sigma_\mathbb{A}) = \frac{\tau(G)}{s(\mathbf{X})} \prod_{v < \infty_F} \lambda_v(St(\mathbf{X}(o_{F_v})))^{-1}.$$

Proof. Write the coset decomposition

$$G(\mathbb{A}_F) = \bigcup_{i=1}^{s(\mathbf{X})} [G, \mathbf{X}]\tau_i.$$

Then

$$\tau(G) = \sum_{i=1}^{s(\mathbf{X})} \int_{G(F)\backslash[G,\mathbf{X}]\tau_i} \lambda_G = s(\mathbf{X}) \int_{G(F)\backslash[G,\mathbf{X}]} \lambda_G.$$

Consider the double coset decomposition

$$[G, \mathbf{X}] = \bigcup G(F)\gamma_\mathbb{A}\sigma_\mathbb{A}^{-1} St(\mathbf{X})\sigma_\mathbb{A}.$$

Then

$$\int_{G(F)\backslash[G,\mathbf{X}]} \lambda_G = \sum_{\gamma_{\mathbb{A}}} \int_{G(F)\backslash G(F)\gamma_{\mathbb{A}}\sigma_{\mathbb{A}}^{-1}St(\mathbf{X})\sigma_{\mathbb{A}}\gamma_{\mathbb{A}}^{-1}} \lambda_G.$$

Since one has the following fundamental domain

$$G(F)\backslash G(F)\gamma_{\mathbb{A}}\sigma_{\mathbb{A}}^{-1}St(\mathbf{X})\sigma_{\mathbb{A}}\gamma_{\mathbb{A}}^{-1}$$
$$\cong ((G(F) \cap (\gamma_{\mathbb{A}}\sigma_{\mathbb{A}}^{-1}St(\mathbf{X})\sigma_{\mathbb{A}}\gamma_{\mathbb{A}}^{-1}))\backslash G(F_\infty)) \times (\prod_{v<\infty_F} \gamma_v\sigma_v^{-1}St(\mathbf{X}(o_{F_v}))\sigma_v\gamma_v^{-1})$$

where $\sigma_{\mathbb{A}} = (\sigma_v)$ and $\gamma_{\mathbb{A}} = (\gamma_v)$, one obtains that

$$\int_{G(F)\backslash G(F)\gamma_{\mathbb{A}}\sigma_{\mathbb{A}}St(\mathbf{X})\sigma_{\mathbb{A}}^{-1}\gamma_{\mathbb{A}}^{-1}} \lambda_G = M(\mathbf{X} \cdot \sigma_{\mathbb{A}}\gamma_{\mathbb{A}}^{-1}) \prod_{v<\infty_F} \lambda_v(St(\mathbf{X}(o_{F_v})))$$

by Theorem 5.5 in Chapter 5 of [22] and the assumption that G has no non-trivial F-characters. Combining the above together, one completes the proof. □

The following proposition gives the arithmetic interpretation about $r(\mathbf{X} \cdot 1_{\mathbb{A}})$.

Proposition 3.3. *If* $[G, \mathbf{X}] = G(F)St(\mathbf{X})$ *and the map* $G(F) \xrightarrow{p} X(F)$ *in (1.1) is surjective, then*

$$r(\mathbf{X} \cdot 1_{\mathbb{A}}) = \sum_i m(\mathbf{X} \cdot \tau_i^{-1})$$

where

$$\mathbf{X}(o_F) = \bigcup_i y_i \cdot \Gamma \quad and \quad P \cdot \tau_i = y_i$$

for $\tau_i \in G(F)$.

Proof. Since the natural map

$$H(F)\backslash p^{-1}(\mathbf{X} \cdot 1_{\mathbb{A}}) \cap [G, \mathbf{X}]/St(\mathbf{X}) \to H(\mathbb{A}_F)\backslash p^{-1}(\mathbf{X} \cdot 1_{\mathbb{A}}) \cap (H(\mathbb{A}_F)[G, \mathbf{X}])/St(\mathbf{X})$$

is surjective and the fiber of an element

$$H(\mathbb{A}_F)\tau St(\mathbf{X}) \quad \text{with } \tau \in G(F) \text{ and } P \cdot \tau \in \mathbf{X}(o_F)$$

is given by

$$H(F)\backslash(H(\mathbb{A}_F)\tau St(\mathbf{X}) \cap G(F)St(\mathbf{X}))/St(\mathbf{X})$$
$$\cong H(F)\backslash(H(\mathbb{A}_F) \cap G(F)St(\mathbf{X})\tau^{-1})/(H(\mathbb{A}_F) \cap \tau St(\mathbf{X})\tau^{-1})$$

by Lemma 2.6, the result follows from Prop.2.9, 2) and Prop.2.12. □

The assumption that $[G, \mathbf{X}] = G(F)St(\mathbf{X})$ will be satisfied if any simple factor of semi-simple part of G is not compact by Theorem 7.28 in [5]. Let

$$h(\mathbf{X}) := [H(\mathbb{A}_F) : H(\mathbb{A}_F) \cap [G, \mathbf{X}]].$$

Proposition 3.4. *If*

$$p^{-1}(\mathbf{X} \cdot \sigma_{\mathbb{A}}) = \bigcup_{u_{\mathbb{A}}} H(\mathbb{A}_F)u_{\mathbb{A}}(\sigma_{\mathbb{A}}^{-1}St(\mathbf{X})\sigma_{\mathbb{A}})$$

for $\sigma_{\mathbb{A}} \in G(\mathbb{A}_F)$, *then*

$$r(\mathbf{X} \cdot \sigma_{\mathbb{A}}) = \frac{\tau(H)}{h(\mathbf{X})} \sum_{u_{\mathbb{A}}} (\prod_{v<\infty_F} \nu_v(H(F_v) \cap (u_v\sigma_v^{-1}St(\mathbf{X}(o_{F_v}))\sigma_v u_v^{-1}))^{-1})$$

where $u_{\mathbb{A}} = (u_v)$ *runs over the above double coset decomposition satisfying* $u_{\mathbb{A}} \in H(\mathbb{A}_F)[G, \mathbf{X}]$.

Proof. The contribution of each piece $H(\mathbb{A}_F)u_{\mathbb{A}}(\sigma_{\mathbb{A}}^{-1}St(\mathbf{X})\sigma_{\mathbb{A}})$ to $r(\mathbf{X} \cdot \sigma_{\mathbb{A}})$ is given by

$$g := \sum_{\gamma_{\mathbb{A}}} m(\mathbf{X} \cdot \sigma_{\mathbb{A}}\gamma_{\mathbb{A}}^{-1})$$

where $\gamma_{\mathbb{A}}$ runs over the double coset decomposition

$$H(F)\backslash H(\mathbb{A}_F)u_{\mathbb{A}}(\sigma_{\mathbb{A}}^{-1}St(\mathbf{X})\sigma_{\mathbb{A}})/\sigma_{\mathbb{A}}^{-1}St(\mathbf{X})\sigma_{\mathbb{A}} \quad \text{and} \quad \gamma_{\mathbb{A}} \in [G, \mathbf{X}].$$

By Lemma 2.6, one has

$$g = \sum_{\gamma_{\mathbb{A}}} m(\mathbf{X} \cdot \sigma_{\mathbb{A}}u_{\mathbb{A}}^{-1}\gamma_{\mathbb{A}}^{-1})$$

where $\gamma_{\mathbb{A}}$ runs over the double coset decomposition

$$H(F)\backslash(H(\mathbb{A}_F) \cap u_{\mathbb{A}}^{-1}[G, \mathbf{X}])/(H(\mathbb{A}_F) \cap ((u_{\mathbb{A}}\sigma_{\mathbb{A}}^{-1})St(\mathbf{X})(u_{\mathbb{A}}\sigma_{\mathbb{A}}^{-1})^{-1}))$$

and

$$g \neq 0 \quad \Leftrightarrow \quad u_{\mathbb{A}} \in H(\mathbb{A}_F)[G, \mathbf{X}].$$

For any $\gamma_{\mathbb{A}} \in H(\mathbb{A}_F)$, one has the following fundamental domain

$$H(F)\backslash H(F)(H(\mathbb{A}_F) \cap (\gamma_{\mathbb{A}}u_{\mathbb{A}}\sigma_{\mathbb{A}}^{-1}St(\mathbf{X})\sigma_{\mathbb{A}}u_{\mathbb{A}}^{-1}\gamma_{\mathbb{A}}^{-1}))$$
$$\cong((H(F) \cap (\gamma_{\mathbb{A}}u_{\mathbb{A}}\sigma_{\mathbb{A}}^{-1}St(\mathbf{X})\sigma_{\mathbb{A}}u_{\mathbb{A}}^{-1}\gamma_{\mathbb{A}}^{-1}))\backslash H(F_{\infty}))$$
$$\times \prod_{v<\infty_F} (H(F_v) \cap (\gamma_v u_v \sigma_v^{-1}St(\mathbf{X}(o_{F_v}))\sigma_v u_v^{-1}\gamma_v^{-1}))$$

where $\sigma_{\mathbb{A}} = (\sigma_v)$, $u_{\mathbb{A}} = (u_v)$ and $\gamma_{\mathbb{A}} = (\gamma_v)$. Then

$$m(\mathbf{X} \cdot \sigma_{\mathbb{A}}u_{\mathbb{A}}^{-1}\gamma_{\mathbb{A}}^{-1}) = \nu_H(H(F)\backslash H(F)(H(\mathbb{A}_F) \cap (\gamma_{\mathbb{A}}u_{\mathbb{A}}\sigma_{\mathbb{A}}^{-1}St(\mathbf{X})\sigma_{\mathbb{A}}u_{\mathbb{A}}^{-1}\gamma_{\mathbb{A}}^{-1}))$$
$$\cdot \prod_{v<\infty_F} \nu_v(H(F_v) \cap (\gamma_v u_v \sigma_v^{-1}St(\mathbf{X}(o_{F_v}))\sigma_v u_v^{-1}\gamma_v^{-1}))^{-1}$$
$$= \nu_H(H(F)\backslash H(F)\gamma_{\mathbb{A}}(H(\mathbb{A}_F) \cap (u_{\mathbb{A}}\sigma_{\mathbb{A}}^{-1}St(\mathbf{X})\sigma_{\mathbb{A}}u_{\mathbb{A}}^{-1}))$$
$$\cdot \prod_{v<\infty_F} \nu_v(H(F_v) \cap (u_v \sigma_v^{-1}St(\mathbf{X}(o_{F_v}))\sigma_v u_v^{-1}))^{-1}.$$

Therefore

$$g = \nu_H(H(F)\backslash(H(\mathbb{A}_F) \cap u_{\mathbb{A}}^{-1}[G, \mathbf{X}])) \prod_{v<\infty_F} \nu_v(H(F_v) \cap (u_v \sigma_v^{-1}St(\mathbf{X}(o_{F_v}))\sigma_v u_v^{-1}))^{-1}$$

$$= \frac{\tau(H)}{h(\mathbf{X})} \prod_{v<\infty_F} \nu_v(H(F_v) \cap (u_v \sigma_v^{-1}St(\mathbf{X}(o_{F_v}))\sigma_v u_v^{-1}))^{-1}$$

and the proof is complete. $\square$

Corollary 3.5. *For any $\sigma_{\mathbb{A}} \in G(\mathbb{A}_F)$ and $\tau_{\mathbb{A}} \in H(\mathbb{A}_F)[G, \mathbf{X}]$, one has*

$$r(\mathbf{X} \cdot \sigma_{\mathbb{A}}) = r(\mathbf{X} \cdot \sigma_{\mathbb{A}}\tau_{\mathbb{A}}).$$

Proof. If

$$p^{-1}(\mathbf{X} \cdot \sigma_{\mathbb{A}}) = \bigcup_{u_{\mathbb{A}}} H(\mathbb{A}_F)u_{\mathbb{A}}(\sigma_{\mathbb{A}}^{-1}St(\mathbf{X})\sigma_{\mathbb{A}}),$$

then

$$p^{-1}(\mathbf{X} \cdot \sigma_{\mathbb{A}}\tau_{\mathbb{A}}) = \bigcup H(\mathbb{A}_F)u_{\mathbb{A}}\tau_{\mathbb{A}}((\sigma_{\mathbb{A}}\tau_{\mathbb{A}})^{-1}St(\mathbf{X})\sigma_{\mathbb{A}}\tau_{\mathbb{A}}).$$

Since

$$u_{\mathbb{A}} \in H(\mathbb{A}_F)[G, \mathbf{X}] \quad \Leftrightarrow \quad u_{\mathbb{A}}\tau_{\mathbb{A}} \in H(\mathbb{A}_F)[G, \mathbf{X}],$$

one obtains the result by replacing $u_{\mathbb{A}}$ with $u_{\mathbb{A}}\tau_{\mathbb{A}}$ and $\sigma_{\mathbb{A}}$ with $\sigma_{\mathbb{A}}\tau_{\mathbb{A}}$ in Prop. 3.4. □

Let

$$\chi: \quad G(\mathbb{A}_F)/H(\mathbb{A}_F)[G,\mathbf{X}] \longrightarrow \mathbb{C}^{\times} \tag{3.6}$$

be a character. Then

$$\chi = \prod_{v \in \Omega_F} \chi_v$$

where χ_v is induced by

$$\chi_v: G(F_v) \to G(\mathbb{A}_F)/H(\mathbb{A}_F)[G,\mathbf{X}] \xrightarrow{\chi} \mathbb{C}^{\times}$$

for any $v \in \Omega_F$ and $\chi_v = 1$ for $v \in \infty_F$. Moreover, each χ_v also induces the locally constant function $\overline{\chi}_v$ on $X(F_v)$ by setting

$$\overline{\chi}_v(x_v) = \chi_v(p^{-1}(x_v))$$

for any $x_v \in X(F_v)$, where p is the map induced by (1.1). This $\overline{\chi}_v$ is well-defined since χ_v is trivial over $H(F_v)$.

