SURJECTIVITY OF CERTAIN WORD MAPS ON $\mathrm{PSL}(2,\mathbb{C})$ AND $\mathrm{SL}(2,\mathbb{C})$

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Abstract. Let $n \geq 2$ be an integer and F_n the free group on n generators, $F^{(1)}, F^{(2)}$ its first and second derived subgroups. Let K be an algebraically closed field of characteristic zero. We show that if $w \in F^{(1)} \setminus F^{(2)}$, then the corresponding word map $\mathrm{PSL}(2,K)^n \to \mathrm{PSL}(2,K)$ is surjective. We also describe certain words maps that are surjective on $\mathrm{SL}(2,\mathbb{C})$.

1. Introduction

The surjectivity of word maps on groups became recently a vivid topic: the review on the latest activities may be found in [21], [18], [3], [16].

Let $w \in F_n$ be an element of the free group F_n on $n > 1$ generators $g_1, \ldots, g_n$:

$$w = \prod_{i=1}^{k} g_{n_i}^{m_i},\ 1 \leq n_i \leq n.$$

For a group G by the same letter w we shall denote the corresponding word map $w : G^n \to G$ defined as a non-commutative product by the formula

$$w(x_1, \ldots, x_n) = \prod_{i=1}^{k} x_{n_i}^{m_i}. \tag{1}$$

We call $w(x_1, \ldots, x_n)$ both *a word in n letters* if considered as an element of a free group and *a word map in n letters* if considered as the corresponding map $G^n \to G$.

We assume that it is reduced, i.e. $n_i \neq n_{i+1}$ for every $1 \leq i \leq k-1$ and $m_i \neq 0$ for $1 \leq i \leq k$.

Let K be a field and H a connected semisimple algebraic linear group. If w is not the identity then by Theorem of A Borel ([6]) the regular map of (affine) K-algebraic varieties

$$w : H^n \to H,\ (h_1, \ldots, h_n) \mapsto w(x_1, \ldots, x_n)$$

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is *dominant*, i.e., its image is a Zariski dense subset of H. Let us consider the group $G = H(K)$ and the image

$$w_G := w(G^n) = \{z \in G \mid z = w(x_1, \dots, x_n) \text{ for some } (x_1, \dots, x_n) \in G^n\}.$$

We say that a word (a word map) w is *surjective* on G if $w_G = G$.

In [17], [18] formulated is the following Question.

Problem 7 of [17]**, Question 2.1 (i) of** [18]**.** Assume that w is not a power of another reduced word and $G = H(K)$ a connected semisimple algebraic linear group.

Is w surjective when $K = \mathbb{C}$ is a field of complex numbers and H is of adjoint type?

According to[18], Question 2.1(i) is still open, even in the simplest case $G = PSL(2, \mathbb{C})$, even for words in two letters.

We consider word maps on groups $G = \mathrm{SL}(2, K)$ and $\tilde{G} = \mathrm{PSL}(2, K)$. Put

$$F := F_n, \; F^{(1)} = [F, F], \; F^{(2)} = [F^{(1)}, F^{(1)}].$$

As usual, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ stand for the ring of integers and fields of rational, real and complex numbers respectively. $\mathbb{A}(K)^m_{x_1,\dots,x_m}$ or, simply, $\mathbb{A}^m$, stands for the $n-$dimensional affine space over a field K with coordinates $x_1, \dots, x_m$. If $K = \mathbb{C}$, we use $\mathbb{C}^m_{x_1,\dots,x_m}$.

Let $w \in F$. For a corresponding word map on $G = \mathrm{SL}(2, K)$ we check the following properties of the image w_G.

Properties 1.1.

- **a:** w_G contains all semisimple elements x with $tr(x) \neq 2$;
- **b:** w_G contains all unipotent elements x with $tr(x) = 2$;
- **c:** w_G contains all minus unipotent elements x with $tr(x) = -2$ and $x \neq -id$;
- **d:** w_G contains $-id$.

The word map w is surjective on $G = \mathrm{SL}(2, K)$ if all Properties 1.1 are met. For surjectivity on $\tilde{G} = \mathrm{PSL}(2, K)$ it is sufficient that only Properties 1.1 **a** and **b** are valid.

Definition 1.2. (cf.[2]) We say that the word map w is almost surjective on $G = \mathrm{SL}(2, K)$ if it has Properties 1.1 **a**,**b**, and **c**, i.e $w_G \supset \mathrm{SL}(2, K) \setminus -\{id\}$.

The goal of the paper is to describe certain words $w \in F$ such that the corresponding word maps are surjective or almost surjective on G and/or $\tilde{G}$.

Assume that the field K is algebraically closed. If $w(x_1, \dots, x_d) = x_i^n$ then w is surjective on G if and only if n is odd (see ([10], [11]). Indeed, the element

$$x = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

is not a square in SL$(2, K)$. Since only the elements with $tr(x) = -2$ may be outside w_G ([10], [11]), the induced by w word map $\tilde{w}$ is surjective on $\tilde{G}$.

Consider a word map (1). For an index $j \leq n$ let $S_j = \sum_{n_i=j} m_i$.

If, say, $S_1 \neq 0$, then $w(x_1, id, \ldots, id) = x_1^{S_1}$, hence word w is surjective on PSL$(2, K)$. If $S_j = 0$ for all $1 \leq j \leq d$, then $w \in F^{(1)} = [F, F]$. In Section 5 we prove (see Corollary 5.4) the following

Theorem 1.3. *The word map defined by a word $w \in F^{(1)} \setminus F^{(2)}$ is surjective on* PSL$(2, K)$ *if K is an algebraically closed field with $char(K) = 0$.*

The proof makes use of a variation on the Magnus Embedding Theorem, which is stated in Section 3 and proven in Section 4.

In Section 6, Section 7, and Section 8 we consider words in two variables, i.e. $n = 2$. In this case we give explicit formulas for $w(x, y)$, where $x, y \in$ SL$(2, \mathbb{C})$ are upper triangular matrices. Using explicit formulas in Section 7 and Section 8 we provide criteria for surjectivity and almost surjectivity of a word map on $G =$ SL$(2, \mathbb{C})$. In Section 7 these criteria are formulated in terms of properties of exponents $a_i, b_i,\ i = 1 \ldots, k$, of a word

$$w(x, y) = \prod_{i=1}^{k} x^{a_i} y^{b_i}, \tag{2}$$

where $a_i \neq 0$ and $b_i \neq 0$, for all $i = 1, ..., k$. A sample of such criteria is

Corollary 1.4. *If all b_i are positive, then the word map w is either surjective or the square of another word $v \neq id$.*

In Section 8 we connect the almost surjectivity of a word map with a property of the corresponding trace map. The last sections contain explicit examples.

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2. Semisimple elements

Let K be an algebraically closed field with $char(K)=0$, and $G=\mathrm{SL}(2,K)$. Consider a word map $w: G^n\to G$:

$$w(x_1,\dots,x_n)=\prod_{i=1}^{k} x_{n_i}^{m_i}.$$

We consider G as an affine set

$$G=\{ad-bc=1\}\subset \mathbb{A}^4_{a,b,c,d}.$$

The following Lemma is , may be , known, but the authors do not have a proper reference.

Lemma 2.1. *A regular non-constant function on G^n omits no values in K.*

Proof. Since all the sets are affine, a function f regular on G^k is a restriction of a polynomial P_f onto G^k. We use induction on k.

Step 1. $k=1$.

$$G=\{ad-bc=1\}\subset \mathbb{A}^4_{a,b,c,d}$$

is an irreducible quadric. Assume that $f\in K[G]$ omits a value. Let $p: G\to \mathbb{A}^1_a$ be a projection defined by $p(a,b,c,d)=a$. If $a\neq 0$ then fiber $F_a:=p^{-1}(a)\cong \mathbb{A}^2_{b,c}$, is an affine space with coordinates b,c because $d=\frac{1+bc}{a}$. Since f omits a value, the restriction $f\Big|_{F_a}$ is constant for every $a\neq 0$. 'Therefore it is constant on every fiber (note that the fiber $a=0$ is conneceted). On the other hand, f has to be constant along the curve

$$C=\{(a,0,1,1)\}\cong \mathbb{A}^1_a(K).$$

Since curve $C\subset G$ intersects every fiber F_a of projection p, function f is constant on G.

Step 2. Assume that the statement of the Lemma is valid for all $k\le n$. Let $f\in K[G^n]$ omit a value. We have: $G^n=M\times N$, where $M=G^{n-1}$ and $N=G$. Let $p: G^n\to N$ be a natural projection. Then, by induction assumption, f is constant along every fiber of this projection. Take $x\in M$ and consider the set $C=x\times N\subset G^n$. Then$f\Big|_C=const$ and C intersects every fiber of p. Hence, f is constant. ∎

Proposition 2.2. *For every word $w(x_1,\dots,x_k)\neq id$ the image w_G contains every element $z\in G$ with $a:=tr(z)\neq\pm 2$.*

Proof. We consider $G^n \subset \mathbb{A}(K)^{4n}$ as the product $(1 \leq i \leq n)$ of

$$G_i = \{a_i d_i - b_i c_i = 1\} \subset \mathbb{A}^4_{a_i, b_i, ic_i, d_i}.$$

The function $f(a_1, b_1, c_1, d_1, \dots, a_n, b_n, c_n, d_n) = tr(w(x_1, \dots, x_n))$ is a polynomial in $4n$ variables with integer coefficients, i.e $f \in K[G^n]$. According to Lemma 2.1, it takes on all the values in K.

Thus for every value $A \in K$ there is element $u = w(y_1, \dots, y_n) \in w_G$ such that $tr(u) = A$.

