GEOMETRICALLY FORMAL HOMOGENEOUS METRICS OF POSITIVE CURVATURE

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ABSTRACT. A Riemannian manifold is called geometrically formal if the wedge product of harmonic forms is again harmonic, which implies in the compact case that the manifold is topologically formal in the sense of rational homotopy theory. A manifold admitting a Riemannian metric of positive sectional curvature is conjectured to be topologically formal. Nonetheless, we show that among the homogeneous Riemannian metrics of positive sectional curvature a geometrically formal metric is either symmetric, or a metric on a rational homology sphere.

Compact manifolds of positive sectional curvature form an intriguing field of study. On the one hand, there are few known examples, and on the other hand the two main conjectures in the subject, the two Hopf conjectures, are still wide open.

The most basic examples of positive curvature are the rank one symmetric spaces $\mathbb{S}^n$, $\mathbb{CP}^n$, $\mathbb{HP}^n$ and $\mathrm{Ca}\mathbb{P}^2$. Homogeneous spaces of positive curvature have been classified [Be, BB]: there are the homogeneous flag manifolds due to Wallach, $W^6 = \mathrm{SU}(3)/T^2$, $W^{12} = \mathrm{Sp}(3)/\mathrm{Sp}(1)^3$ and $W^{24} = \mathsf{F}_4/\mathrm{Spin}(8)$, the Berger spaces $B^7 = \mathrm{SO}(5)/\mathrm{SO}(3)$ and $B^{13} = \mathrm{SU}(5)/\mathrm{Sp}(2) \cdot \mathrm{S}^1$, and the Aloff–Wallach spaces $W_{p,q}^7 = \mathrm{SU}(3)/\mathrm{diag}(z^p, z^q, \bar{z}^{p+q})$ with $\gcd(p,q) = 1$, $p \geq q > 0$. See e.g. [Zi2] for a detailed discussion. Furthermore, we have the biquotient examples due to Eschenburg [E1, E2] and Bazaikin [Baz] and the more recent cohomogeneity one example in [De, GVZ].

All the known examples have the following remarkable properties: They are rationally elliptic spaces, i.e. their rational homotopy groups $\pi_i(M) \otimes \mathbb{Q}$ vanish from a certain degree i on, and the even dimensional ones have positive Euler characteristic. For general simply-connected positively (or more generally non-negatively) curved manifolds, the Bott-Grove-Halperin conjecture claims rational ellipticity, whilst the Hopf conjecture asserts that their Euler characteristic is positive in even dimensions.

A (simply-connected) topological space is called (topologically) *formal* if its rational homotopy type is a formal consequence of its rational cohomology algebra, or, equivalently in the case of a manifold, if its real cohomology algebra is weakly equivalent to its de Rham algebra. It is a classical result of rational homotopy theory that rationally elliptic spaces with positive Euler characteristic are formal, see e.g. [FHT]. In fact, one easily sees that all known examples of positive curvature are formal, in even as well as in odd dimensions. It is thus natural to conjecture that positively curved manifolds are formal in general.

We mention here that the situation is different in non-negative curvature. Homogeneous spaces G/H naturally admit non-negative curvature and are rationally elliptic. If $\mathrm{rk}\, H =$

rk G they have positive Euler characteristic and are hence formal. On the other hand, in [Am, KT3] one finds many examples of non-formal homogeneous spaces.

Other classical examples of formal spaces are compact symmetric spaces and compact Kähler manifolds. In the case of symmetric spaces this simply follows from the fact that harmonic forms are parallel. Thus in [Ko1] the notion of geometric formality was introduced: A Riemannian metric is *geometrically formal* if wedge products of harmonic forms are again harmonic. On a compact manifold the Hodge decomposition implies that a manifold admitting a geometrically formal metric is also topologically formal. See [Ba] and [Ko2] for some recent results on geometrically formal metrics in dimension 3 and 4, and [Ko1, KT1, KT2, KT3, OP, GN] for obstructions to geometric formality.

There are very few known examples of compact geometrically formal manifolds. In fact, to our knowledge they all belong to the following classes (see [Ko1, Ko2, KT3, Ba])

- a Riemannian metric all of whose harmonic forms are parallel,
- a homogeneous metric on a manifold whose rational cohomology is isomorphic to the cohomology of $\mathbb{S}^p \times \mathbb{S}^q$ with either p and q both odd, or p even and q odd with $p > q$,
- Riemannian products of the above and finite quotients by a group of isometries.

In the homogeneous case geometric formality is an obvious consequence of homogeneity, since harmonic forms must be invariant under the id component of the isometry group. Homogeneous spaces which have the rational cohomology of the product of spheres are classified in [Kr], and in [KT3] it was shown that many of them are not homotopy equivalent to symmetric spaces. There are other metrics where all harmonic forms are parallel, besides the compact symmetric spaces. For example, any metric on a rational homology sphere or a Kähler metric on a rational $\mathbb{CP}^n$, e.g. the twistor space of the quaternionic symmetric space $G_2 / SO(4)$. If one allows the manifold not to be simply connected, there are many such examples, e.g. fake $\mathbb{CP}^2$ and $\mathbb{CP}^4$, see [GN], which are compact quotients of complex hyperbolic space. Although these spaces may be called topologically formal, this property usually has not the strong consequences known from rational homotopy theory unless the space is nilpotent. For quotients of products, as for example $(M \times \mathbb{R}^n)/\Gamma$ with M geometrically formal, one simply observes that geometric formality is a local property.