Define

$$N_v(\mathbf{X}, \overline{\chi}_v) = \int_{\mathbf{X}(o_{F_v})} \overline{\chi}_v d_v$$

for $v < \infty_F$. One can establish the mass formula associated to χ.

Theorem 3.7. *For any χ in (3.6), one has*

$$\frac{1}{R(\mathbf{X} \cdot 1_{\mathbb{A}})} \sum_{\sigma_{\mathbb{A}} \in G(\mathbb{A}_F)/H(\mathbb{A}_F)[G,\mathbf{X}]} \chi(\sigma_{\mathbb{A}}) \cdot r(\mathbf{X} \cdot \sigma_{\mathbb{A}}^{-1}) = \frac{\tau(H)s(\mathbf{X})}{\tau(G)h(\mathbf{X})} \prod_{v < \infty_F} N_v(\mathbf{X}, \overline{\chi}_v).$$

Proof. By Corollary 3.5, the left hand side is well-defined. Let

$$p^{-1}(\mathbf{X} \cdot 1_{\mathbb{A}}) = \bigcup_{u_{\mathbb{A}}} H(\mathbb{A}_F) u_{\mathbb{A}} St(\mathbf{X}).$$

Then

$$p^{-1}(\mathbf{X} \cdot \sigma_{\mathbb{A}}^{-1}) = \bigcup_{u_{\mathbb{A}}} H(\mathbb{A}_F) u_{\mathbb{A}} \sigma_{\mathbb{A}}^{-1} (\sigma_{\mathbb{A}} St(\mathbf{X}) \sigma_{\mathbb{A}}^{-1})$$

and

$$r(\mathbf{X} \cdot \sigma_{\mathbb{A}}^{-1}) = \frac{\tau(H)}{h(\mathbf{X})} \sum_{u_{\mathbb{A}}} (\prod_{v < \infty_F} \nu_v(H(F_v) \cap (u_v St(\mathbf{X}(o_{F_v})) u_v^{-1}))^{-1})$$

with $u_{\mathbb{A}} \sigma_{\mathbb{A}}^{-1} \in H(\mathbb{A}_F)[G,\mathbf{X}]$ by Prop.3.4. Since the measures

$$d_X = \prod_{v \in \Omega_F} d_v, \quad \nu_H = \prod_{v \in \Omega_F} \nu_v \quad \text{and} \quad \lambda_G = \prod_{v \in \Omega_F} \lambda_v$$

match together, one has

$$\lambda_v(u_v St(\mathbf{X}(o_{F_v})) u_v^{-1}) = d_v(P \cdot (u_v St(\mathbf{X}(o_{F_v})) u_v^{-1})) \cdot \nu_v(H(F_v) \cap (u_v St(\mathbf{X}(o_{F_v})) u_v^{-1}))$$

for $v < \infty_F$. Therefore

$$\nu_v(H(F_v) \cap (u_v St(\mathbf{X}(o_{F_v})) u_v^{-1}))^{-1} = d_v(P \cdot u_v St(\mathbf{X}(o_{F_v}))) \cdot \lambda_v(St(\mathbf{X}(o_{F_v})))^{-1}$$

for $v < \infty_F$. This implies that

$$\chi(\sigma_{\mathbb{A}}) \cdot r(\mathbf{X} \cdot \sigma_{\mathbb{A}}^{-1}) = \frac{\tau(H)s(\mathbf{X})}{\tau(G)h(\mathbf{X})} R(\mathbf{X} \cdot 1_{\mathbb{A}}) \sum_{u_{\mathbb{A}}} \prod_{v < \infty_F} \int_{P \cdot u_v St(\mathbf{X}(o_{F_v}))} \overline{\chi}_v d_v$$

by Lemma 3.2 with $u_{\mathbb{A}} \in H(\mathbb{A}_F)[G, \mathbf{X}]\sigma_{\mathbb{A}}$. One concludes that

$$\sum_{\sigma_{\mathbb{A}} \in G(\mathbb{A}_F)/H(\mathbb{A}_F)[G,\mathbf{X}]} \chi(\sigma_{\mathbb{A}}) \cdot r(\mathbf{X} \cdot \sigma_{\mathbb{A}}^{-1}) = \frac{\tau(H)s(\mathbf{X})}{\tau(G)h(\mathbf{X})} R(\mathbf{X} \cdot 1_{\mathbb{A}}) \prod_{v < \infty_F} \int_{\mathbf{X}(o_{F_v})} \overline{\chi}_v dv$$

and the proof is complete. $\qquad\square$

Corollary 3.8.

$$r(\mathbf{X} \cdot 1_{\mathbb{A}}) = \frac{\tau(H)}{\tau(G)} R(\mathbf{X} \cdot 1_{\mathbb{A}}) \sum_{\chi} (\prod_{v < \infty_F} N_v(\mathbf{X}, \overline{\chi}_v))$$

where χ runs over all characters in (3.6).

Proof. It follows from

$$[G(\mathbb{A}_F) : H(\mathbb{A}_F)[G, \mathbf{X}]] = \frac{s(\mathbf{X})}{h(\mathbf{X})}$$

and

$$\sum_{\chi} \chi(\sigma_{\mathbb{A}}) = \begin{cases} [G(\mathbb{A}_F) : H(\mathbb{A}_F)[G, \mathbf{X}]] & \text{if } \sigma_{\mathbb{A}} \in H(\mathbb{A}_F)[G, \mathbf{X}] \\ 0 & \text{otherwise} \end{cases}$$

where χ runs over all characters in (3.6). $\qquad\square$

The immediate application of the above result is to test the existence of the integral points on $\mathbf{X}$.

Corollary 3.9. *If $[G, \mathbf{X}] = G(F)St(\mathbf{X})$ and the map $G(F) \xrightarrow{p} X(F)$ in (1.1) is surjective, then*

$$\mathbf{X}(o_F) \neq \emptyset \quad \Leftrightarrow \quad \{(x_v) \in \prod_{v < \infty_F} \mathbf{X}(o_{F_v}) : \prod_{v < \infty_F} \overline{\chi}_v(x_v) = 1, \ \forall \chi \text{ in } (3.6)\} \neq \emptyset.$$

Proof. By Proposition 3.3, one has

$$\mathbf{X}(o_F) = \emptyset \quad \Leftrightarrow \quad r(\mathbf{X} \cdot 1_{\mathbb{A}}) = 0.$$

Therefore

$$\mathbf{X}(o_F) = \emptyset \quad \Leftrightarrow \quad \sum_{\chi} \prod_{v < \infty_F} N_v(\mathbf{X}, \overline{\chi}_v) = 0$$

by Corollary 3.8. Decompose

$$\prod_{v < \infty_F} \mathbf{X}(o_{F_v}) = D_0 \cup D_1 \cup \cdots \cup D_l$$

as disjoint closed subsets such that $\chi|_{D_i}$ is constant for all χ with $0 \leq i \leq l$ and

$$D_0 = \{(x_v) \in \prod_{v < \infty_F} \mathbf{X}(o_{F_v}) : \prod_{v < \infty_F} \overline{\chi}_v(x_v) = 1, \ \forall \chi \text{ in } (3.6)\}.$$

For any $i \neq 0$, there is χ in (3.6) such that $\chi(D_i) \neq 1$. This implies that

$$\sum_{\chi} \int_{D_i} \chi d_X = (\sum_{\chi} \chi(D_i)) \int_{D_i} d_X = 0.$$

Therefore

$$\sum_{\chi} \prod_{v < \infty_F} N_v(\mathbf{X}, \overline{\chi}_v) = \frac{s(\mathbf{X})}{h(\mathbf{X})} \int_{D_0} d_X.$$

One concludes that

$$\mathbf{X}(o_F) = \emptyset \quad \Leftrightarrow \quad \int_{D_0} d_X = 0.$$

Since D_0 is an open compact subset of $\prod_{v<\infty_F} \mathbf{X}(o_{F_v})$, one has

$$\int_{D_0} d_X = 0 \quad \Leftrightarrow \quad D_0 = \emptyset$$

and the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

To conclude this section, we will point out that all characters in (3.6) can be interpreted as the elements in $Br(X)$. Indeed, one can view χ as an element ξ in $Br(X)$ satisfying

$$\xi(x_\mathbb{A} \cdot s_\mathbb{A}) = \xi(x_\mathbb{A})$$

for all $x_\mathbb{A} \in X(\mathbb{A}_F)$ and $s_\mathbb{A} \in St(\mathbf{X})$ by Theorem 8.2 in [5], Theorem 2.8 and Corollary 3.5 in [2].

4. Counting the integral points via equi-distribution

We keep the same notation as that in the previous sections.

Lemma 4.1. *Suppose $X = H\backslash G$ where both G and H are the connected reductive groups without non-trivial characters over F. For any finite subgroup B of $Br(X)/Br(F)$, one has*

$$\sum_{\xi \in B} \left(\prod_{v<\infty_F} \int_{\mathbf{X}(o_{F_v})} \xi d_v \cdot \int_{X(F_\infty,T)} \xi d_{\infty_F} \right) = \sharp B \int_{(\prod_{v<\infty_F} \mathbf{X}(o_{F_v}) \times X(F_\infty,T))^B} d_X$$

where

$$\left(\prod_{v<\infty_F} \mathbf{X}(o_{F_v}) \times X(F_\infty,T) \right)^B = \left(\prod_{v<\infty_F} \mathbf{X}(o_{F_v}) \times X(F_\infty,T) \right) \cap (\mathbf{X} \cdot 1_\mathbb{A})^B$$

and $\mathbf{X}$ is a separated scheme of finite type over o_F such that $\mathbf{X} \times_{o_F} F = X$.

Proof. Since G and H have no non-trivial F-characters, one concludes that the infinite product in the above lemma is convergent. Decompose

$$\prod_{v<\infty_F} \mathbf{X}(o_{F_v}) \times X(F_\infty,T) = \bigcup_i X_i$$

as a finite disjoint union such that all X_i's are all closed and ξ takes the constant value on X_i for all $\xi \in B$.

