CYCLOTOMIC NUMERICAL SEMIGROUPS

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ABSTRACT. Given a numerical semigroup S, we let $\mathrm{P}_S(x) = (1-x)\sum_{s\in S} x^s$ be its *semigroup polynomial*. We study *cyclotomic numerical semigroups*; these are numerical semigroups S such that $\mathrm{P}_S(x)$ has all its roots in the unit disc. We conjecture that S is a cyclotomic numerical semigroup if and only if S is a complete intersection numerical semigroup and present some evidence for it.

Aside from the notion of cyclotomic numerical semigroups we introduce the notion of *cyclotomic exponents* and *polynomially related numerical semigroups*. We derive some properties and give some applications of these new concepts.

1. INTRODUCTION

A *numerical semigroup* S is a submonoid of $\mathbb{N}$ (the set of nonnegative integers) under addition, with finite complement in $\mathbb{N}$. The nonnegative integers not in S are its *gaps*, and the largest integer not in S is its *Frobenius number*, $\mathrm{F}(S)$. The number of gaps of S, also known as the *genus* of S, is denoted by $\mathrm{g}(S)$. A numerical semigroup admits a unique minimal generating system; its cardinality is called its *embedding dimension* $\mathrm{e}(S)$, and its elements *minimal generators*. The smallest positive integer in S is called the *multiplicity* of S, and it is denoted by $\mathrm{m}(S)$ (see for instance [24] for an introduction to numerical semigroups).

For $A \subseteq \mathbb{N}$, we use $\langle A\rangle$ to denote the set of integers of the form $\sum_{a\in A}\lambda_a a$ where $a \in A$, $\lambda_a \in \mathbb{N}$ and all but finitely many λ_a are equal to zero. The set $\langle A\rangle$ is a numerical semigroup if and only if the greatest common divisor of the elements of A equals 1.

To a numerical semigroup S we can associate $\mathrm{H}_S(x) := \sum_{s\in S} x^s$, its *Hilbert series* (sometimes called the generating function associated to S), and $\mathrm{P}_S(x) = (1-x)\sum_{s\in S} x^s$, its *semigroup polynomial*. Since all elements larger than $\mathrm{F}(S)$ are in S, $\mathrm{H}_S(x)$ is not a polynomial, but $\mathrm{P}_S(x)$ is. On noting that $\mathrm{H}_S(x) = (1-x)^{-1} - \sum_{s\notin S} x^s$, we see that

$$\mathrm{P}_S(x) = 1 + (x-1)\sum_{s\notin S} x^s, \tag{1}$$

where $s \notin S$ denotes the sum over the numbers in $\mathbb{N}\backslash S$. Observe that $\mathrm{P}_S(x)$ is a monic polynomial of degree $\mathrm{F}(S)+1$.

Recall (see, for instance, Damianou [6]) that a *Kronecker polynomial* is a monic polynomial with integer coefficients having all its roots in the unit disc. We define a numerical semigroup to be *cyclotomic* if its semigroup polynomial is a Kronecker polynomial. The following result of Kronecker and the fact that $\mathrm{P}_S(1) \neq 0$ allow us to give an alternative more explicit definition (readers not so familiar with cyclotomic polynomials are referred to Section 2.1).

Lemma 1 (Kronecker, 1857, cf. [6]). *If f is a Kronecker polynomial with $f(0) \neq 0$, then all roots of f are actually on the unit circle and f factorizes over the rationals as a product of cyclotomic polynomials.*

2010 *Mathematics Subject Classification.* 20M14, 11C08, 11B68.

Key words and phrases. Numerical semigroups, cyclotomic polynomials, semigroup polynomials, cyclotomic numerical semigroups, cyclotomic exponents, polynomially related semigroups.

The second author is supported by the projects MTM2010-15595, MTM2014-55367-P, FQM-343, FQM-5849, and FEDER funds.

Definition 1. *A numerical semigroup is* cyclotomic *if its semigroup polynomial factorizes over the rational numbers as a product of cyclotomic polynomials, that is, if we can write*

$$\mathrm{P}_S(x) = \prod_{d\in\mathscr{D}} \Phi_d(x)^{e_d}, \tag{2}$$

with $\mathscr{D}$ a finite set of positive integers and every e_d a positive integer.

Remark 1. *Since cyclotomic polynomials are irreducible, the exponents e_d are unique.*

Remark 2. *On using that Φ_n is selfreciprocal for $n > 1$ and that Φ_1 does not divide P_S, we infer that if S is cyclotomic, then P_S is selfreciprocal.*

We can now formulate the main problem we like to address:

Problem 1. *Find an intrinsic characterization of the numerical semigroups S for which S is cyclotomic, that is, one which does not involve P_S or its roots in any way.*

Our conjectural solution of this problem involves two classes of numerical semigroups: the symmetric ones and the complete intersection ones.

Recall that a numerical semigroup S is said to be *symmetric* if $S \cup (\mathrm{F}(S) - S) = \mathbb{Z}$, thus symmetry is an example of an intrinsic characterization of S.

Theorem 1. *If S is cyclotomic, then it must be symmetric.*

Proof. Using (1) it is not difficult to conclude (see Moree [21]) that S is symmetric if and only if P_S is selfreciprocal. By Remark 2, P_S is selfreciprocal. □

The converse is however not true, as illustrated at the end of Section 3.

New let us recall the definition of *complete intersection* numerical semigroups. If S is minimally generated by $\{n_1,\dots,n_e\}$, then the monoid morphism $\phi : \mathbb{N}^e \to S$, $\phi(a_1,\dots,a_e) = \sum_{i=1}^{e} a_i n_i$ is an epimorphism. Consequently S is isomorphic, as a monoid, to $\mathbb{N}^e/\ker\phi$, where $\ker\phi = \{(a,b) \in \mathbb{N}^e \times \mathbb{N}^e : \phi(a) = \phi(b)\}$ is the kernel congruence of ϕ. It turns out that $\ker\phi$ is finitely generated (as a congruence) and that the minimum number of generators is at least $\mathrm{e}(S) - 1$ (see for instance [24]). Then S is a *complete intersection* numerical semigroup if $\ker\phi$ is minimally generated by $\mathrm{e}(S) - 1$ pairs. It is well-known that every complete intersection numerical semigroup is symmetric (see for instance [24, Corollary 9.17]).

The following observation is not deep, see Section 5.

Lemma 2. *Every complete intersection numerical semigroup is cyclotomic.*

The next lemma sums up the above[1].

Lemma 3. *We have the following inclusions of numerical semigroups*

$$\{\textit{complete intersection}\} \subseteq \{\textit{cyclotomic}\} \subseteq \{\textit{symmetric}\}.$$

Let $\mathscr{I}_k$, $\mathscr{C}_k$ and $\mathscr{S}_k$ denote the set of numerical semigroups that have Frobenius number k and are complete intersection, cyclotomic and symmetric, respectively. Using the GAP package [8] it is seen that $\mathscr{I}_k = \mathscr{C}_k$ for $k \le 70$. If in the sequel we state that a polynomial is Kronecker or not, this was always established using this package (using the Graeffe method based on [4]).

We conjecture that the first two sets in Lemma 3 are actually equal (that is, $\mathscr{I}_k = \mathscr{C}_k$ for every $k \ge 1$).

Conjecture 1. *A numerical semigroup is cyclotomic if and only if it is a complete intersection numerical semigroup.*

The second set, however, is strictly contained in the third one (we have, e.g., $\mathscr{C}_9 \subsetneq \mathscr{S}_9$, see Section 3). Here we make the following conjecture.

[1] The referee suggested the following mnemonic as an easy way to remember the order of inclusions. Look at the initials of the words and sort them alphabetically: CI ⊆ CY ⊆ SY.

Conjecture 2. *Let $e \geq 4$. There exists a symmetric numerical semigroup of embedding dimension e that is not cyclotomic.*

In the rest of the paper, the main focus is on theoretical contributions towards solving Conjecture 1 and some related problems.

Throughout, the letters p, q and r are used to indicate primes.

For a pedestrian introduction to both cyclotomic polynomials and numerical semigroups, the reader is referred to Moree [21].

2. Tools

2.1. **Cyclotomic polynomials.**

In this section we discuss relevant (elementary) properties of cyclotomic polynomials. A nice introduction to cyclotomic polynomials is Thangadurai [27].