Let now $z \in G, \ A := tr(z) \neq \pm 2$. Since $tr(z) = tr(u)$, z is conjugate to u, i.e there is $v \in G$ such that $vuv^{-1} = z$. Hence

$$z = w(vy_1v^{-1}, \dots, vy_nv^{-1}).$$

□

It follows that in order to check whether the word map w is surjective on G (or on $\tilde{G}$) it is sufficient to check whether the elements z with $tr(z) = \pm 2$ (or the elements z with $tr(z) = 2$, respectively) are in the image. For that we need a version of the Embedding Magnus Theorem.

3. Variation on Magnus Embedding Theorem: Statements

Let $n \geq 2$ be an integer and $\Lambda_n = \mathbb{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$ be the ring of Laurent polynomials in n independent variables $t_1, \dots, t_n$ over $\mathbb{Z}$. Let $F = F_n$ be a free group of rank n with generators $\{g_1, \dots, g_n\}$. Recall: we write $F^{(1)}$ for the derived subgroup of F and $F^{(2)}$ for the derived subgroup of $F^{(1)}$. We have

$$F^{(2)} \subset F^{(1)} \subset F;$$

both $F^{(1)}$ and $F^{(2)}$ are normal subgroups in F. The quotient $A := F/F^{(1)} = \mathbb{Z}^n$ is a free abelian group of rank n with (standard) generators $\{e_1, \dots, e_n\}$ where each e_i is the image of g_i $(1 \leq i \leq n)$. It is well known that the group ring $\mathbb{Z}[A]$ of A is canonically isomorphic to Λ_n: under this isomorphism each

$$e_i \in A \subset \mathbb{Z}[A]$$

goes to

$$t_i \in \mathbb{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}] = \Lambda_n.$$

We write R_n for the ring of polynomials

$$\Lambda_n[s_1, \dots, s_n] = \mathbb{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}; s_1, \dots, s_n]$$

in n independent variables $s_1, \dots, s_n$ over Λ_n. If R is a commutative ring with 1 then we write $T(R)$ for the group of invertible 2×2 matrices of the form

$$\begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix}$$

with $a \in R^*, b \in R$ and $ST(R)$ for the group of unimodular 2×2 matrices of the form

$$\begin{bmatrix} a & 0 \\ b & a^{-1} \end{bmatrix}$$

with $a \in R^*, b \in R$. We have

$$T(R) \subset \mathrm{GL}(2,R),\ ST(R) \subset \mathrm{SL}(2,R).$$

Every homomorphism $R \to R'$ of commutative rings (with 1) induces the natural group homomorphisms

$$T(R) \to T(R'),\ ST(R) \to ST(R'),$$

which are injective if $R \to R'$ is injective.

The following assertion (that is based on the properties of the famous Magnus embedding [19]) was proven in [25, Lemma 2].

Theorem 3.1. *The assignment*

$$g_i \mapsto \begin{bmatrix} t_i & 0 \\ s_i & t_i^{-1} \end{bmatrix}\ (1 \le i \le n)$$

extends to a group homomorphism

$$\mu_W : F \to ST(\Lambda_n)$$

with kernel $F^{(2)}$ and therefore defines an embedding

$$F/F^{(2)} \hookrightarrow ST(R_n) \subset \mathrm{SL}(2, R_n).$$

It follows from Theorem 3.1 that if K is a field of characteristic zero, whose transcendence degree over $\mathbb{Q}$ is, at least, $2n$ then there is an embedding

$$F/F^{(2)} \hookrightarrow ST(K) \subset \mathrm{SL}(2, K).$$

(In particular, it works for $K = \mathbb{R}$, $\mathbb{C}$ or the field $\mathbb{Q}_p$ of p-adic numbers [25].) The aim of the following considerations is to replace in this statement the lower bound $2n$ by n.

Theorem 3.2. *The assignment*

$$g_i \mapsto \begin{bmatrix} t_i & 0 \\ 1 & t_i^{-1} \end{bmatrix}\ (1 \le i \le n)$$

extends to a group homomorphism

$$\mu_1 : F \to ST(\Lambda_n)$$

with kernel $F^{(2)}$ and therefore defines an embedding

$$F/F^{(2)} \hookrightarrow ST(\Lambda_n) \subset \mathrm{SL}(2, \Lambda_n).$$

Remark 3.3. Let

$$\mathrm{ev}_1 : R_n = \Lambda_n[s_1, \dots, s_n] \to \Lambda_n$$

be the Λ_n-algebra homomorphism that sends all s_i to 1 and let

$$\mathrm{ev}_1{}^* : ST(R_n) \to ST(\Lambda_n)$$

be the group homomorphism induced by ev_1. Then μ_1 coincides with the composition

$$\mathrm{ev_1}^*\mu_W : F \to ST(R_n) \to ST(\Lambda_n).$$

Corollary 3.4. *Let K be a field of characteristic zero. Suppose that the transcendence degree of K over $\mathbb{Q}$ is, at least, n. Then there is a group embedding*

$$F/F^{(2)} \hookrightarrow ST(K) \subset \mathrm{SL}(2, K).$$

Proof of Theorem 3.2 is based on the following observation.

Lemma 3.5. *Let K be a field of characteristic zero. Suppose that the transcendence degree of K over $\mathbb{Q}$ is, at least, n and let $\{u_1, \dots, u_n\} \subset K$ be an n-tuple of algebraically independent elements (over $\mathbb{Q}$). Let $\mathbb{Q}(u_1, \dots, u_n)$ be the subfield of K generated by $\{u_1, \dots, u_n\}$ and let us consider K as the $\mathbb{Q}(u_1, \dots, u_n)$-vector space. Let $\{y_1, \dots, y_n\} \subset K$ be a n-tuple that is linearly independent over $\mathbb{Q}(u_1, \dots, u_n)$. Let R be the subring of K generated by $3n$ elements $u_1, u_1^{-1}, \dots, u_n, u_n^{-1}; y_1, \dots, y_n$.*

Then the assignment

$$g_i \mapsto \begin{bmatrix} u_i & 0 \\ y_i & 1 \end{bmatrix} \ (1 \le i \le n) \in T(R)$$

extends to a group homomorphism

$$\mu : F \to T(R) \subset T(K)$$

with kernel $F^{(2)}$ and therefore defines an embedding

$$F/F^{(2)} \hookrightarrow T(R) \subset T(K).$$

Example 3.6. Let K be the field $\mathbb{Q}(t_1, \dots, t_n)$ of rational functions in n independent variables $t_1, \dots, t_n$ over $\mathbb{Q}$. One may view Λ_n as the subring of K generated by $2n$ elements $t_1, t_1^{-1}, \dots, t_n, t_n^{-1}$. By definition, the n-tuple $\{t_1, \dots, t_n\} \subset K$ is algebraically independent (over $\mathbb{Q}$). Clearly, the n-tuple

$$\{u_1 = t_1^2, \dots, u_i = t_i^2, \dots, u_n = t_n^2\} \subset K$$

is also algebraically independent. Then the n elements

$$y_1 = t_1, \dots, y_i = t_i, \dots, y_n = t_n$$

are linearly independent over the (sub)fileld $\mathbb{Q}(t_1^2, \dots, t_n^2) = \mathbb{Q}(u_1, \dots, u_n)$. Indeed, if a rational function

$$f(t_1, \dots, t_n) = \sum_{i=1}^{n} t_i \cdot f_i$$

where all $f_i \in \mathbb{Q}(t_1^2, \dots, t_n^2)$ then

$$2t_1 f_1 = f(t_1, t_2, \dots, t_n) - f(-t_1, t_2, \dots, t_n), \dots,$$

$$2t_i f_i = f(t_1, \dots, t_i, \dots, t_n) - f(t_1, \dots, -t_i, \dots, t_n), \dots,$$

$$2t_nf_n = f(t_1, \dots, t_i, \dots, t_n) - f(t_1, \dots, t_i, \dots, -t_n).$$

This proves that if $f = 0$ then all f_i are also zero, i.e., the set $\{t_1, \dots, t_n\}$ is linearly independent over $\mathbb{Q}(t_1^2, \dots t_n^2)$.

By definition, R coincides with the subring of K generated by $3n$ elements

$$t_1^2, t_1^{-2}, \dots, t_n^2, t_n^{-2}; t_1, \dots, t_n.$$

This implies easily that $R = \Lambda_n$. Applying Lemma 3.5, we conclude the Example by the following statement.

The assignment

$$g_i \mapsto \begin{bmatrix} t_i^2 & 0 \\ t_i & 1 \end{bmatrix} \ (1 \le i \le n) \in T(\Lambda_n)$$

extends to a group homomorphism

$$\mu : F \to T(R) = T(\Lambda_n)$$

with kernel $F^{(2)}$ *and therefore defines an embedding*

$$F/F^{(2)} \hookrightarrow T(\Lambda_n).$$

We prove Lemma 3.5, Theorem 3.2 and Corollary 3.4 in Section 4.