If there is $\xi \in B$ such that ξ does not take the trivial value on X_i, then

$$\sum_{\xi \in B} \int_{X_i} \xi d_X = \left(\sum_{\xi \in B} \xi(X_i) \right) \int_{X_i} d_X = 0$$

and the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

We further assume the following equi-distribution property

$$\sharp\{y \in x \cdot \Gamma : |y|_{\infty_F} \leq T\} \sim \frac{\nu_{x,\infty_F}(\Gamma_{H_x}\backslash H_x(F_\infty))}{\lambda_{\infty_F}(\Gamma\backslash G(F_\infty))} d_{\infty_F}(x \cdot G(F_\infty) \cap X(F_\infty,T)) \quad (4.2)$$

as $T \to \infty$, where H_x is the stabilizer of x in G, $\Gamma_{H_x} = H_x(F) \cap \Gamma$ and ν_{x,∞_F} is the induced Tamagawa measure over H_x. This assumption has been proved in the following various situations.

1). G is an anisotropic torus and X is a trivial torsor of G. The proof is exactly similar to the proof of Dirichlet Unit Theorem, which is given in [28] (see also Theorem 5.12 in [22]).

2). G is simply connected, almost F-simple, $G(F_\infty)$ is not compact and X is symmetric. Such result was first proved in [6], soon simplified in [7] over $\mathbb{Q}$ and extended to general number fields in [1].

3). H is a maximal proper connected reductive group over $\mathbb{Q}$. Such result was proved in [8], which extends the result in [6] and [7] over $\mathbb{Q}$.

Theorem 4.3. *Suppose $X = H\backslash G$ where G is semi-simple and simply connected and $G'(F_\infty)$ is not compact for any non-trivial simple factor G' of G and H is connected reductive without non-trivial F-characters. Then*

$$N(\mathbf{X}, T) \sim \sum_{\xi \in (Br(X)/Br(F))} (\prod_{v < \infty_F} N_v(\mathbf{X}, \xi)) \cdot N_{\infty_F}(X, T, \xi)$$

as $T \to \infty$ under the assumption (4.2), where $\mathbf{X}$ is a separated scheme of finite type over o_F such that $\mathbf{X} \times_{o_F} F = X$.

Proof. Decompose

$$\mathbf{X}(o_F) = \bigcup_i (\mathbf{X}(o_F) \cap x_i St(\mathbf{X}))$$

with $x_i \in \mathbf{X}(o_F)$ and

$$\mathbf{X}(o_F) \cap x_i St(\mathbf{X}) = \bigcup_j (y_j^{(i)} G(F) \cap x_i St(\mathbf{X})) = \bigcup_j (y_j^{(i)} G(F) \cap y_j^{(i)} St(\mathbf{X}))$$

with $y_j^{(i)} \in \mathbf{X}(o_F)$ and

$$y_j^{(i)} G(F) \cap x_i St(\mathbf{X}) = \bigcup_k z_k^{(i,j)} \Gamma$$

with $z_k^{(i,j)} \in \mathbf{X}(o_F)$. Therefore

$$N(\mathbf{X}, T) \sim \sum_i \sum_j \sum_k \frac{\nu_{\infty_F}^{(i,j,k)}(\Gamma_{i,j,k} \backslash H_{i,j,k}(F_\infty))}{\lambda_{\infty_F}(\Gamma \backslash G(F_\infty))} d_{\infty_F}(z_k^{(i,j)} \cdot G(F_\infty) \cap X(F_\infty, T))$$

as $T \to \infty$ by the assumption (4.2), where $H_{i,j,k}$ is the stabilizer of $z_k^{(i,j)}$ in G, $\Gamma_{i,j,k} = H_{i,j,k}(F) \cap \Gamma$ and $\nu^{(i,j,k)}$ is the Tamagawa measure of $H_{i,j,k}$.

Since there is $g_k \in G(F)$ such that $z_k^{(i,j)} = y_j^{(i)} \cdot g_k$, one has the isomorphism between $H_{i,j,k}$ and the stabilizer $H_{i,j}$ of $y_j^{(i)}$ over F given by the conjugation with g_k. This implies that

$$\nu_{\infty_F}^{(i,j,k)}(\Gamma_{i,j,k} \backslash H_{i,j,k}(F_\infty)) = \nu_{\infty_F}^{(i,j)}(g_k \Gamma g_k^{-1} \cap H_{i,j}(F) \backslash H_{i,j}(F_\infty))$$

where $\nu^{(i,j)}$ is the Tamagawa measure over $H_{i,j}$.

Since there is $s_k \in St(\mathbf{X})$ such that $z_k^{(i,j)} = y_j^{(i)} \cdot s_k$, one obtains

$$\nu_{\infty_F}^{(i,j)}(g_k \Gamma g_k^{-1} \cap H_{i,j}(F) \backslash H_{i,j}(F_\infty))$$

$$= \nu^{(i,j)}(H_{i,j}(F) \backslash H_{i,j}(F)(H_{i,j}(\mathbb{A}_F) \cap g_k St(\mathbf{X}) g_k^{-1}))$$

$$\cdot \prod_{v < \infty_F} \nu_v^{(i,j)}(H_{i,j}(F_v) \cap g_k St(\mathbf{X}(o_{F_v})) g_k^{-1})^{-1}$$

$$= \nu^{(i,j)}(H_{i,j}(F) \backslash H_{i,j}(F) h_k (H_{i,j}(\mathbb{A}_F) \cap St(\mathbf{X})))$$

$$\cdot \prod_{v < \infty_F} \nu_v^{(i,j)}(H_{i,j}(F_v) \cap St(\mathbf{X}(o_{F_v})))^{-1}$$

where $h_k = g_k \cdot s_k^{-1} \in H_{i,j}(\mathbb{A}_F)$. Since $G(\mathbb{A}_F) = G(F)St(\mathbf{X})$ by the strong approximation property of G, we have

$$\bigcup_k H_{i,j}(F)h_k(H_{i,j}(\mathbb{A}_F) \cap St(\mathbf{X})) = H_{i,j}(\mathbb{A}_F)$$

by Proposition 2.12. Therefore

$$N(\mathbf{X},T) \sim \sum_i \frac{d_{\infty_F}(x_i \cdot G(F_\infty) \cap X(F_\infty,T))}{\lambda_{\infty_F}(\Gamma \backslash G(F_\infty))}$$

$$(\sum_j \tau(H_{i,j}) \prod_{v < \infty_F} \nu_v^{(i,j)}(H_{i,j}(F_v) \cap St(\mathbf{X}(o_{F_v})))^{-1})$$

as $T \to \infty$, where $\tau(H_{i,j})$ is the Tamagawa number of $H_{i,j}$.

Since

$$\nu_v^{(i,j)}(H_{i,j}(F_v) \cap St(\mathbf{X}(o_{F_v})))^{-1} = \frac{d_v(y_j^{(i)}St(\mathbf{X}(o_{F_v})))}{\lambda_v(St(\mathbf{X}(o_{F_v})))} = \frac{d_v(x_iSt(\mathbf{X}(o_{F_v}))}{\lambda_v(St(\mathbf{X}(o_{F_v})))}$$

and $H_{i,j}$ is an inner form of H, one concludes that $\tau(H_{i,j}) = \tau(H)$ by (5.1.1) in [14] and

$$N(\mathbf{X},T) \sim \sharp Pic(H) \sum_i \frac{d_{\infty_F}(x_i \cdot G(F_\infty) \cap X(F_\infty,T)) \prod_{v < \infty_F} d_v(x_iSt(\mathbf{X}(o_{F_v}))}{\lambda_{\infty_F}(\Gamma \backslash G(F_\infty)) \prod_{v < \infty_F} \lambda_v(St(\mathbf{X}(o_{F_v})))}$$

as $T \to \infty$ by Proposition 2.11 and Theorem 10.1 in [24].

By the strong approximation property for G and [15] and Proposition 2.10 (ii) in [4], one has

$$\lambda_{\infty_F}(\Gamma \backslash G(F_\infty)) \prod_{v < \infty_F} \lambda_v(St(\mathbf{X}(o_{F_v})))$$

$$= \lambda_G(G(F) \backslash G(F)St(\mathbf{X})) = \lambda_G(G(F) \backslash G(\mathbb{A}_F)) = 1$$

and $\sharp Pic(H) = \sharp(Br(X)/Br(F))$. Then

$$N(\mathbf{X},T) \sim \sharp(Br(X)/Br(F)) \int_{(\prod_{v < \infty_F} \mathbf{X}(o_{F_v}) \times X(F_\infty,T))^{Br(X)}} d_X$$

as $T \to \infty$ by Proposition 2.9, 1). The result follows from Lemma 4.1. $\qquad \square$

For general reductive groups, one has the similar result with some restriction.

Theorem 4.4. *Suppose G and H are connected reductive groups without non-trivial characters over F and $X = H \backslash G$. If $G'(F_\infty)$ is not compact for any non-trivial simple factor G' of the semi-simple part of G and the map $G(F) \xrightarrow{p} X(F)$ in (1.1) is surjective, then*

$$N(\mathbf{X},T) \sim \frac{\tau(H)}{\tau(G)}(\sum_\chi \prod_{v < \infty_F} N_v(\mathbf{X},\overline{\chi}_v)) \cdot d_{\infty_F}(X(F_\infty,T))$$

as $T \to \infty$ under the assumption (4.2), where χ runs over all characters in (3.6) and $\mathbf{X}$ is a separated scheme of finite type over o_F such that $\mathbf{X} \times_{o_F} F = X$.

Proof. By Theorem 7.28 in [5], one has $[G, \mathbf{X}] = G(F)St(\mathbf{X})$. By Proposition 3.3 and the assumption (4.2), one has

$$N(\mathbf{X},T) \sim \frac{d_{\infty_F}(X(F_\infty,T))}{\lambda_{\infty_F}(\Gamma \backslash G(F_\infty))} r(\mathbf{X} \cdot 1_\mathbb{A})$$

as $T \to \infty$. Since

$$\lambda_{\infty_F}(\Gamma \backslash G(F_\infty)) \prod_{v < \infty_F} \lambda_v(St(\mathbf{X}(o_{F_v}))) = \lambda_G(G(F)\backslash[G, \mathbf{X}]) = \frac{\tau(G)}{s(\mathbf{X})},$$

one has

$$N(\mathbf{X}, T) \sim \frac{\tau(H)}{\tau(G)}(\sum_\chi \prod_{v < \infty_F} N_v(\mathbf{X}, \overline{\chi}_v)) \cdot d_{\infty_F}(X(F_\infty, T))$$

as $T \to \infty$ by Corollary 3.8 and Lemma 3.2. $\qquad \square$

5. Application to norm equations

In this section, we will apply Theorem 4.4 to study the asymptotic formula of the number of integral solutions for the scheme $\mathbf{X}$ over o_F whose generic fiber $X = \mathbf{X} \times_{o_F} F$ is a trivial torsor of anisotropic torus. Let G be a torus and U be an open subgroup of $G(\mathbb{A}_F)$. A character

$$\varrho : \quad G(\mathbb{A}_F)/G(F)U \longrightarrow \mathbb{C}^\times \tag{5.1}$$

can be written as $\varrho = \prod_{v \in \Omega_F} \varrho_{F_v}$ where ϱ_{F_v} is induced by

$$\varrho_{F_v} : G(F_v) \to G(\mathbb{A}_F)/G(F)U \xrightarrow{\varrho} \mathbb{C}^\times$$

for any $v \in \Omega_F$. Let X be a trivial torsor with a fixed rational point $P \in X(F)$ and $p : G \to X$ be a morphism induced by P. Then each ϱ_{F_v} also induces a continuous function $\overline{\varrho}_v$ on $X(F_v)$ by setting

$$\overline{\varrho}_v(x_v) = \varrho_{F_v}(p^{-1}(x_v))$$

for any $x_v \in X(F_v)$ with $v \in \Omega_F$. One can reformulate Theorem 4.4 in more flexible way.

Lemma 5.2. *Let G is an anisotropic torus and X is a trivial torsor of G. If $\mathbf{X}$ is a separated scheme of finite type over o_F such that $\mathbf{X} \times_{o_F} F = X$ and U is an open subgroup of $St(\mathbf{X})$, then*

$$N(\mathbf{X}, T) \sim \frac{1}{\tau(G)} \sum_\varrho (\prod_{v < \infty_F} N_v(\mathbf{X}, \overline{\varrho}_v)) N_\infty(X, T, \overline{\varrho}_\infty)$$

as $T \to \infty$, where $\tau(G)$ is the Tamagawa number of G and

$$N_v(\mathbf{X}, \overline{\varrho}_v) = \int_{\mathbf{X}(o_{F_v})} \overline{\varrho}_v dv$$

for any $v < \infty_F$ and

$$N_\infty(X, T, \overline{\varrho}_\infty) = \int_{X(F_\infty, T)} \overline{\varrho}_\infty d\infty_F$$

for $T \gg 0$ and ϱ runs over all characters in (5.1).

