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SOLUTIONS OF THE CONGRUENCE $\sum_{k=1}^{n} k^{f(n)} \equiv 0 \pmod{n}$

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Abstract. In this paper we characterize, in terms of the prime divisors of n, the pairs (k,n) for which n divides $\sum_{j=1}^{n} j^k$. As an application, we derive some results on the sets $\mathcal{M}_f := \{n \geq 1 : f(n) > 1 \text{ and } \sum_{j=1}^{n} j^{f(n)} \equiv 0 \pmod{n}\}$ for some choices of f.

1. Introduction

In the literature on power sums $S_k(n) := \sum_{j=1}^{n} j^k$ the following congruence is well known

Proposition 1. (von Staudt [19], 1840). *Let $k, n \geq 1$ be integers with k even. We have that*

$$S_k(n) \equiv -\sum_{p|n,\ p-1|k} \frac{n}{p} \pmod{n}.$$

This result motivates us to study $S_k(n) \pmod{n}$ and, more generally, to study $S_{f(n)}(n) \pmod{n}$ for different arithmetic functions f (see [11] for some results in this spirit). Thus, if $p-1 \mid f(p)$, for every prime p, we have that the congruence $S_{f(n)}(n) \equiv -1 \pmod{n}$ holds for every $n = p$ prime and it is interesting to find the composite numbers which also satisfy it. In this direction we have the *Giuga numbers* (see [1]), which are composite numbers such that $S_{\phi(n)}(n) \equiv -1 \pmod{n}$, the *strong Giuga numbers*, which are composite numbers such that $S_{n-1}(n) \equiv -1 \pmod{n}$ (Giuga's conjecture [3] states that there are no strong Giuga numbers. Tipu [20] estimates the number of strong Giuga numbers up to x to be $O(x^{1/2} \log x)$ while Luca, Pomerance and Shparlinski [9] improve this to $O(x^{1/2}/\log^2 x)$), or the *K-strong Giuga numbers*, which are composite numbers such that $S_{K(n-1)}(n) \equiv -1 \pmod{n}$ (see [5]).

In this paper we characterize, in terms of the prime divisors of n, the pairs (k,n) for which n divides $S_k(n)$. This characterization is given in the following theorem.

Theorem 1. *Let $k, n \geq 1$ be integers. Then, $n \mid S_k(n)$ if and only if one of the following holds:*

i) *n is odd and $p - 1 \nmid k$ for every prime divisor p of n.*
ii) *n is a multiple of 4 and $k > 1$ is odd.*

Moreover, inspired by Giuga's ideas we investigate the congruence $S_{f(n)}(n) \equiv 0 \pmod{n}$ for some functions f. This work started in [4], where the case $f(n) = (n-1)/2$ was considered. The case of arithmetic functions f such that $p-1 \nmid f(p)$ for every prime p is of special interest.

In what follows we will consider the natural numbers

$$\mathcal{M}_f := \{n \geq 1 : f(n) > 1 \text{ and } S_{f(n)}(n) = \sum_{j=1}^{n} j^{f(n)} \equiv 0 \pmod{n}\} \tag{1}$$

associated to an arbitrary function $f : \mathbb{N} \longrightarrow \mathbb{N}$. The reader might wonder why the definition involves $f(n) > 1$, rather than $f(n) \geq 1$. The reason for this is that by Theorem 1 the case $f(n) = 1$ is somewhat exceptional.

Here we study the sets $\mathcal{M}_f$ in the case $f(n) = an + b$, the affine case, and in some cases such that $\mathcal{M}_f$ contains all prime numbers. We have characterized the elements of these sets and, in some cases, we have computed their asymptotic density.

In [6] the related problem of studying the sets $\{n : S_{Qn}(n) \equiv n \pmod{Qn}\}$ for certain very special Q ('weak primary pseudoperfect numbers') is studied

2. A proof of Theorem 1

In this section we will establish Theorem 1. It will be convenient to work with

$$S_k(n) := \sum_{j=1}^{n} j^k \text{ and } S_k^*(n) := \sum_{j=1}^{n-1} j^k.$$

In particular we will characterize the pairs (k, n) such that n divides $S_k(n)$. If $k = 0$, clearly $S_k(n) = n$ and there is no problem to study. Thus, in what follows we will assume $k > 0$.

We will start this section with three simple lemmas.

Lemma 1. *Let p be a prime and let $k > 0$ be an integer. Then, we have*

$$S_k(p) \equiv \begin{cases} -1 \pmod{p} & \text{if } p-1 \mid k; \\ 0 \pmod{p} & \text{if } p-1 \nmid k. \end{cases}$$

Proof. See [7] for the standard proof using primitive roots, or [10] for a recent elementary proof. □

The next lemma extends Lemma 2 in Moree [17], where it is proved that (2) holds if p is odd or $p = 2$ and r is even.

Lemma 2. *Let λ and r be positive integers and p be a prime. We have*

$$S_r(p^{\lambda+1}) \equiv pS_r(p^\lambda) \pmod{p^{\lambda+1}}, \tag{2}$$

unless $\lambda = 1$, $p = 2$, r is odd and $r \geq 3$ in which case we have $0 \equiv S_r(4) \not\equiv 2S_r(2) \equiv 2 \pmod{4}$.

