QUASI-ARITHMETICITY OF LATTICES IN $\mathrm{PO}(n,1)$

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Abstract. We show that the non-arithmetic lattices in $\mathrm{PO}(n,1)$ of Belolipetsky and Thomson [BT11], obtained as fundamental groups of closed hyperbolic manifolds with short systole, are quasi-arithmetic in the sense of Vinberg, and, by contrast, the well-known non-arithmetic lattices of Gromov and Piatetski-Shapiro are *not* quasi-arithmetic. A corollary of this is that there are, for all $n \geqslant 2$, non-arithmetic lattices in $\mathrm{PO}(n,1)$ that are not commensurable with the Gromov–Piatetski-Shapiro lattices.

Contents

1. Introduction

1.1. **Background, motivation and discussion.** The reader will find precise statements of theorems in §1.3 (p.4), following the definitions given in §1.2. Beforehand, we give some background and context.

The study of locally symmetric spaces is often re-framed as the study of discrete subgroups of semisimple Lie groups and in particular of those groups acting with finite co-volume (which are usually called 'lattices'). A well known source of examples of lattices in semisimple Lie groups are the 'arithmetic' lattices; i.e., subgroups of algebraic groups (over $\mathbb{Q}$) that are defined over $\mathbb{Z}$. That the arithmetic subgroups of semisimple Lie groups have finite co-volume was shown by A. Borel and Harish-Chandra [BHC62], who also obtained co-compactness criteria. (The co-compactness criteria were also obtained by G. Mostow and T. Tamagawa [MT62].) If the real rank of a semisimple Lie group G is at least 2, then by results of G. Margulis it is known that any lattice in G is arithmetic [Mar91, (A), p.298]. For the case of real rank 1 Lie groups, and in particular for $\mathrm{PO}(n,1)$ (the isometry group of real hyperbolic n-space), it was not until the work of M. Gromov and I. Piatetski-Shapiro [GPS87] that one knew that there are non-arithmetic lattices in $\mathrm{PO}(n,1)$, for every $n \geqslant 2$, alongside the previously known arithmetic ones. The examples of Gromov and

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Piatetski-Shapiro arise as fundamental groups of finite-volume hyperbolic manifolds, constructed by 'gluing' together pieces of non-commensurable arithmetic hyperbolic manifolds along isometric totally geodesic boundaries. Before this point, examples of non-arithmetic lattices in $\mathrm{PO}(n, 1)$ were known for some small n, and these arise as reflection groups whose non-arithmeticity may be deduced from criteria determined by È. Vinberg [Vin67] [VS93, 3.2, p.227].

In his work on the arithmeticity of these reflection groups, Vinberg introduced the class of 'quasi-arithmetic' lattices (in a given Lie group) [Vin67, p.437]. (Note that 'quasi-arithmeticity' is different from 'sub-arithmeticity': cf. §3.3.) Every arithmetic group is quasi-arithmetic, whilst on the other hand Vinberg himself gave examples of lattices that are quasi-arithmetic but *not* arithmetic (lattices that we will call 'properly quasi-arithmetic'). However, Vinberg's examples are only given for dimensions no greater than 4, and it appears that his definition has not since been considered a great deal in the literature, save once, to the author's knowledge [HLMA92]. Vinberg has, however, produced more examples of quasi-arithmetic groups in dimension 2 (though without explicitly using this terminology) [Vin]. Vinberg computes the rings of definition of these groups and infinitely many of these are not equal to $\mathbb{Z}$ and so hence infinitely many of his examples are non-arithmetic.

M. Norfleet has also recently produced a family of examples of Fuchsian groups, which, being contained in $\mathrm{PSL}_2(\mathbb{Q})$, are quasi-arithmetic [Nor15].

It was recently shown by M. Belolipetsky and S. Thomson [BT11] that one may obtain a class of non-arithmetic lattices in $\mathrm{PO}(n, 1)$, by a construction of closed hyperbolic manifolds with short closed geodesics:

Theorem 1.1 (B.–T. [BT11]). *Let $\varepsilon > 0$ and let $n \geqslant 2$. Then there exists a closed hyperbolic n-manifold M such that M contains a non-contractible closed geodesic of length less than ε. Moreover, for ε small enough (depending on n and some parameters in the construction of M), M is a non-arithmetic hyperbolic manifold.*

This construction involves 'gluing' in the spirit of Gromov and Piatetski-Shapiro, but in these more recent examples the manifolds being glued together are still commensurable with one another and so one might expect that the 'non-arithmeticity' that arises should be 'weaker' than that of Gromov and Piatetski-Shapiro. Here, we show that the non-arithmetic examples of Belolipetsky and Thomson are, in fact, properly quasi-arithmetic lattices (§2.2).