Proof. Decompose

$$G(\mathbb{A}_F) = \bigcup_{i=1}^n a_i G(F) St(\mathbf{X}) \quad \text{and} \quad G(F)St(\mathbf{X}) = \bigcup_{j=1}^m b_j G(F)U$$

where we choose $b_j \in St(\mathbf{X})$ for $1 \leq j \leq m$. Then

$$(\prod_{v < \infty_F} N_v(\mathbf{X}, \overline{\varrho}_v)) N_\infty(X, T, \overline{\varrho}_\infty) = \sum_{i=1}^{n} \varrho(a_i) \sum_{j=1}^{m} \varrho(b_j) \int_{D_{i,j}} d_X$$

where

$$D_{i,j} = (X(F_\infty, T) \times \prod_{v < \infty_F} \mathbf{X}(o_{F_v})) \bigcap p(a_i b_j G(F) U)$$

for $1 \leq i \leq n$ and $1 \leq j \leq m$. Since $b_j \in St(\mathbf{X})$, one has that

$$\int_{D_{i,1}} d_X = \cdots = \int_{D_{i,m}} d_X$$

for $1 \leq i \leq n$. If the restriction of ϱ to $G(F) St(\mathbf{X}) / G(F) U$ is not trivial, then

$$\sum_{j=1}^{m} \varrho(b_j) = 0.$$

Therefore

$$\sum_{\varrho} (\prod_{v < \infty_F} N_v(\mathbf{X}, \overline{\varrho}_v)) N_\infty(X, T, \overline{\varrho}_\infty) = \sum_{\chi} \prod_{v < \infty_F} N_v(\mathbf{X}, \overline{\chi}_v) \cdot d_{\infty_F}(X(F_\infty, T))$$

where χ runs over all characters in (3.6) and the result follows from Theorem 4.4. $\square$

In practice, one can use the idea in [34] and [35] to embed a torus to certain standard torus $\prod_\lambda R_{E_\lambda/F}(\mathbb{G}_m)$ and apply the results in [34] and [35] to get more computable formulae. We explain this point by considering the more concrete scheme $\mathbf{X}$ over o_F defined by

$$N_{K/F}(\alpha_1 x_1 + \cdots + \alpha_n x_n) = m \tag{5.3}$$

where K/F is a finite extension of degree n and $\alpha_1, \cdots, \alpha_n \in o_K$ which are F-linear independent and $m \in o_F \setminus \{0\}$. Then $X = \mathbf{X} \times_{o_F} F$ is a torsor of the norm one torus $G = R^1_{K/F}(\mathbb{G}_m)$. Let

$$n_\infty = \sum_{v \in \infty_F} (n_v - 1)$$

where n_v is the number of places of K above v. If $\mathbf{X}(o_F) \neq \emptyset$ and $n_\infty > 0$, then there exists a constant $c > 0$ such that

$$N(\mathbf{X}, T) \sim c(\log T)^{n_\infty}$$

as $T \to \infty$ by the proof of generalized Dirichlet Unit Theorem (see Theorem 5.12 of Chapter 5 in [22]). We will explain how to determine the above constant c explicitly.

Let $\{\sigma_1, \cdots, \sigma_n\}$ be the set of all embedding of K over F and $\Delta = det(\sigma_i(\alpha_j))_{n \times n}$.

Lemma 5.4. *Let*

$$u = \sum_{j=1}^{n} \alpha_j x_j \quad and \quad \sigma_i(u) = \sum_{j=1}^{n} \sigma_i(\alpha_j) x_j$$

for $1 \leq i \leq n$. Then

$$\omega = m^{-1} \Delta^{-1} \sigma_1(u) \cdot d\sigma_2(u) \wedge d\sigma_3(u) \wedge \cdots \wedge d\sigma_n(u)$$

is an invariant differential form of X.

Proof. Since
$$\omega = \Delta^{-1} \cdot \frac{d\sigma_2(u)}{\sigma_2(u)} \wedge \cdots \wedge \frac{d\sigma_n(u)}{\sigma_n(u)},$$
one has that ω is an invariant differential form of $X \times_F K$.

Let M_j be the minor for entry $\sigma_1(\alpha_j)$ in $(\sigma_i(\alpha_j))_{n \times n}$. Then we have

$$d\sigma_2(u) \wedge \cdots \wedge d\sigma_n(u) = \sum_{j=1}^{n} M_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n. \tag{5.5}$$

Since
$$\sum_{i=1}^{n} \frac{d\sigma_i(u)}{\sigma_i(u)} = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} \mathrm{Tr}_{K/F}(\alpha_i/u) \cdot dx_i = 0,$$
one obtains that

$$dx_1 = -\mathrm{Tr}_{K/F}(\alpha_1/u)^{-1} (\sum_{i=2}^{n} \mathrm{Tr}_{K/F}(\alpha_i/u) \cdot dx_i). \tag{5.6}$$

Replacing dx_1 in (5.5) with (5.6), we have
$$d\sigma_2(u) \wedge \cdots \wedge d\sigma_n(u)$$
$$= \mathrm{Tr}_{K/F}(\alpha_1/u)^{-1} (\sum_{j=1}^{n} (-1)^{1+j} M_j \, \mathrm{Tr}_{K/F}(\alpha_j/u)) \cdot dx_2 \wedge \cdots \wedge dx_n.$$

Since
$$\sum_{j=1}^{n} (-1)^{1+j} M_j \cdot \sigma_1(\alpha_j) = \Delta \quad \text{and} \quad \sum_{j=1}^{n} (-1)^{1+j} M_j \cdot \sigma_i(\alpha_j) = 0$$
for $2 \le i \le n$, one obtains

$$\mathrm{Tr}_{K/F}(\alpha_1/u)^{-1} \sum_{j=1}^{n} (-1)^{1+j} M_j \, \mathrm{Tr}_{K/F}(\alpha_j/u)$$

$$= \mathrm{Tr}_{K/F}(\alpha_1/u)^{-1} (\sum_{j=1}^{n} (-1)^{1+j} M_j \cdot \sigma_1(\alpha_j)) \sigma_1(u)^{-1}$$

$$+ \sum_{i=2}^{n} (\sum_{j=1}^{n} (-1)^{1+j} M_j \cdot \sigma_i(\alpha_j)) \sigma_i(u)^{-1}$$

$$= \mathrm{Tr}_{K/F}(\alpha_1/u)^{-1} \sigma_1(u)^{-1} \Delta.$$

Hence
$$d\sigma_2(u) \wedge \cdots \wedge d\sigma_n(u) = \mathrm{Tr}_{K/F}(\alpha_1/u)^{-1} \sigma_1(u)^{-1} \Delta \cdot dx_2 \wedge \cdots \wedge dx_n$$
and
$$\omega = m^{-1} \mathrm{Tr}_{K/F}(\alpha_1/u)^{-1} \cdot dx_2 \wedge \cdots \wedge dx_n.$$
This implies that ω is a differential form of X. $\qquad \square$

We will use this volume form ω to calculate $d_{\infty_F}(X(F_\infty, T))$.

Lemma 5.7. *(1) Suppose v is a real place.*

If there are $r \ge 1$ real places and s complex places of K over v, then

$$d_v(X(F_v, T)) \sim 2^{r-1}(2\pi)^s \frac{n^{r+s-1}}{(r+s-1)!} |N_{K/F}(\Delta)|_v^{-1} \cdot (\log T)^{r+s-1}$$

as $T \to \infty$.

If all places over v are complex, then

$$d_v(X(F_v, T)) \sim (2\pi)^{s-1} \frac{n^{s-1}}{(s-1)!} |N_{K/F}(\Delta)|_v^{-1} \cdot (\log T)^{s-1}$$

as $T \to \infty$ with $s = \frac{n}{2}$.

(2) Suppose v be a complex place. Then

$$d_v(X(F_v, T)) \sim (2\pi)^{n-1} \frac{n^{n-1}}{(n-1)!} |N_{K/F}(\Delta)|_v^{-1} \cdot (\log T)^{n-1}$$

as $T \to \infty$.

Proof. We only prove the case that v be a real place and $r > 0$. The rest of cases follows from the exact the same arguments. Assume $\sigma_1, \cdots, \sigma_r$ are real embedding and $\sigma_{r+1}, \bar{\sigma}_{r+1}, \cdots, \sigma_{r+s}, \bar{\sigma}_{r+s}$ are complex embedding. Let $B = [-1, 1]^n \subset \mathbb{R}^n$. By changing coordinate $z_i = \sigma_i(u)$ for $1 \le i \le r + s$ with $\mathbb{R}^n \to \mathbb{R}^r \times \mathbb{C}^s$, one has

$$d_v(X(F_v, T)) = \int_{T \cdot B \cap X(F_v)} |\omega| = \int_{T \cdot B' \cap X(F_v)} |\omega'|$$

where

$$\omega' = |N_{K/F}(\Delta)|_v^{-1} \frac{dz_2}{|z_2|} \wedge \cdots \wedge \frac{dz_r}{|z_r|} \wedge \frac{dz_{r+1}}{|z_{r+1}|} \wedge \frac{d\bar{z}_{r+1}}{|\bar{z}_{r+1}|} \wedge \cdots \wedge \frac{dz_{r+s}}{|z_{r+s}|} \wedge \frac{d\bar{z}_{r+s}}{|\bar{z}_{r+s}|}$$

and B' is the image of B in $\mathbb{R}^r \times \mathbb{C}^s$ under this change coordinate.

For any $\delta > 0$, we define

$$B(\delta) = \{(x_1, \cdots, x_r, x_{r+1}, \cdots, x_{r+s}) \in \mathbb{R}^r \times \mathbb{C}^s : |x_i| \le \delta \text{ for all } 1 \le i \le r + s\}.$$

Since there exist $\delta_1 > \delta_2 > 0$ such that $B(\delta_2) \subset B' \subset B(\delta_1)$ and

$$\int_{B(T \cdot \delta_2) \cap X(F_v)} |\omega'| \le d_v(X(F_v, T)) \le \int_{B(T \cdot \delta_1) \cap X(F_v)} |\omega'|,$$

one only needs to show that

$$\int_{B(T \cdot \delta) \cap X(F_v)} |\omega'| \sim 2^{r-1} (2\pi)^s \frac{n^{r+s-1}}{(r+s-1)!} |N_{K/F}(\Delta)|_v^{-1} \cdot (\log T)^{r+s-1}$$

as $T \to \infty$ for any $\delta > 0$.

Since $B(T \cdot \delta) \cap X(F_v)$ has 2^{r-1} connected components and the integral over each connected component is the same. Therefore

$$\int_{B(T \cdot \delta) \cap X(F_v)} |\omega'| = 2^{r-1} \int_{B_{X,T}} |\omega'|$$

where $B_{X,T}$ is the connected component of $B(T \cdot \delta) \cap X(F_v)$ such that the first r-real coordinates are positive.