Proof. Note that it is equivalent to prove the statement with $S_r(\cdot)$ replaced by $S_r^*(\cdot)$. Since the statement clearly holds for $r = 1$ we may assume that $r \geq 2$. Every $0 \leq j < p^{\lambda+1}$ can be uniquely written as $j = \alpha p^\lambda + \beta$ with $0 \leq \alpha < p$ and $0 \leq \beta < p^\lambda$. Hence we obtain by invoking the binomial theorem

$$S_r^*(p^{\lambda+1}) = \sum_{\alpha=0}^{p-1} \sum_{\beta=0}^{p^\lambda-1} (\alpha p^\lambda + \beta)^r \equiv p \sum_{\beta=0}^{p^\lambda-1} \beta^r + rp^\lambda \sum_{\alpha=0}^{p-1} \alpha \sum_{\beta=0}^{p^\lambda-1} \beta^{r-1} \pmod{p^{2\lambda}}.$$

Since the first single sum equals $S_r^*(p^\lambda)$, we see that (2) holds if and only if $\frac{r}{2}p(p-1)S_{r-1}^*(p^\lambda) \equiv 0 \pmod{p}$. Now suppose that the latter congruence does not hold. Then we must have $p = 2$, $2 \nmid r$ and $r \geq 3$. Since $2 \mid S_{r-1}^*(2^\lambda)$ for $\lambda \geq 2$ we must have $\lambda = 1$. The proof is easily completed on noting that for $r \geq 3$ and odd we have $S_r(4) \equiv 1^r + 3^r \equiv 0 \pmod{4}$. □

As so often in number theory, 'two is the oddest of primes' and needs special treatment

Lemma 3. *Let $e, k \geq 1$. We have $2^e \mid S_k(2^e)$ if and only if $k \geq 3$ is odd and $e \geq 2$.*

Proof. Follows on combining the previous two lemmas. □

In fact, using Lemma 2 it is easy to evaluate $S_k(2^e)$ modulo 2^e (where we ignore the trivial case $e = 1$). We give the result for completeness' sake.

Lemma 4. *Let $e > 1$. Then*

$$S_k(2^e) \equiv \begin{cases} 0 \pmod{2^e} & \text{if } k \text{ is odd;} \\ 2^{e-1} \pmod{2^e} & \text{if } k > 1 \text{ is even.} \end{cases}$$

Proof of Theorem 1. If $b \mid n$, then clearly $S_k(n) \equiv \frac{n}{b} S_k(b) \pmod{b}$. Now let $n = \prod_{i=1}^{s} p_i^{e_i}$ be the canonical prime factorisation of n. Noting that $p_i \nmid n p_i^{-e_i}$ we infer from $S_k(n) \equiv \frac{n}{p_i^{e_i}} S_k(p_i^{e_i}) \pmod{p_i^{e_i}}$ that

$$n \mid S_k(n) \text{ if and only if } p_i^{e_i} \mid S_k(p_i^{e_i}), \text{ for } i = 1, 2, \ldots, s. \tag{3}$$

If p_i is odd, then it follows on combining Lemma 1 and Lemma 2 that

$$p_i^{e_i} \mid S_k(p_i^{e_i}) \text{ if and only if } p_i - 1 \nmid k. \tag{4}$$

Using this and Lemma 3 we see that $n \mid S_k(n)$ if and only if
i) n is odd and $p - 1 \nmid k$ for every odd prime divisor p of n;
or
ii) n is a multiple of 4, $k > 1$ is odd and $p - 1 \nmid k$ for every odd prime divisor p of n.

Note that in i) the second 'odd' is a consequence of the first 'odd'. Likewise in ii) the condition that k is odd implies that $p - 1 \nmid k$ for every odd prime divisor p of n. On leaving out the redundant parts of i) and ii) the proof is completed. □

3. Some remarks concerning Theorem 1

3.1. **The Erdős-Moser equation.** Erdős conjectured around 1950 that the Diophantine equation

$$S_k(n-1) = n^k \tag{5}$$

has only the solution $1 + 2 = 3$ corresponding to $(k, n) = (1, 3)$. Note that if (k, n) satisfies $S_k(n-1) = n^k$, then $n \mid S_k(n)$. The first results on this problem were obtained by original but entirely elementary methods by Leo Moser [18], cf. [17]. He showed that if (5) has a further solution with $k > 1$, then k is even and $n > 10^{10^6}$. He showed that either $n \equiv 0 \pmod{8}$ or $n \equiv 3 \pmod{8}$. Note that by Theorem 1 we can actually deduce that $n \equiv 3 \pmod{8}$ and $p \mid n$ implies $p - 1 \nmid k$. A slightly improved and extended version of Moser's results was given by the second author as Theorem 4 in [16]. This also incorporates that $n \equiv 3 \pmod{8}$ (explicitly) and $p \mid n$ implies $p - 1 \nmid k$ (implicitly). The implicit fact follows from [16, (8)] which states that

$$\sum_{(p-1) \mid k,\ p \mid n} \frac{1}{p} \in \mathbb{Z} \tag{6}$$

and the remark that a sum of reciprocals of distinct primes can never be a positive integer. Moser's proof rests on deriving four equations similar to (6) (these are the four mathemagical rabbits in the title of [16]). The baby mathemagical rabbit (6) he apparently overlooked.

Theorem 1 can also be used to get some information on the generalized Erdős-Moser equation $S_k(n-1) = an^k$, with a a fixed positive integer. Here it is not difficult to show that if there is a solution with $k > 1$, then k must be even. By Theorem 1 we then infer that if

(a, n, k) is a solution with $k > 1$, then n is odd and $p \mid n$ implies $p - 1 \mid k$. These are known results, see Moree [13].

3.2. **The Carlitz-von Staudt theorem.** Proposition 1 deals only with the case k even. Carlitz [2] considered the case k is odd and claimed that $n \mid S_k(n)$ in that case. The second author [12] pointed out that this is false. It is true, however, that $S_k(n) = rn/2$ with r an integer. The following lemma from a preprint of Kellner [8] gives the parity of r.

Lemma 5. *Let $k \geq 3$ be odd. We have $S_k(n) = rn/2$ with r an integer. Here r is odd if $n \equiv 2 \pmod 4$ and r is even otherwise.*

Proof. Since k is odd, we have $j^k \equiv -(n-j)^k \pmod n$ for every integer j.