4. Variation on the Magnus Embedding Theorem: Proofs

Proof of Lemma 3.5. Let

$$\Lambda \subset \mathbb{Q}(u_1, \dots, u_n) \subset K$$

be the subring generated by $2n$ elements $u_1, u_1^{-1}, \dots, u_n, u_n^{-1}$. Since u_i are algebraically independent over K, the assignment

$$t_i \mapsto u_i, \ t_i^{-1} \mapsto u_i^{-1}$$

extends to a ring isomorphism $\Lambda_n \cong \Lambda$. The linear independence of y_i's over $\mathbb{Q}(u_1, \dots, u_n)$ implies that $M = \Lambda \cdot y_1 + \dots + \Lambda \cdot y_n \subset R \subset K$ is a free Λ-module of rank n. On the other hand, let

$$U \subset \Lambda^* \subset \mathbb{Q}(u_1, \dots, u_n)^* \subset K^*$$

be the multiplicative (sub)group generated by all u_i. The assignment $g_i \mapsto u_i$ extends to the surjective group homomorphism

$$\delta : F \twoheadrightarrow U$$

with kernel $F^{(1)}$ and gives rise to the group isomorphism

$$A \cong U,$$

which sends e_i to u_i and allows us to identify the group ring $\mathbb{Z}[U]$ of U with $\Lambda \cong \Lambda_n = \mathbb{Z}[A]$. Notice that M carries the natural structure of free $\mathbb{Z}[U]$-module of rank n defined by

$$\lambda(m) := \lambda \cdot m \in K \ \forall \lambda \in \mathbb{Z}[U] = \Lambda \subset K, m \in M \subset K.$$

We have

$$\mu(F) \subset \begin{bmatrix} U & 0 \\ M & 1 \end{bmatrix} \subset T(R) \subset \mathrm{GL}_2(R).$$

It follows from [26, Lemma 1(c) on p. 175] that $\ker(\mu)$ coincides with the derived subgroup of $\ker(\delta)$. Since $\ker(\delta) = F^{(1)}$, we conclude that $\ker(\mu) = F^{(2)}$ and we are done. □

Proof of Theorem 3.2. Let us return to the situation of Example 3.6. In particular, the group embedding (we know that it is an embedding, thanks to already proven Lemma 3.5)

$$\mu : F \hookrightarrow T(\Lambda_n) \subset \mathrm{GL}_2(\Lambda_n)$$

is defined by

$$\mu(g_i) = \begin{bmatrix} t_i^2 & 0 \\ t_i & 1 \end{bmatrix} \in T(\Lambda_n)$$

for all g_i.

Let us consider the group homomorphism

$$\kappa : F \to \Lambda_n^*, \ g_i \mapsto t_i.$$

Since t_i are algebraically independent, they are multiplicatively independent and

$$\ker(\kappa) = F^{(1)}.$$

I claim that $\mu_1 : F \to ST(\Lambda_n)$ coincides with the group homomorpism

$$g \mapsto \kappa(g)^{-1} \cdot \mu(g).$$

Indeed, we have for all g_i

$$\kappa(g_i)^{-1} \cdot \mu(g_i) = t_i^{-1} \cdot \begin{bmatrix} t_i^2 & 0 \\ t_i & 1 \end{bmatrix} = \begin{bmatrix} t_i & 0 \\ 1 & t_i^{-1} \end{bmatrix} = \mu_1(g_i) \subset ST(\Lambda_n),$$

which proves our claim. Recall that we need to check that $\ker(\mu_1) = F^{(2)}$. In order to do that, first notice that $\mu_1(g)$ is of the form $\begin{bmatrix} \kappa(g) & 0 \\ * & \kappa(g)^{-1} \end{bmatrix}$ for all $g \in F$ just because it is true for all $g = g_i$. This implies that

$$\ker(\mu_1) \subset \ker(\kappa) = F^{(1)}.$$

But $\mu = \mu_1$ on $F^{(1)}$. This implies that

$$\ker(\mu_1) = \ker(\mu) \bigcap F^{(1)}.$$

However, as we have seen in Example 3.6,

$$\ker(\mu) = F^{(2)} \subset F^{(1)}.$$

This implies that

$$\ker(\mu_1) = F^{(2)} \bigcap F^{(1)} = F^{(2)}$$

and we are done. □

Proof of Corollary 3.4. There exists an n-tuple $\{x_1, \dots, x_n\} \subset K$ that is algebraically independent over $\mathbb{Q}$. The assignment

$$t_i \mapsto x_i, \ t_i^{-1} \mapsto x_i^{-1}$$

extends to an *injective* ring homomorphism

$$\Lambda_n = \mathbb{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}] \hookrightarrow K.$$

This implies that $ST(\Lambda_n)$ is isomorphic to a subgroup of $ST(K)$. Thanks to Theorem 3.2, $F/F^{(2)}$ is isomorphic to a subgroup of $ST(\Lambda_n)$. This implies that $F/F^{(2)}$ is isomorphic to a subgroup of $ST(K)$. □

Remark. Similar arguments prove the following generalization of Theorem 3.2.

Theorem 4.1. *Let $\{b_1, \dots, b_n\}$ be an n-tuple of nonzero integers. Then the assignment*

$$g_i \mapsto \begin{bmatrix} t_i & 0 \\ b_i & t_i^{-1} \end{bmatrix} \ (1 \le i \le n)$$

extends to a group homomorphism $F \to ST(\Lambda_n)$ with kernel $F^{(2)}$.

5. Word maps and unipotent elements

Lemma 5.1. *Let w be an element of $F^{(1)}$ that does not belong to $F^{(2)}$. Then there exists a nonzero Laurent polynomial*

$$\mathcal{L}_w = \mathcal{L}_w(t_1, \dots t_n) \in \mathbb{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}] = \Lambda_n$$

such that

$$\mu_1(w) = \begin{bmatrix} 1 & 0 \\ \mathcal{L}_w & 1 \end{bmatrix}.$$

Proof. We have seen in the course of the proof of Theorem 3.2 that for all $g \in F$

$$\mu_1(g) = \begin{bmatrix} \kappa(g) & 0 \\ * & \kappa(g)^{-1} \end{bmatrix} \in ST(\Lambda_n).$$

This means that there exists a Laurent polynomial $\mathcal{L}_g \in \Lambda_n$ such that

$$\mu_1(g) = \begin{bmatrix} \kappa(g) & 0 \\ \mathcal{L}_g & \kappa(g)^{-1} \end{bmatrix}.$$

We have also seen that if $g \in F^{(1)}$ then $\kappa(g) = 1$. Since $w \in F^{(1)}$,

$$\mu_1(w) = \begin{bmatrix} 1 & 0 \\ \mathcal{L}_w & 1 \end{bmatrix}$$

with $\mathcal{L}_w \in \Lambda_n$. On the other hand, by Theorem 3.2, $\ker(\mu_1) = F^{(2)}$. Since $w \notin F^{(2)}$, $\mathcal{L}_w \neq 0$ in Λ_n. □

Corollary 5.2. *Let w be an element of $F^{(1)}$ that does not belong to $F^{(2)}$. Suppose that $\mathbf{a} = \{a_1, \dots, a_n\}$ is an n-tuple of nonzero rational numbers such that*

$$c := \mathcal{L}_w(a_1, \dots, a_n) \neq 0.$$

(Since $\mathcal{L}_w \neq 0$, such an n-tuple always exists.) Let us consider the group homomorphism

$$\gamma_{\mathbf{a}} : F \to ST(\mathbb{Q}) \subset \mathrm{SL}(2, \mathbb{Q}),\ g_i \mapsto \begin{bmatrix} a_i & 0 \\ 1 & a_i^{-1} \end{bmatrix} := Z_i.$$

Then

$$\gamma_{\mathbf{a}}(w) = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} = w(Z_1, \dots, Z_n).$$

is a unipotent matrix that is not the identity matrix.

Proof. One has only to notice that $\gamma_{\mathbf{a}}$ is the composition of μ_1 and the homomorphism $ST(\Lambda_n) \to ST(\mathbb{Q})$ induced by the ring homomorphism

$$\Lambda_n \to \mathbb{Q},\ t_i \mapsto a_i, t_i^{-1} \mapsto a_i^{-1}.$$

□

Corollary 5.3. *Let w be an element of $F^{(1)}$ that does not belong to $F^{(2)}$. Let K be a field of characteristic zero. Then for every unipotent matrix $X \in \mathrm{SL}(2, K)$ there exists a group homomorphism $\psi_{w,X} : F \to \mathrm{SL}(2, K)$ such that*

$$\psi_{w,X}(w) = X.$$

In other words, there exist $Z_1, \dots, Z_n \in \mathrm{SL}(2, K)$ such that $w(Z_1, \dots, Z_n) = X$.

Proof. We have

$$\mathbb{Q} \subset K,\ \mathrm{SL}(2, \mathbb{Q}) \subset \mathrm{SL}(2, K) \lhd \mathrm{GL}(2, K).$$

We may assume that X is *not* the identity matrix. Let $\mathbf{a} = \{a_1, \dots, a_n\}$ and $\gamma_{\mathbf{a}}$ be as in Corollary 5.2. Recall that $c = \mathcal{L}_w(a_1, \dots, a_n) \neq 0$. Then there exists a matrix $S \in \mathrm{GL}(2, K)$ such that

$$X = S \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} S^{-1}.$$

Let us consider the group homomorphism

$$\psi_{w,X} : F \to \mathrm{SL}(2, K),\ g \mapsto S\gamma_a(g)S^{-1}.$$

Then $\psi_{w,X}$ sends w to

$$S\gamma_{\mathbf{a}}(w)S^{-1} = S \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} S^{-1} = X. \tag{3}$$

□

Corollary 5.4. *(Theorem 1.3) Let w be an element of $F^{(1)}$ that does not belong to $F^{(2)}$. Let K be an algebraically closed field of characteristic zero. Then the word map w is surjective on* $\mathrm{PSL}(2,K)$.

Proof. Consider w as a word map on $G = \mathrm{SL}(2,K)$. Due to Corollary 5.3 the image w_G contains all unipotents. According to Proposition 2.2 the image contains all the semisimple elements as well. Thus, the word map w has the Properties 1.1 **a** and **b**. It follows that it is surjective on $\mathrm{PSL}(2,K)$. □

Remark 5.5. In [12] the words from $F^{(1)} \setminus F^{(2)}$ are proved to be surjective on $SU(n)$ for an infinite set of integers n.