We let

$$\Phi_n(x) = \prod_{j=1,\,(j,n)=1}^{n} \left(1 - e^{\frac{2\pi i j}{n}}\right)$$

denote the n-th cyclotomic polynomial. It is well-known that it has integer coefficients. Furthermore it is monic of degree $\varphi(n)$ (where φ denotes Euler's totient function) and irreducible over $\mathbb{Q}$ (see, e.g., Weintraub [29]). Over the rational numbers $x^m - 1$ factorizes into irreducibles as

$$x^m - 1 = \prod_{d \mid m} \Phi_d(x). \tag{3}$$

This equation implies that Φ_d divides $x^n - 1$ if and only if d divides n. By Möbius inversion we infer from (3) that

$$\Phi_n(x) = \prod_{d \mid n} (x^d - 1)^{\mu(n/d)}, \tag{4}$$

where $\mu(n)$ denotes the Möbius function. It follows, for example from (4), that if p and q are distinct primes, then

$$\Phi_{pq}(x) = \frac{(x^{pq} - 1)(x - 1)}{(x^p - 1)(x^q - 1)}. \tag{5}$$

Using (4) it is easily shown that

$$\Phi_{pn}(x) = \Phi_n(x^p) \text{ if } p \mid n. \tag{6}$$

On invoking the fact that $\sum_{d \mid n} \mu(d) = 0$ for $n > 1$, we infer from (4) that

$$\Phi_n(x) = \prod_{d \mid n} (1 - x^d)^{\mu(n/d)}, \tag{7}$$

and, on using the identity $\sum_{d \mid n} d\mu(n/d) = \varphi(n)$, we deduce that

$$x^{\varphi(n)} \Phi_n\left(\frac{1}{x}\right) = \Phi_n(x). \tag{8}$$

Hence Φ_n is selfreciprocal for $n > 1$. Note that $\Phi_1(x) = x - 1$ is not selfreciprocal.

It is a well-known fact, see, e.g., Lang [18, p. 74], that

$$\Phi_n(1) = \begin{cases} 0 & \text{if } n = 1; \\ p & \text{if } n = p^m; \\ 1 & \text{otherwise.} \end{cases} \tag{9}$$

Note that $\Phi_1(-1) = -2$ and $\Phi_2(-1) = 0$. For $n > 2$ we have

$$\Phi_n(-1) = \begin{cases} p & \text{if } n = 2p^m; \\ 1 & \text{otherwise,} \end{cases} \tag{10}$$

which follows from (6) and the observation that $\Phi_{2m}(x) = \Phi_m(-x)$ if $m > 1$ is odd (for a different proof, see [11]).

2.2. **Semigroup polynomials.** In this section we establish some basic, yet useful facts relating a numerical semigroup to its polynomial.

Lemma 4. *Let S be a numerical semigroup and assume that $\mathrm{P}_S(x) = a_0 + a_1x + \cdots + a_kx^k$. Then, for $s \in \{0, \ldots, k\}$,*

$$a_s = \begin{cases} 1 & \text{if } s \in S \text{ and } s-1 \notin S; \\ -1 & \text{if } s \notin S \text{ and } s-1 \in S; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof easily follows from the fact that $\mathrm{P}_S(x) = (1-x)\mathrm{H}_S(x)$ and that a coefficient of $x\mathrm{H}_S(x)$ is 1 if and only if its degree minus one belongs to S. □

Corollary 1. *The nonzero coefficients of $\mathrm{P}_S(x)$ alternate between 1 and -1.*

Lemma 5. *Let $S \neq \mathbb{N}$ be a numerical semigroup. Then*
a) $\mathrm{P}_S(1) = 1$;
b) $\mathrm{P}'_S(1) = \mathrm{g}(S)$.
Proof.
a) Immediate from (1).
b) The condition $S \neq \mathbb{N}$ is equivalent to $1 \notin S$, ensuring $\mathrm{m}(S) \geq 2$. On using Lemma 4 together with the fact that the first nonzero element of S is $\mathrm{m}(S)$ and the last gap of S is $\mathrm{F}(S)$, we note that there exist $2 \leq k_1 < \cdots < k_{2n+1}$ such that

$$\mathrm{P}_S(x) = 1 - x + x^{k_1} - x^{k_2} + \cdots - x^{k_{2n}} + x^{k_{2n+1}}. \tag{11}$$

In fact, $k_1 = \mathrm{m}(S)$ and $k_{2n+1} = \mathrm{F}(S) + 1$. Lemma 4 tells us that

$$\mathbb{N} \backslash S = [1, k_1 - 1] \cup [k_2, k_3 - 1] \cup \ldots \cup [k_{2n}, k_{2n+1} - 1] \tag{12}$$

Differentiating (11) yields

$$\mathrm{P}'_S(x) = (-1 + k_1x^{k_1-1}) + \cdots + (-k_{2n}x^{k_{2n}-1} + k_{2n+1}x^{k_{2n+1}-1})$$

and evaluating at 1 gives

$$\mathrm{P}'_S(1) = (k_1 - 1) + (k_3 - k_2) + \cdots + (k_{2n+1} - k_{2n}). \tag{13}$$

The conclusion now follows on comparing (12) and (13). □

Note that

$$\mathrm{P}_S(-1) = 1 + 2g(S) - 4\sum_{\substack{s \notin S \\ 2|s}} 1 \tag{14}$$

is an odd integer and hence nonzero.

Lemma 6. *If S is a numerical semigroup satisfying*

$$\sum_{s \notin S,\ 2\nmid s} 1 < \sum_{s \notin S,\ 2|s} 1,$$

then S is not cyclotomic.

Proof. On using (14) the latter inequality is seen to be equivalent with $\mathrm{P}_S(-1) < 0$. Now assume that S is cyclotomic. Then by (10) always $\Phi_n(-1) \geq 0$ and hence $\mathrm{P}_S(-1) \geq 0$. This contradiction finishes the proof. □

Example 1. Take $S = \langle 3,5,7\rangle$. It has one odd gap and two even gaps. By Lemma 6, S is not cyclotomic. Observe that S is not even symmetric.

The converse of Lemma 6 is not true. The numerical semigroup $S = \langle 5,6,7,8\rangle$ is not cyclotomic; there are three odd gaps in S and two even gaps.

We took all numerical semigroups S that are symmetric and not complete intersection with Frobenius number $\le k$ and determined how often on average Lemma 6 applies. Our computations (with $k \le 69$) indicate that likely an average exists and is in $[0.8,0.85]$.

2.3. **Apéry sets and semigroup polynomials.** The *Apéry set* of S with respect to a nonzero $m \in S$ is defined as

$$\mathrm{Ap}(S;m) = \{s \in S\colon s-m \notin S\}.$$

Note that

$$S = \mathrm{Ap}(S;m) + m\mathbb{N} \tag{15}$$

and that $\mathrm{Ap}(S;m)$ consists of a complete set of residues modulo m. Thus we have

$$\mathrm{H}_S(x) = \sum_{w\in \mathrm{Ap}(S;m)} x^w \sum_{i=0}^{\infty} x^{mi} = \frac{1}{1-x^m} \sum_{w\in \mathrm{Ap}(S;m)} x^w, \tag{16}$$

cf. [23, (4)].

Apéry sets can also be defined in a natural way for integers m not in the semigroup (see for instance [7] or [13]), but in this case $\#\mathrm{Ap}(S;m) \neq m$.

Proposition 1. *Let S be a numerical semigroup and m be a positive integer. Then $\#\mathrm{Ap}(S;m) = m$ if and only if $m \in S$.*

Proof. For $i \in \{0,\dots,m-1\}$ set $w_i = \min\{s \in S\colon s \equiv i \pmod m\}$. By definition, $w_0 = 0$ and $\{w_0,\dots,w_{m-1}\} \subseteq \mathrm{Ap}(S;m)$. Hence $\#\mathrm{Ap}(S;m) \ge m$, and equality holds if and only if $\{w_0,w_1,\dots,w_{m-1}\} = \mathrm{Ap}(S;m)$.

If $m \in S$, [24, Lemma 2.4] asserts that $\{w_0,w_1,\dots,w_{m-1}\} = \mathrm{Ap}(S;m)$.

Now assume that $\{w_0,w_1,\dots,w_{m-1}\} = \mathrm{Ap}(S;m)$. Then, for every $i \in \{0,\dots,m-1\}$ and every $k \in \mathbb{N}$, $w_i + km \in S$. In particular $w_0 + m = m \in S$. □

Example 2. Let S be a numerical semigroup minimally generated by $\{a,b\}$. Assume that u,v are integers with $0 \le u < b$ and $1 = ua + vb$. By Lemma 4, the number of ones in P_S equals $\#\mathrm{Ap}(S;1)$ and, in view of [7, Theorem 14], we have that $\#\mathrm{Ap}(S;1) = u(a+v)$ (compare with [21, Corollary 1]). Given $0 < \gamma < 1/2$, let $\mathrm{C}_\gamma(x)$ denote the number of numerical semigroups $S = \langle p,q\rangle$ with p,q primes and $m = pq \le x$ such that $\#\mathrm{Ap}(\langle p,q\rangle;1) \le m^{1/2+\gamma}$. Bzdęga [5] was the first to obtain sharp upper and lower bounds for this quantity. Fouvry [12], using deep methods from analytic number theory, even obtained an asymptotic for $\mathrm{C}_\gamma(x)$ in the range $\gamma \in \left(\frac{12}{25},\frac{1}{2}\right)$.