By using the the polar coordinates

$$\begin{cases} z_i = e^{\rho_i} & \text{for } i = 1, \cdots, r \\ z_j = e^{\rho_j + \mathbf{i}\theta_j} & \text{with } 0 \le \theta_j < 2\pi \text{ for } j = r+1, \cdots r+s, \end{cases}$$

one obtains that

$$\int_{B_{X,T}} |\omega'| = |N_{K/F}(\Delta)|_v^{-1} (2\pi)^s \int_{V(T)} \left(\bigwedge_{i=2}^r d\rho_i\right) \wedge \left(\bigwedge_{j=1}^s d(2\rho_{r+j})\right)$$

where

$$V(T) = \{(\rho_i) \in (-\infty, \; \log(T \cdot \delta)]^{r+s} : \; \sum_{i=1}^{r} \rho_i + \sum_{j=1}^{s} 2\rho_{r+j} = \log |m|_v\}.$$

By substituting ρ_i by $\rho_i \log T$ for $1 \le i \le r+s$, one has

$$\int_{V(T)} (\bigwedge_{i=2}^{r} d\rho_i) \wedge (\bigwedge_{j=1}^{s} d(2\rho_{r+j})) \sim (\log T)^{s+r-1} \int_{V} (\bigwedge_{i=2}^{r} d\rho_i) \wedge (\bigwedge_{j=1}^{s} d(2\rho_{r+j}))$$

as $T \to \infty$, where

$$V = \{(\rho_i) \in (-\infty, 1]^{r+s} : \; \sum_{i=1}^{r} \rho_i + \sum_{j=1}^{s} 2\rho_{r+j} = 0\}$$

$$= \{(u_i) \in (-\infty, 0]^{r+s} : \; \sum_{i=1}^{r} u_i + \sum_{j=1}^{s} 2u_{r+j} = -n\}$$

$$= \{(u_2, \cdots, u_{r+s}) \in (-\infty, 0]^{r+s-1} : \; \sum_{i=2}^{r} u_i + \sum_{j=1}^{s} 2u_{r+j} \ge -n\}$$

with $u_i = \rho_i - 1$ for $i = 1, \cdots, r+s$. Therefore

$$\int_{V} (\bigwedge_{i=2}^{r} d\rho_i) \wedge (\bigwedge_{j=1}^{s} d(2\rho_{r+j})) = n^{r+s-1}/(r+s-1)!$$

by the standard computation. The proof is complete. $\qquad\square$

If $F = \mathbb{Q}$ and $\alpha_1, \cdots, \alpha_n \in o_K$ are linear independent over $\mathbb{Q}$ such that

$$L = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n$$

is an order of K, one can associate the narrow ring class field H_L corresponding to the order L with the Artin reciprocity isomorphism

$$\psi_{H_L/K} : \; \mathbb{I}_K/K^\times(K_\infty^+ \prod_{p<\infty} L_p^\times) \xrightarrow{\cong} \mathrm{Gal}(H_L/K) \tag{5.8}$$

where K_∞^+ is the connected component of 1 inside K_∞, L_p is the p-adic completion of L inside $K_p = K \otimes_\mathbb{Q} \mathbb{Q}_p$ and $L_p^\times$ is the unit group of L_p for any prime p.

Let

$$G = R_{K/\mathbb{Q}}^1(\mathbb{G}_m) \quad \text{and} \quad \mathbf{G}(\mathbb{Z}_p) = \{\xi \in L_p^\times : \; N_{K_p/\mathbb{Q}_p}(\xi) = 1\}$$

for any prime p. Set $G(\mathbb{R})^+$ to be the connected component of 1 inside

$$G(\mathbb{R}) = \{x \in K_\infty^\times : \; N_{K_\infty/\mathbb{R}}(x) = 1\}.$$

The homomorphism induced by the natural inclusion

$$\lambda_K : G(\mathbb{A}_\mathbb{Q})/G(\mathbb{Q})(G(\mathbb{R})^+ \prod_{p<\infty} \mathbf{G}(\mathbb{Z}_p)) \longrightarrow \mathbb{I}_K/K^\times(K_\infty^+ \prod_{p<\infty} L_p^\times) \tag{5.9}$$

is well-defined. Moreover, if $x_\mathbb{A} \in ker(\lambda_K)$, then there are

$$\alpha \in K^\times \quad \text{and} \quad y_\mathbb{A} \in K_\infty^+ \prod_{p<\infty} L_p^\times$$

such that $x_\mathbb{A} = \alpha \cdot y_\mathbb{A}$. Therefore

$$N_{K/\mathbb{Q}}(\alpha^{-1}) = N_{K/\mathbb{Q}}(\alpha^{-1}x_\mathbb{A}) = N_{K/\mathbb{Q}}(y_\mathbb{A}) \in \mathbb{Q} \cap (\mathbb{R}^+ \times \prod_p \mathbb{Z}_p^\times) = \{1\}$$

where $\mathbb{R}^+$ is the set of positive reals. This implies that λ_K is injective.

Consider the Diophantine equation (5.3) $\mathbf{X}$ over $\mathbb{Z}$. Then $\mathbf{X}(\mathbb{Z}) \neq \emptyset$ if and only if there is

$$(x_{1,p}, \cdots, x_{n,p})_{p \leq \infty} \in \prod_{p \leq \infty} \mathbf{X}(\mathbb{Z}_p)$$

such that

$$\psi_{H_L/K}((\sum_{i=1}^n \alpha_i x_{i,p})_{p \leq \infty}) = 1$$

where

$$\sum_{i=1}^n \alpha_i x_{i,p} \in K_p = K \otimes_{\mathbb{Q}} \mathbb{Q}_p$$

by Corollary 1.6 in [34].

Lemma 5.10. *Let $K/\mathbb{Q}$ be a finite extension of degree n and $\mathbf{X}$ be the scheme defined by the norm equation*

$$N_{K/\mathbb{Q}}(\alpha_1 x_1 + \cdots + \alpha_n x_n) = m$$

with $\alpha_1, \cdots, \alpha_n \in o_K$ such that $L = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n$ is an order of K and $m \in \mathbb{Z}$ with $m \neq 0$. If $X = \mathbf{X} \times_{\mathbb{Z}} \mathbb{Q}$ is a trivial torsor of $G = R^1_{K/\mathbb{Q}}(\mathbb{G}_m)$, then

$$N(\mathbf{X}, T) \sim \frac{h_G}{h_L \cdot \tau(G)} \sum_{\phi} \prod_{p \ primes} N_p(\mathbf{X}, \phi) N_\infty(X, T, \phi)$$

as $T \to \infty$, runs over all characters of $\mathrm{Gal}(H_L/K)$ and

$$h_G = [G(\mathbb{A}_{\mathbb{Q}}) : G(\mathbb{Q})(G(\mathbb{R})^+ \prod_{p<\infty} \mathbf{G}(\mathbb{Z}_p))] \quad and \quad h_L = [\mathbb{I}_K : K^\times(K^+_\infty \prod_{p<\infty} L^\times_p)]$$

with the above notation and $\tau(G)$ is the Tamagawa number of G and

$$N_p(\mathbf{X}, \phi) = \int_{\mathbf{X}(\mathbb{Z}_p)} \phi(\psi_{H_L/K}(\sum_{i=1}^n \alpha_i x_{i,p})) d_p(x_{1,p}, \cdots, x_{n,p})$$

for all primes p and

$$N_\infty(X, T, \phi) = \int_{X(\mathbb{R}, T)} \phi(\psi_{H_L/K}(\sum_{i=1}^n \alpha_i x_{i,\infty})) d_\infty(x_{1,\infty}, \cdots, x_{n,\infty})$$

with the Artin map $\psi_{H_L/K}$ in (5.8).

Proof. Fix $(\varsigma_1, \cdots, \varsigma_n) \in X(\mathbb{Q})$. One can apply Lemma 5.2 by using

$$\overline{\varrho}_p(x_{1,p}, \cdots, x_{n,p}) = \varrho_{F_v}((\sum_{i=1}^n \alpha_i x_{i,p})(\sum_{i=1}^n \alpha_i \varsigma_i)^{-1})$$

for any $(x_{1,p}, \cdots, x_{n,p}) \in X(\mathbb{Q}_p)$ with all primes $p \leq \infty$.

Since the natural inclusion λ_K in (5.9) is injective, one concludes that

$$N(\mathbf{X}, T) \sim \frac{h_G}{h_L \cdot \tau(G)} \sum_{\phi} \prod_{p \leq \infty} N_p(\mathbf{X}, \phi) N_\infty(X, T, \phi)$$

as $T \to \infty$, where ϕ runs over all characters of $\mathrm{Gal}(H_L/K)$ by the Artin reciprocity law (5.8). $\qquad \square$

We further specialize to the case that $K/\mathbb{Q}$ is a real quadratic extension with the discriminant D and $L = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ is the maximal order o_K of K. Let $P(D)$ be the set of all prime divisors of D and

$$m = (-1)^s p_1^{e_1} \cdots p_f^{e_f} \prod_{q_i \in P(D)} q_i^{t_i}$$

where $p_1, \cdots, p_f$ are the distinct primes and prime to D. Let

$$Q_1 = \{p_i : \ (\frac{D}{p_i}) = 1 \text{ with } 1 \le i \le f\}$$

and

$$Q_2 = \{p_i : \ (\frac{D}{p_i}) = -1 \text{ with } 1 \le i \le f\}.$$

Since $\mathrm{Gal}(K/\mathbb{Q})$ acts on the two real places transitively, one real place is ramified in H_L/K if and only if the other real place is ramified in H_L/K. Since the global element -1 in K gives identity in $\mathrm{Gal}(H_L/K)$ via the Artin map, one obtains that the complex conjugations over these two real places are the same by using the product of the local Artin maps when a real place v of K is ramified in H_L/K. Let σ_{-1} be the complex conjugation if a real place of K is ramified in H_L/K; otherwise σ_{-1} is the identity in $\mathrm{Gal}(H_L/K)$.

If $p_i \in Q_1$, then p_i splits into two primes $\mathfrak{p}_1$ and $\mathfrak{p}_2$ in K. Since the global element p_i gives identity in $\mathrm{Gal}(H_L/K)$ via the Artin map, one obtains that the Frobenius of $\mathfrak{p}_1$ and the Frobenius of $\mathfrak{p}_2$ are inverse to each other in $\mathrm{Gal}(H_L/K)$ by using the product of the local Artin maps. Let σ_{p_i} be the Frobenius of $\mathfrak{p}_1$ in $\mathrm{Gal}(H_L/K)$. Then the Frobenius of $\mathfrak{p}_2$ in $\mathrm{Gal}(H_L/K)$ is $\sigma_{p_i}^{-1}$. Define

$$\delta_{p_i}(\phi) = \begin{cases} (e_i + 1)\phi(\sigma_{p_i})^{e_i} & \text{if } \phi(\sigma_{p_i}^2) = 1 \\ (1 - \phi(\sigma_{p_i})^{2e_i+2})(\phi(\sigma_{p_i})^{e_i} - \phi(\sigma_{p_i})^{e_i+2})^{-1} & \text{otherwise.} \end{cases}$$

for any character ϕ of $\mathrm{Gal}(H_L/K)$. Then $\delta_{p_i}(\phi)$ is independent of the choice of the place $\mathfrak{p}_1$ of K.

If $p_i \in Q_2$, then p_i is inert in K. Then the Frobenius σ_{p_i} of p_i satisfies

$$\sigma_{p_i} = \psi_{H_L/K, p_i}(p_i) = \prod_{v \ne p_i} \psi_{H_L/K, v}(p_i)^{-1}$$

by the reciprocity law. Since H_L/K is unramified at all finite places, we have

$$\sigma_{p_i} = \prod_{v | \infty} \psi_{H_L/K, v}(p_i)^{-1} \cdot \prod_{v < \infty, v \ne p_i} \psi_{H_L/K, v}(p_i)^{-1} = 1 \cdot 1 = 1.$$

If $p_i \in P(D)$, the Frobenius of the unique place of K over p_i in $\mathrm{Gal}(H_L/K)$ is denoted by σ_{p_i}.

Proposition 5.11. *Suppose $K/\mathbb{Q}$ is a real quadratic extension and $L = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ is the maximal order o_K of K. With the above notation, we define*

$$c_m = \sum_{\phi} \phi(\sigma_{sgn(m)}) \cdot \prod_{p_i \in P(D)} \phi(\sigma_{p_i})^{t_i} \cdot \prod_{p_i \in Q_1} \delta_{p_i}(\phi)$$

where ϕ runs over all characters of $\mathrm{Gal}(H_L/K)$ and

$$\sigma_{sgn(m)} = \begin{cases} \sigma_{-1} & \text{if } m < 0 \\ 1 & \text{if } m > 0. \end{cases}$$

Then the norm equation

$$N_{K/\mathbb{Q}}(\alpha_1 x + \alpha_2 y) = m$$

is solvable over $\mathbb{Z}$ if and only if this equation is solvable over $\mathbb{Z}_p$ for all primes $p \leq \infty$ and $c_m \neq 0$. Moreover

$$N(\mathbf{X}, T) \sim \frac{2c_m}{h_K^+ \sqrt{D}} \cdot \frac{\log T}{\log \epsilon}$$

as $T \to \infty$, where ϵ is the unique minimal unit of o_K such that $\epsilon > 1$ and $N_{K/\mathbb{Q}}(\epsilon) = 1$ and

$$h_K^+ = [\mathbb{I}_K : K^\times (K_\infty^+ \prod_{v < \infty_K} o_{K_v}^\times)]$$

with the connected component K_∞^+ of 1 in $K_\infty^\times$ is the narrow class number of K.