Case n is even: All terms of the sum cancel each other modulo n except for the middle term $(n/2)^k$. We infer that $S_k(n) = rn/2$ with $r \equiv (n/2)^{k-1} \pmod n$. It follows that r is even if $4 \mid n$ and r is odd if $n \equiv 2 \pmod 4$.

Case n is odd: The sum $S_k(n)$, having no middle term, vanishes modulo n and hence r is even. □

Using this lemma we can give a general version of Proposition 1

Proposition 2. *Let $k, n \geq 1$ be integers, then*

$$S_k(n) \equiv \begin{cases} -\sum_{\substack{p \mid n \\ p-1 \mid k}} \frac{n}{p} \pmod n, & \text{if } k \text{ is even;} \\ n/2 \pmod n, & \text{if } k = 1 \text{ and } n \text{ is even;} \\ n/2 \pmod n, & \text{if } k > 1 \text{ is odd and } n \equiv 2 \pmod 4; \\ 0 \pmod n, & \text{otherwise.} \end{cases}$$

Proof. If k is even this is the classical result given in Proposition 1. If $k = 1$ it is clear that $S_k(n) = n(n+1)/2$ so, $S_k(n) \equiv n/2 \pmod n$ if n is even and $S_k(n) \equiv 0 \pmod n$ if n is odd. The remaining cases follow immediately from Lemma 5. □

Lemma 5 can be sharpened. In [16] the second author showed that in fact $S_k(n) = tn(n+1)/2$. We now determine the parity of t

Proposition 3. *Let $k \geq 3$ be odd. We have $S_k(n) = tn(n+1)/2$ with t an integer. Here t is odd if $n \equiv 1, 2 \pmod 4$ and t is even otherwise.*

Proof. Since k is odd, we have $j^k \equiv -(n-j)^k \pmod n$ and $j^k \equiv -(n-j+1)^k \pmod{n+1}$ for every integer j.

Case n is even: In this case we have that $S_k(n) \equiv (n/2)^k \pmod n$ and $S_k(n) \equiv 0 \pmod{n+1}$. Since $\gcd(n, n+1) = 1$, we infer that $S_k(n) = tn(n+1)/2$ with $t \equiv (n/2)^{k-1} \pmod 2$. It follows that t is even if $4 \mid n$ and t is odd if $n \equiv 2 \pmod 4$.

Case n is odd: In this case we have that $S_k(n) \equiv 0 \pmod n$ and $S_k(n) \equiv ((n+1)/2)^k \pmod{n+1}$. Since $\gcd(n, n+1) = 1$, we infer that $S_k(n) = tn(n+1)/2$ with $t \equiv ((n+1)/2)^{k-1} \pmod 2$. It follows that t is even if $4 \mid n+1$ and t is odd if $n \equiv 1 \pmod 4$. □

4. The affine case

In this section we will focus on the case where f is an *affine function*; i.e., a linear function. In what follows we will denote $an + b$ by $f_{a,b}(n)$. Recall the definition (1) of $\mathcal{M}_f$. In what follows it will be easier to characterize $\mathbb{N}_f \setminus \mathcal{M}_f$ instead of $\mathcal{M}_f$ itself, where

$$\mathbb{N}_f = \{n \in \mathbb{N} : f(n) > 1\}.$$

Let us introduce some further notation. Given $(a, b) \in \mathbb{N} \times \mathbb{Z}$, we will consider the set

$$\mathcal{P}_{a,b} := \{p \text{ odd prime} : b \equiv 0 \pmod{\gcd(a, p-1)}\}.$$

and if $(a, b, p) \in \mathbb{N} \times \mathbb{Z} \times \mathcal{P}_{a,b}$ we define

$$\mu_{a,b}(p) := \min\{x \in \mathbb{N} : xpa \equiv -b \pmod{p-1}\}.$$

Note that in case p is an odd prime the equation $xpa \equiv -b \pmod{p-1}$ has a solution if and only if p is in $\mathcal{P}_{a,b}$. For notational convenience we shorten $\{n \in \mathbb{N} : n \equiv c \pmod d\}$ to $\{c \pmod d\}$. The intersection of a set S with $\mathbb{N}_f$ will be denoted by S_f. With this notation in mind we can prove the following result.

Theorem 2. *Let $(a, b) \in \mathbb{N} \times \mathbb{Z}$. Put $p_a := (p-1)/\gcd(a, p-1)$ and $f(n) := a + bn$. Then*

i) *If a and b are even,*

$$\mathbb{N}_f \setminus \mathcal{M}_f = \{2\mathbb{N}\}_f \ \cup \bigcup_{p \in \mathcal{P}_{a,b}} \{p\mu_{a,b}(p) \pmod{p \cdot p_a}\}_f.$$

ii) *If a and b are odd,*

$$\mathbb{N}_f \setminus \mathcal{M}_f = \{2 \pmod 4\}_f \ \cup \bigcup_{p \in \mathcal{P}_{a,b}} \{p\mu_{a,b}(p) \pmod{p \cdot p_a}\}_f.$$

iii) *If a is even and b is odd, then*

$$\mathbb{N}_f \setminus \mathcal{M}_f = \{2 \pmod 4\}_f.$$

iv) *If a is odd and b is even, then*

$$\mathbb{N}_f \setminus \mathcal{M}_f = \{2\mathbb{N}\}_f.$$

Proof. Suppose that $n \in \mathbb{N}_f$. Then $f(n) > 1$. By Theorem 1 we have $n \nmid S_f(n)$ if and only if
a) n is odd and $p - 1 \mid f(n)$ for some odd prime divisor p of n;
b) $n \equiv 2 \pmod 4$;
or
c) n is a multiple of 4, $f(n)$ is even.