Example 3. Let m and q be positive integers such that $m \ge 2q+3$ and let

$$S = \langle m, m+1, qm+2q+2, \dots, qm+(m-1)\rangle.$$

Then by [24, Lemma 4.22] S is symmetric with multiplicity m and embedding dimension $m-2q$. It is easy to deduce that $\mathrm{Ap}(S;m) = \{0, m+1, 2m+2, \dots, qm+q, qm+2q+2, \dots, qm+(m-1), (q+1)(m+1), \dots, (2q+1)(m+1)\}$. On invoking (16) and carrying out the computations, we obtain an explicit formula for the semigroup polynomial:

$$\mathrm{P}_S(x) = \sum_{k=0}^{q} x^{km} + x^{q(m+2)+2} \sum_{k=0}^{q+1} x^{km} - x \sum_{k=0}^{2q+1} x^{k(m+1)}.$$

3. Conjecture 1 holds for embedding dimension ≤ 3

Using the fact that every symmetric numerical semigroup S with embedding dimension $\mathrm{e}(S) \leq 3$ is a complete intersection ([14]), it is easy to see that the following result holds.

Lemma 7. *For all numerical semigroups S with $\mathrm{e}(S) \leq 3$, we have*

$$\textit{complete intersection} \Leftrightarrow \textit{cyclotomic} \Leftrightarrow \textit{symmetric}.$$

Example 4. Let $S = \langle 4,6,9 \rangle$. We find that S is symmetric and hence it must be cyclotomic. Indeed, we have $\mathrm{P}_S(x) = x^{12} - x^{11} + x^8 - x^7 + x^6 - x^5 + x^4 - x + 1 = \Phi_6(x)\Phi_{12}(x)\Phi_{18}(x)$.

Corollary 2. *Conjecture 1 holds true for all numerical semigroups S such that $\mathrm{e}(S) \leq 3$.*

The analogous version of Lemma 7 is not true if $\mathrm{e}(S) = 4$ as shown, for instance, by the numerical semigroup $S = \langle 6,7,10,11 \rangle$ that is obtained by setting $m = 6$ and $q = 1$ in Example 3. The semigroup polynomial then equals $\mathrm{P}_S(x) = x^{16} - x^{15} + x^{10} - x^8 + x^6 - x + 1$, which is not Kronecker. Further, we suspect that the numerical semigroups described in Example 3 are not cyclotomic for embedding dimension ≥ 4. We did an exhaustive search in this family of numerical semigroups up to multiplicity 30, and indeed, only those with embedding dimension three were cyclotomic.

It turns out that the smallest Frobenius number that can occur for a symmetric numerical semigroup that is not cyclotomic is 9. There is only one such semigroup, namely $S = \langle 5,6,7,8 \rangle$, where we have $\mathrm{P}_S(x) = x^{10} - x^9 + x^5 - x + 1$. For Frobenius number 11, we have two symmetric numerical semigroups that are not cyclotomic: $\langle 5,7,8,9 \rangle$ and $\langle 6,7,8,9,10 \rangle$. (Recall that a symmetric numerical semigroup has an odd Frobenius number, see, for instance, [24].)

Problem 2. *Prove that the numerical semigroups S given in Example 3 for which $\mathrm{e}(S) \geq 4$ are not cyclotomic, or find a counterexample.*

4. On the factorization of $\mathrm{P}_S(x)$ into irreducibles

In this section we consider a cyclotomic numerical semigroup S and try to infer some restrictions on the possible factorizations of $\mathrm{P}_S(x)$ into cyclotomic polynomials. Lemma 8 is obtained on substituting $x = 1$, and Lemma 9 on substituting $x = -1$ in (2).

Lemma 8. *Let S be cyclotomic and $\mathscr{D}$ be as in Definition 1. If $d \in \mathscr{D}$, then $d > 1$ and d is not a prime power.*

Proof. By (1) and (2) we have $1 = \mathrm{P}_S(1) = \prod_{d \in \mathscr{D}} \Phi_d(1)^{e_d}$, and hence $e_1 = 0$. The proof is completed on using that $\Phi_d(1) \notin \{-1,1\}$ for those d that are prime powers (see (9)). $\square$

Recall that $\mathrm{P}_S(-1) = 1 + 2\mathrm{g(S)} - 4\sum_{\mathrm{s} \notin \mathrm{S},\, 2|\mathrm{s}} 1$. This implies in particular that $\mathrm{P}_S(-1)$ is odd.

Lemma 9. *Let S be a cyclotomic numerical semigroup and $p > 2$ a prime. Then*

$$p \mid \mathrm{P}_S(-1) \Leftrightarrow \Phi_{2p^k}(x) \mid \mathrm{P}_S(x)$$

for some $k \geq 1$.

Proof. "$\Leftarrow$". The assumption $\Phi_{2p^k}(x) \mid \mathrm{P}_S(x)$ implies that $\Phi_{2p^k}(-1) \mid \mathrm{P}_S(-1)$. Now invoke (10) and (14).
"$\Rightarrow$". We must have $p \mid \Phi_n(-1)$ for some n and $\Phi_n(x) \mid \mathrm{P}_S(x)$. By Lemma 8 we must have $n > 2$ (in fact $n \geq 6$) and n is not a power of two. By (10) it now follows that $n = 2p^k$ for some $k \geq 1$. $\square$

Example 5. Let $S = \langle 6,9,11 \rangle$. Then

$$\mathrm{P}_S(x) = x^{26} - x^{25} + x^{20} - x^{19} + x^{17} - x^{16} + x^{15} - x^{13} + x^{11} - x^{10} + x^9 - x^7 + x^6 - x + 1 = \Phi_{18}(x)\Phi_{33}(x).$$

Observe that $18 = 2 \cdot 3^2$ and $\mathrm{P}_S(-1) = 3$.

Remark 3. *If p divides $\mathrm{P}_S(-1)$ exactly, then there is a unique positive integer k such that $\Phi_{2p^k}(x)$ divides $\mathrm{P}_S(x)$ exactly.*

5. GLUINGS OF NUMERICAL SEMIGROUPS

In this section we will use gluings to infer that every complete intersection numerical semigroup is cyclotomic and hence Lemma 2.

Let T, T_1 and T_2 be submonoids of $\mathbb{N}$. We say that T is the *gluing* of T_1 and T_2 if

(1) $T = T_1 + T_2$;
(2) $\operatorname{lcm}(d_1, d_2) \in T_1 \cap T_2$, with $d_i = \gcd(T_i)$ for $i \in \{1,2\}$,

and we will write $T = T_1 +_d T_2$, with $d = \operatorname{lcm}(d_1, d_2)$.

Every nontrivial submonoid T of $\mathbb{N}$ is isomorphic as a monoid to $T/\gcd(T)$, which is a numerical semigroup. Hence, in the above definition if $T = S$ is a numerical semigroup, and $S = T_1 +_d T_2$, then $T_i = d_i S_i$, with $S_i = T_i/d_i$, and $\gcd(d_1, d_2) = \gcd(S) = 1$. Hence $\operatorname{lcm}(d_1, d_2) = d_1 d_2$, which leads to $d_i \in S_j$ for $\{i, j\} = \{1,2\}$.

For a_1, a_2 integers greater than 2 with $\gcd(a_1, a_2) = 1$, it is shown in [1] that

$$\mathrm{H}_{a_1 S_1 +_{a_1 a_2} a_2 S_2}(x) = (1 - x^{a_1 a_2}) \mathrm{H}_{S_1}(x^{a_1}) \mathrm{H}_{S_2}(x^{a_2}). \tag{17}$$

For the particular case $S = \langle a_1, a_2 \rangle = a_1\mathbb{N} +_{a_1 a_2} a_2\mathbb{N}$, we obtain (see also [21])

$$\mathrm{H}_{\langle a_1, a_2 \rangle}(x) = \frac{1 - x^{a_1 a_2}}{(1 - x^{a_1})(1 - x^{a_2})}, \tag{18}$$

and by using (3), we get

$$\mathrm{P}_{\langle a_1, a_2 \rangle}(x) = \frac{(1-x)(1 - x^{a_1 a_2})}{(1 - x^{a_1})(1 - x^{a_2})} = \prod_{d | a_1 a_2,\ d \nmid a_1,\ d \nmid a_2} \Phi_d(x). \tag{19}$$

Hence

$$\mathrm{P}_{a_1 S_1 +_{a_1 a_2} a_2 S_2}(x) = \frac{(1-x)(1 - x^{a_1 a_2})}{(1 - x^{a_1})(1 - x^{a_2})} \mathrm{P}_{S_1}(x^{a_1}) \mathrm{P}_{S_2}(x^{a_2}) = \mathrm{P}_{\langle a_1, a_2 \rangle}(x) \mathrm{P}_{S_1}(x^{a_1}) \mathrm{P}_{S_2}(x^{a_2}). \tag{20}$$

Delorme in [10] proved (but with a different notation) that a numerical semigroup S is a complete intersection if and only if S is either $\mathbb{N}$ or the gluing of two complete intersection numerical semigroups. If we proceed recursively and $A = \{a_1, \ldots, a_t\}$ is a minimal generating system of S, we will find positive integers $g_1, \ldots, g_{t-1}$ such that

$$S = a_1\mathbb{N} +_{g_1} a_2\mathbb{N} + \cdots +_{g_{t-1}} a_t\mathbb{N}.$$

By using [1, Theorem 20], we obtain

$$\mathrm{P}_S(x) = (1-x) \prod_{i=1}^{t-1} (1 - x^{g_i}) \prod_{i=1}^{t} (1 - x^{a_i})^{-1}, \tag{21}$$

and we deduce that every complete intersection numerical semigroup is cyclotomic, and hence we have proved Lemma 2, that is, one of the directions of Conjecture 1.