Proof. Let $\mathbf{X}$ be the scheme over $\mathbb{Z}$ defined by the above norm equation. Since

$$\prod_{p \leq \infty} \mathbf{X}(\mathbb{Z}_p) \neq \emptyset,$$

one gets that $X = \mathbf{X} \times_{\mathbb{Z}} \mathbb{Q}$ is a trivial torsor of $G = R_{K/\mathbb{Q}}^1(\mathbb{G}_m)$. In order to apply Lemma 5.10, one needs to compute the integrals in each term.

If $(p, m) = 1$, then $\psi_{H_L/K}$ takes the trivial value over $\mathbf{X}(\mathbb{Z}_p)$ and

$$N_p(\mathbf{X}, \phi) = \int_{\mathbf{X}(\mathbb{Z}_p)} d_p = \int_{\mathbf{G}(\mathbb{Z}_p)} d_p$$

for all character ϕ of $\mathrm{Gal}(H_L/K)$.

If $p_i \in P(D)$, then

$$\psi_{H_L/K}(\alpha_1 x_{p_i} + \alpha_2 y_{p_i}) = \sigma_{p_i}^{t_i}$$

for all $(x_{p_i}, y_{p_i}) \in \mathbf{X}(\mathbb{Z}_{p_i})$. Therefore

$$N_{p_i}(\mathbf{X}, \phi) = \phi(\sigma_{p_i})^{t_i} \int_{\mathbf{G}(\mathbb{Z}_{p_i})} d_{p_i}$$

for all character ϕ of $\mathrm{Gal}(H_L/K)$.

If $p_i \in Q_1$, then p splits into two places $\mathfrak{p}_1$ and $\mathfrak{p}_2$ in $K/\mathbb{Q}$. Let σ_{p_i} be the Frobenius of H_L/K at $\mathfrak{p}_1$. Therefore

$$N_{p_i}(\mathbf{X}, \phi) = \sum_{i=0}^{e_i} \int_{z_1 z_2 = m, v_{p_i}(z_1) = i} \phi(\sigma_{p_i}^i \cdot \sigma_{p_i}^{-(e_i - i)}) d_{p_i}$$

$$= \sum_{i=0}^{e_i} \phi(\sigma_{p_i})^{2i - e_i} \int_{\mathbf{G}(\mathbb{Z}_p)} d_{p_i} = \delta_{p_i}(\phi) \int_{\mathbf{G}(\mathbb{Z}_{p_i})} d_{p_i}$$

for all character ϕ of $\mathrm{Gal}(H_L/K)$.

If $p = \infty$ and a real place over ∞ is ramified in H_L/K, then

$$\psi_{H_L/K}(\alpha_1 x_\infty + \alpha_2 y_\infty) = \begin{cases} \sigma_{-1} & \text{when } \alpha_1 x_\infty + \alpha_2 y_\infty < 0 \\ 1 & \text{otherwise} \end{cases}$$

over such a real place. Therefore

$$N_\infty(X, T, \phi) = \begin{cases} \phi(\sigma_{-1}) d_\infty(X(\mathbb{R}, T)) & \text{if } m < 0 \\ d_\infty(X(\mathbb{R}, T)) & \text{if } m > 0 \end{cases}$$

for all character ϕ of $\mathrm{Gal}(H_L/K)$.

Let
$$h_G = [G(\mathbb{A}_{\mathbb{Q}}) : G(\mathbb{Q})(G(\mathbb{R})^+ \prod_{p < \infty} \mathbf{G}(\mathbb{Z}_p))]$$
where $G(\mathbb{R})^+$ is the connected component of 1. Then
$$\tau(G) = h_G \prod_{p < \infty} \int_{\mathbf{G}(\mathbb{Z}_p)} d_p \cdot \int_{\mathbf{G}(\mathbb{Z})\backslash G(\mathbb{R})} d_\infty$$
where $\tau(G)$ is the Tamagawa number of G and
$$\mathbf{G}(\mathbb{Z}) = G(\mathbb{Q}) \cap (G(\mathbb{R})^+ \prod_{p < \infty} \mathbf{G}(\mathbb{Z}_p)).$$

By using the invariant differential in Lemma 5.4, one obtains that
$$\int_{\mathbf{G}(\mathbb{Z})\backslash G(\mathbb{R})} d_\infty = \frac{2}{\sqrt{D}} \log \epsilon$$
where ϵ is the unique minimal unit of o_K satisfying with $\epsilon > 1$ and $N_{K/\mathbb{Q}}(\epsilon) = 1$. By Lemma 5.7, we have
$$d_\infty(X(\mathbb{R}, T)) \sim \frac{4}{D} \cdot \log T$$
as $T \to \infty$. The result follows from combining the above computation and Lemma 5.10 and Proposition 3.3. $\qquad\qquad\square$

By Proposition 5.11, the negative Pell equation
$$x^2 - \delta y^2 = -1$$
where δ is a square-free positive integer with $\delta \not\equiv 1 \mod 4$ is solvable over $\mathbb{Z}$ if and only if σ_{-1} is trivial in $\mathrm{Gal}(H_L/K)$, which is equivalent to that the narrow class number h_K^+ of K is equal to the the class number h_K of K. This is the well-known classical result. In this case, one has
$$N(\mathbf{X}, T) \sim \delta^{-\frac{1}{2}} \frac{\log T}{\log \epsilon}$$
as $T \to \infty$ by Proposition 5.11, where $\epsilon = x_0 + y_0\sqrt{\delta}$ with the integral solution (x_0, y_0) of $x^2 - \delta y^2 = 1$ such that $\epsilon > 1$ and ϵ is minimal.

Now we provide a more explicit example. For any integer m, one can write
$$m = (-1)^{s_0} 2^{s_1} 17^{s_2} p_1^{e_1} \cdots p_g^{e_g} \quad \text{and} \quad \Pi(m) = \{p_1, \cdots, p_g\}.$$
Decompose $\Pi(m)$ into the disjoint union of the following subsets
$$\Pi_1 = \{p \in \Pi(m) : (\frac{2}{p}) = (\frac{17}{p}) = -1\} \text{ and } \Pi_2 = \{p \in \Pi(m) : (\frac{34}{p}) = -1\}$$
$$\Pi_3 = \{p \in \Pi(m) : (\frac{2}{p}) = (\frac{17}{p}) = 1 \text{ and } (\frac{-7 + 4\sqrt{2}}{p}) = 1\}$$
$$\Pi_4 = \{p \in \Pi(m) : (\frac{2}{p}) = (\frac{17}{p}) = 1 \text{ and } (\frac{-7 + 4\sqrt{2}}{p}) = -1\}.$$
Let
$$m_1 = (-1)^{s_0} \prod_{p_i \in \Pi_1} p_i^{e_i}.$$

Example 5.12. *With the above notation, the equation $x^2 - 34y^2 = m$ is solvable over $\mathbb{Z}$ if and only if $m_1 \equiv \pm 1 \mod 8$, $(\frac{m_1}{17}) = 1$, $(\frac{34}{p_i}) = 1$ for odd e_i and*

(1) there is an odd e_i for some $p_i \in \Pi_1$. In this case

$$N(\mathbf{X}, T) \sim \frac{1}{2\sqrt{34}} \prod_{p_i \in \Pi(m) \setminus \Pi_2} (1 + e_i) \cdot \frac{\log T}{\log(35 + 6\sqrt{34})}$$

as $T \to \infty$.

(2) $\Pi_1 \neq \emptyset$ and all e_i are even for $p_i \in \Pi_1$. In this case

$$N(\mathbf{X}, T) \sim \frac{r}{2\sqrt{34}} \prod_{p_i \in \Pi_3 \cup \Pi_4} (1 + e_i) \cdot \frac{\log T}{\log(35 + 6\sqrt{34})}$$

as $T \to \infty$, where

$$r = (-1)^{s_0 + s_2 + \frac{1}{2}\sum_{p_i \in \Pi_1} e_i + \sum_{p_i \in \Pi_4} e_i} + \prod_{p_i \in \Pi_1} (1 + e_i).$$

(3) $\Pi_1 = \emptyset$ and $\sum_{p_i \in \Pi_4} e_i \equiv s_0 + s_2 \mod 2$. In this case

$$N(\mathbf{X}, T) \sim \frac{1}{\sqrt{34}} \prod_{p_i \in \Pi_3 \cup \Pi_4} (1 + e_i) \cdot \frac{\log T}{\log(35 + 6\sqrt{34})}$$

as $T \to \infty$.

Proof. The narrow Hilbert class field H_L of $K = \mathbb{Q}(\sqrt{34})$ is given by

$$H_L = K(\sqrt{-7 + 4\sqrt{2}}) \quad \text{with} \quad \mathrm{Gal}(H_L/K) \cong \mu_4 = \{\pm 1, \pm \mathbf{i}\}.$$

By the notation in Proposition 5.11, we have

$$\sigma_p = \begin{cases} 1 & \text{if } p \in \{2\} \cup \Pi_2 \cup \Pi_3 \\ -1 & \text{if } p = \{-1, 17\} \cup \Pi_4 \\ \pm \mathbf{i} & \text{if } p \in \Pi_1 \end{cases}$$

and $\epsilon = 35 + 6\sqrt{34}$. Let ϕ be the generator of the character group of $\mathrm{Gal}(H_L/K)$.
    If $\varphi = 1$ or ϕ^2, then

$$\delta_p(\varphi) = (1 + e_i)\varphi(\sigma_{p_i})^{e_i}$$

for $p_i \in \Pi_1 \cup \Pi_3 \cup \Pi_4$.
    If $\varphi = \phi$ or ϕ^3, then

$$\delta_p(\varphi) = \begin{cases} \frac{1}{2}(\mathbf{i}^{e_i} + (-\mathbf{i})^{e_i}) & \text{if } p_i \in \Pi_1 \\ (1 + e_i)\varphi(\sigma_{p_i})^{e_i} & \text{if } p_i \in \Pi_3 \cup \Pi_4. \end{cases}$$

Therefore

$$c_m = \sum_{\phi} \phi(\sigma_{-1}) \prod_{p_i \in P(34)} \phi(\sigma_{p_i})^{e_i} \prod_{p_i \in \Pi(m) \setminus \Pi_2} \delta_{p_i}(\phi)$$

$$= \prod_{p_i \in \Pi(m) \setminus \Pi_2} (1 + e_i) + \prod_{p_i \in \Pi_1} (-1)^{e_i}(1 + e_i) \cdot \prod_{p_i \in \Pi_3 \cup \Pi_4} (1 + e_i)$$

$$+ 2(-1)^{s_0 + s_2} \prod_{p_i \in \Pi_1} \frac{1}{2}(\mathbf{i}^{e_i} + (-\mathbf{i})^{e_i}) \cdot \prod_{p_i \in \Pi_3} (1 + e_i) \cdot \prod_{p_i \in \Pi_4} (-1)^{e_i}(1 + e_i).$$

Since $(\frac{n_1}{17}) = 1$ by the local solvability condition, one has

$$\prod_{p_i \in \Pi_1} (-1)^{e_i} = 1.$$

If there is an odd e_i for some $p_i \in \Pi_1$, then

$$c_m = 2 \prod_{p_i \in \Pi(m) \backslash \Pi_2} (1 + e_i).$$

If all e_i are even for all $p_i \in \Pi_1$ or $\Pi_1 = \emptyset$, then

$$c_m = 2 \prod_{p_i \in \Pi(m) \backslash \Pi_2} (1 + e_i) + 2(-1)^{s_0 + s_2 + \frac{1}{2} \sum_{p_i \in \Pi_1} e_i + \sum_{p_i \in \Pi_4} e_i} \prod_{p_i \in \Pi_3 \cup \Pi_4} (1 + e_i).$$

The result follows from Proposition 5.11 and Example 5.5 in [34]. $\qquad\square$

Comparing this example with Example 5.5 in [34], one can find that both conditions for existence the integral points look slight different. This is because the **X**-admissible subgroups in [34, Theorem 1.10] are not unique. In other word, the finite subgroups of $Br(X)$ for testing the existence of the integral points are not unique.

