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HIGHER TODA BRACKETS AND MASSEY PRODUCTS

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Abstract. We provide a uniform definition of higher order Toda brackets in a general setting, covering the known cases of long Toda brackets for topological spaces and Massey products for differential graded algebras, among others.

Introduction

Toda brackets and Massey products have played an important role in homotopy theory ever since they were first defined in [Mas] and [To1, To2]: in applications, such as [Ad2, BJM, MP], and in a more theoretical vein, as in [Ad1, Ba3, He, Kri, Mar, Sa, Sp1]. There are a number of variants (see, e.g., [Al, HKM, Mi, PS] and [Ba1, §3.6.4]), as well as higher order versions including [Kl, Kra, KM, Mau, Mo, P1, P2, Re, Sp2, W]. In recent years they have appeared in many other areas of mathematics, including symplectic geometry, representation theory, deformation theory, topological robotics, number theory, mathematical physics, and algebraic geometry (see [BT, BKS, FW, G, Ki, La, LS, Ri]).

Toda brackets were originally defined for diagrams of the form

$$S^n \xrightarrow{f} S^p \xrightarrow{g} S^k \xrightarrow{h} X\ , \tag{0.1}$$

with $g \circ f$ and $h \circ g$ nullhomotopic.

If we choose nullhomotopies $F : g \circ f \sim 0$ and $G : h \circ g \sim 0$, they fit into a diagram of cones as in Figure 0.2:

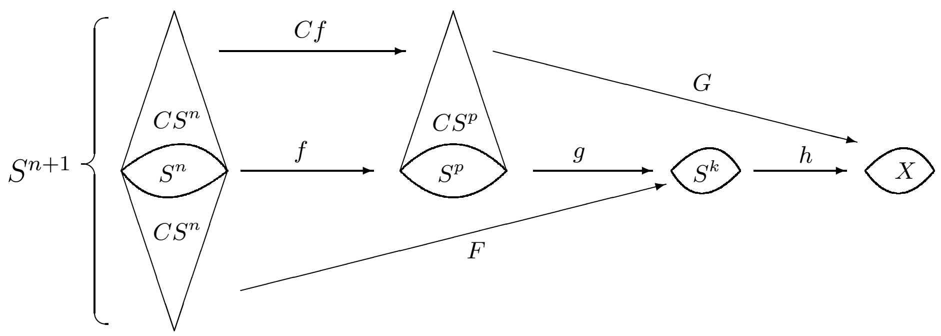

Figure 0.2. The Toda bracket construction

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This yields an element $\langle h,g,f\rangle$ in $[S^{n+1},\,X]$, called the *Toda bracket.* The value we get depends on the choices of nullhomotopies F and G, so it is not uniquely determined. The Toda bracket is thus more properly a certain double coset of $h_{\#}\pi_{n+1}(S^k)+\Sigma f^{\#}\pi_{p+1}(X)$.

If we view $[h]$ as an element in π_*X, while $[g]$ is seen as a primary homotopy operation acting trivially on $[f]$ and $[h]\circ[g]=0$ is a relation among primary operations, we can think of the Toda bracket as a secondary homotopy operation. Similarly, a diagram of the form

$$X \xrightarrow{f} K(G,n) \xrightarrow{g} K(G',p) \xrightarrow{h} K(G'',k) \tag{0.3}$$

with $g\circ f\sim 0\sim h\circ g$ defines a secondary cohomology operation in the sense of [Ad2].

On the other hand, the *Massey product* in cohomology – defined whenever we have three classes $\alpha,\beta,\gamma\in H^*X$ with $\alpha\cdot\beta=0=\beta\cdot\gamma$ – is a different type of secondary cohomology operation which does not fit into this paradigm.

All three examples have higher order versions, though the precise definitions are not always self-evident or unique (cf. [W] and [Mau, Kl]). Nevertheless, these higher order operations play an important role in homotopy theory – for instance, in enhancing our theoretical understanding of spectral sequences (cf. [BB]) and in providing a conceptual full invariant for homotopy types of spaces (see [Ta] and [BJT2]).

The main goal of this note is to explain that higher order Toda brackets and higher Massey products have a uniform description, covering all cases known to the authors (including both the homotopy and cohomology versions).

The setting for our general notion of higher Toda brackets is any category $\mathcal{C}$ enriched in a suitable monoidal category $\mathcal{M}$. In fact, the minimal context in which higher Toda brackets can be defined is just an enrichment in a monoidal category equipped with a certain structure of “null cubes”, encoded by the existence of an augmented path space functor $PX\to X$ satisfying certain properties (abstracted from those enjoyed by the usual path fibration of topological spaces). We call such an $\mathcal{M}$ a *monoidal path category* – see Section 1.

In this context we can define the notion of a higher order chain complex: that is, one in which the identity $\partial\partial=0$ holds only up to a sequence of coherent homotopies (see Section 2). This suffices to allow us to *define* the values of the corresponding higher order Toda bracket (see Section 3, where higher Massey products are also discussed).