It is the main result of this article that geometric formality is also rare in positive curvature:

THEOREM. *A homogeneous geometrically formal metric of positive curvature is either symmetric or a metric on a rational homology sphere.*

In [KT3],Theorem 25, it was shown that a metric on a non-trivial $\mathbb{S}^2$ bundle over $\mathbb{CP}^2$ cannot be formal. This includes the 6 dimensional flag manifold W^6, as well as the inhomogenous Eschenburg biquotient. We will show that any homogeneous metric on the other two flag manifolds W^{12} and W^{24} cannot be geometrically formal. Of course, every metric on a sphere is geometrically formal, and every homogeneous metric on $\mathbb{CP}^{2n}$, $\mathbb{HP}^n$ and $\mathrm{Ca}\mathbb{P}^2$ is symmetric. The Berger space B^7 is geometrically formal as well, since it is a rational homology sphere. This leaves the Berger space B^{13}, the Aloff–Wallach spaces, and the homogeneous metrics on $\mathbb{CP}^{2n+1}$. For the Aloff–Wallach spaces, it was shown in [KT3] that

the normal homogeneous metric is not geometrically formal, but this metric does not have positive curvature.

The recent example of positive curvature in [De, GVZ] is a rational homology sphere and hence geometrically formal. It would be interesting to know if the only other known examples of positive curvature, i.e. the 7 dimensional Eschenburg spaces and 13 dimensional Bazaikin spaces, can admit geometrically formal metrics. They have the same cohomology as $W_{p,q}$ and B^{13}, but our methods do not apply in this case since the isometry group is too small.

It would also be interesting to have some other examples of homogeneous spaces where some of the homogeneous metrics are geometrically formal. Although the methods in this paper can be used to check this, an example seems to be difficult to find. Any relationship in the cohomology ring puts strong restrictions on a geometrically formal metric.

To prove the theorem we use the elementary fact that the de Rham cohomology is isomorphic to the finite dimensional algebra of invariant forms, and hence closed and harmonic forms can be computed explicitly. The Berger space B^{13} has the rational cohomology of $\mathbb{CP}^2 \times \mathbb{S}^9$ and the Aloff–Wallach space $W_{p,q}$ that of $\mathbb{S}^2 \times \mathbb{S}^5$. Hence there is a unique harmonic 2-form η and to be geometrically formal implies that η^3 resp. η^2 must be 0 as a form. It turns out that even among the closed invariant forms there are none whose power is 0. For W^{12} and W^{24} there are relations in the cohomology ring that contradict geometric formality. In the case of $\mathbb{CP}^{2n+1}$, the situation is more interesting. Here the condition is that η^k must be harmonic for all k. But already the harmonic 4-form changes with the metric and is the square of the harmonic 2-form only if the metric is symmetric. We point out that this metric is also almost Kähler, hence gives examples of such metrics which are not geometrically formal.

In Section 1 we explain some background about homogeneous spaces and their cohomology. In Section 2 we deal with B^{13} and in Section 3 with the Aloff–Wallach spaces. In Section 4 we discuss W^{12} and W^{24}, and in Section 5 $\mathbb{CP}^{2n+1}$.

1. Preliminaries

We first discuss the methods we will use to prove our main theorem.

Let $M = G/H$ be a homogeneous space with H the stabilizer group at a base point $p_0 \in M$. Using a fixed biinvariant metric Q on the Lie algebra $\mathfrak{g}$, we define an orthogonal splitting

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} \quad \text{with identification} \quad \mathfrak{m} \simeq T_{p_0}M$$

induced by the action fields X^* via $X \in \mathfrak{m} \to X^*(p_0)$. The action of H on $T_{p_0}M$ then becomes the adjoint action Ad_H on $\mathfrak{m}$. Choose an Ad_H invariant and Q orthogonal decomposition

$$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_k$$

such that $\mathrm{Ad}_{H|\mathfrak{m}_0} = \mathrm{Id}$ and $\mathrm{Ad}_{H|\mathfrak{m}_i}$ is irreducible. A metric of the form

$$g = g_{0|\mathfrak{m}_0} + \lambda_1 Q_{|\mathfrak{m}_1} + \lambda_2 Q_{|\mathfrak{m}_2} + \ldots + \lambda_k Q_{|\mathfrak{m}_k}$$

with g_0 an inner product on $\mathfrak{m}_0$ and λ_i positive constants, is then a G invariant metric on M. If the the Ad_H representations $\mathfrak{m}_i$, $i = 1 \ldots k$, are all inequivalent, every G invariant

metric has this form. If $\mathfrak{m}_i \simeq \mathfrak{m}_j$ the inner products between $\mathfrak{m}_i$ and $\mathfrak{m}_j$ can be described by $1, 2$ or 4 arbitrary constants, depending on wether the representations are real, complex, or quaternionic.

We will use the elementary fact that the DeRham cohomology is isomorphic to the cohomology of G invariant forms. By homogeneity this in turn is isomorphic to

$$H^*_{DR}(M) \simeq \left((\Lambda^* \mathfrak{m})^H, d\right)$$

of forms on $\mathfrak{m}$ invariant under the isotropy action. The differential of a k-form $\omega \in (\Lambda^k \mathfrak{m})^H$ is again H invariant and can be computed via the following formula:

$$(1.1) \qquad d\omega(u_1, \ldots, u_{k+1}) = \sum_{i<j}(-1)^{i+j}\omega([u_i, u_j]_\mathfrak{m}, u_1, \ldots, \hat{u}_i, \ldots, \hat{u}_j \ldots u_{k+1})$$

for $u_i \in \mathfrak{m}$, where $[u_i, u_j]_\mathfrak{m}$ denotes the projection of $[u_i, u_j]$ into $\mathfrak{m}$. On $\Lambda^* \mathfrak{m}$ we use the inner product that makes $e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_r}$, $i_1 < i_2 < \ldots < i_r$ into an orthonormal basis of $\Lambda^r \mathfrak{m}$ for any orthonormal basis e_i of $\mathfrak{m}$. We denote the codifferential by δ. Since $\langle \omega, d\eta \rangle = \langle \delta\omega, \eta \rangle$, a G invariant form $\omega \in (\Lambda^r \mathfrak{m})^H$ is harmonic if and only if

$$d\omega = 0 \quad \text{and} \quad \langle \omega, d\eta \rangle = 0 \ \text{ for all } \ \eta \in (\Lambda^{r-1} \mathfrak{m})^H.$$

This reduces the computation of the DeRham cohomology and the harmonic forms to a finite dimensional purely Lie algebraic computation. The equations are in fact linear in the coefficients of ω in some basis, and quadratic in the coefficients of the metric.