We will give a complete proof of i), the other cases being similar.

Since by assumption a and b are even, $f(n)$ is even and hence, by b) and c), we have that $\{2\mathbb{N}\}_f \subseteq \mathbb{N}_f \setminus \mathcal{M}_f$. Now, assume that $n \notin \mathcal{M}_f$ is odd. Then by a) there must exist an odd prime $p \mid n$ such that $p - 1 \mid an + b$. Since $an \equiv 0 \pmod{ap}$ and $an \equiv -b \pmod{p-1}$ it follows that p is in $\mathcal{P}_{a,b}$ and $an \in \{A + s \cdot \operatorname{lcm}(ap, p-1) : s \geq 0\}$ with

$$A = \min\{x \in \mathbb{N} : x \equiv 0 \pmod{ap}, x \equiv -b \pmod{p-1}\}.$$

Using that $A = ap\mu_{a,b}(p)$ we find that

$$n \in \{\frac{A}{a} + \frac{s}{a}\operatorname{lcm}(ap, p-1) : s \geq 0\} = \{p\mu_{a,b}(p) \pmod{p \cdot p_a}\}.$$

On taking the requirement $f(n) > 1$ into account we obtain that $n \in \{p\mu_{a,b}(p) \pmod{p \cdot p_a}\}_f$ for some $p \in \mathcal{P}_{a,b}$ is necessary and sufficient for an odd n to be in $\mathbb{N}_f \setminus \mathcal{M}_f$. □

Here and throughout, we denote by $\delta(A)$ (resp. $\underline{\delta}(A)$, $\overline{\delta}(A)$) the asymptotic (resp. lower, upper asymptotic) density of an integer sequence A. Recall that

$$\delta(A) = \lim_{N \to \infty} \frac{\operatorname{card}([0, N] \cap A)}{N},$$

while $\underline{\delta}(A)$ and $\overline{\delta}(A)$ are obtained using the lower or upper limit in the previous expression.

We will be interested in computing the asymptotic density of the sets $\mathcal{M}_{f_{a,b}}$, at least for some particular values of a and b. To do so we must first show that this density exists and the following lemma will be our main tool.

Lemma 6. *Let $\mathcal{A} := \{a_k\}_{k\in\mathbb{N}}$ and $\{c_k\}_{k\in\mathbb{N}}$ be two sequences of positive integers and*

$$\mathcal{B}_k := \{a_k + (s-1)c_k : s \in \mathbb{N}\}.$$

If $\sum_{k=1}^{\infty} c_k^{-1}$ is convergent and $\mathcal{A}$ has zero asymptotic density, then $\bigcup_{k=1}^{\infty} \mathcal{B}_k$ has an asymptotic density with $\delta(\bigcup_{k=1}^{\infty} \mathcal{B}_k) = \lim_{n\to\infty} \delta(\bigcup_{k=1}^{n} \mathcal{B}_k)$ and

$$\delta(\bigcup_{k=1}^{\infty} \mathcal{B}_k) - \delta(\bigcup_{k=1}^{n} \mathcal{B}_k) \leq \sum_{i=n+1}^{\infty} \frac{1}{c_i}.$$

Proof. Let us denote $B_n := \bigcup_{k=n+1}^{\infty} \mathcal{B}_k$ and $\vartheta(n,N) := \mathrm{card}([0,N] \cap B_n)$. Then

$$\vartheta(n,N) \leq \mathrm{card}([0,N] \cap \mathcal{A}) + N \sum_{k=n+1}^{\infty} \frac{1}{c_k}.$$

From this, we get

$$\overline{\delta}(B_n) = \limsup \frac{\vartheta(n,N)}{N} \leq \limsup \frac{\mathrm{card}([0,N] \cap \mathcal{A})}{N} + \sum_{k=n+1}^{\infty} \frac{1}{c_k} = \sum_{k=n+1}^{\infty} \frac{1}{c_k}.$$

Now, for every n, $\bigcup_{k=1}^{n} \mathcal{B}_k$ has an asymptotic density and the sequence $\delta_n := \delta\left(\bigcup_{k=1}^{n} \mathcal{B}_k\right)$ is non-decreasing and bounded (by 1), thus convergent. Consequently

$$\delta_n \leq \underline{\delta}\left(\cup_{k=1}^{\infty}\mathcal{B}_k\right) \leq \overline{\delta}\left(\cup_{k=1}^{\infty}\mathcal{B}_k\right) = \overline{\delta}\left(\cup_{k=1}^{n}\mathcal{B}_k \cup B_n\right) \leq \delta_n + \overline{\delta}(B_n) \leq \delta_n + \sum_{k=n+1}^{\infty} c_k^{-1},$$

and taking into account that $\sum_{k=n+1}^{\infty} c_j^{-1}$ converges to zero, it is enough to take limits in order to finish the proof. □

With the help of this lemma the following proposition is easy to prove.

Proposition 4. *If $(a,b) \in \mathbb{N} \times \mathbb{Z}$, then the set $\mathcal{M}_{f_{a,b}}$ has an asymptotic density $\delta(\mathcal{M}_{f_{a,b}})$.*

Proof. As $\delta(\mathbb{N}_{f_{a,b}}) = 1$ it is enough to see that $\mathbb{N}_{f_{a,b}} \setminus \mathcal{M}_{f_{a,b}}$ has an asymptotic density.