For $S = \langle n_1, \ldots, n_e \rangle$, according to [26, (1)], the only nonzero terms of $\mathscr{K}(x) = \mathrm{H}_S(x) \prod_{i=1}^{e} (1 - x^{n_i})$ are those of degrees $n \in S$ such that the *Euler characteristic* of the *shaded set* of n, $\Delta_n = \{L \subset \{n_1, \ldots, n_e\} : n - \sum_{s \in L} s \in S\}$, is not zero, that is, $\chi_S(n) := \sum_{L \in \Delta_n} (-1)^{\# L} \neq 0$. We have been trying to determine whether $\mathscr{K}(x)$ factors as $\prod_{b \in \mathrm{Betti}(S)} (1 - x^b)^{m_b}$, where $\mathrm{Betti}(S)$ is the set of the *Betti numbers* of S, i.e., the elements $n \in S$ for which the underlying graph of Δ_n is not connected (the graph whose vertices are the elements $n_i \in \{n_1, \ldots, n_e\}$ such that $n - n_i \in S$, and $n_i n_j$ is an edge whenever $i, j \in \{1, \ldots, e\}$, $i \neq j$ and $n - (n_i + n_j) \in S$; see [24, §7.3]) and $m_b \in \mathbb{N}$. This is what actually happens in (21). We will detail our efforts done in this regard in Section 6.1, after introducing some further tools.

5.1. **Free semigroups.** Let S be a numerical semigroup generated by $\{n_1,\dots,n_t\}$. We say that S is *free* if either $S=\mathbb{N}$ or it is the gluing of the free semigroup $\langle n_1,\dots,n_{t-1}\rangle$ and $\langle n_t\rangle$ (see [3]). The way we enumerate the generators is relevant. For instance S is free for the arrangement $\{n_1=4,n_2=6,n_3=9\}$, but not for $\{n_1=4,n_2=9,n_3=6\}$.

Example 6. Let S be an embedding dimension three symmetric numerical semigroup. Then S is free and it has a system of generators of the form $\langle am_1,am_2,bm_1+cm_2\rangle$, with $a,b,c\in\mathbb{N}$ such that $a\geq 2$, $b+c\geq 2$ and $\gcd(a,bm_1+cm_2)=1$ ([24, Theorem 10.6]). It follows that S can be expressed as $S=a\langle m_1,m_2\rangle+_{a(bm_1+cm_2)}(bm_1+cm_2)\mathbb{N}$. From (21) we get

$$\mathrm{P}_S(x)=\frac{(1-x)(1-x^{a(bm_1+cm_2)})(1-x^{am_1m_2})}{(1-x^{am_1})(1-x^{am_2})(1-x^{bm_1+cm_2})}.$$

If $S=\langle n_1,n_2,n_3\rangle$ is nonsymmetric with embedding dimension three, then it can be deduced from [26] and [2] (see also [23, Theorem 4]) that

$$\mathrm{P}_S(x)=\frac{(1-x)(1-x^{c_1n_1}-x^{c_2n_2}-x^{c_3n_3}+x^{f_1+n_1+n_2+n_3}+x^{f_2+n_1+n_2+n_3})}{(1-x^{n_1})(1-x^{n_2})(1-x^{n_3})},$$

where

- $c_i=\min\{m\in\mathbb{N}\setminus\{0\}\colon mn_i\in\langle n_j,n_k\rangle\}$ for all $\{i,j,k\}=\{1,2,3\}$,
- $f_1=\mathrm{F}(S)$ and $f_2\neq f_1$ is such that $f_2+S\setminus\{0\}\subset S$ (f_1 and f_2 are the *pseudo-Frobenius numbers* of S; their expression can be found for instance in [2, Corollary 11]).

Formulas for symmetric and pseudo-symmetric embedding dimension four can be derived from [2, Section 4], and the number of nonzero coefficients of $\mathrm{H}_S(x)\prod_{i=1}^4(1-x_i^n)$ is 12 and 14, respectively. (Recall that S is *pseudo-symmetric* if $\mathrm{F}(S)$ is even and for every $x\in\mathbb{Z}\setminus S$, either $x=\mathrm{F}(S)/2$ or $\mathrm{F}(S)-x\in S$.) From [26] it follows that the number of nonzero coefficients of $\mathrm{H}_S(x)\prod_{i=1}^4(1-x_i^n)$ is not bounded when S ranges over all numerical semigroups of embedding dimension 4.

Special families of free numerical semigroups are the *telescopic* ones (free with respect to the arrangement $n_1<n_2<\cdots<n_t$, [17]), numerical semigroups associated to irreducible plane curve singularities ([30]) and *binomial semigroups* (they will be considered in Example 7).

Let $n\geq 2$ and $(a_1,a_2,\dots,a_n)$ be a sequence of relatively prime positive integers. For every $k\in\{1,\dots,n\}$, let $d_k=\gcd(a_1,\dots,a_k)$. For $k\in\{2,\dots,n\}$, let $c_k=d_{k-1}/d_k$. Let S_k be the semigroup generated by $a_1,\dots,a_k$. We say that the sequence $(a_1,a_2,\dots,a_n)$ is *smooth* if $c_ka_k\in S_{k-1}$ for every $k=2,\dots,n$.

Observe that a numerical semigroup S is generated by a smooth sequence if and only if S is free. Also $c_ka_k\in S_{k-1}$ is equivalent to $\frac{a_k}{d_k}\in\frac{1}{d_{k-1}}S_{k-1}$ (and $\frac{1}{d_{k-1}}S_{k-1}$ is a numerical semigroup). Notice that $S_k=S_{k-1}+a_k\mathbb{N}$. With the notation of gluing, we have $\frac{1}{d_k}S_k=c_k\left(\frac{1}{d_{k-1}}S_{k-1}\right)+_{c_k\frac{a_k}{d_k}}\frac{a_k}{d_k}\mathbb{N}$. By using (21), we recover the following result.

Lemma 10 (Leher [19, Corollary 8])**.** *Let $n\geq 2$ and $(a_1,a_2,\dots,a_n)$ be a smooth sequence. Let S be the numerical semigroup generated by $a_1,\dots,a_n$. We have*

$$\mathrm{P}_S(x)=(1-x)\prod_{i=2}^n(1-x^{c_ia_i})\prod_{i=1}^n(1-x^{a_i})^{-1},$$

which factorizes as

$$\mathrm{P}_S=\Phi_1\prod_{d|a_1}\Phi_d^{-1}\prod_{i=2}^n\prod_{d|c_ia_i,\ d\nmid a_i}\Phi_d. \tag{22}$$

Corollary 3. *Let S be the numerical semigroup generated by the smooth sequence $(a_1,\dots,a_n)$ with $n\geq 2$. Then*

a) $\mathrm{F}(S)=\sum_{i=2}^n c_ia_i-\sum_{i=1}^n a_i$ *(this formula can also be derived from* [10] *or* [15]*).*

b) S is symmetric.

c) S is cyclotomic.

Example 7 (Binomial semigroups). Consider $B_m(a,b) := \langle a^m, ba^{m-1}, \ldots, b^{m-1}a, b^m\rangle$, where $a,b>1$ are relatively prime. Putting $a_k = a^{m-k}b^k$, $k\in\{0,\ldots,m\}$, we see that the sequence $(a_0,\ldots,a_m)$ is smooth (with $c_k = a$ for $k\in\{1,\ldots,m\}$ and $c_k a_k = ba_{k-1}\in\langle a_0,\ldots,a_{k-1}\rangle$). By Corollary 3 it follows that

$$\mathrm{F}(B_m(a,b)) = \sum_{k=1}^{m} a^{m+1-k}b^k - \sum_{k=0}^{m} a^{m-k}b^k.$$

Further, we have

$$\mathrm{P}_{B_n(a,b)}(x) = (1-x)\prod_{k=1}^{m}(1-x^{a^{m+1-k}b^k})\prod_{k=0}^{m}(1-x^{a^{m-k}b^k})^{-1}.$$

In particular, let $B = B_n(p,q)$ be a binomial numerical semigroup with p and q different primes. From (22) we infer that

$$\mathrm{P}_B = \Phi_1(\Phi_1\Phi_p\cdots\Phi_{p^n})^{-1}\prod_{k=1}^{n}\prod_{j=0}^{k}\Phi_{p^{n+1-k}q^j} = \prod_{k=1}^{n}\prod_{j=1}^{k}\Phi_{p^{n+1-k}q^j} = \prod_{l=2}^{n+1}\prod_{\substack{i+j=l\\ 1\le i,j\le l}}\Phi_{p^iq^j}. \tag{23}$$

6. Cyclotomic exponents and a first step in proving the conjecture

The reader might wonder whether the expression in the right-hand side of (21) is unique. It is easy to see the answer is yes and indeed a little more can be shown, see Moree [20, Lemma 1].