In order to simplify the computation of the differentials $d\omega$ we observe the following. Let e_i be a basis of $\mathfrak{g}$ where each basis vector lies either in $\mathfrak{h}$ or $\mathfrak{g}$ and denote, by abuse of notation, the dual basis of 1-forms again by e_i. Although the 1-forms e_i are in general not $\mathrm{Ad}(H)$ invariant, and hence do not represent forms on G/H, we can nevertheless formally use (1.1) to compute de_i. By using the product rule we can then compute $d\omega$ for any r-form $\omega = \sum e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_r}$, in particular the $\mathrm{Ad}(H)$ invariant forms. To see that the formula in (1.1) satisfies the product rule, observe that we could replace $[u_i, u_j]_\mathfrak{m}$ by $[u_i, u_j]$ since the $\mathfrak{h}$ component will evaluate to 0. But then it becomes the usual formula for the Lie algebra cohomology of $\mathfrak{g}$ and hence satisfies a product rule. Notice though that in this generality $d^2\omega$ does not have to be 0, unless ω is H invariant. This is due to the fact that the proof that it vanishes, in the case of the Lie algebra cohomology, uses the Jacobi identity which does not hold if we take the $\mathfrak{m}$ component of all Lie brackets.

2. The Berger space B^{13}

For the 13 dimensional Berger space $B^{13} = \mathrm{SU}(5)/\mathrm{Sp}(2)\cdot\mathrm{S}^1$, the embedding $\mathrm{Sp}(2)\cdot\mathrm{S}^1 \subset \mathrm{SU}(5)$ is given by $\mathrm{diag}(zA, \bar{z}^4)$ for $A \in \mathrm{Sp}(2) \subset \mathrm{SU}(4)$ and $z \in \mathrm{S}^1$. The manifold B^{13} has the same DeRham cohomology as $\mathbb{CP}^2 \times \mathbb{S}^9$. One can see this for example by using the two homogeneous fibrations

$$\mathrm{S}^1 \to \mathrm{SU}(5)/\mathrm{Sp}(2) \to B^{13} \ \text{ and } \ \mathrm{SU}(4)/\mathrm{Sp}(2) \to \mathrm{SU}(5)/\mathrm{Sp}(2) \to \mathrm{SU}(5)/\mathrm{SU}(4)$$

and the fact that $\mathrm{SU}(5)/\mathrm{SU}(4) = \mathbb{S}^9$, $\mathrm{SU}(4)/\mathrm{Sp}(2) = \mathrm{SO}(6)/\mathrm{SO}(5) = \mathbb{S}^5$ and that B^{13} is simply connected. Thus there exists one harmonic 2-form η. Geometric formality requires η^2 to be harmonic, and $\eta^3 = 0$ on the level of forms. We actually do not need to explicitly

compute the harmonic forms since we will show that there are no closed invariant 2-forms ω with $\omega^3 = 0$.

To compute the invariant forms, we first make the following observations. $\mathrm{Sp}(n)$ acts on $\mathbb{H}^n$ via matrix multiplication and $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ via $(A, q)v = Avq^{-1}$ for $A \in \mathrm{Sp}(n), q \in \mathrm{Sp}(1)$ and $v \in \mathbb{H}^n$. It is well known that the algebra $\Lambda^*(\mathbb{H}^n)^{\mathrm{Sp}(n)}$ of invariant forms is generated by the 3 symplectic forms, corresponding to the Kähler forms $\omega_I, \omega_J, \omega_k$ associated to the 3 complex structures coming from right multiplication with $I, J, K \in \mathrm{Sp}(1)$. Right multiplication with $\mathrm{Sp}(1)$ acts on $\mathrm{span}\{I, J, K\} \simeq \mathbb{R}^3$ via matrix multiplication by $\mathrm{SO}(3)$ under the two fold cover $\mathrm{Sp}(1) \to \mathrm{SO}(3)$. Thus if $\mathrm{S}^1 \subset \mathrm{Sp}(1)$ is given by e^{it}, the algebra

(2.1) $\Lambda^*(\mathbb{H}^n)^{\mathrm{Sp}(n) \cdot \mathrm{S}^1}$ is spanned by ω_I and its powers.

From the inclusions $\mathrm{Sp}(2) \cdot \mathrm{S}^1 \subset \mathrm{SU}(4) \cdot \mathrm{S}^1 = \mathrm{U}(4) \subset \mathrm{SU}(5)$ it easily follows that the decomposition of $\mathfrak{m}$ into irreducibles under the action of $H = \mathrm{Sp}(2) \cdot \mathrm{S}^1$ is given by $\mathfrak{m} = V \oplus W$ with $\dim V = 5$ and $\dim W = 8$. On V the factor S^1 acts trivially and $\mathrm{Sp}(2)$ via matrix multiplication by $\mathrm{SO}(5)$ under the two fold cover $\mathrm{Sp}(2) \to \mathrm{SO}(5)$. On W it acts via $(A, z)v = Avz^{-1}$ with $(A, z) \in \mathrm{Sp}(2) \times \mathrm{S}^1$. It follows that $\Lambda^*(V)^H$ is spanned by a 0-form and a 5-form, the volume form v, and $\Lambda^*(W)^H$ by ω_I and its powers by (2.1). On the other hand $\Lambda^k(V) \otimes \Lambda^l(W)$ with $k, l > 0$ contains no invariant forms since the S^1 factor clearly acts non-trivially. Thus $(\Lambda\mathfrak{m})^H$ is spanned by v and ω_I as an algebra. Since there is only one invariant 2-form, ω_I must be harmonic, and similarly ω_I^2 as well. In order to obtain the cohomology ring of B^{13}, we need $dv \neq 0$, but the only possibility, up to a multiple, is $dv = \omega_I^3$. This implies that $\omega_I^3 \neq 0$ and hence no invariant metric can be geometrically formal.