Cases iii) and iv) above are obvious. In cases i) and ii) it is enough to apply the previous lemma since $\mathbb{N}_{f_{a,b}} \setminus \mathcal{M}_{f_{a,b}}$ is a countable union of arithmetic progressions modulo $p \cdot p_a$ whose initial terms, $p \cdot \mu_{a,b}(p)$, form a set of zero asymptotic density, and the associated series of reciprocal moduli

$$\sum_{p \text{ prime}} \frac{1}{p \cdot p_a} = \sum_{p \text{ prime}} \frac{\gcd(a, p-1)}{p(p-1)}$$

is convergent. □

The rest of this section will be devoted to the study of $\delta(\mathcal{M}_{f_{1,b}})$. If b is even, $\mathcal{M}_{f_{1,b}}$ is exactly the set of odd positive integers $> 1-b$ and its asymptotic density is $\frac{1}{2}$. The case when b is odd is much more interesting. In particular we will see that, in this case, the asymptotic density of $\mathcal{M}_{f_{1,b}}$ is slightly greater than $\frac{1}{2}$. Our density computation will be based on the following corollary of Theorem 2.

Corollary 1. *Put*

$$\mathcal{G}_p^b := \{-bp \pmod{p(p-1)}\}. \tag{7}$$

If $b \in \mathbb{Z}$ is odd, then $\mathbb{N}_{f_{1,b}} \setminus \mathcal{M}_{f_{1,b}} = \bigcup_{p\geq 3}\{\mathcal{G}_p^b\}_{f_{1,b}} \cup \{2 \pmod 4\}_{f_{1,b}}$.

We note that $\delta(\bigcup_{p\geq 3} \mathcal{G}_p^0)$ is the density of the set of integers such that $p(p-1) \mid m$ for some $p \mid m$ with $p \geq 3$. Note that, for b odd,

$$\delta(\mathcal{M}_{f_{1,b}}) = 1 - \delta(\mathbb{N}_{f_{1,b}} \setminus \mathcal{M}_{f_{1,b}}) = 1 - \delta(\bigcup_{p\geq 3} \mathcal{G}_p^b \cup \{2 \pmod 4\}) = \frac{3}{4} - \delta(\bigcup_{p\geq 3} \mathcal{G}_p^b), \tag{8}$$

where we used the observation that $\mathcal{G}_p^b$ consists of odd integers only. The final density in (8) can be computed using the inclusion-exclusion principle. For this it will be necessary to have a good criterion to determine when the intersection of $\mathcal{G}_p^b$ for various odd primes p is empty. For m square-free we have $\mathrm{lcm}[p-1 : p \mid m] = \lambda(m)$, with λ the Carmichael function.

Proposition 5. *Let $\mathcal{P}$ be a finite set of odd primes and put $m := \prod_{p\in\mathcal{P}} p$. Then $\bigcap_{p\in\mathcal{P}} \mathcal{G}_p^b$ is non-empty if and only if $\gcd(m, \phi(m)) \mid b$. If the intersection is non-empty, then the set $\bigcap_{p\in\mathcal{P}} \mathcal{G}_p^b$ is an arithmetic progression having modulus lcm$(m, \lambda(m))$.*

Proof. It is clear that $\bigcap_{p\in\mathcal{P}} \mathcal{G}_b^b$ is non-empty if and only if there exists n such that $n/p \equiv n \equiv -b \pmod{p-1}$ and $p \mid n$ for every $p \in \mathcal{P}$. This happens if and only if there exists n such that $n \equiv -b \pmod{\lambda(m)}$ and $n \equiv 0 \pmod m$. Note that the latter congruences have a solution if and only if $\gcd(m, \lambda(m))$ divides b. To finish the proof it is enough to observe that, m being square-free, $\gcd(m, \lambda(m)) = \gcd(m, \phi(m))$ and to apply the Chinese remainder theorem. □

To compute the density of the set $\mathbb{N} \setminus \mathcal{M}_{f_{1,b}}$ we define, given $\epsilon > 0$, $k := k(\varepsilon)$ to be the smallest integer such that

$$\sum_{j\geq k} \frac{1}{p_j(p_j-1)} < \varepsilon.$$

Thus, with an error of at most ε, the density of the set $\mathbb{N} \setminus \mathcal{M}_{f_{1,b}}$ is the same as the density of $\bigcup_{j<k} \mathcal{G}_{p_j}^b$:

$$\delta\left(\bigcup_{j<k} \mathcal{G}_{p_j}^b\right) < \delta(\mathbb{N} \setminus \mathcal{M}_{f_{1,b}}) < \delta\left(\bigcup_{j<k} \mathcal{G}_{p_j}^b\right) + \varepsilon$$

and, by the inclusion-exclusion principle, we find

$$\delta\left(\bigcup_{j<k} \mathcal{G}_{p_j}^b\right) = \sum_{s\geq 1} \sum_{1\leq i_1<i_2<\cdots<i_s\leq k-1} \frac{\alpha_{i_1,i_2,\ldots,i_s}}{\mathrm{lcm}[p_{i_1}(p_{i_1}-1), \ldots, p_{i_s}(p_{i_s}-1)]},$$

with the coefficient $\alpha_{i_1,i_2,\ldots,i_s}$ being zero if $\bigcap_{t=1}^{s} \mathcal{G}_{p_{i_t}}^b = \emptyset$, and being $(-1)^{s-1}$ otherwise. Alternatively we can write this as

$$\delta\left(\bigcup_{j<k} \mathcal{G}_{p_j}^b\right) = -\sum_{\substack{m>1,\ m\mid p_2p_3\cdots p_{k-1} \\ \gcd(m,\varphi(m)\mid b}} \frac{\mu(m)}{\mathrm{lcm}(m, \lambda(m))}. \tag{9}$$

It is not difficult to see, cf. [4], that the series

$$\sum_{\gcd(m,\varphi(m))|b} \frac{\mu(m)}{\operatorname{lcm}(m,\lambda(m))}$$

converges absolutely. Using this, (8) and (9), we then obtain the following result.