Lemma 11. *Let $f(x) = 1 + a_1x + \cdots + a_dx^d \in \mathbb{Z}[x]$ be a polynomial of degree d (hence $a_d \neq 0$). Let $\alpha_1,\ldots,\alpha_d$ be its roots. Put $s_f(k) = \alpha_1^{-k} + \cdots + \alpha_d^{-k}$. Then the numbers $s_f(k)$ are integers and satisfy the recursion*

$$s_f(k) + a_1s_f(k-1) + \cdots + a_{k-1}s_f(1) + ka_k = 0,$$

with $a_k = 0$ for every $k > d$. Put

$$b_f(k) = \frac{1}{k}\sum_{d|k} s_f(d)\mu\left(\frac{k}{d}\right).$$

Then $b_f(k)$ is an integer. Moreover, we have the formal identity

$$1 + a_1x + \cdots + a_dx^d = \prod_{j=1}^{\infty}(1-x^j)^{b_f(j)}.$$

It is a consequence of this lemma that given a numerical semigroup S, there are unique integers $\mathrm{e}_1, \mathrm{e}_2, \ldots$ such that

$$\mathrm{P}_S(x) = \prod_{j=1}^{\infty}(1-x^j)^{\mathrm{e}_j}. \tag{24}$$

The sequence $\mathbf{e} = \{\mathrm{e}_1, \mathrm{e}_2, \ldots\}$ we call the *cyclotomic exponent sequence* of S.

Problem 3. *Relate the properties of S to its cyclotomic exponent sequence.*

By Lemma 21 we have $\mathrm{e}_1 = 0$ if $S = \langle 1\rangle$ and $\mathrm{e}_1 = 1$ otherwise.

Remark 4. *The identity (24) only holds for $|x| < \delta < 1$, where δ is postive and easily related to the roots of f, see [20] for details.*

Lemma 12. *A numerical semigroup S has a cyclotomic exponent sequence with finitely many nonzero terms if and only if S is a cyclotomic numerical semigroup.*

Proof. "$\Rightarrow$". We can write $\mathrm{P}_S(x)=\prod_{j=1}^{k}(1-x^j)^{\mathrm{e}_j}$ for some k, hence $\mathrm{P}_S(x)$ has only roots of unity as zeros and so S is a cyclotomic numerical semigroup.
"$\Leftarrow$". By Definition 1, the fact that $\mathrm{P}_S(1)\neq 0$ and formula (7) we infer that $\mathrm{P}_S(x)=\prod_{j=1}^{r}(1-x^j)^{f_j}$, with the f_j integers. By the uniqueness of the cyclotomic exponents it now follows that $f_j=\mathrm{e}_j$ and so $\mathrm{e}_j=0$ for all j large enough. □

Lemma 13 makes the above result effective using the arithmetic function $a(n)=\max\{m\colon \varphi(m)\le n\}$. Using the estimate $\varphi(n)\ge (n/3)^{2/3}$ (see, e.g., [4]) we see that we can write $a(n)=\max\{1\le m\le 3n^{3/2}\colon \varphi(m)\le n\}$, giving an algorithm to determine $a(n)$.

Lemma 13. *Let S be a numerical semigroup. Put $r=a(\mathrm{F}(s)+1)$. Then S is cyclotomic if and only if $\mathrm{P}_S(x)=\prod_{j=1}^{r}(1-x^j)^{\mathrm{e}_j}$, with e_j the cyclotomic exponents of P_S.*

Proof. By Lemma 12 we have that S is cyclotomic if and only if $\mathrm{P}_S(x)=\prod_{j=1}^{M}(1-x^j)^{\mathrm{e}_j}$ for some integer M, so that $\mathrm{e}_j=0$ for all $j>M$. Now $\Phi_M(x)$ divides $1-x^M$ and no $1-x^j$ with $1\le j<M$, and so $\Phi_m(x)\mid \mathrm{P}_S(x)$. If $M>a(F(s)+1)$ it would follow that $\varphi(M)>F(s)+1$, and thus the product would have degree $>\mathrm{F}(S)+1$, whereas $\mathrm{P}_S(x)$ has degree $\mathrm{F}(S)+1$. This contradiction shows that $M\le r$. □

The proof of the latter lemma is easily adapted to show the correctness of the following algorithm which determines whether a monic polynomial $f(x)\in\mathbb{Z}[x]$, with $f(0)\neq 0$, is Kronecker or not.

Algorithm 1. *Let $f(x)\in\mathbb{Z}[x]$ with $f(0)\neq 0$ be a monic polynomial of degree d. We are going to write $f(x)$ as $h(x)\prod_{1\le d\le M}\Phi_d(x)^{g_d}$, with $h(x)$ coprime to the product $\prod_{1\le d\le M}\Phi_d(x)^{g_d}$ and $M\le a(d)$. We determine the* gcd *of $f(x)$ and $\Phi_1(x)$. As long as this* gcd *is $\Phi_1(x)$ we divide $\Phi_1(x)$ out and continue until the* gcd *is* 1*. We keep track of the number of divisions and in this way we have determined g_1. We proceed with taking the* gcd *with $\Phi_2(x)$ and repeat the process. In this way we determine the $\prod_{1\le d\le M}\Phi_d(x)^{g_d}$ and $h(x)$. Then $f(x)$ is Kronecker if and only if $h(x)=1$.*

6.1. **Cyclotomic exponents and Betti numbers.** Write $S=\langle n_1,n_2,\ldots,n_e\rangle$, with $e=\mathrm{e}(S)$ and $0<n_1<\cdots<n_e$. Note that

$$(1-x^{n_1})(1-x^{n_2})\cdots(1-x^{n_e})=\sum_{j_1=0}^{1}\sum_{j_2=0}^{1}\cdots\sum_{j_e=0}^{1}(-1)^{j_1+j_2+\cdots+j_e}x^{j_1n_1+j_2n_2+\cdots+j_en_e}.$$

We can thus write

$$\mathrm{P}_S(x)=\frac{1-x}{(1-x^{n_1})\cdots(1-x^{n_e})}\left(\sum_{j_1=0}^{1}\sum_{j_2=0}^{1}\cdots\sum_{j_e=0}^{1}(-1)^{j_1+j_2+\cdots+j_e}x^{j_1n_1+j_2a_2+\cdots+j_en_e+S}\right), \tag{25}$$

where $m+S:=\{m+s:s\in S\}$. On recalling the definition of $\chi_S(n)$ given in Section 5, we can rewrite (25) as

$$\mathrm{P}_S(x)=\frac{1-x}{(1-x^{n_1})\cdots(1-x^{n_e})}\sum_n \chi_S(n)x^n. \tag{26}$$

Note that $\sum_n \chi_S(n)x^n$ is a polynomial since every $n>\mathrm{F}(S)+n_1+\cdots+n_e$ can be written as $\sum a_in_i$ with $a_i\ge 1$ for $1\le i\le e$ and hence $\chi_S(n)=0$; this recovers the formula given in [26]. Alternatively, this can be seen by noting that $\sum_n \chi_S(n)x^n$ is the product of the polynomials $\mathrm{P}_S(x)$ and $(1-x^{n_1})\cdots(1-x^{n_e})/(1-x)$.

As a first step in proving Conjecture 1, the following can be shown. Let $\mu=\min\{n>1:\chi_S(n)\neq 0\}$ and let $\mathfrak{d}(n)$ be the *denumerant* of n, that is,

$$\mathfrak{d}(n)=\#\left\{(a_1,\ldots,a_e)\in\mathbb{N}^e:\sum a_in_i=n\right\}.$$

Lemma 14. *Let $S=\langle n_1,\ldots,n_e\rangle$ be a minimally generated cyclotomic numerical semigroup such that $n_1<\cdots<n_e$ and let $\mathbf{e}=\{\mathrm{e}_1,\mathrm{e}_2,\ldots\}$ be its cyclotomic exponent sequence.*

a) We have $\mathrm{e}_1 = 1$. *If* $\mu > n_e$, *then* $\mathrm{e}_{n_1} = \mathrm{e}_{n_2} = \cdots = \mathrm{e}_{n_e} = -1$. *Further* $\mathrm{e}_\mu = -\chi_S(\mu)$. *If* $1 \le j \le \mu$ *and* $j \notin \{1, n_1, \ldots, n_e, \mu\}$, *then* $\mathrm{e}_j = 0$.
b) If, in addition, there is some $s \in S$, $s \le n_e$ *with* $\mathfrak{d}(s) \ge 2$, *then* $\mathrm{e}_j \ge 0$ *for all* $j > n_e$.