In order to establish a distinction between the two above classes of examples of non-arithmetic lattices, we also show here that the lattices of Gromov and Piatetski-Shapiro are *not* quasi-arithmetic (§3.2).

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1.2. **Definitions and foundational material.** Here we give a definition of quasi-arithmetic lattices and establish some notation for the rest of the article. For a general reference on the theory of arithmetic groups the reader may wish to consult the books of Margulis [Mar91] and Platonov and Rapinchuk [PR94]; whereas for an introduction to the subject, the book of Morris [WM14] (for example) is directed towards the newcomer.

Definition 1.2. Let $\overline{G}$ be a connected semisimple real Lie group without compact factors and with trivial centre. We say that an algebraic group $\mathbf{G}$ is *admissible for* $\overline{G}$ if

(1) $\mathbf{G}$ is defined over $\mathbb{Q}$;
(2) there exists a *surjective* homomorphism $\phi\colon \mathbf{G}(\mathbb{R})^\circ \to \overline{G}$; and
(3) the homomorphism ϕ has compact kernel.

We think of the algebraic group $\mathbf{G}$ as a model for the Lie group $\overline{G}$, whose algebraic structure allows us to obtain, easily, interesting classes of subgroups as described below. From a geometric point of view, we are interested in these subgroups' images in $\overline{G}$.

Recall that a discrete subgroup $\Gamma < \overline{G}$ is a *lattice* if the quotient $\Gamma\backslash\overline{G}$ has finite volume induced by the Haar measure on $\overline{G}$.

Definition 1.3. Suppose that $\overline{G}$ is as in Definition 1.2 and that $\Gamma < \overline{G}$ is a lattice. We say that Γ is *quasi-arithmetic* if

(1) there exists an admissible algebraic group $\mathbf{G}$ for $\overline{G}$; and
(2) there exists a finite-index subgroup $\Gamma' \leqslant \Gamma$ such that $\Gamma' \subseteq \phi\big(\mathbf{G}(\mathbb{Q})\big)$.

We will say that Γ is *arithmetic* if, in addition to being quasi-arithmetic, the following stronger statement holds:

- Γ is commensurable with $\phi\big(\mathbf{G}(\mathbb{Z})\big)$.

If Γ is quasi-arithmetic but not arithmetic then it is called *properly quasi-arithmetic*.

Note that arithmeticity implies quasi-arithmeticity. On the other hand Theorem 1.5 shows that, in PO$(n, 1)$ (for any $n \geqslant 2$), quasi-arithmeticity does not imply arithmeticity (cf. §1.1) and so the class of properly quasi-arithmetic lattices is non-empty in this context.

Remark. The term 'quasi-arithmetic' should not be confused with the term 'subarithmetic', introduced by Gromov and Piatetski-Shapiro. (Cf. §3.3.)

1.2.1. *Hyperbolic space.* We adopt the Lorentz model of hyperbolic n-space, where we first equip $\mathbb{R}^{n+1}$ with the quadratic form f_n, given in the standard basis by

$$f_n(x) = -x_0^2 + x_1^2 + \cdots + x_n^2,$$

and then denote

$$\mathbb{H}^n = \big\{x \in \mathbb{R}^{n+1} \mid f_n(x) = -1 \text{ and } x_0 > 0\big\}$$

with metric $\mathrm{d}_{\mathbb{H}^n}$ derived from the Lorentzian inner product $(\cdot,\cdot)$ associated to the quadratic form f_n by

$$\cosh\big(\mathrm{d}_{\mathbb{H}^n}(x, y)\big) = -(x, y).$$

We then have $\mathrm{Isom}(\mathbb{H}^n) \cong \mathrm{PO}(n, 1) = \mathrm{O}(\mathbb{R}^{n+1}, f_n)/\{\pm 1\}$ where the equality is by definition and where $\mathrm{O}(\mathbb{R}^{n+1}, f_n)$ is the orthogonal group $\{g \in \mathrm{GL}_n(\mathbb{R}) \mid f(g(x)) = f(x) \;\; \forall x \in \mathbb{R}^{n+1}\}$ [Rat06]. The group of orientation-preserving isometries of $\mathbb{H}^n$ (denoted $\mathrm{Isom}^+(\mathbb{H}^n)$) is isomorphic to $\mathrm{PSO}(n, 1)$, which is in turn defined to be the quotient $\mathrm{SO}(\mathbb{R}^{n+1}, f_n)/\{\pm 1\}$: this has index 2 in $\mathrm{PO}(n, 1)$. The groups $\mathrm{PO}(n, 1)$ and $\mathrm{PSO}(n, 1)$ are real Lie groups. If K is a totally real algebraic number field (with a fixed embedding $K \subset \mathbb{R}$), and if f is any quadratic form over K with signature