Theorem 3. *If $b \in \mathbb{Z}$ is odd, then*

$$\delta(\mathcal{M}_{f_{1,b}}) = \frac{3}{4} + \sum_{\substack{m>1,\ 2\nmid m \\ \gcd(m,\varphi(m))|b}} \frac{\mu(m)}{lcm(m,\lambda(m))}.$$

Corollary 2. *We have*

$$\delta(\mathcal{M}_{f_{1,\pm 1}}) = \frac{3}{4} + \sum_{\substack{m>2 \\ \gcd(m,\varphi(m))=1}} \frac{(-1)^{\omega(m)}}{m\lambda(m)},$$

where $\omega(m)$ is the number of distinct prime factors of m.

Proof. Note that $m > 1, 2 \nmid m$ and $\gcd(m,\varphi(m)) \in \{-1,1\}$ if and only if $m > 2$ and $\gcd(m,\varphi(m)) = 1$. The m satisfying these conditions are odd and square-free and thus we have $\gcd(m,\varphi(m)) = \gcd(m,\lambda(m)) = 1$ and hence $\operatorname{lcm}(m,\lambda(m)) = m\lambda(m)$ and $\mu(m) = (-1)^{\omega(m)}$. □

The asymptotic density of $\mathcal{M}_{f_{1,\pm 1}}$ is closely related to that of the set

$$\mathfrak{P} := \{n \geq 1 : 2 \nmid n,\ S_{\frac{n-1}{2}} \equiv 0 \pmod{n}\},$$

which was defined and studied in [4] and where it is shown that

$$\delta(\mathfrak{P}) = \frac{1}{2} + \sum_{\substack{m>2 \\ \gcd(m,\varphi(m)=1}} \frac{(-1)^{\omega(m)}}{2m\lambda(m)} \in [0.379005, 0.379826].$$

On combining this with Corollary 2 we reach the following conclusion.

Proposition 6. *We have $\delta(\mathcal{M}_{f_{1,\pm 1}}) = 2\delta(\mathfrak{P}) - 1/4 \in [0.50801, 0.50966]$.*

Recall that a Carmichael number n is a positive composite integer that satisfies Fermat's Little Theorem: $a^{n-1} \equiv 1 \pmod{n}$ for every a coprime to n. It follows that a Carmichael number n meets Korselt's criterion: it must be square-free with $p-1$ dividing $n-1$ for each prime factor p of n. We will say that a positive integer n is an *anti-Korselt number* if for every p prime divisor of n, $p-1$ does not divide $n-1$.

Lemma 7.
i) An integer n is an anti-Korselt number if and only if $2 \nmid n$ and $n \mid S_{n-1}(n)$.
ii) The set of anti-Korselt numbers $\mathfrak{K}$ has an asymptotic density $\delta(\mathfrak{K})$ satisfying

$$\delta(\mathfrak{K}) = \delta(\mathcal{M}_{f_{1,-1}}) - \frac{1}{4} = 2\delta(\mathfrak{P}) - \frac{1}{2} \in [0.25801, 0.259652].$$

Proof. i) This follows from Theorem 1 and the observation that anti-Korselt numbers are odd.

ii) The density $\delta(\mathfrak{K})$ equals that of the odd integers in $M_{f_{1,-1}}$, and hence, keeping in mind that the sets G_p^{-1}, $p \geq 3$, consist of odd numbers only, we infer from Corollary 1 that

$$\delta(\mathfrak{K}) = \delta(\{n : 2 \nmid n\}) - \delta(\bigcup_{p \geq 3} \mathcal{G}_p^{-1}) = \frac{1}{2} - \delta(\bigcup_{p \geq 3} \mathcal{G}_p^{-1}).$$

By (8) we see that $\delta(\mathfrak{K}) = \delta(\mathcal{M}_{f_{1,-1}}) - 1/4$. Now invoke Proposition 6. □

Lemma 8. *Let $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ be two families of sets such that:*

i) *$\delta(A_i) = \delta(B_i)$.*
ii) *$\delta(A_i \bigcap A_j) \geq \delta(B_i \bigcap B_j)$.*

Then

$$\delta(\bigcup_{i=1}^n A_i) \leq \delta(\bigcup_{i=1}^n B_i),$$

with the inequality being strict if any of the inequalities in ii) is strict.

Proof. We proceed by induction on n. The result for $n = 2$ is trivial. Now, assume that

$$\delta(\bigcup_{i=1}^n A_i) \leq \delta(\bigcup_{i=1}^n B_i).$$

Note that, from condition ii) it follows that

$$\delta(A_{n+1} \bigcap (\bigcup_{i=1}^n A_i)) \geq \delta(B_{n+1} \bigcap (\bigcup_{i=1}^n B_i)).$$

Hence, we have that

$$\delta(\bigcup_{i=1}^{n+1} A_i) = \delta(\bigcup_{i=1}^n A_i) + \delta(A_{n+1}) - \delta(A_{n+1} \bigcap (\bigcup_{i=1}^n A_i)) \leq$$
$$\leq \delta(\bigcup_{i=1}^n B_i) + \delta(B_{n+1}) - \delta(B_{n+1} \bigcap (\bigcup_{i=1}^n B_i)) = \delta(\bigcup_{i=1}^{n+1} B_i),$$

and the result follows. □

Lemma 9. *Let $b \mid b'$ and suppose that m is an odd integer. We have*

$$\delta(\bigcup_{p \mid m} G_p^b) \geq \delta(\bigcup_{p \mid m} G_p^b) \geq \delta(\bigcup_{p \mid m} G_p^0) = -\sum_{d \mid m,\ d > 1} \frac{\mu(d)}{\operatorname{lcm}(d, \lambda(d))}.$$