Proof.
a) Since $\chi_S(0) = 1$ we have $\sum_n \chi_S(n)x^n = 1 + \chi_S(\mu)x^\mu + \cdots$ and the result follows.
b) If there exist both positive e_j and negative e_i with $i, j > n_e$, then we can write

$$\mathrm{P}_S(x) = \frac{1-x}{\prod_{i=1}^{e}(1-x^{n_i})} \cdot \frac{(1-x^{j_1})^{\mathrm{e}_{j_1}}(1-x^{j_2})^{\mathrm{e}_{j_2}}\cdots}{(1-x^{i_1})^{\mathrm{e}_{i_1}}(1-x^{i_2})^{\mathrm{e}_{i_2}}\cdots},$$

where both products in the numerator and denominator are finite, $n_e < j_1 < j_2 < \cdots$, $n_e < i_1 < i_2 < \cdots$, and $\mathrm{e}_{i_k}, \mathrm{e}_{j_l} > 0$. Now, $(1-x^{i_1})^{-\mathrm{e}_{i_1}}(1-x^{i_2})^{-\mathrm{e}_{i_2}}\cdots = (1+x^{i_1}+x^{2i_1}+\cdots)^{\mathrm{e}_{i_1}}\cdots = 1+\beta x^b + O(x^{b+1})$ is an infinite sum, with $b = i_1 > n_e, \beta = \mathrm{e}_{i_1}$, and $(1-x^{j_1})^{\mathrm{e}_{j_1}}(1-x^{j_2})^{\mathrm{e}_{j_2}}\cdots = 1 - \alpha x^a + O(x^{a+1})$ is a finite sum, with $a = j_1 > n_e, \alpha = \mathrm{e}_{j_1}$. Hence, using the easy fact that

$$\prod_{i=1}^{e}(1-x^{n_i})^{-1} = \prod_{i=1}^{e}\sum_{j=0}^{\infty} x^{jn_i} = \sum_{s\in S}\mathfrak{d}(s)x^s, \tag{27}$$

we get

$$\mathrm{H}_S(x) = \left(\sum_{s\in S}\mathfrak{d}(s)x^s\right)(1-\alpha x^a + O(x^{a+1}))(1+\beta x^b + O(x^{b+1})). \tag{28}$$

Note that $a \ne b$, hence we distinguish two cases:
Case 1: $a > b$. Then $(1-\alpha x^a+O(x^{a+1}))(1+\beta x^b+O(x^{b+1})) = 1+\beta x^b+O(x^{b+1})$, so that, after multiplication by $\sum \mathfrak{d}(s)x^s$, the power x^b does not get reduced and appears, with some coefficient, in the Hilbert series, hence $b \in S$. But then this coefficient will be at least $\mathfrak{d}(b)+\beta \ge 1+1 = 2$, contradiction.
Case 2: $a < b$. Then $(1-\alpha x^a + O(x^{a+1}))(1+\beta x^b + O(x^{b+1})) = 1 - \alpha x^a + O(x^{a+1})$, hence $\left(\sum \mathfrak{d}(s)x^s\right)(1 - \alpha x^a + O(x^{a+1}))(1+\beta x^b + O(x^{b+1})) = \sum_{s\in S, s<a}\mathfrak{d}(s)x^s + \cdots$ and, by assumption, there is some $s \le n_e < a$ with $\mathfrak{d}(s) \ge 2$, leading to a coefficient greater than 1 in the Hilbert series, contradiction.

Note that having $\mathrm{e}_i \le 0$ for all $i > n_e$ is impossible, as we would then get $\mathrm{H}_S(x) = \left(\sum \mathfrak{d}(s)x^s\right)\left(\sum_j r_j x^j\right)$ with $r_j \ge 0$. On expanding this, one can find coefficients larger than 1 in $\mathrm{H}_S(x)$. Therefore we can only have $\mathrm{e}_j \ge 0$ for all $j > n_e$. □

It follows that we can express the Hilbert series of a numerical semigroup S satisfying the conditions of Lemma 14 as

$$\mathrm{H}_S(x) = \frac{(1-x^{d_1})^{\mathrm{e}_1}\cdots(1-x^{d_k})^{\mathrm{e}_k}}{(1-x^{n_1})\cdots(1-x^{n_e})}, \tag{29}$$

where $n_e < d_1 < d_2 < \cdots < d_k$ and $\mathrm{e}_i \ge 1$, $i = 1, \ldots, k$ (with $\sum_{i=1}^{k}\mathrm{e}_i = m-1$, which follows on noting that $\mathrm{H}_S(x)$ must have a simple pole at $x = 1$). The conditions of Lemma 14 are rather restrictive. However, solely from a factorization such as (29), it is easy to prove the following.

Lemma 15. *Let* $S = \langle n_1, \ldots, n_e\rangle$ *be a minimally generated numerical semigroup such that* (29) *holds. Then* $d_i \in S$ *for all* $i = 1, \ldots, k$ *and* $d_1 = \min\{s : s \in \mathrm{Betti}(S)\}$.

Proof. Rewrite (29) as

$$(1-\mathrm{e}_1x^{d_1}+\cdots)\cdots(1-\mathrm{e}_kx^{d_k}+\cdots) = \sum_{s\in S}x^s\prod_{i=1}^{e}(1-x^{n_i}). \tag{30}$$

The right-hand side of (30) is of the form $\sum_{s\in S} r(s)x^s$, for some $r(s) \in \mathbb{Z}$, while the left equals $1 - \mathrm{e}_1x^{d_1} + O(x^{d_1+1})$. Then $d_1 \in S$. Next, when expanding the left-hand side, the power x^{d_2} either gets cancelled by a power $x^{\alpha d_1}$, for some $\alpha\in\mathbb{N}$, or appears in the sum with a nonzero coefficient. Either case, $d_2\in S$ and we can repeat the same argument to show that $d_3,\dots,d_k\in S$. Combining (27) and (30) yields

$$\mathrm{H}_S(x)=(1-\mathrm{e}_1x^{d_1}+\cdots)\sum\mathfrak{d}(s)x^s=\sum_{s\in S,s<d_1}\mathfrak{d}(s)x^s+(\mathfrak{d}(d_1)-\mathrm{e}_1)x^{d_1}+\cdots,$$

hence d_1 is the first element $s\in S$ with $\mathfrak{d}(s)\geq 2$. We prove that this implies $d_1=\min\{s:s\in\mathrm{Betti}(S)\}$. Note that, by definition, $\mathfrak{d}(s)\geq 2$ for any $s\in\mathrm{Betti}(S)$. Therefore it suffices to prove that $d_1\in\mathrm{Betti}(S)$.

Let $d_1=a_1n_1+\cdots+a_en_e=b_1n_1+\cdots+b_en_e$ be two different representations of d_1 in terms of the generators, with $a_i,b_i\in\mathbb{N}$. If there is $1\leq i\leq e$ such that $a_i,b_i>0$, then $d_1-n_i\in S$ and $\mathfrak{d}(d_1-n_i)\geq 2$, contradiction. But this implies that the underlying graph of Δ_{d_1} is disconnected, i.e., $d_1\in\mathrm{Betti}(S)$. Indeed, take any two distinct representations $d_1=a_1n_{i_1}+\cdots+a_kn_{i_k}=b_1n_{j_1}+\cdots+b_ln_{j_l}$, where $a_i,b_j>0$, $k,l\geq 1$ and $\{i_1,\dots,i_k\}\cap\{j_1,\dots,j_l\}=\emptyset$. Then there can be no edge between n_{i_α} and n_{j_β} in the underlying graph of Δ_{d_1}. Otherwise, if say, $n_{i_1}n_{j_1}$ is an edge, then $n:=d_1-n_{i_1}-n_{j_1}\in S$ and thus $d_1-n_{i_1}=(a_1-1)n_{i_1}+a_2n_{i_2}+\cdots+a_kn_{i_k}=n+n_{j_1}$ admits at least two distinct representations, contradiction. Hence the vertices n_{i_α} and respectively n_{j_β} lie in distinct connected components. Consequently, the underlying graph of Δ_{d_1} is disconnected, that is, $d_1\in\mathrm{Betti}(S)$. □

7. Polynomially related numerical semigroups

We say that a numerical semigroup S is *polynomially related to* the numerical semigroup T, and denote this by $S\leq_P T$, if there exist $f(x)\in\mathbb{Z}[x]$ and an integer $w\geq 1$ such that

$$\mathrm{H}_S(x^w)f(x)=\mathrm{H}_T(x). \tag{31}$$

From (31) we infer that

$$\mathrm{P}_S(x^w)f(x)=\mathrm{P}_T(x)(1+x+\cdots+x^{w-1}). \tag{32}$$

Note that (31) and (32) are equivalent formulations of S being polynomially related to T.