$(n,1)$ then we may consider the algebraic group $\mathbf{SO}_f$, and we have a Lie group isomorphism $\mathbf{SO}_f(\mathbb{R})^\circ \to \mathrm{PSO}(n,1)$. We think of $\mathrm{PSO}(n,1)$ as a fixed concrete realisation of $\mathrm{Isom}^+(\mathbb{H}^n)$ and $\mathbf{SO}_f(\mathbb{R})$ as an algebraic 'model' over K.

In what follows we will have $G = \mathrm{Isom}(\mathbb{H}^n) = \mathrm{PO}(n,1)$. Then, the group $\overline{G} = G^\circ$ will satisfy the hypothesis of Definition 1.2. We will not be considering non-orientable lattices here, though we will need to consider some isometries lying in $G \smallsetminus G^\circ$.

1.2.2. *Standard arithmetic lattices.* As before, we still have $G = \mathrm{Isom}(\mathbb{H}^n)$ and $\overline{G} = G^\circ$. If $\mathbf{G}$ is some algebraic group over a number field K, then we can form a related algebraic group over $\mathbb{Q}$, denoted $\mathbf{Res}_{K/\mathbb{Q}}(\mathbf{G})$, such that $\mathbf{G}(K) \cong \mathbf{Res}_{K/\mathbb{Q}}(\mathbf{G})(\mathbb{Q})$. If K is totally real, of degree d over $\mathbb{Q}$, then we have $\mathbf{G}(K \otimes_{\mathbb{Q}} \mathbb{R}) \cong \mathbf{Res}_{K/\mathbb{Q}}(\mathbf{G})(\mathbb{R})$; moreover since $K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^d$ we obtain a direct product

$$\mathbf{Res}_{K/\mathbb{Q}}(\mathbf{G})(\mathbb{R}) \cong \prod_{\sigma \in \mathrm{Gal}(K/\mathbb{Q})} \mathbf{G}^\sigma(\mathbb{R}). \tag{1}$$

The algebraic group $\mathbf{Res}_{K/\mathbb{Q}}(\mathbf{G})$ is called the Weil restriction of $\mathbf{G}$, and we call $\mathbf{Res}_{K/\mathbb{Q}}$ the restriction of scalars functor [PR94, p.50] [Mar91, I.1.7].

Definition 1.4. Fix some $n \geqslant 2$. Let K be a totally real number field, and let f be a (K-valued) non-degenerate quadratic form on K^{n+1}. Let $\mathbf{SO}_f$ be the algebraic K-group determined by $\{x \in \mathrm{O}(K^{n+1}, f) \mid \det(x) = 1\}$. We say that (K, f) is an *admissible field-form pair* if the group $\mathbf{Res}_{K/\mathbb{Q}}(\mathbf{SO}_f)$ is admissible for $\overline{G}$. Equivalently, by (1), the pair (K,f) is admissible if $\mathbf{SO}_f(K \otimes_{\mathbb{Q}} \mathbb{R})^\circ$ is isomorphic to $\overline{G}$, modulo a compact kernel.

(One could do away with reference to K as this field is implicit in the definition of f. We keep K for emphasis on the field of definition, especially as we will later have occasion to consider extensions of scalars of the form $K \otimes L$.)

In more concrete terms, we may fix some basis for K^{n+1}, so realising f as a homogeneous quadratic polynomial with coefficients in K. We also regard K as embedded in $\mathbb{R}$. Then for (K,f) to be admissible for $\mathrm{PO}(n,1)$ requires that f has signature $(n,1)$ on $\mathbb{R}^{n+1}$ and that for all elements $\sigma \in \mathrm{Gal}(K/\mathbb{Q}) \smallsetminus \{\mathrm{id}\}$ the conjugate forms f^σ (obtained by applying σ to the coefficients of f) are positive definite.