3. THE WALLACH SPACES $W_{p,q}$

Let $H = \mathrm{S}_k^1 = \mathrm{diag}(e^{ik_1 t}, e^{ik_2 t}, e^{ik_3 t}) \subset G = \mathrm{SU}(3)$ where k_i are fixed integers with $\sum k_i = 0$. The quotient $G/H = \mathrm{SU}(3)/\mathrm{S}_k^1$ was studied by Aloff–Wallach [AW] who showed that it admits a homogeneous metric with positive sectional curvature if none of the k_i is 0. We will show that in fact none of the homogeneous metrics, even in this special case, can be geometrically formal. This was shown to be the case for the metric induced by the biinvariant metric on $\mathrm{SU}(3)$, but this metric does not have positive curvature.

It is well known that the rational cohomology ring of $\mathrm{SU}(3)/\mathrm{S}_k^1$ is that of $\mathbb{S}^2 \times \mathbb{S}^5$, but they can be differentiated by a torsion group in H^4. Thus there exists one harmonic 2-form η. To be geometrically formal, we need $\eta^2 = 0$ on the level of forms. As in the previous case, we will again show that there are no closed 2-forms with square 0, although the computation in this case is much more involved.

We choose the following basis for the Lie algebra of $\mathrm{SU}(3)$. To describe it, let E_{ij} be the matrix which has a 1 in row i and column j, and 0 otherwise. Set

$$
\begin{array}{lll}
E_1 = E_{12} - E_{21}, & E_2 = -E_{13} + E_{31}, & E_3 = E_{23} - E_{32} \\
F_1 = iE_{12} + iE_{21}, & F_2 = iE_{13} + iE_{31}, & F_3 = iE_{23} + iE_{32} \\
H_1 = iE_{11} - iE_{22}, & H_2 = -iE_{11} + iE_{33}, & H_3 = iE_{22} - iE_{33}.
\end{array}
$$

We also choose the biinvariant metric on $\mathfrak{su}(3)$ given by $\langle A, B \rangle = -\frac{1}{2}\operatorname{tr}(AB)$ in which E_i, F_i are orthonormal. Furthermore, H_i have unit length, are orthogonal to E_i, F_i, and $\langle H_i, H_j \rangle = -\frac{1}{2}$. For the Lie brackets we have:

$$[H_i, E_i] = 2F_i, \qquad [H_i, F_i] = -2E_i, \qquad [H_i, E_j] = -F_j, \qquad [H_i, F_j] = E_j$$
$$[E_i, E_j] = E_k, \qquad [F_i, F_j] = -E_k, \qquad [E_i, F_j] = -F_k, \qquad [E_i, F_i] = 2H_i$$

where i, j, k is a cyclic permutation of $1, 2, 3$. For the decomposition of $\mathfrak{g}$ we choose

$$\mathfrak{g} = \mathfrak{h} + V_0 + V_1 + V_2 + V_3$$

where

$$V_0 = \operatorname{span}(\varepsilon), \ V_i = \operatorname{span}(E_i, F_i) \text{ for } i = 1, \ldots, 3.$$

Here ε needs to be Q-orthogonal to $\mathfrak{h}$ and of unit length, i.e.

$$\varepsilon = \sum r_i H_i \text{ with } (r_1 - r_2)k_1 + (r_3 - r_1)k_2 + (r_2 - r_3)k_3 = 0 \text{ and } \sum r_i^2 - \sum r_i r_j = 1.$$

The subspaces V_i are invariant under the isotropy action by H. On V_0 it acts trivially and on V_i as:

$$\operatorname{Ad}(\operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}))E_1 = (\theta_1 - \theta_2)F_1, \qquad \operatorname{Ad}(\operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}))F_1 = -(\theta_1 - \theta_2)E_1$$
$$\operatorname{Ad}(\operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}))E_2 = (\theta_3 - \theta_1)F_2, \qquad \operatorname{Ad}(\operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}))F_2 = -(\theta_3 - \theta_1)E_2$$
$$\operatorname{Ad}(\operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}))E_3 = (\theta_2 - \theta_3)F_3, \qquad \operatorname{Ad}(\operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}))F_3 = -(\theta_2 - \theta_3)E_3$$

where $\theta_i = k_i \cdot t$.

For the differential forms we use the basis of 1-forms dual to the basis ε, E_i, F_i and by abuse of notation use the same letters. Using (1.1) and the above Lie brackets one easily obtains the following exterior derivatives of 1-forms:

w	$\mathrm{d}w$
E_1	$E_2 \wedge E_3 - F_2 \wedge F_3 + s_1 F_1 \wedge \varepsilon$
E_2	$E_3 \wedge E_1 - F_3 \wedge F_1 + s_2 F_2 \wedge \varepsilon$
E_3	$E_1 \wedge E_2 - F_1 \wedge F_2 + s_3 F_3 \wedge \varepsilon$
F_1	$-E_2 \wedge F_3 - F_2 \wedge E_3 - s_1 E_1 \wedge \varepsilon$
F_2	$-E_3 \wedge F_1 - F_3 \wedge E_1 - s_2 E_2 \wedge \varepsilon$
F_3	$-E_1 \wedge F_2 - F_1 \wedge E_2 - s_3 E_3 \wedge \varepsilon$
ε	$s_1 E_1 \wedge F_1 + s_2 E_2 \wedge F_2 + s_3 E_3 \wedge F_3$

TABLE 3.1. Differentials of one-forms

where

$$s_i = 2r_i - r_j - r_k \text{ with } i, j, k \text{ distinct.}$$

In this computation we need to use the fact that

$$(H_1)_{\mathfrak{m}} = Q(H_1, \varepsilon)\varepsilon = Q\big(H_1, \sum r_i H_i\big)\varepsilon = (r_1 - \tfrac{1}{2}(r_2 + r_3))$$

and hence $(2H_i)_{\mathfrak{m}} = s_i \varepsilon$.