Proof. We consider the families $\{\mathcal{G}_p^{b'}\}$, $\{\mathcal{G}_p^b\}$ and $\{\mathcal{G}_p^b\}$ (recall Corollary 1). Since $\mathcal{G}_p^{b'}$, $\mathcal{G}_p^b$ and $\mathcal{G}_p^b$ are arithmetic progressions of the same modulus $p(p-1)$, it follows that $\delta(\mathcal{G}_p^{b'}) = \delta(\mathcal{G}_p^{b'}) = \delta(\mathcal{G}_p^0)$. Also observe that, if $p \neq q$ are primes and $\mathcal{G}_p^{b'} \cap \mathcal{G}_q^{b'} = \emptyset$, then also $\mathcal{G}_p^b \cap \mathcal{G}_q^b = \emptyset$. On the other hand, if $\mathcal{G}_p^{b'} \cap \mathcal{G}_q^{b'} \neq \emptyset$, then $\mathcal{G}_p^b \cap \mathcal{G}_q^b$ is either empty or has the same density as $\mathcal{G}_p^{b'} \cap \mathcal{G}_q^{b'}$. Note that the intersection $\mathcal{G}_p^0 \cap \mathcal{G}_q^0$ is never empty. On applying Lemma 8 the two inequalities are established. The final identity holds by an argument similar to the one used to establish equation (9), where we use again that that the intersection $\mathcal{G}_p^0 \cap \mathcal{G}_q^0$ is never empty. □

Corollary 3. *If $b \mid b'$ and m is odd, then*

$$\delta(\mathcal{M}_{f_{1,b}}) \leq \delta(\mathcal{M}_{f_{1,b'}}) \leq \sum_{d|m} \frac{\mu(d)}{\mathrm{lcm}(d,\lambda(d))} - \frac{1}{4}.$$

On applying Proposition 6 and Corollary 3 with m the product of the first 22 odd primes, we obtain

$$0.508 < \delta(\mathcal{M}_{f_{1,1}}) < \delta(\mathcal{M}_{f_{1,b}}) < 0.647.$$

It is easy to observe using Lemma 8 that if b is odd and $|b| > 1$, then $\delta(\mathcal{M}_{f_{1,b}}) > \delta(\mathcal{M}_{f_{1,1}})$. Let $\kappa(n) = \prod_{p|n} p$ denote the squarefree kernel of n. By Theorem 3 it follows that if $\kappa(b) = \kappa(b')$, then $\delta(\mathcal{M}_{f_{1,b'}}) = \delta(\mathcal{M}_{f_{1,b}})$. Moreover, if $\kappa(b) \mid \kappa(b')$ and $\kappa(b) < \kappa(b')$, then $\delta(\mathcal{M}_{f_{1,b'}}) > \delta(\mathcal{M}_{f_{1,b}})$.

5. $\mathcal{M}_f$ containing the prime numbers

In this section we will characterize the set $\mathcal{M}_f$ for some functions f such that $f(p) = \frac{p-1}{2}$ for every odd prime. Note that in this case $\mathcal{M}_f$ contains all odd primes $p > 3$. In particular, we will focus on $f = \frac{\varphi}{2}$ and $f = \frac{\lambda}{2}$, where φ and λ denote the Euler and Carmichael function, respectively.

Proposition 7. *We have $\mathcal{M}_{\frac{\varphi}{2}} = \{p^k : p \text{ odd prime}\} \setminus \{3\}$.*

Proof. Note that $\frac{\varphi(p^k)}{2} > 1$ if and only if $p^k \neq 3$. Hence, $3 \notin \mathcal{M}_{\frac{\varphi}{2}}$ and in what follows we assume that $p^k \neq 3$.

If p is an odd prime and $k \in \mathbb{N}$, $\frac{\varphi(p^k)}{2} = \frac{p^{k-1}(p-1)}{2}$ and $\gcd\left(\frac{p^{k-1}(p-1)}{2}, p-1\right) < p-1$. Consequently we can apply Theorem 1 to get that $p^k \in \mathcal{M}_{\frac{\varphi}{2}}$.

Now, if n is odd and there exist distinct odd primes p, q dividing n, it readily follows that $p-1$ divides $\frac{\varphi(n)}{2}$ so Theorem 1 i) applies and it follows that $n \notin \mathcal{M}_{\frac{\varphi}{2}}$. Thus, if there is an odd $n \in \mathcal{M}_{\frac{\varphi}{2}}$ it must be a prime power exceeding 3.

Finally, if $n \in \mathcal{M}_{\frac{\varphi}{2}}$ is even, Theorem 1 ii) implies that 4 divides n and also that $\frac{\varphi(n)}{2}$ is odd and exceeding 1. Since these statements are contradictory the result follows. □

In what follows we will use the notation $\nu_2(m) := \max\{k \in \mathbb{N} : 2^k \text{ divides } m\}$.

Proposition 8. *Let $n = 2^m p_1^{r_1} \cdots p_s^{r_s}$ with $s > 0$. Then $3 \neq n \in \mathcal{M}_{\frac{\lambda}{2}}$ if and only if one of these conditions holds:*

i) *$m = 0$ and $\nu_2(p_i - 1) = \nu_2(p_j - 1)$ for every i, j.*
ii) *$m = 2$ or 3, $\nu_2(p_i - 1) = 1$ for every i and $\frac{n}{2^m} \neq 3$.*

Proof. Note that $\frac{\lambda(n)}{2} > 1$ if and only if $n \neq 3$. Hence, $3 \notin \mathcal{M}_{\frac{\lambda}{2}}$ and in what follows we assume that $n \neq 3$.