If (K,f) is an admissible pair, and if $\mathcal{O}_K$ denotes the ring of integers in K, then it is well-known that $\mathbf{SO}_f(\mathcal{O}_K)$ can be realised as an arithmetic lattice in $\mathrm{PO}(n,1)$ [Mar91, (3.2.7)]. This follows from (1), since $\mathbf{Res}_{K/\mathbb{Q}}(\mathbf{SO}_f)(\mathbb{Z})$ is a lattice in $\mathbf{Res}_{K/\mathbb{Q}}(\mathbf{SO}_f)(\mathbb{R})$ and on projecting to the non-compact factor (which is the *only* such factor by admissibility of (K,f)), we obtain a lattice in $\mathbf{SO}_f(\mathbb{R})$, isomorphic to $\mathbf{SO}_f(\mathcal{O}_K)$. To realise a lattice in $\mathrm{PO}(n,1)$, note that we have a homomorphism $\mathbf{SO}_f(\mathbb{R})^\circ \to \mathrm{PO}(n,1)$ with finite co-kernel.

We call $\mathbf{SO}_f(\mathcal{O}_K)$ the *standard arithmetic lattice* associated to the pair (K,f). Lattices of this type may also be called arithmetic lattices of the simplest type [VS93, p.217]. By abuse of notation we will also refer to $\mathrm{PO}_f(\mathcal{O}_K)$ as the associated standard arithmetic lattice, when we wish to work with the image of this lattice in $\mathrm{PO}_f(\mathbb{R})$.

1.3. **Summary and results.** In §2.1, we will recall the construction of I. Agol, generalised by Belolipetsky and Thomson, of hyperbolic manifolds with short closed geodesics, and then show that their associated lattices are quasi-arithmetic. This leads to the following theorem:

Theorem 1.5. *For any admissible field-form pair* (K, f) *there are infinitely many commensurability classes of* properly *quasi-arithmetic lattices* $\Gamma < \mathrm{PO}(n, 1)$ *arising from* $\mathbf{SO}_f(K)$.

(A proof of Theorem 1.5 is given on p.7.)

In §3 we will examine the construction of Gromov and Piatetski-Shapiro and show that the lattices so obtained are not quasi-arithmetic:

Theorem 1.6. *Let* Γ *be a Gromov–Piatetski-Shapiro lattice. Then* Γ *is not quasi-arithmetic.*

The definition and construction of a 'Gromov–Piatetski-Shapiro lattice' is given in §3.1. Theorems 1.5 and 1.6 together give us the following:

Corollary 1.7. *There are infinitely many commensurability classes of non-arithmetic lattices in* $\mathrm{PO}(n, 1)$ *(for* $n \geqslant 2$*) that are not commensurable with Gromov–Piatetski-Shapiro lattices.*

2. Hyperbolic manifolds with short systole

If M is a non-simply-connected closed Riemannian manifold, then the *systole* of M is by definition the (length of) the shortest closed non-trivial curve in M. One may show that the systole of such an M is always positive [Kat07, p.39].

In this section we will revisit a recent construction of closed hyperbolic n-manifolds with 'short' systole, and then see that the corresponding lattices (i.e., fundamental groups) in $\mathrm{PO}(n, 1)$ are quasi-arithmetic.

2.1. Constructing hyperbolic manifolds with short systole.

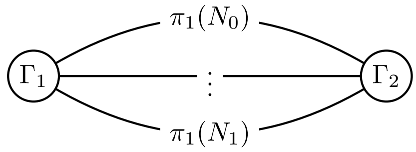
Let $n \geqslant 2$ and $\varepsilon > 0$. We will describe how one can construct a closed hyperbolic n-manifold with systole at most ε, as per the construction of Belolipetsky and Thomson [BT11]. This will allow us to establish some notation. The method is a generalisation of a construction by I. Agol in dimension 4 [Ago06]. (It was also pointed out by N. Bergeron, F. Haglund and D. Wise that some of their own methods allowed Agol's 4-dimensional construction to be generalised to every dimension using subgroup separability arguments [BHW11, Remark on p.17].)

Fix some admissible pair (K, f) with $K \neq \mathbb{Q}$, and let Λ be a torsion-free subgroup of the associated standard arithmetic lattice in $\mathrm{PO}(n, 1)$, so that $\Lambda\backslash\mathbb{H}^n$ is a compact hyperbolic n-manifold N. So, for some Galois embedding $\sigma\colon K \to \mathbb{R}$, the form f^σ on $K^{n+1} \otimes_{\sigma(K)} \mathbb{R}$ has signature $(n, 1)$ and we can isometrically identify $\mathbb{H}^n$ with the more convenient model

$$\mathbb{H}^n_f = \left\{x \in K^{n+1} \otimes_{\sigma(K)} \mathbb{R} \mid f(x) < 0\right\}/\sim,$$