As explained above, these one-forms are not all well defined on G/H but are useful for computing the exterior derivative of 2-forms via the product formula for forms.

The discussion now depends on the values of the 3 integers k_i and we differentiate between 3 cases.

3.1. **All three k_i are distinct.** Assume that $H = S_k^1 = \mathrm{diag}(e^{ik_1 t}, e^{ik_2 t}, e^{ik_3 t})$ with all k_i distinct. Since the differences $k_i - k_j$ are then also all distinct, the actions of H on V_i are all non-trivial and inequivalent. Hence an invariant metric depends on 4 parameters. The only invariant 1-form is ε, and the only invariant 2-forms are the volume forms of V_i, i.e. $\omega_i = E_i \wedge F_i$. Without having to compute which forms $\omega = \sum a_i \omega_i$ are closed, it is clear that an invariant metric cannot be geometrically formal since $\omega^2 = 0$ implies that $a_i = 0$ for all i.

Remark. One easily sees that the form $\omega = \sum a_i \omega_i$ is closed iff $\sum a_i = 0$ and harmonic if in addition $\sum a_i s_i t_i^2 = 0$, where t_i is the length of E_i and F_i.

3.2. **One of the k_i vanishes.** Here we can assume, since cyclic permutations of the k_i and changing the sign of all 3 does not change the homogeneous space, that $(k_1, k_2, k_3) = (0, -1, 1)$. These are in fact precisely those Wallach spaces which do not admit an invariant metric with positive curvature. Nevertheless we will now show that even here there are no geometrically formal metrics. The action of Ad_H on V_i is a rotation of speed 1 on V_1 and V_2, and speed 2 on V_3. Thus the space of invariant metrics is 6-dimensional. ε is still the only invariant one form, but now we have 5 invariant 2-forms:

$$\omega_i = E_i \wedge F_i, \ i = 1, 2, 3, \ \text{and} \ \omega_4 = E_1 \wedge E_2 + F_1 \wedge F_2, \ \omega_5 = F_1 \wedge E_2 - E_1 \wedge F_2.$$

For ε we choose

$$\varepsilon = (H_1 + 2H_2)/\sqrt{3} = \mathrm{diag}(-i, -i, 2i)/\sqrt{3} \ \text{and hence} \ (s_1, s_2, s_3) = (0, 3, -3).$$

From Table 3.1 we easily obtain the exterior derivatives of the invariant 2-forms:

w	$\mathrm{d}w$
ω_1	$E_1 \wedge E_2 \wedge F_3 - E_1 \wedge E_3 \wedge F_2 + E_2 \wedge E_3 \wedge F_1 - F_1 \wedge F_2 \wedge F_3$
ω_2	$E_1 \wedge E_2 \wedge F_3 - E_1 \wedge E_3 \wedge F_2 + E_2 \wedge E_3 \wedge F_1 - F_1 \wedge F_2 \wedge F_3$
ω_3	$E_1 \wedge E_2 \wedge F_3 - E_1 \wedge E_3 \wedge F_2 + E_2 \wedge E_3 \wedge F_1 - F_1 \wedge F_2 \wedge F_3$
ω_4	$3\,\omega_5 \wedge \varepsilon$
ω_5	$3\,\omega_4 \wedge \varepsilon$

TABLE 3.2. Differentials of 2-forms for $(k_1, k_2, k_3) = (0, -1, 1)$

Thus the 2-form $\omega = \sum a_i \omega_i$ is closed if and only if $\sum a_i = 0$ and $a_4 = a_5 = 0$, and as in the previous case it follows that $\omega^2 = 0$ implies $a_i = 0$ for all i.

3.3. **Two of the k_i are equal.** Up to permutations, we can assume that $(k_1, k_2, k_3) = (-2, 1, 1)$. Thus Ad_H acts with speed 3 on V_1 and V_2, but with opposite orientation, and trivially on V_3 and V_0. The metric is thus arbitrary on $V_0 \oplus V_3$. Since the action on V_1 and V_2 are also equivalent, an invariant metric depends on 10 parameters. Now the invariant 1-forms are ε, E_3 and F_3, and the invariant 2-forms are:

$$\omega_i = E_i \wedge F_i,\ i = 1, 2, 3,\ \omega_4 = E_1 \wedge E_2 - F_1 \wedge F_2,\ \omega_5 = F_1 \wedge E_2 + E_1 \wedge F_2,\ \omega_6 = E_3 \wedge \varepsilon,\ \omega_7 = F_3 \wedge \varepsilon.$$

For ε we choose

$$\varepsilon = H_3 \text{ and hence } (s_1, s_2, s_3) = (-1, -1, 2).$$

The differentials for the invariant 2-forms are:

w	$\mathrm{d}w$
ω_1	$E_1 \wedge E_2 \wedge F_3 - E_1 \wedge E_3 \wedge F_2 + E_2 \wedge E_3 \wedge F_1 - F_1 \wedge F_2 \wedge F_3$
ω_2	$E_1 \wedge E_2 \wedge F_3 - E_1 \wedge E_3 \wedge F_2 + E_2 \wedge E_3 \wedge F_1 - F_1 \wedge F_2 \wedge F_3$
ω_3	$E_1 \wedge E_2 \wedge F_3 - E_1 \wedge E_3 \wedge F_2 + E_2 \wedge E_3 \wedge F_1 - F_1 \wedge F_2 \wedge F_3$
ω_4	$-2(E_1 \wedge F_1 + E_2 \wedge F_2) \wedge F_3 + 2\omega_5 \wedge \varepsilon$
ω_5	$-2(E_1 \wedge F_1 + E_2 \wedge F_2) \wedge E_3 - 2\omega_4 \wedge \varepsilon$
ω_6	$(E_1 \wedge F_1 + E_2 \wedge F_2) \wedge E_3 + \omega_4 \wedge \varepsilon$
ω_7	$(E_1 \wedge F_1 + E_2 \wedge F_2) \wedge F_3 - \omega_5 \wedge \varepsilon$