Example 8. Put $S_1=\langle p,q\rangle$ and $S_3=\langle p^3,q\rangle$. By (19) we have $\Phi_{pq}\Phi_{p^2q}\Phi_{p^3q}=\mathrm{P}_{S_3}$. Recall that $\mathrm{P}_{S_1}=\Phi_{pq}$. We have

$$\mathrm{P}_{S_1}(x)\Phi_{p^2q}(x)\Phi_{p^3q}(x)=\mathrm{P}_{S_3}(x),\ \mathrm{P}_{S_1}(x^p)\Phi_{pq}(x)\Phi_{p^3q}(x)=\mathrm{P}_{S_3}(x),\ \mathrm{P}_{S_1}(x^{p^2})\Phi_{pq}(x)\Phi_{p^2q}(x)=\mathrm{P}_{S_3}(x),$$

giving three different polynomial relations between S_1 and S_3.

Lemma 16. *Being polynomially related defines a partial order on the set of numerical semigroups.*

Proof. Obviously a numerical semigroup is polynomially related to itself. Further, being polynomially related is clearly transitive. Using part d) of Lemma 17 we see that $\mathrm{F}(S)<\mathrm{F}(T)$ unless $S=T$. This implies that being polynomially related defines an antisymmetric binary relation on the set of numerical semigroups. □

Problem 4. *Find necessary and sufficient conditions for S to be polynomially related to T.*

In proving the following result we make repeatedly use of the fact that $\mathrm{P}_S(1)=1$ and $\mathrm{P}_S'(1)=\mathrm{g}(S)$ (see Lemma 5.)

Lemma 17. *Suppose that $\mathrm{H}_S(x^w)f(x)=\mathrm{H}_T(x)$ holds with S,T numerical semigroups. Then*

a) $f(0)=1$.
b) $f(1)=w$.
c) $f'(1)=w(\mathrm{g}(T)-w\mathrm{g}(S)+(w-1)/2)$.
d) $\mathrm{F}(T)=w\mathrm{F}(S)+\deg(f)$.
e) If w is even, then $f(-1)=0$.
f) If w is odd, then $f(-1)=\mathrm{P}_T(-1)/\mathrm{P}_S(-1)$.
g) If T is cyclotomic, then so is S.
h) If S is cyclotomic, then T is cyclotomic if and only if f is Kronecker.

Proof.
a) We have $\mathrm{P}_S(0) = \mathrm{P}_T(0) = 1$.
b) On substituting $x = 1$ in the identity (32) and noting that $\mathrm{P}_S(1) = \mathrm{P}_T(1) = 1$, we obtain $f(1) = w$.
c) The identity (32) yields (on differentiating both sides) that

$$\mathrm{P}'_S(x^w)wx^{w-1}f(x) + \mathrm{P}_S(x^w)f'(x) = \mathrm{P}'_T(x)(1 + x + \cdots + x^{w-1}) + \mathrm{P}_T(x)\sum_{j=0}^{w-2}(j+1)x^j.$$

The claim now easily follows on setting $x = 1$ and invoking part b).
d) Use that $\deg(\mathrm{P}_S) = \mathrm{F}(S) + 1$.
e)+f) Note that $\mathrm{P}_S(-1) \neq 0$ and substitute $x = -1$ in (32).
g)+h) Obvious. □

The next result gives more specific information about f in case f has nonnegative coefficients only.

Lemma 18. *Suppose that S and T are numerical semigroups. Then $\mathrm{H}_S(x^w)f(x) = \mathrm{H}_T(x)$ for some integer $w \geq 1$ and $f \in \mathbb{N}[x]$ if and only if there are $0 = e_1 < e_2 < \cdots < e_w$ such that $f(x) = \sum_{i=1}^{w} x^{e_i}$ and every $t \in T$ can be written in a unique way as*

$$t = e_i + s \cdot w,\ 1 \leq i \leq w,\ s \in S.$$

Proof. "⇒". If f were to have a coefficient greater than 1, this would lead to a coefficient greater than 1 in H_T, which is not possible. By Lemma 17 we have $f(0) = 1$ and $f(1) = w$, and hence it follows that $f(x) = \sum_{i=1}^{w} x^{e_i}$ with $0 = e_1 < \cdots < e_w$. The identity $\sum_{i=1}^{w} x^{e_i} \sum_{s\in S} x^{sw} = \mathrm{H}_T(x)$ yields that every element $t \in T$ can be written as $t = e_i + s \cdot w$, with $1 \leq i \leq w$ and $s \in S$. Since every nonzero coefficient of H_T is 1, this writing way of t must be unique.
"⇐". Obvious. □

Compare the expression of t in the above lemma with [24, Lemma 2.6].

Remark 5. *By Lemma 17 we have $\sum_{i=1}^{w} e_i = w(\mathrm{g}(T) - w\mathrm{g}(S) + (w-1)/2)$.*

Corollary 4.

a) *We have $\langle p^a, q^b\rangle \leq_P \langle p^m, q^n\rangle$ if $1 \leq a \leq m$ and $1 \leq b \leq n$.*
b) *We have $\langle p^a, q^b\rangle \leq_P B_n(p,q)$ if $a, b \geq 1$ and $2 \leq a + b \leq n+1$.*
c) *Let V be a numerical semigroup generated by $\{n_1, \ldots, n_k\}$. Let $d = \gcd(n_1, \ldots, n_{k-1})$ and set $U = S(n_1/d, \ldots, n_{k-1}/d, n_k)$. The numerical semigroup U is polynomially related to V.*

Proof.
a) This is a consequence of the identity

$$\mathrm{P}_{\langle p^m, q^n\rangle}(x) = \prod_{\substack{1\leq\alpha\leq m\\ 1\leq\beta\leq n}} \Phi_{p^\alpha q^\beta}(x), \tag{33}$$

which is a consequence of (19).
b) Results on comparing (33) with the factorization of P_B given in Example 7.
c) It is easy to see (cf. [24, Lemma 2.16]) that $\mathrm{Ap}(V; n_k) = d\,\mathrm{Ap}(U; n_k)$. By using this identity and (16) we derive

$$\mathrm{H}_U(x^d)\left(\frac{1 - x^{n_k d}}{1 - x^{n_k}}\right) = \mathrm{H}_V(x).$$ □

7.1. **An application.** We will use our insights into polynomially related numerical semigroups to establish the following result.

Theorem 2. *Let $p \neq q$ be primes and m, n positive integers. The quotient*

$$Q(x) := \mathrm{P}_{\langle p^m, q^n\rangle}(x)/\Phi_{p^m q^n}(x)$$

is monic, is in $\mathbb{Z}[x]$ and has constant coefficient 1. Its nonzero coefficients alternate between 1 and -1.

Proof. On using that $\mathrm{P}_S(x) = (1-x)\mathrm{H}_S(x)$ and the identity (19), we infer that

$$\mathrm{H}_{\langle p^m, q^n\rangle}(x) = \mathrm{H}_{\langle p,q\rangle}(x^{p^{m-1}q^{n-1}}) \sum_{j=0}^{q^{n-1}-1} x^{jp^m} \sum_{k=0}^{p^{m-1}-1} x^{kq^n}. \tag{34}$$

The identity (33) yields that $Q(x)$ is a polynomial in $\mathbb{Z}[x]$. On noticing that

$$\mathrm{P}_{\langle p,q\rangle}(x^{p^{m-1}q^{n-1}}) = \Phi_{p^m q^n}(x)$$

we obtain from (34) that

$$Q(x) = \frac{1-x}{1-x^{p^{m-1}q^{n-1}}} \sum_{j=0}^{q^{n-1}-1} x^{jp^m} \sum_{k=0}^{p^{m-1}-1} x^{kq^n}.$$

The set

$$\{\alpha p^m + \beta q^n : 0 \le \alpha \le q^{n-1}-1,\ 0 \le \beta \le p^{m-1}-1\}$$

forms a complete residue system modulo $p^{m-1}q^{n-1}$ and it follows that around $x = 0$ we can write $Q(x) = (1-x)\sum_{s\in S'} x^s$ for some set S' containing zero and all large enough integers. From this it follows that $Q(x)$ is a monic polynomial and that the nonzero coefficients of $Q(x)$ alternate between 1 and -1. □

Remark 6. *An alternative, much more conceptual proof of the identity* (34) *is obtained on using the following lemma; one notes that on writing down the Hilbert series for both sides of* (35), *we obtain the identity* (34).

Lemma 19. *Let $T = \langle p^m, q^n\rangle$ and $S = \langle p, q\rangle$. Every element of T can be uniquely written as*

$$t = \alpha p^m + \beta q^n + s p^{m-1}q^{n-1}, 0 \le \alpha \le q^{n-1}-1, 0 \le \beta \le p^{m-1}-1, s \in S. \tag{35}$$

Proof. Suppose that $t \in T$. Then

$$t = ap^m + bq^n = (q^{n-1}a_1 + \alpha)p^m + (p^{m-1}b_1 + \beta)q^n, \tag{36}$$

with $0 \le \alpha \le q^{n-1}-1$ and $0 \le \beta \le p^{m-1}-1$. Put $s = a_1 p + b_1 q$. Clearly $s \in S$. From (36) we then infer that $t = \alpha p^m + \beta q^n + s p^{m-1}q^{n-1}$, as required. The congruence class of t modulo $p^{m-1}q^{n-1}$ determines α and β uniquely. Since α and β are determined uniquely, so is s. □

Theorem 2 can be alternatively proven on invoking the following more general result together with Lemma 19.