where $x \sim \lambda x$ for all $x \in K^{n+1} \otimes_{\sigma(K)} \mathbb{R}$ and all $\lambda \in \mathbb{R} \smallsetminus 0$. In this model, we have $\mathrm{Isom}(\mathbb{H}^n_f) \cong \mathrm{PO}_f(K \otimes_{\sigma(K)} \mathbb{R})$. We will continue to refer to $\mathbb{H}^n$ rather than $\mathbb{H}^n_f$. Let H_0 and H_1 be two disjoint 'K-rational' hyperplanes in $\mathbb{H}^n_f$; that is, choosing two vectors v_0 and v_1 in K^{n+1} with $f(v_i) > 0$ for $i = 0, 1$, let $H_i = \langle v_i\rangle^{\perp_f} \cap \mathbb{H}^n$, and suppose that $H_0 \cap H_1 = \emptyset$. If one is interested in obtaining a manifold with short systole then one chooses the v_i so that the H_i are at most hyperbolic distance $\varepsilon/2$ apart. The projections $\pi(H_i) \subseteq N$ are immersed totally geodesic hypersurfaces in N, but they might not be embedded and they might intersect eachother. However, by replacing Λ by a suitable finite-index (congruence) subgroup this can be avoided, so

FIGURE 1. The graph of groups $\mathcal{G}$ whose fundamental group is equal to that of the fundamental group of M. There are between 2 and 4 edges, one or two for N_0 and one or two for N_1.



that H_i ($i = 0, 1$ respectively) projects to a totally geodesic embedded hypersurface $N_i \subset N$ ($i = 0, 1$ respectively), and so that $N_0 \cap N_1 = \emptyset$ [BT11, Lem. 3.1].

We now cut along the two hyperplanes N_i to obtain a manifold with boundary. Keeping the connected component M' that contains the common perpendicular geodesic segment c between the two H_i, we double along the boundary B of M' to obtain a manifold $M = DM'$ [Lee13, p.226]. Then M contains the closed geodesic Dc of length at most ε. We will suppose that B has ℓ connected components, and depending on whether or not the cutting separates the manifold N, we have $2 \leqslant \ell \leqslant 4$.

Thus M is a compact hyperbolic n-manifold that can be written as $\Gamma\backslash\mathbb{H}^n$ for some lattice $\Gamma \in \mathrm{PO}_f(K \otimes_{\sigma(K)} \mathbb{R})$. If ε is small enough then the manifold M (having systole at most ε) is non-arithmetic [BT11, 5.1].

2.2. **Quasi-arithmeticity of short systole manifolds.** Let M be as in §2.1, with sufficiently short systole that it is non-arithmetic, and write $M = \Gamma\backslash\mathbb{H}^n$. We still suppose (K, f) to be an admissible pair for $\mathrm{PO}(n, 1)$. We prove the following:

Proposition 2.1. *The group Γ can be generated by elements in $\mathrm{PO}_f(K)$ and so is quasi-arithmetic.*

Proof. By construction, the manifold M is a union $M_1 \cup_B M_2$ (where M_i is isometric to M'), so it has a symmetry τ that interchanges its two parts M_1 and M_2. By this decomposition we see that the lattice $\Gamma \cong \pi_1(M)$ splits as the fundamental group of a graph of groups $\mathcal{G}$, where the graph $\mathcal{G}$ is the graph with two nodes and edges between the two nodes corresponding to each boundary component along which the double is taken.

That is, the edge groups are the fundamental groups of the boundary components and the vertex groups are the (isomorphic) groups Γ_1 and Γ_2. (See Fig. 1.)

So, Γ is generated by the elements of the two fundamental groups $\Gamma_i = \pi_1(M_i)$ (for $i = 1, 2$), as well as some extra elements corresponding to the new homotopy classes that appear as a result of the gluing. (See Serre [Ser80] for the properties of graphs of groups.)

In what follows we describe how to concretely realise Γ as a subgroup of $\mathrm{PO}_f(K)$.

Claim. We may assume that Γ_1 can be identified with its image in Λ, on identifying M_1 with $M' \subset N$.