TABLE 3.3. Differentials of 2-forms for $(k_1, k_2, k_3) = (-2, 1, 1)$

Thus a 2-form $\omega = \sum a_i \omega_i$ is closed if and only if

$$a_1 + a_2 + a_3 = 0,\ -2a_4 + a_7 = 0,\ -2a_5 + a_6 = 0,$$

in other words, $a_1 + a_2 + a_3 = 0, a_7 = 2a_4$ and $a_6 = 2a_5$. This leaves us with a 4-dimensional space of closed forms in degree 2. One easily sees that the square of such a closed form is 0 iff all a_i vanish.

This finishes the proof for the Aloff–Wallach spaces.

4. The flag manifolds W^{12} and W^{24}

The cohomology ring of the the 3 flag manifolds W^6, W^{12} and W^{24} is well know, and can be computed by using Borel's method for the cohomology ring of a homogeneous space G/H, see e.g. [Bo1, Bo2]. In our case this is particularly simple since $\mathrm{rk}\,H = \mathrm{rk}\,G$ and since we can restrict ourselves to real coefficients.

The result is that it is generated by 3 elements $a_1, a_2, a_3 \in H^{2k}(M, R)$, where $k = 1, 2, 4$ for the 3 different flag manifolds. The relationships come from the Weyl group invariant

polynomials, i.e., the symmetric polynomials in a_i vanish. If we choose the generators $x = a_1 + a_2$ and $y = a_1 - a_2$ the cohomology ring is:

$$H^*(M, R) = \{x, y \mid x^3 = 0, y^2 = -3x^2\}$$

with basis x, y in dimension $2k$, as well as x^2, xy in dimension $4k$, and the fundamental class y^3 in dimension $6k$. The two relationships $x^3 = 0$ and $y^2 = -3x^2$ put strong restrictions on a geometrically formal metric. For W^6, the method in [KT3] used the fact that y must be a symplectic form, whereas x has a kernel, contradicting $y^2 = -3x^2$. This proof does not seem to work when $k > 1$. Instead, we restrict ourselves to homogeneous metrics and use the algebra of invariant forms.

For all three flag manifolds G/H we have the splitting

$$\mathfrak{m} = V_1 \oplus V_2 \oplus V_3$$

into Ad_H irreducibles, with $\dim V_i = 2k$. Using representation theory, one easily sees that there are no invariant forms in degree $< 2k$. In degree $2k$ we clearly have the Ad_H invariant volume forms ω_i of the modules V_i. Some differential must be nonzero since $b_{2k} = 2$. For $\mathrm{SU}(3)/\mathrm{T}^2$ and $\mathrm{Sp}(3)/\mathrm{Sp}(1)^3$ we also have inner automorphisms (e.g $\mathrm{Ad}(E_{12} - E_{21})$ for $\mathrm{SU}(3)/\mathrm{T}^2$) which interchange the 3 modules V_i. For $F_4/\mathrm{Spin}(8)$ we have the triality automorphism of $\mathrm{Spin}(8)$. This outer automorphisms of $\mathrm{Spin}(8)$ also extends to inner automorphisms of F_4, see e.g. [WZ], Theorem 3.2, and takes V_1 to V_2, V_2 to V_3, and V_3 to V_1. Thus there exist diffeomorphisms of G/H which interchange the volume forms ω_i, which implies that $d\omega_i \neq 0$ for all i. By rescaling ω_i if necessary we can assume that $\omega = \sum a_i \omega_i$ is closed iff $a_1 + a_2 + a_3 = 0$.

From the description of the forms ω_i it is also clear that $\omega_i^2 = 0$, that $\omega_i \wedge \omega_j$, $i < j$ are linearly independent, and that $vol = \omega_1 \wedge \omega_2 \wedge \omega_3$ is a volume form. Thus $\omega^3 = 6a_1a_2a_3 vol$ is nonzero iff all three a_i are nonzero. Hence x must be one of 3 forms, depending which a_i vanishes. Assume that say $a_3 = 0$ and hence $x = \omega_1 - \omega_2$ up to a multiple. Then $y = \sum a_i \omega_i$ for some nonzero a_i with $\sum a_i = 0$. The $2k$ dimensional classes x and y are the only closed invariant forms and are hence harmonic. But then the relation $y^2 = -3x^2$ in cohomology must also hold on the level of $4k$-forms. Since

$$x^2 = -2\omega_1 \wedge \omega_2 \text{ and } y^2 = 2a_1a_2\omega_1 \wedge \omega_2 + 2a_1a_3\omega_1 \wedge \omega_3 + 2a_2a_3\omega_2 \wedge \omega_3,$$

it follows that $a_1a_3 = a_2a_3 = 0$, which implies that y is a multiple of x. But this is not possible. This finishes the proof for the 3 Wallach flag manifolds.

5. THE COMPLEX PROJECTIVE SPACE $\mathbb{CP}^{2n+1}$

For $\mathbb{CP}^{2n+1} = \mathrm{SU}(2n+2)/S(\mathrm{U}(2n+1)\,\mathrm{U}(1))$ it is well known that the set of homogeneous metrics can be described as follows, see e.g. [Zi1]. First, observe that $\mathrm{Sp}(n+1) \subset \mathrm{SU}(2n+2)$ acts transitively on $\mathbb{CP}^{2n+1}$ with stabilizer $\mathrm{Sp}(n) \cdot \mathrm{S}^1$. From the inclusions $\mathrm{Sp}(n) \cdot \mathrm{S}^1 \subset \mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \subset \mathrm{Sp}(n+1)$ we obtain the twistor fibration $\mathbb{S}^2 \to \mathbb{CP}^{2n+1} \to \mathbb{HP}^n$ and every homogeneous metric is a Riemannian submersion where one scales the metric induced by a biinvariant metric on $\mathrm{Sp}(n)$ with a factor t on the fiber.