6. Examples for semi-simple groups

As application, we will explain that the asymptotic formula of the number of integral solutions in Theorem 1.1 of [8] is the same as the Hardy-Littlewood expectation in sense of [3] although the Brauer-Manin obstruction is not trivial in general.

Example 6.1. *Let $p(\lambda)$ be an irreducible monic polynomial of degree $n \geq 2$ over $\mathbb{Z}$ and $\mathbf{X}$ be the scheme over $\mathbb{Z}$ defined by the following equations with variables $x_{i,j}$ for $1 \leq i, j \leq n$*

$$det(\lambda I_n - (x_{i,j})) = p(\lambda)$$

and $X = \mathbf{X} \times_{\mathbb{Z}} \mathbb{Q}$. Then

$$N(\mathbf{X}, T) \sim \left(\prod_{p < \infty} \int_{\mathbf{X}(\mathbb{Z}_p)} d_p \right) \cdot \int_{\mathbf{X}(\mathbb{R}, T)} d_\infty$$

as $T \to \infty$.

Proof. Since both SL_n and GL_n act on X by $x \circ g = gxg^{-1}$, one has that X is the homogeneous space of SL_n and GL_n with a rational point

$$v = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{pmatrix} \in X(\mathbb{Q})$$

where $p(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_0$. Moreover, the stabilizers of v inside SL_n and GL_n are $S \cong R^1_{K/\mathbb{Q}}(\mathbb{G}_m)$ and $R_{K/\mathbb{Q}}(\mathbb{G}_m)$ respectively, where $K = \mathbb{Q}(\theta)$ and θ is a root of $p(\lambda)$. Then we have the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & S & \longrightarrow & SL_n & \longrightarrow & X & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & R_{K/\mathbb{Q}}(\mathbb{G}_m) & \longrightarrow & GL_n & \longrightarrow & X & \longrightarrow & 1
\end{array}$$

For any field extension $k/\mathbb{Q}$, one obtains the following commutative diagram

$$
\begin{array}{ccccccc}
& & 1 & & 1 & & \\
& & \downarrow & & \downarrow & & \\
1 \longrightarrow & S(k) & \longrightarrow & SL_n(k) & \longrightarrow & X(k) \xrightarrow{\delta_1} & H^1(k,S) \\
& \downarrow & & \downarrow & & \| & \\
1 \longrightarrow & (k \otimes_{\mathbb{Q}} K)^{\times} & \longrightarrow & GL_n(k) & \longrightarrow & X(k) \longrightarrow & 1 \\
& \downarrow{\scriptstyle N_{K/\mathbb{Q}}} & & \downarrow{\scriptstyle det} & & & \\
1 \longrightarrow & k^{\times} & = \!\!\!= \!\!\!= & k^{\times} & & & \\
& \downarrow{\scriptstyle \delta_2} & & \downarrow & & & \\
& H^1(k,S) & & 1 & & &
\end{array}
$$

by Galois cohomology, Shapiro's Lemma and Hilbert 90.

For any $x \in X(k)$, there is a $g \in GL_n(k)$ such that $gvg^{-1} = x$ by the second line of the above diagram. Let $y \in R_{K/\mathbb{Q}}(\mathbb{G}_m)(\bar{k})$ such that $N_{K/\mathbb{Q}}(y) = \det(g)$. Then $gy^{-1} \in SL_n(\bar{k})$ and

$$
(gy^{-1})v(gy^{-1})^{-1} = g(y^{-1}vy)g^{-1} = gvg^{-1} = x
$$

and

$$
\delta_1(v) = \sigma(y^{-1}g)(y^{-1}g)^{-1} = \sigma(y^{-1})\sigma(g)g^{-1}y = \sigma(y^{-1})y = \delta_2(\det(g)^{-1}).
$$

Therefore

$$
\begin{aligned}
\delta_1 : \ X(k) &\longrightarrow H^1(k,S) \cong k^{\times}/N_{K/k}((K \otimes_{\mathbb{Q}} k)^{\times}) \\
x &\mapsto \det(g)^{-1} \cdot N_{K/k}((K \otimes_{\mathbb{Q}} k)^{\times})
\end{aligned}
\tag{6.2}
$$

Let L be the Galois closure of $K/\mathbb{Q}$, $\Lambda = \mathrm{Gal}(L/\mathbb{Q})$ and $\Upsilon = \mathrm{Gal}(L/K)$. Write $\widehat{S} = Hom(S, \mathbb{G}_m)$ be the character group of S. Then one has the short exact sequence of Λ-module

$$
0 \longrightarrow \mathbb{Z} \longrightarrow Ind_{\Upsilon}^{\Lambda}(\mathbb{Z}) \longrightarrow \widehat{S} \longrightarrow 0
$$

which gives the long exact sequence

$$
\rightarrow H^1(\Lambda, Ind_{\Upsilon}^{\Lambda}(\mathbb{Z})) \rightarrow H^1(\Lambda, \widehat{S}) \rightarrow H^2(\Lambda, \mathbb{Z}) \rightarrow H^2(\Lambda, Ind_{\Upsilon}^{\Lambda}(\mathbb{Z}))
$$

where

$$
H^1(\Lambda, Ind_{\Upsilon}^{\Lambda}(\mathbb{Z})) = H^1(\Upsilon, \mathbb{Z}) = 0
$$

and

$$
H^2(\Lambda, \mathbb{Z}) = H^1(\Lambda, \mathbb{Q}/\mathbb{Z}) = Hom(\Lambda, \mathbb{Q}/\mathbb{Z})
$$

and

$$
H^2(\Lambda, Ind_{\Upsilon}^{\Lambda}(\mathbb{Z})) = H^2(\Upsilon, \mathbb{Z}) = H^1(\Upsilon, \mathbb{Q}/\mathbb{Z}) = Hom(\Upsilon, \mathbb{Q}/\mathbb{Z})
$$

by Shapiro's Lemma. Since

$$
0 \rightarrow H^1(\Lambda, \widehat{S}) \xrightarrow{inf} H^1(\mathbb{Q}, \widehat{S}) \xrightarrow{res} H^1(L, \widehat{S})
$$

and

$$
H^1(L, \widehat{S}) = Hom(\mathrm{Gal}(\bar{\mathbb{Q}}/L), \widehat{S}) = 0
$$

where $\bar{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$, one concludes that

$$
Pic(S) = H^1(\mathbb{Q}, \widehat{S}) = H^1(\Lambda, \widehat{S}) = ker(\widehat{\Lambda} \rightarrow \widehat{\Upsilon})
$$

by Theorem of §4.3 of Chapter 2 in [30], where

$$\widehat{\Lambda} = Hom(\Lambda, \mathbb{Q}/\mathbb{Z}) \quad \text{and} \quad \widehat{\Upsilon} = Hom(\Upsilon, \mathbb{Q}/\mathbb{Z}).$$

By Proposition 2.10 of [4], one only needs to consider that $Pic(S)$ is not trivial. For any $\chi \in ker(\widehat{\Lambda} \to \widehat{\Upsilon})$ with $\chi \neq 1$, one gets a non-trivial abelian extension $A/\mathbb{Q}$ inside $L/\mathbb{Q}$ such that $ker(\chi) = \text{Gal}(L/A)$. By Minkowski's Theorem (see (2.17) of Chapter III in [18]), there is a prime p such that p is ramified in $A/\mathbb{Q}$. By (1.7) Proposition of Chapter V and (5.6) Proposition of Chapter VI in [18], there is an idele $(u_l)_l$ of $\mathbb{Q}$ defined by $u_l = 1$ for $l \neq p$ and $u_p \in \mathbb{Z}_p^\times$ for $l = p$ such that

$$\chi(\psi_{A/\mathbb{Q}}((u_l)_l)) \neq 1$$

where $\psi_{A/\mathbb{Q}}$ is the Artin map.

Let $\xi \in Br(X)$ be the image of χ under the map

$$\delta_{tors}(SL_n) : Pic(S) \to Br(X)$$

defined in P.314 of [4]. By Proposition 2.10 in [4], one has that ξ is not trivial in $Br(X)/Br(\mathbb{Q})$. Applying the change of variables on $X(\mathbb{Q}_p)$ by using the diagonal matrix $diag(u_p^{-1}, 1, \cdots, 1)$

$$(x_{i,j}) \mapsto diag(u_p^{-1}, 1, \cdots, 1) \cdot (x_{i,j}) \cdot diag(u_p, 1, \cdots, 1)$$

which leaves $\mathbf{X}(\mathbb{Z}_p)$ stable, one obtains that

$$\int_{\mathbf{X}(\mathbb{Z}_p)} \xi d_p = \chi(\psi_{A/\mathbb{Q}}((u_l)_l)) \int_{\mathbf{X}(\mathbb{Z}_p)} \xi d_p$$

by (6.2) and the diagram (3.1) in [4]. Therefore

$$\int_{\mathbf{X}(\mathbb{Z}_p)} \xi d_p = 0$$

and the result follows from Theorem 4.3 and Proposition 2.10 (ii) in [4].   □

We will answer a question raised by Borovoi how to compute the ratio of the number of the integral solutions with the Hardy-Littlewood expectation for Example 6.3 in [3].

Example 6.3. *Let* $a \in \mathbb{Z}$ *with* $a \neq 0$ *and* $\mathbf{X}_a$ *be a scheme over* $\mathbb{Z}$ *defined by the equation*

$$det(x_{i,j})_{n \times n} = a \quad \text{with } x_{i,j} = x_{j,i} \text{ for all } 1 \leq i, j \leq n$$

with $X_a = \mathbf{X}_a \times_\mathbb{Z} \mathbb{Q}$ *and* $n \geq 3$. *Define*

$$c_n(a) = \lim_{T \to \infty} \frac{N(\mathbf{X}_a, T)}{(\prod_{p < \infty} \int_{\mathbf{X}_a(\mathbb{Z}_p)} d_p) \cdot \int_{\mathbf{X}_a(\mathbb{R},T)} d_\infty}.$$

Then

$$c_n(a) = 1 + \prod_{p|2a} \frac{\int_{\mathbf{X}_a(\mathbb{Z}_p)} h \, d_p}{\int_{\mathbf{X}_a(\mathbb{Z}_p)} d_p} \cdot \lim_{T \to \infty} \frac{\int_{\mathbf{X}_a(\mathbb{R},T)} h \, d_\infty}{\int_{\mathbf{X}_a(\mathbb{R},T)} d_\infty}$$

where $h : X(\mathbb{Q}_p) \longrightarrow \{\pm 1\}$ *is the Hasse-Witt symbol function defined in P.167 of* [20]. *In particular*

$$c_n(a) = \begin{cases} 1 & \text{if } a > 0 \text{ and } n \equiv 2 \mod 4 \\ \frac{1}{2} & \text{if } a = 1 \text{ and } n = 3. \end{cases}$$

Proof. Since SL_n acts on X_a by $x \circ g = g'xg$ where g' is the transpose of g, one has that X_a is the homogeneous space of SL_n with a rational point of diagonal matrix $v = diag(a, 1, \cdots, 1) \in X_a(\mathbb{Q})$ and the stabilizer of v is the special orthogonal group $SO(v)$ defined by $diag(a, 1, \cdots, 1)$. Moreover, one gets that

$$Pic(SO(v)) \cong \mathbb{Z}/(2)$$

by Proposition 2.5 and 2.6 in [4]. The non-trivial element E of $Pic(SO(v))$ gives the central extension of $SO(v)$ by $\mathbb{G}_m$ satisfying the following diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mu_2 & \longrightarrow & Spin & \longrightarrow & SO(v) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & id\downarrow & & \\
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & E & \longrightarrow & SO(v) & \longrightarrow & 1
\end{array}
$$

because $Spin$ is almost simple. Applying the Galois cohomology, one has

$$
\begin{array}{ccc}
H^1(k, SO(v)) & \xrightarrow{\ h'\ } & {}_2Br(k) \\
id\downarrow & & \downarrow \\
H^1(k, SO(v)) & \longrightarrow & Br(k)
\end{array}
\qquad (6.4)
$$

for any field extension $k/\mathbb{Q}$, where h' is the Hasse-Witt invariant by (31.41) in [13]. By Proposition 2.10 in [4], one has that $\delta_{tors}(SL_n)(E)$ is the non-trivial element ξ of $Br(X_a)/Br(\mathbb{Q})$. Moreover, the evaluation of ξ over $X_a(k)$ is equal to the Hasse-Witt invariant of the corresponding torsor by Proposition 2.9 in [4] and the above commutative diagram (6.4). By the commutative diagram (3.1) in [4] and Theorem 4.3 and the Hilbert reciprocity law (see Chapter VII in [20]), one concludes that

$$c_n(a) = 1 + \frac{\left(\prod_{p<\infty} \int_{\mathbf{X}_a(\mathbb{Z}_p)} h \, d_p\right)}{\left(\prod_{p<\infty} \int_{\mathbf{X}_a(\mathbb{Z}_p)} d_p\right)} \cdot \lim_{T\to\infty} \frac{\int_{\mathbf{X}_a(\mathbb{R},T)} h \, d_\infty}{\int_{\mathbf{X}_a(\mathbb{R},T)} d_\infty}$$

where $h : X_a(\mathbb{Q}_p) \longrightarrow \{\pm 1\}$ is the Hasse-Witt symbol function. If $p \nmid 2a$, then $h(x) = 1$ by 92:1 in [20] and the first part of the result follows.