Proof of claim. A priori it is not in general possible to make this latter identification; in particular if the hypersurfaces N_i do not each separate N. But, there exists, nevertheless, a double cover $\pi\colon N' \to N$ such that each $\pi^{-1}(N_i)$ *does* separate N

[GPS87, 2.8C]. Then, the piece obtained by cutting along both $\pi^{-1}(N_i)$ is isometric to the original M', and $\pi_1(M') \hookrightarrow \pi_1(N')$. Since $N' \to N$ is a finite cover, the discrete subgroup $\pi_1(N') < \pi_1(N)$ is still arithmetic. (We may need to take a cover for each hypersurface N_i; but these then have a common cover, obtained by intersecting their fundamental groups, which is still a finite cover of N.) □

Now, write I_0 for the reflection in the hyperplane H_0: we have $\Gamma_2 = I_0^{-1}\Gamma_1 I_0$, and we note that $I_0 \in \mathrm{PO}_f(K)$ since H_0 has a normal vector $v_0 \in K^{n+1}$ (cf. §2.1).

The group Γ_i is a discrete convex-co-compact group acting on $\mathbb{H}^n$ (for both $i = 1$ and $i = 2$). Choosing some basepoint $x_0 \in H_0$ we may construct Dirichlet fundamental domains $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ (about x_0) for Γ, Γ_1 and Γ_2 respectively. By the doubling construction it is evident that $\mathcal{F}$ will be invariant under the reflection I_0. Thus we can view $\mathcal{F}$ as a union $F_1 \cup F_2$ of the two pieces exchanged by I_0, and each F_i satisfies the inclusion $F_i \subseteq \mathcal{F}_i$. We also have the inclusion $\mathcal{F} \subseteq \mathcal{F}_i$ $(i = 1, 2)$ since $\Gamma_i < \Gamma$. The intersection $\mathcal{F}_1 \cap \mathcal{F}_2$ is a Dirichlet fundamental domain (at x_0) for the group $\langle \Gamma_1, \Gamma_2 \rangle$ and so we have $\mathcal{F} \subseteq \mathcal{F}_1 \cap \mathcal{F}_2$ since $\langle \Gamma_1, \Gamma_2 \rangle \leqslant \Gamma$.

The domains $\mathcal{F}_i$ may be decomposed into two parts separated by H_0. By $\tilde{\mathcal{F}}_i$ we mean the part containing the ends corresponding to the boundary components whose lifts to $\mathbb{H}^n$ are H_1 and $\gamma_j(H_j)$ $(j = 0, 1)$ if either of the latter exist (cf. §2.1). If H_0 intersects a bounding hyperplane of $\mathcal{F}_i$ then it must do so orthogonally and hence we have the inclusion $I_0(\tilde{\mathcal{F}}_i) \subseteq \mathcal{F}_i \smallsetminus \tilde{\mathcal{F}}_i$. Let the set J be the index set $\{1, \ldots, \ell - 1\}$, and if $2 \in J$ or $3 \in J$ then denote the two hyperplanes $H_2 = \gamma_0(H_0)$ and $H_3 = \gamma_1(H_1)$. Write I_j for the reflection in H_j (whenever H_j exists), and write H_j^- for the half-space bounded by H_j and containing H_0. The set $\mathcal{F}$ is the intersection

$$\mathcal{F} = (\mathcal{F}_1 \cap \mathcal{F}_2) \cap \bigcap_{j \in J} H_j^- \cap \bigcap_{j \in J} I_0(H_j^-),$$

and the group Γ may be given by the generators

$$\Gamma = \left\langle \Gamma_1, \quad I_0^{-1}\Gamma_1 I_0, \quad I_j^{-1} I_0 \ \ (j \in J) \right\rangle. \tag{2}$$

Geometrically, the conjugate copy of Γ_1 by I_0 corresponds to matching up two copies of the fundamental domain $\mathcal{F}_1$ at H_0, and the elements $I_j^{-1} I_0$ correspond to the gluing isometries for the remaining sides of the fundamental domain $\mathcal{F}$ corresponding to the H_1 and $\gamma_j(H_j)$; equivalently the ends of $\mathcal{F}_1$ and $\mathcal{F}_2$ away from H_0.

We finally note that since the H_j are K-rational hyperplanes, their corresponding reflections I_j do indeed lie in $\mathrm{PO}_f(K)$. □

Proof of Theorem 1.5 (cf. p.5). We have already seen in Prop. 2.1 that any lattice constructed as in §2.1 is quasi-arithmetic. That there are infinitely many commensurability classes of such lattices has already been demonstrated in the literature [BT11, 5.2], and follows from that fact that when these lattices are non-arithmetic, their commensurator is also a lattice in $\mathrm{PO}_f(\mathbb{R})$ [Mar91, (B), p.298]: as the systole length in the construction of §2.1 decreases, one obtains a sequence of non-arithmetic lattices Γ_m (for $m \in \mathbb{N}$), and if these were commensurable then Margulis' theorem would imply that we have some maximal lattice Γ with $\Gamma_m \subset \Gamma$ for every m. But since the Γ_m have decreasing systole lengths, this is not possible. □

3. Gromov–Piatetski-Shapiro manifolds

Gromov and Piatetski-Shapiro's construction of non-arthmetic lattices in $\mathrm{PO}(n,1)$, from hybrid hyperbolic manifolds, is well-known and so we do not revisit it in complete detail; but we present below enough of their construction to examine the quasi-arithmeticity properties of the resulting lattices. The reader interested in the details of the construction should find the original article accessible even to the non-expert [GPS87].