Of course, on $\mathbb{CP}^{2n+1}$ we have only one harmonic 2-form α_H, and a metric is geometrically formal if α_H^k are again harmonic for $k = 1, 2, \ldots 2n+1$. We will show that already for α_H^2 this is only the case for the symmetric metric.

We need to explicitly express the invariant form in some basis. We choose the embedding of $\mathrm{Sp}(n)$ in $\mathrm{Sp}(n+1)$ as the upper block embedding, i.e. the stabilizer of the last basis vector in its action on $\mathbb{H}^n$. We first describe the basis of its orthogonal complement. Recall that $E_{i,j}$ is the matrix which has a 1 in row i and column j, and 0 otherwise. Set

$$e_1 = iE_{n+1,n+1}, \quad e_2 = jE_{n+1,n+1}, \quad e_3 = kE_{n+1,n+1}, \quad Y_\alpha = E_{\alpha,n+1} - E_{n+1,\alpha}$$

$$Y_{\alpha,1} = iE_{\alpha,n+1} + iE_{n+1,\alpha}, \quad Y_{\alpha,2} = jE_{\alpha,n+1} + jE_{n+1,\alpha}, \quad Y_{\alpha,3} = kE_{\alpha,n+1} + kE_{n+1,\alpha}$$

where α goes from 1 to n.

Let $H = \mathrm{Sp}(n) \cdot \mathrm{S}^1 \subset \mathrm{Sp}(n+1)$ where $\mathrm{S}^1 = e^{it} \subset \mathrm{Sp}(1)$. Then the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ splits as

$$\mathfrak{m} = V \oplus W = \mathbb{C} \oplus \mathbb{H}^n \text{ with } V = \mathrm{span}(e_2, e_3) \text{ and } W = \mathrm{span}(Y_\alpha, Y_{\alpha,1}, Y_{\alpha,2}, Y_{\alpha,3}), \ \alpha = 1, \ldots n$$

and $\mathrm{diag}(A, e^{it}) \in H$ acts on $\mathfrak{m}$ as $(z, v) \to (e^{2it}z, Ave^{-it})$. H acts irreducibly on V and W and hence the metric depends on 2 parameters. We denote by $\langle \ , \ \rangle_t$ the metric on $\mathfrak{m}$ where e_i have length t and the basis vectors in W have length 1. Extended to a homogeneous metric on G/H, the symmetric metric then corresponds to $t = 1$.

For the $\mathfrak{m}$ component of the Lie brackets of vectors in $\mathfrak{m}$ we have:

$$[e_2, e_3]_\mathfrak{m} = 0 \ , \ [e_i, Y_\alpha]_\mathfrak{m} = -Y_{\alpha,i} \ , \ [e_i, Y_{\alpha,i}]_\mathfrak{m} = Y_\alpha \ , \ [e_i, Y_{\alpha,j}]_\mathfrak{m} = Y_{\alpha,k} \ , \ [Y_\alpha, Y_{\alpha,i}]_\mathfrak{m} = -2e_i$$

$$[Y_{\alpha,i}, Y_{\alpha,j}]_\mathfrak{m} = 2e_k \ , \ [Y_\alpha, Y_\beta]_\mathfrak{m} = [Y_\alpha, Y_{\beta,i}]_\mathfrak{m} = [Y_{\alpha,i}, Y_{\beta,i}]_\mathfrak{m} = [Y_{\alpha,i}, Y_{\beta,j}]_\mathfrak{m} = 0$$

where i, j, k is a cyclic permutation of $1, 2, 3$ and α, β are distinct.

As in the previous case, we first compute the differentials of 1-forms:

w	$\mathrm{d}w$
Y_α	$e_2 \wedge Y_{\alpha,2} + e_3 \wedge Y_{\alpha,3}$
$Y_{\alpha,1}$	$e_2 \wedge Y_{\alpha,3} - e_3 \wedge Y_{\alpha,2}$
$Y_{\alpha,2}$	$-e_2 \wedge Y_\alpha \ + e_3 \wedge Y_{\alpha,1}$
$Y_{\alpha,3}$	$-e_2 \wedge Y_{\alpha,1} - e_3 \wedge Y_\alpha$
e_2	$\sum_\alpha (-2Y_\alpha \wedge Y_{\alpha,2} + 2Y_{\alpha,3} \wedge Y_{\alpha,1})$
e_3	$\sum_\alpha (-2Y_\alpha \wedge Y_{\alpha,3} + 2Y_{\alpha,1} \wedge Y_{\alpha,2}).$

TABLE 5.1. Differentials of one-forms on $\mathbb{CP}^{2n+1}$

We now determine the H invariant forms. Clearly, in $\Lambda^*(V)$ we only have the volume element $v = e_2 \wedge e_3 \in \Lambda^2(V)$. Recall that the algebra $\Lambda^*(\mathbb{H}^n)^{\mathrm{Sp}(n)}$ of invariant forms is generated by the 3 symplectic forms, corresponding to the Kähler forms $\omega_I, \omega_J, \omega_k$ associated

to the 3 complex structures coming from right multiplication with $I, J, K \in \mathrm{Sp}(1)$ on $W = \mathbb{H}^n$. Thus $\Lambda^*(W)^{\mathrm{Sp}(n)}$ is generated by:

$$\omega_I = \sum_\alpha (Y_\alpha \wedge Y_{\alpha,1} - Y_{\alpha,2} \wedge Y_{\alpha,3}) , \omega_J = \sum_\alpha (Y_\alpha \wedge Y_{\alpha,2} - Y_{\alpha,3} \wedge Y_{\alpha,1}), \ \omega_K = \sum_\alpha (Y_\alpha \wedge Y_{\alpha,3} - Y_{\alpha,1} \wedge Y_{\alpha,2})$$

and hence

$$\Lambda^*(\mathfrak{m})^{\mathrm{Sp}(n)} \text{ is generated by } \ e_1, \ e_2, \ \omega_I, \ \omega_J, \ \omega_K.$$