Theorem 5.6. *Let w be an element of $F^{(1)}$ that does not belong to $F^{(2)}$. Let G be a connected semisimple linear algebraic group over a field K of characteristic zero. If $u \in G(K)$ is a unipotent element then there exists a group homomorphism $F \to G(K)$ such that the image of w coincides with u. In other words, there exist $Z_1, \ldots, Z_n \in G(K)$ such that $w(Z_1, \ldots, Z_n) = u$.*

Proof. Let $\mathbf{a} = \{a_1, \ldots, a_n\}$, $\gamma_{\mathbf{a}}$ and $c = \mathcal{L}_w(a_1, \ldots, a_n) \neq 0$ be as in Corollary 5.2. By Lemma 5.7 below, there exists a group homomorphism $\phi : ST(K) \to G(K)$ such that $u = \phi(\mathbf{u}_1)$ for

$$\mathbf{u}_1 = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \in ST(K).$$

Now the result follows from Corollary 5.2: the desired homomorphism is the composition

$$\phi\, \gamma_{\mathbf{a}} : F \to ST(K) \to G(K).$$

□

Lemma 5.7. *Let K be a field of characteristic zero, G a connected semisimple linear algebraic K-group of positive dimension, and u a unipotent element of $G(K)$. Then for every nonzero $c \in K$ there is a group homomorphism $\phi : ST(K) \to G(K)$ such that u is the image of*

$$\mathbf{u}_1 = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \in ST(K).$$

Proof. Let us identify the additive algebraic K-group $\mathbb{G}_a$ with the closed subgroup H of all matrices of the form $v(t) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$ in $\mathrm{SL}(2)$. Its Lie subalgebra $\mathrm{Lie}(H)$ is the one-dimensional K-vector subspace $\mathrm{Lie}(H) = \{\lambda \mathbf{x}_0 \mid \lambda \in K\}$ of $\mathfrak{sl}_2(K)$ generated by the matrix

$$\mathbf{x}_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \subset \mathfrak{sl}_2(K).$$

Here we view the K-Lie algebra $\mathfrak{sl}_2(K)$ of 2×2 traceless matrices as the Lie algebra of the algebraic K-group SL(2). Moreover, $\exp(\lambda \mathbf{x}_0) = v(\lambda)$ for all $\lambda \in K$.

We may view G as a closed algebraic K-subgroup of the matrix group $\mathrm{GL}(N) = \mathrm{GL}(V)$, where V is an $N-$dimensional K-vector space for a suitable positive integer N. Then

$$u \in G(K) \subset \mathrm{Aut}_K(V) = \mathrm{GL}(N, K).$$

Thus the K-Lie algebra $\mathrm{Lie}(G)$ becomes a certain *semisimple* Lie subalgebra of $\mathrm{End}_K(V)$. Here we view $\mathrm{End}_K(V)$ as the Lie algebra $\mathrm{Lie}(\mathrm{GL}(V))$ of the K-algebraic group $\mathrm{GL}(V)$. As usual, we write

$$\mathrm{Ad} : G(K) \to \mathrm{Aut}_K(\mathrm{Lie}(G))$$

for the adjoint action of G. We have

$$\mathrm{Ad}(g)(u) = gug^{-1}$$

for all

$$g \in G(K) \subset \mathrm{Aut}_K(V) \text{ and } u \in \mathrm{Lie}(G) \subset \mathrm{End}_K(V).$$

Since u is a unipotent element, the linear operator $u - 1 : V \to V$ is a nilpotent. Let us consider the nilpotent linear operator

$$x = \log(u) := \sum_{i=1}^{\infty} (-1)^{i+1} \frac{(u-1)^i}{i} \in \mathrm{End}_K(V)$$

([7, Sect 7, p. 106], [23, Sect.23, p. 336]) and the corresponding homomorphism of algebraic K-groups

$$\varphi_u : H \to \mathrm{GL}(V),\ v(t) \mapsto \exp(tx) = v(0) + tx + \dots .$$

In particular, since $\mathbf{u}_1 = v(1)$,

$$\varphi_u(\mathbf{u}_1) = u.$$

Clearly, the differential of φ_u

$$d\varphi_u : \mathrm{Lie}(H) \to \mathrm{Lie}(\mathrm{GL}(V)) = \mathrm{End}_K(V)$$

is defined as

$$d\varphi_u(\lambda \mathbf{x}_0) = \lambda x \ \forall \lambda \in K$$

and sends $\mathbf{x}_0$ to $x \in \mathrm{Lie}(\mathrm{GL}(V))$. Since $\varphi_u(m) = u^m \in G(K)$ for all integers m and G is closed in $\mathrm{GL}(V)$ in Zariski topology, the image $\varphi_u(H)$ of H lies in G and therefore one may view φ_u as a homomorphism of algebraic K-groups

$$\varphi_u : H \to G.$$

This implies that

$$d\varphi_u(\mathrm{Lie}(H)) \subset \mathrm{Lie}(G);$$

in particular, $x \in \mathrm{Lie}(G)$.

There exists a *cocharacter*

$$\Phi : \mathbb{G}_m \to G \subset \mathrm{GL}(V)$$

of K-algebraic group G such that for each $\beta \in K^* = \mathbb{G}_m(K)$

$$\mathrm{Ad}(\Phi(\beta))(x) = \beta^2 x$$

(see [20, Sect. 6, pp. 402–403]. Here $\mathbb{G}_m$ is the multiplicative algebraic K-group.) This means that for all $\lambda \in K$

$$\Phi(\beta)\lambda x\Phi(\beta)^{-1} = \mathrm{Ad}(\Phi(\beta))(\lambda x) = \lambda\beta^2 x = \beta^2\lambda x \in \mathrm{Lie}(G) \subset \mathrm{End}_K(V),$$

which implies that

$$\Phi(\beta)(\exp(\lambda x))\Phi(\beta)^{-1} = \exp\left(\Phi(\beta)\lambda x\Phi(\beta)^{-1}\right) = \exp(\beta^2\lambda x).$$

It follows that

$$\Phi(\beta)\left(\exp\left(\frac{\lambda}{c}x\right)\right)\Phi(\beta)^{-1} = \exp\left(\beta^2\frac{\lambda}{c}x\right).$$

Recall that $ST(K)$ is a *semi-direct product* of its normal subgroup $H(K)$ and the torus

$$T_1(K) = \left\{\begin{bmatrix}\beta^{-1} & 0\\ 0 & \beta\end{bmatrix},\ \beta \in K^*\right\} \subset ST(K).$$

In addition,

$$\begin{bmatrix}\beta^{-1} & 0\\ 0 & \beta\end{bmatrix}\begin{bmatrix}1 & 0\\ \lambda & 1\end{bmatrix}\begin{bmatrix}\beta^{-1} & 0\\ 0 & \beta\end{bmatrix}^{-1} = \begin{bmatrix}1 & 0\\ \beta^2\lambda & 1\end{bmatrix}\ \forall\lambda \in K, \beta \in K^*.$$

It follows from [8, Ch. III, Prop. 27 on p. 240] that there is a group homomorphism

$$\phi : ST(K) \to G(K)$$

that sends each $\begin{bmatrix}1 & 0\\ \lambda & 1\end{bmatrix}$ to $\exp(\frac{\lambda}{c}x)$ and each $\begin{bmatrix}\beta^{-1} & 0\\ 0 & \beta\end{bmatrix}$ to $\Phi(\beta)$. Clearly, ϕ sends $\mathbf{u}_1 = \begin{bmatrix}1 & 0\\ c & 1\end{bmatrix}$ to $\exp(\frac{c}{c}x)) = \exp(x) = u$. $\square$

6. Words in two letters on $\mathrm{PSL}(2, \mathbb{C})$

In this section we consider words in two letters. We provide the explicit formulas for $w(x, y)$, where x, y are upper triangular matrices. This enables to extract some additional information on the image of words in two letters. .

Consider a word map $w(x, y) = x^{a_1}y^{b_1}\dots x^{a_k}y^{b_k}$, where $a_i \neq 0$ and $b_i \neq 0$ for all $i = 1, ..., k$. Let $A(w) = \sum_{i=1}^k a_i$, $B(w) = \sum_{i=1}^k b_i$. Let $w : \tilde{G}^2 \to \tilde{G}$ be the induced word map on $G = \mathrm{SL}(2, \mathbb{C})$.

If $A(w) = B(w) = 0$, then $w \in F^{(1)} = [F, F]$. Since $F^{(1)}$ is a free group generated by elements $w_{n,m} = [x^n, y^m]$, $n \neq 0$, $m \neq 0$ ([22], Chapter 1, §1.3), the word w with $A(w) = B(w) = 0$ may be written as a (noncommutative) product (with $s_i \neq 0$)

$$w = \prod_1^r w_{n_i,m_i}^{s_i}. \tag{4}$$

Moreover, the shortest (reduced) representation of this kind is unique. We denote by $S_w(n,m)$ the number of appearances of $w_{n,m}$ in representation (4) of w and by $R_w(n,m)$ the sum of exponents at all the appearances. We denote by $Supp(w)$ the set of all pairs (n,m) such that $w_{n,m}$ appears in the product. For example, if $w = w_{1,1}w_{2,2}^5w_{1,1}^{-1}$, then

$$Supp(w) = \{(1,1),(2,2)\}; S_w(1,1) = 2, S_w(2,2) = 1,$$
$$R_w(1,1) = 0, R_w(2,2) = 5.$$

The subgroup

$$F^{(2)} = [F^{(1)}, F^{(1)}] = \{w \in F^{(1)} | R_w(n,m) = 0 \text{ for all } (n,m) \in Supp(w)\}.$$

Example 6.1. The Engel word $e_n = \underbrace{[...[x,y],y],...y]}_{n \quad times}$ belongs to $F^{(1)} \setminus F^{(2)}$ (see also [12]).

Indeed, the direct computation shows that

(5)
$$yw_{n,m} = yx^ny^mx^{-n}y^{-m} = yx^ny^{-1}x^{-n}\cdot x^nyy^mx^{-n}y^{-m}y^{-1}\cdot y = w_{n,1}^{-1}w_{n,m+1}y,$$

(6)
$$yw_{n,m}^{-1} = y\cdot y^mx^ny^{-m}x^{-n} = y^{(m+1)}x^ny^{-(m+1)}x^{-n}\cdot x^nyx^{-n}y^{-1}\cdot y = w_{n,m+1}^{-1}w_{n,1}y.$$

It follows that

$$yw_{1,m}^sy^{-1} = (w_{1,1}^{-1}w_{1,m+1})^s. \tag{7}$$

Let us prove by induction that $|R_{e_n}(1,n)| = 1,\ S_{e_n}(1,n) = 1$ and $S_{e_n}(r,m) = 0$ if $r \neq 1$ or $m > n$, i.e.

$$e_n = (\prod_1^s w_{1,m_i}^{s_i})w_{1,n}^{\varepsilon}(\prod_1^t w_{1,k_j}^{t_j}) \tag{8}$$

for some integers $t \geq 0,\ s \geq 0,\ m_i < n,\ k_j < n$, and where $\varepsilon = \pm 1$.