After recalling the construction we turn to proving that these lattices are not quasi-arithmetic (§3.2).

3.1. **The construction of Gromov and Piatetski-Shaprio.** In order to obtain non-arithmetic lattices $\Gamma < \mathrm{PO}(n,1)$ Gromov and Piatetski-Shapiro first consider two torsion-free co-compact arithmetic lattices $\tilde{\Gamma}_1 < \mathrm{PO}_{f_1}(\mathbb{R})$ and $\tilde{\Gamma}_2 < \mathrm{PO}_{f_2}(\mathbb{R})$ over a field K, such that the two quotient manifolds $\tilde{M}_i = \tilde{\Gamma}_i\backslash\mathbb{H}^n$ $(i = 1,2)$ each contain a co-dimension 1 closed submanifold $M_0^{(i)}$ $(i = 1,2)$ with an isometry $\psi\colon M_0^{(1)} \to M_0^{(2)}$. They show that if the forms f_1 and f_2 are not similar over K then the manifolds $\tilde{M}_1$ and $\tilde{M}_2$ are not commensurable: we assume that the forms are indeed not similar. The manifold M obtained by cutting each of the $\tilde{M}_i$ along $M_0^{(i)}$ and gluing the two together via the boundary isometry ψ, would be a cover of M_1 and M_2, if M was arithmetic [GPS87, 0.2]. However in light of the *non*-commensurability of M_1 and M_2, the glued manifold M cannot be a common cover of these spaces and so must be non-arithmetic. Thus $\Gamma = \pi_1(M)$ is a non-arithmetic lattice in $\mathrm{PO}(n,1)$.

Remark. The manifold M is called a *hybrid* manifold, obtained by a process of *interbreeding*. For the manifolds in §2.1 we call the process *inbreeding* since the two manifolds glued together arise from the same arithmetic lattice.

Definition 3.1. By a *Gromov–Piatetski-Shapiro lattice* (GPS lattice for short) is meant a non-arithmetic lattice $\Gamma < \mathrm{PO}(n,1)$ obtained as $\pi_1(M)$ by the above procedure; that is, by taking two non-similar admissible quadratic forms f_1 and f_2 over a common field K, containing a common subform f_0 giving rise to an embedded codimension 1 hypersurface.

3.2. **Non-quasi-arithmeticity of GPS lattices.** In this section we will prove Theorem 1.6 (cf. p.5); i.e., that GPS lattices are not quasi-arithmetic.

Let $\Gamma < \mathrm{PO}(n,1)$ be a GPS lattice. So, there is a hyperbolic manifold M with $\Gamma = \pi_1(M)$, such that $M = M_1 \cup_{M_0} M_2$ with M_1 and M_2 arising as manifolds with boundary from closed arithmetic manifolds $\tilde{M}_i = \Lambda_i\backslash\mathbb{H}^n$. As in §3.1, suppose that $\Lambda_i \leqslant \mathrm{PO}_{f_i}(\mathcal{O}_K)$ (where we identify $\mathrm{PO}_{f_i}(\mathbb{R})$ with its image in $\mathrm{PO}(n,1)$ via an isomorphism ϕ_i over some finite extension of K). Write $\Gamma_i = \pi_1(M_i)$ for $i = 1,2$. Then each Γ_i is Zariski dense in $\mathrm{PO}(n,1) = \mathrm{PO}_{f_n}(\mathbb{R})$ [GPS87, 0.1].