Theorem 3. *Suppose that S and T are numerical semigroups with $\mathrm{H}_S(x^w)f(x) = \mathrm{H}_T(x)$ for some $w \ge 1$ and $f \in \mathbb{N}[x]$. Put $Q(x) = \mathrm{P}_T(x)/\mathrm{P}_S(x^w)$. Then $Q(0) = 1$, $Q(x)$ is a monic polynomial and its nonzero coefficients alternate between 1 and -1.*

Proof. By Lemma 18 we can write $f(x) = \sum_{i=1}^{w} x^{e_i}$. Since T contains all integers sufficiently large, it follows that $e_1, \ldots, e_w$ form a complete residue system modulo w. By (32) we see that

$$Q(x) = \frac{f(x)}{1-x^w}(1-x).$$

Around $x = 0$ we have $f(x)/(1-x^w) = \sum_{z\in Z} x^z$ for some infinite set of integers Z. Since $e_1, \ldots, e_w$ form a complete residue system modulo w, it follows that all integers large enough are in Z. From this we then infer that $Q(x)$ is a monic polynomial. Note that $Q(0) = f(0) = 1$ by Lemma 17 and so $0 \in Z$. For any set $Z' \subseteq \mathbb{N}$ containing 0, the nonzero coefficients in $(1-x)\sum_{z\in Z'} x^z$ alternate between 1 and -1. □

8. Cyclotomic numerical semigroups of prescribed height and depth

It follows from Lemma 8 and the identity (3) that if S is a cyclotomic numerical semigroup, then $\mathrm{P}_S(x) \mid (x^m-1)^e$ for some integers m and e.

We say that a numerical semigroup S is cyclotomic of *depth* d and *height* h if $\mathrm{P}_S(x) \mid (x^d-1)^h$, where both d and h are chosen minimally, that is, $\mathrm{P}_S(x)$ does not divide $(x^n-1)^{h-1}$ for any n and it does not divide $(x^{d_1}-1)^h$ for any $d_1 < d$.

On noting that $\Phi_m(x) \mid (x^n-1)$ if and only if $m \mid n$ one arrives at the following conclusion.

Lemma 20. *Suppose that S is a cyclotomic numerical semigroup with P_S factorizing as in* (2)*, namely*

$$\mathrm{P}_S(x) = \prod_{i=1}^{s} \Phi_{d_i}(x)^{e_{d_i}},$$

where d_i and e_{d_i} are positive integers. Then S is of depth $\mathrm{lcm}(d_1, d_2, \ldots, d_s)$ *and of height* $\max\{e_1, \ldots, e_s\}$.

Example 9. Consider the binomial semigroup $B = B_n(p,q)$ defined in Example 7. By Lemma 20 and on recalling the factorization (23), we see that B is of depth $d = p^{n+1}q^{n+1}$ and of height $h = 1$.

Problem 5. *Classify all cyclotomic numerical semigroups having a prescribed depth and height.*

In the other direction we might ask for divisors of x^n-1 that are semigroup polynomials. Various authors studied the coefficients of divisors of x^n-1 [9, 16, 22, 25, 28]. By Corollary 1 we know that if a divisor $f(x)$ of x^n-1 is of the form $\mathrm{P}_S(x)$, then its nonzero coefficients alternate between 1 and -1.

We start with considering Problem 5 for height $h = 1$. We will need the following trivial observation.

Lemma 21. *If $S \neq \langle 1 \rangle$, then* $\mathrm{P}_S(x) \equiv 1 - x \pmod{x^2}$.

Proof. If $S \neq \langle 1 \rangle$, then $0 \in S$ and $1 \notin S$ and hence $\sum_{s\in S} x^s \equiv 1 \pmod{x^2}$. □

Theorem 4. *Let p, q and r be pairwise distinct primes. Suppose S is cyclotomic of depth $d = pqr$ and height $h = 1$. Then $S = \langle pr, q \rangle$ up to a cyclic permutation of p, q, r.*

Proof. Suppose that $\mathrm{P}_S(x) \mid x^{pqr}-1$ for some S. Then by (3) and Lemma 8 we have $\mathrm{P}_S = \Phi_{pq}^{k_1}\Phi_{qr}^{k_2}\Phi_{pr}^{k_3}\Phi_{pqr}^{k_4}$ with $0 \le k_i \le 1$. Since the problem is symmetric in p, q and r, we may assume without loss of generality that $k_1 \ge k_2 \ge k_3$. Note that, modulo x^2, $f(x) = 1 + (k_4 - k_1 - k_2 - k_3)x$. On invoking Lemma 21 we now deduce that $(k_1, k_2, k_3, k_4) \in \{(1,0,0,0), (1,1,0,1)\}$. The first case we can exclude, as this leads to a depth $d = pq$. By (19) we have $\Phi_{pq}\Phi_{qr}\Phi_{pqr} = \mathrm{P}_{\langle pr,q\rangle}$. □

Theorem 5. *Suppose T is a cyclotomic numerical semigroup of depth $d = p^n q$ and height $h = 1$. Then $T = \langle p^n, q \rangle$.*

The proof makes use of the following lemma.

Lemma 22. *Let $k \ge 1$ be an integer, $0 \le e_i \le 1$ ($i \in \{1, \ldots, k-1\}$) arbitrary and $e_k = 1$. Suppose that*

$$\Phi_{pq}^{e_1}\Phi_{p^2q}^{e_2}\cdots\Phi_{p^kq}^{e_k} = \mathrm{P}_T, \tag{37}$$

with T a numerical semigroup. Then $e_i = 1$ for $1 \le i \le k$ and $T = \langle p^k, q \rangle$.

Proof. In case $e_i = 1$ for $1 \le i \le k$ the identity (37) holds with $S = \langle p^k, q \rangle$ by (19) with $a_1 = p^k$ and $a_2 = q$. Since, modulo x^2, $\Phi_{p^mq} = 1$ for $m \ge 2$ and $\Phi_{pq} = 1 - x$, we infer that $e_1 = 1$. Suppose now we are not in the case where $e_i = 1$ for $1 \le i \le k$, hence the largest integer j_1 with $e_{j_1} = 1$ satisfies $1 \le j_1 < k$. We let j_2 be the smallest integer such that $j_2 > j_1$ and $e_{j_2} = 1$. Since $e_k = 1$, j_2 exists. We now rewrite the left-hand side of (37) as

$$\mathrm{P}_{\langle p^{j_1},q\rangle}(x)\Phi_{pq}(x^{p^{j_2}})^{e_{j_2}}\cdots\Phi_{pq}(x^{p^k})^{e_k},$$

which by (5) equals, modulo $x^{p^{j_2}+1}$,

$$\mathrm{P}_{\langle p^{j_1},q\rangle}(x)(1 - x^{p^{j_2}}).$$

From this and (37) we infer that

$$\sum_{s\in\langle p^{j_1},q\rangle} x^s(1-x^{p^{j_2}})\equiv \mathrm{H}_T(x) \pmod{x^{p^{j_2}+1}}.$$

It follows that $p^{j_1}\in T$ and $p^{j_2}\notin T$ and hence T is not a numerical semigroup, contradicting our assumption. □

Proof of Theorem 5. By (3) with $m=p^nq$ and Lemma 8 we deduce that

$$\mathrm{P}_T=\Phi_{pq}^{e_1}\Phi_{p^2q}^{e_2}\cdots\Phi_{p^nq}^{e_n}, \tag{38}$$

with $0\le e_i\le 1$. Since, modulo x^2, $\Phi_{p^iq}=1$ for $i\ge 2$ and $\Phi_{pq}=1-x$, we infer that $e_1=1$. Note that $e_n=1$, for otherwise $d\mid p^{n-1}q$. The proof is concluded with the help of Lemma 22. □

9. Acknowledgements

This project was started in August 2013, when the first author carried out an internship at the Max Planck Institute for Mathematics, under the guidance of the third author. He would like to thank the staff for the opportunity given and the very pleasant and productive stay. In January 2014 the first and second author were invited to the Max Planck Institute for Mathematics to work on this paper. They thank the staff for hospitality. The first and third author thank the second author for inviting them to the INdAM meeting on numerical semigroups (Cortona, 2014), which gave them a lot of additional motivation and inspiration. In September 2015 Caterina Viola did an internship at the Max Planck Institute for Mathematics, being supervised by the third author. The authors would like to thank her for the inspiring discussions which gave them further insight into the project. Last, not least, the authors thank the referee for the helpful remarks and suggested modifications.

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