Indeed $e_1 = w_{1,1}$. Assume that the claim is valid for all $k \leq n$. We have $e_{n+1} = e_nye_n^{-1}y^{-1}$. Using (8), we get

$$e_{n+1} = e_n(\prod_t^1 yw_{1,k_j}^{-t_j}y^{-1})yw_{1,n}^{-\varepsilon}y^{-1}(\prod_s^1 yw_{1,m_i}^{-s_i}y^{-1}). \tag{9}$$

Applying (7) to every factor of this product, we obtain that e_{n+1} has the needed form. Thus the claim will remain to be valid for $n+1$.

Since $|R_{e_n}(1,n)| = 1,\ e_n \notin F^{(2)}$.

Let us take

$$x = \begin{pmatrix} \lambda & c \\ 0 & \frac{1}{\lambda} \end{pmatrix}, \tag{10}$$

$$y = \begin{pmatrix} \mu & d \\ 0 & \frac{1}{\mu} \end{pmatrix}, \tag{11}$$

Then

$$x^n = \begin{pmatrix} \lambda^n & c \cdot h_{|n|}(\lambda) sgn(n) \\ 0 & \frac{1}{\lambda^n} \end{pmatrix}, \tag{12}$$

$$y^m = \begin{pmatrix} \mu^m & d \cdot h_{|m|}(\mu) sgn(m) \\ 0 & \frac{1}{\mu^m} \end{pmatrix}, \tag{13}$$

Here *sgn* is the *signum* function, and (see [1], Lemma 5.2) for $n \geq 1$

$$h_n(\zeta) = \frac{\zeta^{2n} - 1}{\zeta^{n-1}(\zeta^2 - 1)}. \tag{14}$$

.

Note that $h_n(1) = n$.
Direct computations show that

$$x^n y^m = \begin{pmatrix} \lambda^n \mu^m & d \cdot \lambda^n sgn(m) h_{|m|}(\mu) + c \cdot sgn(n) h_{|n|}(\lambda) \mu^{-m} \\ 0 & \lambda^{-n} \mu^{-m} \end{pmatrix}. \tag{15}$$

$$x^{-n} y^{-m} = \begin{pmatrix} \lambda^{-n} \mu^{-m} & -d \cdot \lambda^{-n} sgn(m) h_{|m|}(\mu) - c \cdot sgn(n) h_{|n|}(\lambda) \mu^m \\ 0 & \lambda^n \mu^m \end{pmatrix}. \tag{16}$$

$$w_{n,m}(x, y) = \begin{pmatrix} 1 & f(c, d, n, m) \\ 0 & 1 \end{pmatrix}, \tag{17}$$

where

$$f(c, d, n, m) = c h_{|n|}(\lambda) sgn(n) \lambda^n (1 - \mu^{2m}) + d h_{|m|}(\mu) sgn(m) \mu^m (\lambda^{2n} - 1). \tag{18}$$

Hence,

$$w(x, y) = \prod_1^r w_{n_i, m_i}^{s_i}(x, y) = \begin{pmatrix} 1 & F_w(c, d, \lambda, \mu) \\ 0 & 1 \end{pmatrix}, \tag{19}$$

where

$$F_w(c, d, \lambda, \mu) = \sum_1^r s_i f(c, d, n_i, m_i) = c\Phi_w(\lambda, \mu) + d\Psi_w(\lambda, \mu)$$

and

$$\Phi_w(\lambda, \mu) = \sum_{(\alpha, \beta) \in Supp(w)} R_w(\alpha, \beta) sgn(\alpha)(1 - \mu^{2\beta}) \frac{(\lambda^{2|\alpha|} - 1)\lambda^\alpha}{\lambda^{|\alpha| - 1}(\lambda^2 - 1)}, \tag{20}$$

(21) $$\Psi_w(\lambda,\mu) = \sum_{(\alpha,\beta)\in Supp(w)} R_w(\alpha,\beta)sgn(\beta)(\lambda^{2\alpha}-1)\frac{(\mu^{2|\beta|}-1)\mu^{\beta}}{\mu^{|\beta|-1}(\mu^2-1)}.$$

(Since the order of factors in w is not relevant for (20) and (21) , we use here α, β instead of n_i, m_i to simplify the formulas).

Proposition 6.2. *Rational functions $\Phi(\lambda,\mu)$ and $\Psi(\lambda,\mu)$ are non-zero linearly independent rational functions.*

Remark 6.3. It is evident from the Magnus Embedding Theorem that at least one of functions $\Phi(\lambda,\mu)$ and $\Psi(\lambda,\mu)$ is not identical zero. It follows from Proposition 6.2 that the same is valid for both of them.

Proof.

Lemma 6.4. *If $\Phi_w(\lambda,\mu) \equiv 0$ then $R_w(\alpha,\beta) = 0$ for all $(\alpha,\beta) \in Supp(w)$.*

Proof. We use induction by the number $|Supp(w)|$ of elements in $Supp(w)$ for the word w. If $Supp(w)$ contains only one pair (α,β), then there is nothing to prove, because

$$\Phi(\lambda,\mu) = R_w(\alpha,\beta)h_{|\alpha|}(\lambda)sgn(\alpha)\lambda^{\alpha}(1-\mu^{2\beta}).$$

Assume that for words v with $|Supp(v)| = l$ it is proved. Let w be such a word that $|Supp(w)| = l+1$.

Let $n := max\{\alpha \mid (\alpha,\beta) \in Supp(w)\}$.

Case 1. $n > 0$.

We have

$$\Phi_w(\lambda,\mu) = \sum_{(\alpha,\beta)\in Supp(w)} R_w(\alpha,\beta)sgn(\alpha)(1-\mu^{2\beta})\frac{(\lambda^{2|\alpha|}-1)\lambda^{\alpha}}{\lambda^{|\alpha|-1}(\lambda^2-1)} =$$

$$\sum_{(\alpha,\beta)\in Supp(w)} R_w(\alpha,\beta)sgn(\alpha)(1-\mu^{2\beta})\lambda^{a-|a|+1}(1+\lambda^2+\cdots+\lambda^{2(|\alpha|-1)}).$$

It means that the coefficient of $\lambda^{2|n|-1}$ in rational function $\Phi_w(\lambda,\mu)$ is

$$p(\mu) = \sum_{(n,\beta)\in Supp(w)} R_w(n,\beta)(1-\mu^{2\beta}).$$

Hence, if $\Phi_w(\lambda,\mu) \equiv 0$, then $p(\mu) \equiv 0$, and all $R_w(n,\beta) = 0$ for all β.

That means that $\Phi_w(\lambda,\mu) = \Phi_v(\lambda,\mu)$, where v is such a word that may be obtained from $w(x,y) = \prod_1^r w_{n_i,m_i}^{s_i}(x,y)$ by taking away every appearance of $w_{n,\beta}$:

$$v = \prod_{\substack{1\\ n_i\neq n}}^{r} w_{n_i,m_i}^{s_i}(x,y).$$

But $|Supp(v)| \leq l$ and by induction assumption $R_v(\alpha, \beta) = 0$ for all $(\alpha, \beta) \in Supp(v)$. Thus Lemma is valid for w in this case.

Case 2. $n < 0$. Let $-n' := min\{\alpha \ |(\alpha, \beta) \in Supp(w)\}$. We proceed as in Case 1 with $-n'$ instead of n : the coefficient of $\lambda^{-2n'+1}$ is $q(\mu) = \sum_{(-n',\beta)\in Supp(w)} R_w(-n', \beta)(1 - \mu^{2\beta})$. If $\Phi_w(\lambda, \mu) \equiv 0$, then $q(\mu) \equiv 0$, and all $R_w(-n', \beta) = 0$ for all β. Once more, we may replace w by a word v with $|Supp(v)| \leq l$. □

Clearly, the similar statement is valid for $\Psi_w(\lambda, \mu)$.

The functions Φ and Ψ are linearly independent, because Φ is odd with respect to λ and even with respect to μ, while Ψ has opposite properties.

□

Proposition 6.5. *Assume that the word* $w \in F^{(1)} \setminus F^{(2)}$ *and that* $\Phi_w(1, i) \neq 0$, *where* $i^2 = -1$. *Then* $-id \in w_G$, *where* $G = \mathrm{SL}(2, \mathbb{C})$.

Proof. Assume that $\Phi(1, i) \neq 0$. From (20) we get:

$$\Phi_w(1, i) = \sum_{(\alpha,\beta)\in Supp(w),\beta \ odd} 2R_w(\alpha, \beta)\alpha. \tag{22}$$

Take

$$x = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

$$y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then

$$[x, y] = \begin{pmatrix} a^2 & 0 \\ 0 & a^{-2} \end{pmatrix}$$

Thus, if

$$w = \prod_1^r w_{n_j,m_j}^{s_j},$$

then

$$w(x, y) = \prod_{m_j \ odd} \begin{pmatrix} a^{2n_j s_j} & 0 \\ 0 & a^{-2n_j s_j} \end{pmatrix} = \begin{pmatrix} a^N & 0 \\ 0 & a^{-N} \end{pmatrix},$$

where $N = 2 \sum_{m_j \ odd} n_j s_j = \Phi_w(1, i) \neq 0$.

Choose a such that $a^N = -1$. Then $w(x, y) = -id$. □

Remark 6.6. The case $\Psi(i, 1) \neq 0$ may be treated in the similar way, one should only exchange roles of x and y.