If condition i) holds, $n = p_1^{r_1} \cdots p_s^{r_s}$ and $p_i = 2^t q_i + 1$ with q_i even and t not depending on i. In this case $\lambda(n) = \mathrm{lcm}(\varphi(p_1^{r_1}), \ldots, \varphi(p_s^{r_s})) = 2^t \mathrm{lcm}(p_1^{r_1-1} q_1, \ldots, p_s^{r_s-1} q_s) = 2^t L$ with L odd. Consequently $\frac{\lambda(n)}{2} = 2^{t-1} L$ and since L is odd it follows that $p_i - 1$ does not divide $\frac{\lambda(n)}{2}$ and Theorem 1 i) implies that $n \in \mathcal{M}_{\frac{\lambda}{2}}$.

If condition ii) holds, it follows that $\lambda(n) = 2L$ with $L > 1$ odd. Consequently $\lambda(n)/2 = L > 1$ is odd and by Theorem 1 ii) we conclude that $n \in \mathcal{M}_{\frac{\lambda}{2}}$.

Finally, assume that $n = 2^m p_1^{r_1} \cdots p_s^{r_s}$ with $s > 0$ and $p_i = 2^{m_i} q_i + 1$ with q_i odd is such that $n \in \mathcal{M}_{\frac{\lambda}{2}}$. First of all, Theorem 1 implies that $m = 0$ or $m > 1$.

If $m > 1$, Theorem 1 ii) implies that $\frac{n}{2^m} \neq 3$ and also that $\frac{\lambda(n)}{2}$ is odd so $m = 2$ or 3 and $p_i^{r_i - 1}(p_i - 1) = \varphi(p_i^{r_i}) = 2L_i$ with L_i odd; i.e., $p_i - 1 = 2q_i$ with q_i odd as claimed.

If, on the other hand, $m = 0$, Theorem 1 i) implies that $p_i - 1$ does not divide $\frac{\lambda(n)}{2}$ for any i. But if $m_i > m_j$ for some $i \neq j$ we have that $2^{m_i - 1} q_j$ divides $\frac{\lambda(n)}{2}$ and, consequently, $p_j - 1$ divides $\frac{\lambda(n)}{2}$. A contradiction. □

Now, given a positive integer k we define the set

$$\Upsilon_k := \{n \text{ odd } : \nu_2(p-1) = k \text{ for every } p \mid n\}.$$

Note that if $k \neq j$, then Υ_k and Υ_j are disjoint. With this notation, Proposition 8 can be stated as

$$\mathcal{M}_{\frac{\lambda}{2}} = \left(\bigcup_{k=1}^{\infty} \Upsilon_k \cup 4\Upsilon_1 \cup 8\Upsilon_1 \right) \setminus \{3, 12, 24\}. \tag{10}$$

Let $\mathcal{M}_{\frac{\lambda}{2}}(x)$ denote the number of integers $\leq x$ in the set $\mathcal{M}_{\frac{\lambda}{2}}$ and $\Upsilon_j(x)$ the number of integers $\leq x$ in the set Υ_j.

Proposition 9. *Let $k \geq 1$ be an arbitrary integer. We have*

$$\mathcal{M}_{\frac{\lambda}{2}}(x) = \frac{x}{\log x}\Big(c_1 \log^{1/2} x + \sum_{j=2}^{k} c_j \log^{2^{-k}} x + O_k(\log^{2^{-k-1}} x) \Big),$$

with

$$c_1 = \frac{11}{16} \prod_{p \equiv 1 \pmod 4} \left(1 - \frac{1}{p^2}\right)^{1/2} = 0.66896484\cdots$$

and all constants $c_2, \ldots, c_k$ positive. The implied constant in the error term depends at most on k.

Proof. For positive coprime integers a and d, let $N_{a,d}(x)$ denote the number of integers $n \leq x$ that are composed only of primes $p \equiv a \pmod d$. It is a standard result, cf. [15], that

$$N_{a,d}(x) = \frac{c_{a,d}\, x}{\log^{1-1/\varphi(d)} x}\Big(1 + O_d\big(\frac{1}{\log x}\big)\Big), \tag{11}$$

with $c_{a,d}$ a positive constant. For $j \geq 1$ we have, by (11),

$$\Upsilon_j(x) = N_{1+2^j, 2^{j+1}}(x) = \frac{d_j x}{\log^{1-2^{-j}} x}\Big(1 + O_j\big(\frac{1}{\log x}\big)\Big), \tag{12}$$

with d_j a positive constant. One has, cf. [15, pp. 235],

$$d_1 = \frac{1}{2} \prod_{p \equiv 1 \pmod 4} \left(1 - \frac{1}{p^2}\right)^{1/2} = 0.4865198883\cdots \tag{13}$$

Note that

$$\sum_{j=k+1}^{\infty} \Upsilon_j(x) \leq N_{1,2^{k+1}}(x) = O(x \log^{2^{-k-1}-1} x). \tag{14}$$

Since the infinite sets in the decomposition (10) are pairwise disjoint we see from (10) that

$$\mathcal{M}_{\frac{\lambda}{2}}(x) = \Upsilon_1(x) + \Upsilon_1(\frac{x}{4}) + \Upsilon_1(\frac{x}{8}) + \sum_{j=2}^{k} \Upsilon_j(x) + \sum_{j=k+1}^{\infty} \Upsilon_j(x) + O(1).$$

The result now follows from (12) and (14) with $c_1 = 11d_1/8$ and $c_j = d_j$ for $j \geq 2$. □

`Remark`. By Satz 1 of Wirsing [22] we have

$$N_{3,4}(x) \sim \frac{e^{-\gamma/2}}{\sqrt{\pi}} \frac{x}{\log x} \prod_{\substack{p \leq x \\ p \equiv 3 \pmod 4}} (1 - 1/p)^{-1}.$$

On inserting Uchiyama's asymptotic for the latter product (see [21]) and using $\prod_p (1-1/p^2) = 1/\zeta(2) = 6/\pi^2$, one finds that $N_{3,4}(x) \sim d_1 x (\log x)^{-1/2}$ with d_1 as in (13).

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