In this algebra we can identify the H invariant forms by determining the action of the circle in H and diagonalizing it via complexification. On e_2, e_3 the circle $e^{it} \in \mathrm{S}^1 \subset H$ acts via a rotation $R(2t)$ since it is given by conjugation. On the two-plane spanned by Y_α, $Y_{\alpha,1}$ one easily checks that it acts via a rotation $R(-t)$ and on the two-plane spanned by $Y_{\alpha,2}$, $Y_{\alpha,3}$ as a rotation $R(t)$. Hence it acts trivially on ω_I, and on the two-plane spanned by ω_J, ω_K it acts as $R(2t)$. This action is diagonal in the basis $e_2 + ie_3, e_2 - ie_3, \omega_I, \omega_J + i\omega_K, \omega_J - i\omega_K$ and acts via $\theta^2 + (\theta^*)^2 + \mathrm{Id} + \theta^2 + (\theta^*)^2$. Thus we obtain invariant forms, besides ω_I, by taking real and imaginary parts of $(e_2 + ie_3) \wedge (e_2 - ie_3)$ and $(\omega_J + i\omega_K) \wedge (\omega_J - i\omega_K)$ as well as $(e_2 + ie_3) \wedge (\omega_J - i\omega_K)$. This gives us the following basis for the invariant forms in low degrees:

$$\Lambda^2(\mathfrak{m})^H = \mathrm{span}(v, \ \omega_I), \text{ where } \ v = e_2 \wedge e_3$$

and

$$\Lambda^3(\mathfrak{m})^H = \mathrm{span}(\beta_1, \beta_2) \text{ where } \ \beta_1 = e_2 \wedge \omega_J + e_3 \wedge \omega_K \text{ and } \ \beta_2 = e_2 \wedge \omega_K - e_3 \wedge \omega_J$$

and the invariant 4-forms

$$\Lambda^4(\mathfrak{m})^H = \mathrm{span}(\omega_I^2, \ v \wedge \omega_I, \ \omega_J^2 + \omega_K^2).$$

Notice that in the above language $de_2 = -2\omega_J$ and $de_3 = -2\omega_K$, and that we have the relations $v \wedge v = v \wedge \beta_1 = v \wedge \beta_2 = 0$. Using Table 5.1 and the product formula, one easily sees that:

$$dv = 2\beta_2, \ d\omega_I = 2\beta_2, \ d\omega_J = 2e_3 \wedge \omega_I, \ d\omega_K = -2e_2 \wedge \omega_I, \ d\beta_1 = -2(\omega_J^2 + \omega_K^2) - 4v \wedge \omega_I.$$

Thus $\alpha_H = v - \omega_I$ is the only closed 2-form, which is hence harmonic.

We now claim that α_H^2 can only be harmonic for the symmetric metric. For this, we compute the differentials of the 4-forms:

$$d\omega_I^2 = 4\omega_I \wedge \beta_2, \ d(v \wedge \omega_I) = 2\omega_I \wedge \beta_2, \ d(\omega_J^2 + \omega_K^2) = -4\omega_I \wedge \beta_2.$$

Thus we have 2 closed 4-forms:

$$\alpha_H^2 = \omega_I^2 - 2v \wedge \omega_I, \text{ and } \omega_J^2 + \omega_K^2 + 2v \wedge \omega_I,$$

and we need to determine which linear combination is harmonic. For this it needs to be orthogonal to the derivative of the invariant 3-forms, which is $d\beta_1$ since $d\beta_2 = \frac{1}{2}d\omega_I^2 = 0$. Thus the 4-form is harmonic iff

$$\langle a(\omega_I^2 - 2v \wedge \omega_I) + b(\omega_J^2 + \omega_K^2 + 2v \wedge \omega_I), (\omega_J^2 + \omega_K^2) + 2v \wedge \omega_I \rangle_t = 0.$$

Observe that the inner products between the 3 symplectic forms are all the same, say equal to L, and that they are orthogonal to $v \wedge \omega_I$. Furthermore, $\langle v \wedge \omega_I, v \wedge \omega_I \rangle = \langle v, v \rangle \cdot \langle \omega_I, \omega_I \rangle = t^2 L$. Thus we need

$$2aL + 4bL - 4bt^2 L = 2L(a + 2b(1 - t^2)) = 0.$$

But this implies that the only value of t where α_H^2 is harmonic is $t = 1$, i.e. the symmetric metric.

This finishes the proof of our main Theorem.

We note that in the terminology from [Na] we proved that a homogeneous metric on $\mathbb{CP}^n$ which is 2-formal, i.e., the product of harmonic 2-forms is again harmonic, is already symmetric.

We remark further that the metrics with positive sectional curvature are described as follows. For B^{13} and $\mathbb{CP}^{2n+1}$ we consider the fibrations $\mathbb{S}^2 \to \mathbb{CP}^{2n+1} \to \mathbb{HP}^n$ and $\mathbb{RP}^5 \to B^{13} \to \mathbb{CP}^4$ and scale the fibers with t. The metric then has positive curvature iff $0 < t < \frac{4}{3}$. For the more complicated description of the homogeneous positively curved metrics on $W_{p,q}$ see [Pü], for the ones on the flag manifolds see [Va], and for the ones on spheres [VZ].

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