Remark 6.7. Let

$$w = \prod_1^r w_{n_j,m_j}^{s_j},$$

let $gcd(m_j) = k = 2^d s,\ s$ odd. Put $\mu_j = \frac{m_j}{k}$ and

$$u = \prod_1^r w_{n_j,\mu_j}^{s_j}.$$

Note that some of μ_j are odd. Let $z \in \mathrm{SL}(2,\mathbb{C})$ be such that

$$z^k = y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then $w(x,z) = u(x,y)$, hence, if $\Phi_u(1,i) \neq 0$, then $-id \in w_G$.

7. Surjectivity on SL(2, ℂ)

We keep the notation of Section 6.

Lemma 7.1. *Assume that* $w = x^{a_1}y^{b_1}\dots x^{a_k}y^{b_k}, \quad a_i \neq 0,\ b_i \neq 0,\ i = 1,...,k\ A = \sum a_i \neq 0$ *or* $B = \sum b_i \neq 0$ *and* x, y *are defined by* (10), (11) *respectively. Then*

$$w(x,y) = \begin{pmatrix} \lambda^A\mu^B & \tilde{F}_w(c,d,\lambda,\mu) \\ 0 & \lambda^{-A}\mu^{-B} \end{pmatrix}, \tag{23}$$

where

$$\tilde{F}_w(c,d,\lambda,\mu) = c\tilde{\Phi}_w(\lambda,\mu) + d\tilde{\Psi}_w(\lambda,\mu)$$

and

$$\tilde{\Phi}_w(\lambda,\mu) = \sum_1^k sgn(a_i)h_{|a_i|}(\lambda)\frac{\lambda^{\sum_{j<i}a_j}\mu^{\sum_{j<i}b_j}}{\lambda^{\sum_{j>i}a_j}\mu^{\sum_{j\geq i}b_j}}, \tag{24}$$

$$\tilde{\Psi}_w(\lambda,\mu) = \sum_1^k sgn(b_i)h_{|b_i|}(\mu)\frac{\lambda^{\sum_{j\leq i}a_j}\mu^{\sum_{j<i}b_j}}{\lambda^{\sum_{j>i}a_j}\mu^{\sum_{j>i}b_j}}. \tag{25}$$

Proof. We use induction on the complexity k of the word w. Using (15), we get

(26)

$$x^{a_1}y^{b_1} = \begin{pmatrix} \lambda^{a_1}\mu^{b_1} & d\cdot\lambda^{a_1}sgn(b_1)h_{|b_1|}(\mu) + c\cdot sgn(a_1)h_{|a_1|}(\lambda)\mu^{-b_1} \\ 0 & \lambda^{-a_1}\mu^{-b_1} \end{pmatrix}.$$

Thus for $k = 1$ the Lemma is valid. Assume that it is valid for $k' < k$. Let $u = x^{a_1}y^{b_1}\dots x^{a_{k-1}}y^{b_{k-1}}$ and $w = ux^{a_k}y^{b_k}$.

By induction assumption,

$$u(x,y) = \begin{pmatrix} \lambda^{A-a_k}\mu^{B-b_k} & \tilde{F}_u(c,d,\lambda,\mu) \\ 0 & \lambda^{-A+a_k}\mu^{-B+b_k} \end{pmatrix}.$$

From (15) we get

$$x^{a_k}y^{b_k} = \begin{pmatrix} \lambda^{a_k}\mu^{b_k} & d\cdot\lambda^{a_k}sgn(b_k)h_{|b_k|}(\mu)+c\cdot sgn(a_k)h_{|a_k|}(\lambda)\mu^{-b_k} \\ 0 & \lambda^{-a_k}\mu^{-b_k} \end{pmatrix}.$$

Multiplying matrices u and $x^{a_k}y^{b_k}$ we get

$$\tilde{F}_w(c,d,\lambda,\mu) = \lambda^{A-a_k}\mu^{B-b_k}(d\cdot\lambda^{a_k}sgn(b_k)h_{|b_k|}(\mu)$$

$$+c\cdot sgn(a_k)h_{|a_k|}(\lambda)\mu^{-b_k}) + \tilde{F}_u(c,d,\lambda,\mu)\lambda^{-a_k}\mu^{-b_k}.$$

Thus, the induction assumption implies that

$$\tilde{\Phi}_w(\lambda,\mu) = sgn(a_k)h_{|a_k|}(\lambda)\mu^{-b_k}\lambda^{A-a_k}\mu^{B-b_k}+\sum_1^{k-1} sgn(a_i)h_{|a_i|}(\lambda)\frac{\lambda^{\sum_{j<i}a_j}\mu^{\sum_{j<i}b_j}}{\lambda^{\sum_{j=i+1}^k a_j}\mu^{\sum_{j=i}^k b_j}}$$

$$= \sum_1^k sgn(a_i)h_{|a_i|}(\lambda)\frac{\lambda^{\sum_{j<i}a_j}\mu^{\sum_{j<i}b_j}}{\lambda^{\sum_{j>i}a_j}\mu^{\sum_{j\geq i}b_j}}.$$

$$\tilde{\Psi}_w(\lambda,\mu) = sgn(b_k)h_{|b_k|}(\mu)\lambda^{a_k}\lambda^{A-a_k}\mu^{B-b_k}+\sum_1^{k-1} sgn(b_i)h_{|b_i|}(\mu)\frac{\lambda^{\sum_{j\leq i}a_j}\mu^{\sum_{j<i}b_j}}{\lambda^{\sum_{j=i+1}^k a_j}\mu^{\sum_{j=i+1}^k b_j}}$$

$$= \sum_1^k sgn(a_i)h_{|a_i|}(\lambda)\frac{\lambda^{\sum_{j\leq i}a_j}\mu^{\sum_{j<i}b_j}}{\lambda^{\sum_{j>i}a_j}\mu^{\sum_{j>i}b_j}}.$$

□

Denote:

$$A_i = \sum_{j\leq i} a_i; \;\; B_i = \sum_{j<i} b_i,$$

and let C be a curve

$$C = \{\lambda^A\mu^B = -1\} \subset \mathbb{C}^2_{\lambda,\mu}.$$

Multiplying (24) and (25) by $\lambda^A\mu^B$ we see that on C the following relations are valid:

$$\tilde{\Phi}_w(\lambda,\mu)\mid_C = -\sum_1^k sgn(a_i)h_{|a_i|}(\lambda)\lambda^{2A_i-a_i}\mu^{2B_i}, \tag{27}$$

$$\tilde{\Psi}_w(\lambda,\mu)\mid_C = -\sum_1^k sgn(b_i)h_{|b_i|}(\mu)\lambda^{2A_i}\mu^{\sum 2B_i+b_i}. \tag{28}$$

In particular, on C

$$\tilde{\Phi}_w(1,\mu)\mid_C = -\sum_1^k a_i\mu^{2B_i}, \tag{29}$$

$$\tilde{\Psi}_w(\lambda,1)\mid_C = -\sum_1^k b_i\lambda^{2A_i}. \tag{30}$$

Lemma 7.2. *Assume that $A \neq 0$ and the word map w is not surjective. Then*

$$\sum_1^k b_i \gamma^{2A_i} = 0$$

for every root γ of equation

$$q(z) := z^A + 1 = 0.$$

If $B \neq 0$ and the word map w is not surjective, then

$$\sum_1^k a_i \delta^{2B_i} = 0$$

for every root δ of equation

$$p(z) := z^B + 1 = 0.$$

Proof. The matrices z with $tr(z) = 2$ are in the image because $w(x, id) = x^A$, $w(id, y) = y^B$. Assume now that for $K \neq 0$ the matrices

$$\begin{pmatrix} -1 & K \\ 0 & -1 \end{pmatrix} \tag{31}$$

are not in the image. That implies that $\tilde{\Phi}_w(\lambda, \mu) \equiv 0$ and $\tilde{\Psi}_w(\lambda, \mu) \equiv 0$ on the defined above curve

$$C = \{\lambda^A \mu^B = -1\} \subset \mathbb{C}^2_{\lambda,\mu}.$$

If $A \neq 0$ or $B \neq 0$, then, respectively, the pairs $(\gamma, 1)$ and $(1, \delta)$ belong to the curve C. We have to use only (29), (30), respectively . □

Corollary 7.3. *Let $2B_i = k_i B + T_i$, where k_i are integers and $0 \leq T_i < B \neq 0$. If w is not surjective, then for every $0 \leq T < B$*

$$\sum_{i:T_i = T} a_i (-1)^{k_i} = 0. \tag{32}$$

Proof. Indeed in this case

$$0 = \sum_1^k a_i \delta^{2B_i} = \sum_{T=0}^{B-1} \delta^T \Big(\sum_{i:T_i=T} a_i (-1)^{k_i} \Big)$$

for any root δ of equation

$$p(z) = z^B + 1 = 0.$$

Since $p(z)$ has no multiple roots, it implies that $p(z)$ divides the polynomial

$$p_1(z) := \sum_{T=0}^{B-1} z^T \Big(\sum_{i:T_i=T} a_i (-1)^{k_i} \Big).$$

But since degree of $p(z)$ is bigger than degree of $p_1(z)$ that can be only if $p_1(z) \equiv 0$. $\square$

Corollary 7.4. *(Corollary 1.4) If all b_i are positive, then the word map w is either surjective or the square of another word $v \neq id$.*

Proof. In this case $0 \leq 2B_i < 2B$ and sequence B_i is increasing. If w is not surjective, $p_1(z) \equiv 0$ by Corollary 7.3. Thus for every B_i there is B_j such that $2B_i = 2B_j + B$ and $a_i - a_j = 0$.