There are only finitely many orbits of $SL_n(\mathbb{R})$ inside $X_a(\mathbb{R})$ classified by the signatures. Among these orbits, the most significant orbits $Y_+(\mathbb{R})$ and $Y_-(\mathbb{R})$ are given by the exact sequence

$$1 \longrightarrow SO_\pm \longrightarrow SL_n(\mathbb{R}) \xrightarrow{\pi_\pm} Y_\pm(\mathbb{R}) \longrightarrow 1$$

where $SO_\pm$ are the special orthogonal groups defined by $\pm I_n$ respectively. Both $SO_\pm$ are the compact subgroups of $SL_n(\mathbb{R})$. The rest orbits Y_i are given by

$$1 \longrightarrow SO_i \longrightarrow SL_n(\mathbb{R}) \xrightarrow{\pi_i} Y_i(\mathbb{R}) \longrightarrow 1$$

where SO_i are the special orthogonal groups defined by the rest of representatives of the orbits of $SL_n(\mathbb{R})$ whose signatures are different from those of $\pm I_n$. Then such SO_i's are not compact. Since

$$\int_{\mathbf{X}_a(\mathbb{R},T)\cap Y_\pm(\mathbb{R})} d_\infty = \frac{Vol(\pi_\pm^{-1}(\mathbf{X}_a(\mathbb{R},T)\cap Y_\pm(\mathbb{R})))}{Vol(\pi_\pm^{-1}(\mathbf{X}_a(\mathbb{R},T)\cap Y_\pm(\mathbb{R}))\cap SO_\pm)}$$

and

$$\int_{\mathbf{X}_a(\mathbb{R},T)\cap Y_i(\mathbb{R})} d_\infty = \frac{Vol(\pi_i^{-1}(\mathbf{X}_a(\mathbb{R},T)\cap Y_i(\mathbb{R})))}{Vol(\pi_i^{-1}(\mathbf{X}_a(\mathbb{R},T)\cap Y_i(\mathbb{R}))\cap SO_i)},$$

where

$$\lim_{T\to\infty} Vol(\pi_\pm^{-1}(\mathbf{X}_a(\mathbb{R},T)\cap Y_\pm(\mathbb{R}))\cap SO_\pm) = Vol(SO_\pm) < \infty$$

and
$$\lim_{T \to \infty} Vol(\pi_i^{-1}(\mathbf{X}_a(\mathbb{R}, T) \cap Y_i(\mathbb{R})) \cap SO_i) = \infty,$$
one concludes that
$$\lim_{T \to \infty} \frac{\int_{\mathbf{X}_a(\mathbb{R}, T)} h \, d_\infty}{\int_{\mathbf{X}_a(\mathbb{R}, T)} d_\infty} = \begin{cases} 1 & \text{if } a > 0 \text{ and } n \equiv 1 \mod 2 \\ 1 & \text{if } a < 0 \text{ and } n \equiv 3 \mod 4 \\ -1 & \text{if } a < 0 \text{ and } n \equiv 1 \mod 4 \\ 1 & \text{if } a > 0 \text{ and } n \equiv 0 \mod 4 \\ 0 & \text{if } a > 0 \text{ and } n \equiv 2 \mod 4 \end{cases}$$
by computation of Hasse-Witt invariant over $\mathbb{R}$.

For $a = 1$ and $n = 3$, one has
$$c_3(1) = 1 + \frac{\int_{\mathbf{X}_1(\mathbb{Z}_2)} h \, d_2}{\int_{\mathbf{X}_1(\mathbb{Z}_2)} d_2}.$$
By 93:18 (iv) in [20], there are two orbits of $\mathbf{X}_1(\mathbb{Z}_2)$ under the action of $SL_3(\mathbb{Z}_2)$ with the representatives
$$L_{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad L_1 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3^{-1} \end{pmatrix}$$
where $h(L_{-1}) = -1$ and $h(L_1) = 1$. Therefore
$$\int_{\mathbf{X}_1(\mathbb{Z}_2)} h \, d_2 = \frac{Vol(SL_3(\mathbb{Z}_2))}{Vol(SO(L_1))} - \frac{Vol(SL_3(\mathbb{Z}_2))}{Vol(SO(L_{-1}))}$$
and
$$\int_{\mathbf{X}_1(\mathbb{Z}_2)} d_2 = \frac{Vol(SL_3(\mathbb{Z}_2))}{Vol(SO(L_1))} + \frac{Vol(SL_3(\mathbb{Z}_2))}{Vol(SO(L_{-1}))}.$$
By Lemma 1.8.1 in [3], one has
$$\beta_2(L_1, L_1) = Vol(SO(L_1)) \quad \text{and} \quad \beta_2(L_{-1}, L_{-1}) = Vol(SO(L_{-1}))$$
where $\beta_2(L_1, L_1)$ and $\beta_2(L_{-1}, L_{-1})$ are local densities in sense of §5.6 of Chapter 5 [11]. Since
$$\frac{\beta_2(L_1, L_1)}{\beta_2(L_{-1}, L_{-1})} = \frac{1 + 2^{-1}}{1 - 2^{-1}} = 3$$
by Theorem 5.6.3 in [11], one has
$$c_3(1) = 1 + \frac{1 - \frac{\beta_2(L_1, L_1)}{\beta_2(L_{-1}, L_{-1})}}{1 + \frac{\beta_2(L_1, L_1)}{\beta_2(L_{-1}, L_{-1})}} = \frac{1}{2}.$$

$\square$

References

[1] Y.Benoist and H.Oh, *Effective equidistribution of S-integral points on symmetric varieties*, preprint.

[2] M.Borovoi and C.Demarche, *Manin obstruction to strong approximation for homogeneous spaces*, to appear in Commentarii Mathematici Helvetici.

[3] M.Borovoi and Z.Rudnick, *Hardy-Littlewood varieties and semisimple groups*, Invent. Math. **119** (1995), 37 –66.

[4] J-L.Colliot-Thélène and F.Xu, *Brauer-Manin obstruction for integral points of homogeneous spaces and representations by integral quadratic forms*, Compositio Math. **145** (2009), 309–363.

[5] C. Demarche, *Le défaut d'approximation forte dans les groupes linéaires connexes*, Proc.London Math.Soc. **102** (2011), no. 3, 563–597.

[6] W.Duke, Z.Rudnick, and P.Sanark, *Density of integer points on affine homogeneous varieties*, Duke Math. J. **71** (1993), 143 –179.

[7] A.Eskin and C.McMullen, *Mixing, counting, and equidistribution in Lie groups*, Duke Math. J. **71** (1993), 181 –209.

[8] A.Eskin, C.McMullen, and N.Shah, *Unipotent flows and counting lattice points on homogeneous varieties*, Annals of Math. **143** (1996), 253 –199.

[9] D. Harari, *Le défaut d'approximation forte pour les groupes algébriques commutatifs*, Algebra and Number Theory **2** (2008), no. 5, 595–611.

[10] R. Hartshorne, *Algebraic Geometry*, GTM, vol. 52, Springer-Verlag, 1977.

[11] Y. Kitaoka, *Arithmetic of Quadratic Forms*, Cambridge Tracts in Mathematics, vol. 106, Cambridge University Press, 1993.

[12] M.Kneser, *Darstellungsmasse indefiniter quadratischer Formen*, Math. Z. **77** (1961), 188–194.

[13] M.-A. Knus, A.Merkurjev, M.Rost, and J.-P. Tignol, *The Book of Involutions*, Colloquium Publication, vol. 44, American Mathematical Society, 1998.

[14] R.E.Kottwitz, *Stable trace formula: Cuspidal tempered terms*, Duke Math. J. **51** (1984), no. 3, 611–650.

[15] ———, *Tamagawa numbers*, Annals of Math. **127** (1988), 629 –646.

[16] J. S. Milne, *Etale cohomology*, Princeton Press, 1980.

[17] ———, *Arithmetic duality theorems*, Perspectives in Mathematics, vol. 1, Academic Press, 1986.

[18] J.Neukirch, *Algebraic Number Fields*, Grundlehren, vol. 322, Springer, 1999.

[19] J.Neukirch, A.Schmidt, and K.Wingberg, *Cohomology of Number Fields*, Grundlehren, vol. 323, Springer, 2000.

[20] O.T.O'Meara, *Introduction to Quadratic Forms*, Springer, 1973.

[21] T.Ono, *On the Tamagawa number of algebraic tori*, Ann. of Math. **78** (1963), 47–73.

[22] V. P. Platonov and A. S. Rapinchuk, *Algebraic groups and number theory*, Academic Press, 1994.

[23] V.L.Popov and E.B.Vinberg, *II. Invariant Theory*, Algebraic Geometry IV: Linear Algebraic Groups, Invariant Theory; Edited by A.N.Parshin and I.R.Shafarevich, Springer-Verlag, Berlin Heidelberg, 1994.

[24] J.-J. Sansuc, *Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres*, J.reine angew. Math. **327** (1981), 12–80.

[25] W.M. Schmidt, *The density of integer points on homogeneous varieties*, Acta Math. **154** (1985), no. 3-4, 243–296.

[26] R. Schulze-Pillot, *Darstellungsmasse von Spinorgeschlechtern ternärer quadratischer Formen*, J. reine angew. Math. **352** (1984), 114 –132.

[27] J. P. Serre, *Cohomologie Galoisienne*, Lecture Notes in Mathematics, vol. 5, Springer, Berlin, 1965.

[28] J.M. Shyr, *A generalization of Dirichlet's unit theorem*, J. Number Theory **9** (1977), no. 2, 213–217.

[29] A. N. Skorobogatov, *Torsors and rational points*, Cambridge Tracts in Mathematics, vol. 144, Cambridge University Press, 2001.

[30] V.E.Voskresenskii, *Algebraic groups and their birational invariants*, Translations of Mathematical Monographs, vol. 179, AMS, 1998.

[31] A. Weil, *Sur la theórie des formes quadratiques*, Colloque sur la théorie des groupes algébriques, C.B.R.M. Bruxelles (1962), 9-22.

[32] ———, *Basic Number Theory*, Springer, Berlin-Heidelberg-New York, 1974.

[33] ———, *Adeles and Algebraic Groups*, Birkhäuser, Boston, 1982.

[34] D. Wei and F. Xu, *Integral points for multi-norm tori*, Proc. London Math. Soc. **104** (2012), no. 5, 1019 –1044.

[35] ———, *Integral points for groups of multiplicative type*, Adv. Math. **232** (2013), no. 1, 36–56.

[36] F. Xu, *Representation masses of spinor genera*, Duke Math. J. **110** (2001), no. 2, 279–307.

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