Now suppose that there is another admissible pair (K', f') for $\mathrm{PO}(n,1)$ and such that $\Gamma \subset \mathrm{PO}_{f'}(K')$ (where again we identify the corresponding groups of real points by an isomorphism ϕ over an extension of K'). Then we would have the configuration

$$\begin{array}{ccccccccc}
\phi_1^{-1}(\Gamma_1) & \hookrightarrow & \mathrm{PO}_{f_1}(\mathcal{O}_K) & \hookrightarrow & \mathrm{PO}_{f_1}(\mathbb{R}) & & & & \Gamma_1 = \pi_1(M_1) \\
& & & & & \searrow^{\phi_1} & & \swarrow & \\
\Gamma & \hookrightarrow & \mathrm{PO}_{f'}(K') & \hookrightarrow & \mathrm{PO}_{f'}(\mathbb{R}) & \xrightarrow{\phi} & \mathrm{PO}(n,1) & & \\
& & & & & \nearrow_{\phi_2} & & \nwarrow & \\
\phi_2^{-1}(\Gamma_2) & \hookrightarrow & \mathrm{PO}_{f_2}(\mathcal{O}_K) & \hookrightarrow & \mathrm{PO}_{f_2}(\mathbb{R}) & & & & \Gamma_2 = \pi_1(M_2)
\end{array} \quad . \tag{3}$$

Proposition 3.2. *Assuming the above configuration, we have $K = K'$ and for each $i = 1, 2$, there is an isomorphism $\mathrm{PO}_{f'}(K) \cong \mathrm{PO}_{f_i}(K)$.*

Proof. By minimality of K we must have $K \subseteq K'$. Then, examining $\mathrm{PO}_{f'}$, we have

$$\mathrm{PO}_{f'}(K' \otimes \mathbb{R}) \cong \prod_{\sigma \in \mathrm{Gal}(K'/K)} \Big(\prod_{\tau \in \mathrm{Gal}(K/\mathbb{Q})} \mathrm{PO}_{(f_i^{\sigma})^{\tau}}(\mathbb{R}) \Big);$$

and then since $\mathrm{PO}_{f'}$ is admissible there can be only one non-compact factor on the right hand side of this isomorphism. Thus $\mathrm{Gal}(K'/K)$ can contain only precisely one element and so $K = K'$.

In what follows, $\overline{A}^{(F)}$ will denote the Zariski F-closure of a set A with respect to the Zariski F-topology on $\mathrm{PO}_f(F)$, for F a field. That is, $\overline{A}^{(F)}$ is the smallest closed set in $\mathrm{PO}_f(F)$ containing A, in the topology generated by sets of zeroes of polynomials with coefficients in F.

Fix $i = 1$ or $i = 2$. Now, $\phi^{-1}(\Gamma_i)$ is Zariski dense in $\mathrm{PO}_{f'}(K)$; for if not then $\phi\big(\overline{\phi^{-1}(\Gamma_i)}^{(K)}\big)$ would be contained in a K-closed proper subgroup of $\mathrm{PO}(n,1)$ containing Γ_i. But this is also an $\mathbb{R}$-closed subgroup, which is impossible by Zariski density of Γ_i in $\mathrm{PO}(n,1)$. Similarly, $\phi_1^{-1}(\Gamma_i)$ is also Zariski dense in $\mathrm{PO}_{f_i}(K)$. Thus we have an isomorphism $\mathrm{PO}_{f_1}(K) \cong \mathrm{PO}_{f'}(K)$ by composing ϕ^{-1} and ϕ_1.

□

Thus if Γ is a quasi-arithmetic lattice then it must be contained in $\mathrm{PO}_{f_i}(K)$ for both $i = 1, 2$, and these PO_{f_i} must be K-isomorphic, which is impossible since the forms f_i are not similar [GPS87, 2.6].

This concludes the proof of Theorem 1.6. □

3.3. **Sub-arithmeticity.** In the original article describing GPS lattices, the authors define a *sub-arithmetic* group $\Gamma < \mathrm{PO}(n,1)$ to be a discrete group that is Zariski dense and such that for some arithmetic lattice $\Lambda < \mathrm{PO}(n,1)$, we have $|\Gamma : \Lambda \cap \Gamma| < \infty$ (i.e., Γ is virtually contained in an arithmetic lattice) [GPS87, 0.4]. A sub-arithmetic group need not be a lattice and so sub-arithmeticity and quasi-arithmeticity are different notions, albeit both in the spirit of being close to arithmetic.

Both constructions outlined here, of non-arithmetic lattices in $\mathrm{PO}(n,1)$, are based on gluing manifolds with sub-arithmetic fundamental groups. At present the only general methods for constructing non-arithmetic lattices in $\mathrm{PO}(n,1)$ are based on gluing constructions involving sub-arithmetic discrete groups and so the question raised by Gromov and Piatetski-Shapiro—namely, whether there exist non-arithmetic lattices that are *not* constructed from sub-arithmetic subgroups—remains open [GPS87, 0.4].

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