Thus, the sequence of $2B_i$ looks like:

$$0 = 2B_1,\ 2b_1 = 2B_2,\ 2(b_1+b_2) = 2B_3, \dots, 2(b_1+\dots+b_s) = 2B_{s+1} = B,$$

$$2(b_1 + \dots + b_{s+1}) = 2B_{s+2} = B + 2B_2 = B + 2b_1,$$

$$2(b_1 + \dots + b_{s+2}) = 2B_{s+3} = B + 2B_3 = B + 2b_1 + 2b_2, \dots,$$

$$2(b_1 + \dots + b_{2s-1}) = 2B_{2s} = 2B_s + B,$$

$$2(b_1 + \dots + b_{2s}) = 2B_{2s+1} = B + 2B_{s+1} = 2B.$$

It follows that $k = 2s$ and

$$b_{s+1} = B_{s+2} - B_{s+1} = B_2 - B_1 = b_1;$$

$$b_{s+2} = B_{s+3} - B_{s+2} = B_3 - B_2 = b_2;$$

$$b_{2s-1} = B_{2s} - B_{2s-1} = B_s - B_{s-1} = b_{s-1};$$

$$b_k = b_{2s} = B_{2s+1} - B_{2s} = B_{s+1} - B_s = b_s.$$

Thus,

$$b_i = b_{i+s},\ i = 1, \dots, s,\ 2B_i = 2B_{i+s} + B, a_i = a_{i+s}.$$

Therefore the word is the square of $v = x^{a_1} y^{b_1} \dots x^{a_s} y^{b_s}$. $\square$

Corollary 7.5. *If all b_i are negative, then the word map of the word w is either surjective or the square of another word $v \neq id$.*

Proof. We may change y to $z = y^{-1}$ and apply Corollary 7.4 to the word $w(x, z)$. $\square$

Corollary 7.6. *If all a_i are positive, then the word map of the word w is either surjective or the square of another word $v \neq id$.*

Proof. Consider $v = x^{-1},\ z = y^{-1}$, a word

$$w'(z, v) = w(x, y)^{-1} = y^{-b_k} x^{-a_k} \dots y^{-b_1} x^{-a_1} = z^{b_k} v^{a_k} \dots z^{b_1} v^{a_1},$$

and apply Corollary 7.4 to the word $w'(z, v)$. $\square$

8. Trace criteria of almost surjectivity

For every word map $w(x, y) : G^2 \to G$ defined are the trace polynomials $P_w(s, t, u) = tr(w(x, y))$ and $Q_w = tr(w(x, y)y)$ in three variables $s = tr(x),\ t = tr(y), and\ u = tr(xy)$. ([14], [15], [24]).

In other words, the maps

$$\varphi_w : G^2 \to G^2,\ \varphi_w(x, y) = (w(x, y), y)$$

and

$$\psi_w : \mathbb{C}^3_{s,t,u} \to \mathbb{C}^3_{s,t,u},\ \psi_w(s, t, u) = (P_w(s, t, u), t, Q_w(s, t, u))$$

may be included into the following commutative diagram:

$$\begin{array}{ccc} G \times G & \xrightarrow{\varphi} & G \times G \\ \pi \downarrow & & \pi \downarrow \\ \mathbb{C}^3_{s,t,u} & \xrightarrow{\psi} & \mathbb{C}^3_{s,t,u} \end{array} \quad . \tag{33}$$

Moreover, π is a surjective map ([15]). For details, one can be referred to ([5],[3]) .

Since the coordinate t is invariant under ψ, for every fixed value $t = a \in \mathbb{C}$ we may consider the restriction $\psi_a(s, u) = (P_w(s, a, u), Q_w(s, a, u))$ of morphism ψ_w onto the plane $\{t = a\} = \mathbb{C}^2_{s,u}$.

Definition 8.1. We say that $\psi_a(s, u)$ is **Big** if the image $\psi_a(\mathbb{C}^2_{s,u}) = \mathbb{C}^2_{s,u} \setminus T_a$, where T_a is a finite set. We say that the trace map ψ_w of a word $w \in F$ is **Big** if there is a value a such that $\psi_a(s, u)$ is **Big**.

Proposition 8.2. *If the trace map ψ_w of a word $w \in F$ is* **Big** *then the word map $w : G^2 \to G$ is almost surjective.*

Proof. Let a be such a value of t that the map ψ_a is **Big**. Let $S_a = T_a \cup \{(2, a)\} \cup \{(-2, -a)\}$. Consider a line $C_+ = \{s = 2\}$ and $C_- = \{s = -2\} \subset \mathbb{C}^2_{s,u}$. Let $B_+ = C_+ \setminus (C_+ \cap S_a)$; $B_- = C_- \setminus (C_- \cap S_a)$. Since S_a is finite, $B_+ \neq \emptyset, B_- \neq \emptyset$. Moreover, since these curves are outside S_a, we have: $D_+ = \psi^{-1}(B_+) \neq \emptyset,\ D_- = \psi^{-1}(B_-) \neq \emptyset$.

Take $(s_0, u_0) \in D_+$ and $(s_1, u_1) \in D_-$. Then $\psi_w(s_0, a, u_0) = (2, a, b)$ with $a \neq b$; and $\psi_w(s_1, a, u_1) = (-2, a, d)$ with $a \neq -d$. Projection $\pi : G^2 \to \mathbb{C}^3_{s,t,u}$ is surjective, thus there is a pair $(x_0, y_0) \in G^2$ such that $tr(x_0) = s_0,\ tr(y_0) = a,\ tr(x_0y_0) = u_0$. Then $\pi(w(x_0, y_0)) = \psi_w(s_0, a, u_0) = (2, a, b)$. Hence, $tr(w(x_0, y_0)) = 2$, but $w(x_0, y_0) \neq id$, since $tr(w(x_0, y_0)y_0) = b \neq a = tr(y_0)$. Similarly, there is a pair $(x_1, y_1) \in G^2$ such that $tr(x_1) = s_1,\ tr(y_1) = a,\ tr(x_1y_1) = u_1$. Then $\pi(w(x_1, y_1)) = \psi_w(s_1, a, u_1) = (-2, a, d)$. Hence, $tr(w(x_1, y_1)) = -2$, but $w(x_1, y_1) \neq -id$, since $tr(w(x_1, y_1)y_1) = d \neq -a = -tr(y_1)$.

It follows that all the elements $z \neq -id$ with trace 2 and -2 are in the image of the word map w. □

Corollary 8.3. *Assume that the trace map ψ_w of a word w is* **Big**. *Consider a sequence of words defined recurrently in the following way:*

$$v_1(x,y) = w(x,y);\ v_{n+1}(x,y) = w(v_n(x,y),y);$$

Then the word map $v_n : G^2 \to G$ is almost surjective for all $n \geq 1$.

Proof. The trace map $\psi_n = \psi_{v_n}$ of the word map v_n is the n^{th} iteration $\psi_1^{(n)}$ of the trace map $\psi_1 = \psi_w$ (see [5] or [3]). Let us show by induction, that all the maps ψ_n are **Big**. Indeed ψ_1 is **Big** by assumption, hence $(\psi_1)_a(\mathbb{C}^2_{s,u}) = \mathbb{C}^2_{s,u} - T_a$ for some value a and some finite set T_a. Assume now that ψ_{n-1} is **Big**. Let for a value a of t the image $(\psi_{n-1})_a(\mathbb{C}^2_{s,u}) = \mathbb{C}^2_{s,u} \setminus N$ for some finite set N. Hence

$$(\psi_n)_a(\mathbb{C}^2_{s,u}) = (\psi_1)_a((\psi_{n-1})_a(\mathbb{C}^2_{s,u})) = (\psi_1)_a(\mathbb{C}^2_{s,u} \setminus N) \supset$$
$$\supset (\psi_1)_a(\mathbb{C}^2_{s,u}) \setminus (\psi_1)_a(N) = \mathbb{C}^2_{s,u} \setminus (T_a \cup (\psi_1)_a(N)).$$

Thus $(\psi_n)_a$ is **Big** as well for the same value a.

According to Proposition 8.2, the word map v_n is almost surjective. □

Example 8.4. Consider the word $w(x,y) = [yxy^{-1}, x^{-1}]$ and the corresponding sequence

$$v_n(x,y) = [yv_{n-1}y^{-1}, v_{n-1}^{-1}].$$

This is one of the sequences that were used for characterization of finite solvable groups (see [9], [5], [3]).

We have ([5], section 5.1)

$$tr(w(x,y)) = f_1(s,t,u) = (s^2+t^2+u^2-ust-4)(t^2+u^2-ust)+2;$$
$$tr(w(x,y)y) = f_2(s,t,u) = f_1t+(s(st-u)-t)(s^2+t^2+u^2-ust-4)-t;$$

We want to show that for a general value $t = a$ the system of equations

$$f_1(s,a,u) = A \tag{34}$$

$$f_2(s,a,u) = B \tag{35}$$

has solutions for all pairs $(A,B) \in \mathbb{C}^2 \setminus T_a$, where T_a is a finite set.

Consider the system

$$h_1(s,u,a,C) := (s^2+a^2+u^2-usa-4)(a^2+u^2-usa) = A-2 := C, \tag{36}$$

$$h_2(s,u,a,D) := (s(sa-u)-a)(s^2+a^2+u^2-usa-4) = B-a(C+1) := D. \tag{37}$$

Note that the leading coefficient with respect u in h_1 is 1, in h_2 is s. The Magma computations show that the resultant (elimination of u) of $h_1 - C$ and $h_2 - D$ is of the form

$$R(s,a,C,D) = s^4p_1(a,C,D) + s^2p_2(a,C,D) + p_3(a,C,D).$$

It has a non-zero root $s \neq 0$ at any point (a, C, D), where at least two of three polynomials p_1, p_2, p_3 do not vanish. MAGMA computation show that the ideals $J1 =< p_1, p_2 > \subset \mathbb{Q}[a, C, D]$, $J2 =< p_1, p_3 > \subset \mathbb{Q}[a, C, D]$, $J3 =< p_2, p_3 > \subset \mathbb{Q}[a, C, D]$ generated, respectively, by $p_1(a, C, D)$ and $p_2(a, C, D)$, by $p_1(a, C, D)$ and $p_3(a, C, D)$, by $p_2(a, C, D)$ and $p_3(a, C, D)$, are one-dimensional. It follows that for a general value of a the set

$$\{p_1(a, C, D) = p_2(a, C, D) = 0\} \cup \{p_1(a, C, D) = p_3(a, C, D) = 0\} \cup \{p_2(a, C, D) = p_3(a, C, D) = 0\}$$

is a finite subset $N_a \subset \mathbb{C}_{C,D}$. On the other hand, at any point (C, D) outside N_a polynomial $R_a(s) = R(s, a, C, D)$ has a non-zero root, and, therefore system (36), (37) has a solution. Thus, outside the finite set of points $T_a = \{(A = C + 2, B = D + a(C + 1)) \,|(C, D) \in N_a\} \subset \mathbb{C}_{A,B}$, system (34), (35) has a solution as well. Thus, $\psi_w = (f_1, t, f_2)$ is **Big** and all the word maps v_n are almost surjective on G.

Let us cite the Magma computations for $t = a = 1$, where $p = h_1 - C$ and $q = h_2 - D$. R is the resultant of p, q with respect to u.

```
 > r:=u^2+s^2+1-u*s;
>
> p:=(r-4)*(r-s^2)-C;
>
> q:=(r-4)*(s*(s-u)-1)-D;
>
> R:=Resultant(p,q,u);
> R;
-s^4*C^3 - 2*s^4*C^2*D + s^4*C^2 - 2*s^4*C*D^2 + s^4*C*D
 - s^4*D^3 + s^4*D^2 + 4*s^2*C^2*D - 4*s^2*C^2 + 8*s^2*C*D^2
 - 6*s^2*C*D + 6*s^2*D^3 - 8*s^2*D^2 +
    C^2 - 2*C*D^2 + 8*C*D + D^4 - 8*D^3 + 16*D^2
>


>
> p1:=-C^3 - 2*C^2*D + C^2 - 2*C*D^2 + C*D - D^3 + D^2;
> p2:= 4*C^2*D - 4*C^2 + 8*C*D^2 - 6*C*D + 6*D^3 - 8*D^2;
> p3:=C^2 - 2*C*D^2 + 8*C*D + D^4 - 8*D^3 + 16*D^2;
> Factorization(p1);
[
    <C + D - 1, 1>,
    <C^2 + C*D + D^2, 1>
]
> Factorization(p2);
[
    <C^2*D - C^2 + 2*C*D^2 - 3/2*C*D + 3/2*D^3 - 2*D^2, 1>
]
```

```
> Factorization(p3);
[
    <C - D^2 + 4*D, 2>
```

Clearly every pair among polynomials $p1, p2, p3$ has only finite number of common zeros. For example, $p_1 = p_3 = 0$ implies $D^2-5D+1 = 0$ or $(D^2 - 4D)^2 + (D^2 - 4D)D + D^2 = 0$.

Computations show also that the word $w(x, y)$ takes on value $-id$. For example, one make take

$$x = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}, \ y = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

where $t^2 = -1/2$. Here are computations:

```
> R<t>:=PolynomialRing(Q);
>  X:=Matrix(R,2,2,[-1,1,-2,1]);
>  Y:=Matrix(R,2,2,[ 1,t,0,1]);
>   X1:= Matrix(R,2,2,[1,-1,2,-1]);
>  Y1:=Matrix(R,2,2,[1,-t,0,1]);
>
>  Z:=Y*X*Y1;
>
>    p11:=Z[1,1];
>    p12:=Z[1,2];
>    p21:=Z[2,1];
>     p22:=Z[2,2];
>
>  Z1:=Matrix(R,2,2,[p22,-p12,-p21,p11]);
>
>  W:=Z*X1*Z1*X;
>
>    q11:=W[1,1];
>   q12:=W[1,2];
>  q21:=W[2,1];
>   q22:=W[2,2];
>
>
>    q11;
16*t^4 + 8*t^3 + 12*t^2 + 4*t + 1
>   q12;
-8*t^4 - 4*t^2
>  q21;
16*t^3 + 8*t
>   q22;
-8*t^3 + 4*t^2 - 4*t + 1
```

Therefore, $t^2 = -1/2$ implies that $q_{11} = q_{22} = -1, \ q_{12} = q_{21} = 0$.

9. The word $v(x, y) = [[x, [x, y]], [y, [x, y]]]$

In this section we provide an example of a word v that is surjective though it belongs to $F^{(2)}$. The interesting feature of this word is the following: if we consider it as a polynomial in the Lie algebra $\mathfrak{sl}_2$, ($[x, y]$ being the Lie bracket) then it is not surjective ([4], Example 4.9).

Theorem 9.1. *The word* $v(x, y) = [[x, [x, y]], [y[x, y]]]$ *is surjective on* $\mathrm{SL}(2, \mathbb{C})$ *(and, consequently, on* $\mathrm{PSL}(2, \mathbb{C})$*).*

Proof. As it was shown in Proposition 2.2, for every $z \in \mathrm{SL}(2, \mathbb{C})$ with $tr(z) \neq \pm 2$ there are $x, y \in \mathrm{SL}(2, \mathbb{C})^2$ such that $v(x, y) = z$.

Assume now that $a = \pm 2$. We have to show that $-id$ is in the image and that there are matrices x, y in $\mathrm{SL}(2, \mathbb{C})$, such that

$$v(x, y) := \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$$

has the following properties :

- $q_{12} + q_{22} = \pm 2$;
- $q_{12} \neq 0$.

We may look for these pairs among the matrices $x = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$ and $y = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.

In the following MAGMA calculations $C = [x, y]$, $D = [[x, y], x]$, $B = [[x, y], y]$, $A = [D, B]$.

Ideal I in the polynomial ring $\mathbb{Q}[b, c, d, t]$ is defined by conditions $det(x) = 1, tr(A) = 2$. Ideal J in the polynomial ring $\mathbb{Q}[b, c, d, t]$ is defined by conditions $det(x) = 1, tr(A) = -2$. Let $T_+ \subset \mathrm{SL}(2)^2$ and $T_- \subset \mathrm{SL}(2)^2$ be, respectively, the corresponding affine subsets in affine variety $\mathrm{SL}(2)^2$.

The computations show that $q_{12}(b, c, d, t)$ does not vanish identically on T_+ or T_-.

```
 >  Q:=Rationals();
> R<t,b,c,d>:=PolynomialRing(Q,4);
> X:=Matrix(R,2,2,[0,b,c,d]);
> Y:=Matrix(R,2,2,[ 1,t,0,1]);
>  X1:= Matrix(R,2,2,[d,-b,-c,0]);
> Y1:=Matrix(R,2,2,[1,-t,0,1]);
> C:=X*Y*X1*Y1;
>  p11:=C[1,1];
>  p12:=C[1,2];
>  p21:=C[2,1];
>  p22:=C[2,2];
> C1:=Matrix(R,2,2,[p22,-p12,-p21,p11]);
```

```
> D:=C*X*C1*X1;
>
>
> d11:=D[1,1];
> d12:=D[1,2];
> d21:=D[2,1];
> d22:=D[2,2];
> D1:=Matrix(R,2,2,[d22,-d12,-d21,d11]);
>
> B:=C*Y*C1*Y1;
>
>
> b11:=B[1,1];
> b12:=B[1,2];
> b21:=B[2,1];
> b22:=B[2,2];
> B1:=Matrix(R,2,2,[b22,-b12,-b21,b11]);
>
> A:=D*B*D1*B1;
>
> TA:=Trace(A);
>
> q12:=A[1,2];
> I:=ideal<R|b*c+1,TA-2>;
>
> IsInRadical(q12,I);
false
> J:=ideal<R|b*c+1,TA+2>;
>
> IsInRadical(q12,J);
false
>
```

It follows that the function $q_{12}(b, c, d, t)$ does not vanish identically on the sets T_+ and T_-, hence, there are pairs with $tr(v(x, y)) = 2, v(x, y) \neq id$, and $tr(v(x, y)) = -2, v(x, y) \neq -id$.

In order to produce the explicit solutions for $v(x, y) = -id$ and $v(x, y) = z, z \neq -id, tr(z) = -2$, consider the following matrices depending on one parameter d:

$$x = \begin{pmatrix} 1-d & 1 \\ -\frac{2}{3} & d \end{pmatrix},$$

$$y = \begin{pmatrix} 2-3d & 0 \\ 0 & 3d-1 \end{pmatrix}.$$

Since images of the commutator word on GL$(2,\mathbb{C})$ and SL$(2,\mathbb{C})$ are the same, we do not require that $det(x)=1$ or $det(y)=1$. We only assume that $det(x)=d^2-d-2/3\neq 0$ and $det(y)=-9d^2+9d^2-2\neq 0$.

Let

$$A=v(x,y):=\begin{pmatrix} q_{11}(d) & q_{12}(d)\\ q_{21}(d) & q_{22}(d)\end{pmatrix}$$

and $TA=tr(A)$. Magma computations show that

$$q_{11}(d)+1=N_{11}(d^2-d+1/3)H_{11}(d),$$

$$q_{22}(d)+1=N_{22}(d^2-d+1/3)H_{22}(d),$$

$$q_{21}(d)=N_{21}(d-2/3)^2(d-1/2)^3(d-1/3)^2(d^2-d-2/3)(d^2-d+1/3)H_{21}(d),$$

$$q_{12}(d)=N_{21}(d-2/3)^2(d-1/2)^3(d-1/3)^2(d^2-d-2/3)(d^2-d+1/3)H_{12}(d),$$

$$TA+2=N(d^2-d+1/3)H(d),$$

where N_{ij} and N are non-zero rational numbers; H_{ij} and H are polynomials with rational coefficients that are irreducible over $\mathbb{Q}$. Moreover $degH_{21}=degH_{12}=25,\ degH=38$. It follows that if $d^2-d+1/3=0$ then $A=-id$. If d is a root of H that is not a root of H_{21}, then A is a minus unipotent.

□

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