However, in order for these Toda brackets to enjoy the expected properties, such as homotopy invariance, $\mathcal{M}$ must be also be a simplicial model category. In this case there is a model category structure on the category $\mathcal{M}$-Cat of categories enriched in $\mathcal{M}$, due to Lurie, Berger and Moerdijk, and others, in which the weak equivalences are Dwyer-Kan equivalences (see §4.10). This is explained in Section 4, where we prove:

Theorem A. *Higher Toda brackets are preserved under Dwyer-Kan equivalences.*

[See Theorem 4.21 below].

We also show that the usual higher Massey products in a differential graded algebra correspond to our definition (see Proposition 4.35).

In Section 5 we study the case of ordinary Toda brackets for chain complexes, and show their interpretation as secondary Ext-operations.

0.4. **Notation.** The category of sets will be denoted by Set, that of compactly generated topological spaces by Top (cf. [St], and compare [V]), and that of pointed compactly generated spaces by Top_*.

If R is a commutative ring with unit, the category of R-modules will be denoted by Mod_R (though that of abelian groups will be denoted simply by AbGp). The category of non-negatively graded R-modules will be denoted by $\mathsf{grMod}_R^{\geqslant 0}$, with objects $\mathcal{E}_* = \{E_n\}_{n\geq 0}$, and so on.

The category of $\mathbb{Z}$-graded chain complexes over Mod_R will be denoted by Ch_R, with objects $\mathbf{A}_*$, $\mathbf{B}_*$, and so on, where

$$\mathbf{A}_* := (\dots A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} A_{n-2} \xrightarrow{\partial_{n-2}} A_{n-3} \dots) .$$

The category of nonnegatively graded chain complexes over Mod_R will be denoted by $\mathsf{Ch}_R^{\geqslant 0}$. A chain map $f : \mathbf{A}_* \to \mathbf{B}_*$ inducing an isomorphism $f_*\colon H_n\mathbf{A}_* \longrightarrow H_n\mathbf{B}_*$ for all n is called a *quasi-isomorphism.*

Finally, the category of simplicial sets will be denoted by $\mathcal{S}$, and that of pointed simplicial sets by $\mathcal{S}_*$.

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1. Path functors in monoidal categories

Higher order homotopy operations in a pointed model category $\mathcal{C}$, such as Top_*, $\mathcal{S}_*$, or Ch_R, are usually described in terms of higher order homotopies, which can be defined in turn in terms of an enrichment of $\mathcal{C}$ in an appropriate monoidal model category $\mathcal{M}$ (see, e.g., [BJT1]). We here abstract the minimal properties of such an $\mathcal{M}$ needed for the construction of higher operations.

1.1. **Definition.** A *monoidal path category* is a functorially complete and cocomplete pointed monoidal category $\langle \mathcal{M}, \otimes, 1\rangle$, equipped with an *path* endofunctor $P : \mathcal{M} \to \mathcal{M}$ and natural transformations $\mathfrak{p}_X : PX \to X$, $\theta^L : PX \otimes Y \to P(X \otimes Y)$, and $\theta^R : X \otimes PY \to P(X \otimes Y)$.

We require that the following diagrams commute:

(a) Constant path combinations:

$$\begin{array}{ccc} PX \otimes Y & \xrightarrow{\theta^L} & P(X \otimes Y) \\ \downarrow{\scriptstyle \mathfrak{p}_X \otimes \mathrm{Id}_Y} & & \downarrow{\scriptstyle \mathfrak{p}_{X\otimes Y}} \\ X \otimes Y & \xrightarrow{=} & X \otimes Y \end{array} \qquad \begin{array}{ccc} X \otimes PY & \xrightarrow{\theta^R} & P(X \otimes Y) \\ \downarrow{\scriptstyle \mathrm{Id}_X \otimes \mathfrak{p}_Y} & & \downarrow{\scriptstyle \mathfrak{p}_{X\otimes Y}} \\ X \otimes Y & \xrightarrow{=} & X \otimes Y \end{array} \tag{1.2}$$

(b) Coalgebra structure:

$$\begin{array}{ccc} P(PX) & \xrightarrow{P(\mathfrak{p}_X)} & PX \\ \downarrow{\scriptstyle \mathfrak{p}_{PX}} & & \downarrow{\scriptstyle \mathfrak{p}_X} \\ PX & \xrightarrow{\mathfrak{p}_X} & X. \end{array} \tag{1.3}$$

(c) Left and right constants:

(1.4)
$$\begin{array}{ccc} PX \otimes PY & \xrightarrow{\theta^R} & P(PX \otimes Y) \\ {\scriptstyle \theta^L}\downarrow & & \downarrow{\scriptstyle P\theta^L} \\ P(X \otimes PY) & \xrightarrow[P\theta^R]{} & P^2(X \otimes Y) \end{array}$$

(d) From (1.4) we see that there are natural transformations

$$\theta^{(i,j)} \ : \ P^i X \otimes P^j Y \ \to \ P^{i+j}(X \otimes Y)$$

for any $i, j \geq 0$, defined

$$\theta^{(i,j)} \ := \ P^{i+j-1}(\theta^L) \circ \cdots \circ P^j(\theta^L) \circ P^{j-1}(\theta^R) \circ \cdots \circ \theta^R \ .$$

These are required to be associative, in the obvious sense.

(e) If we let $P^n X$ denote the result of applying the functor $P : \mathcal{M} \to \mathcal{M}$ to X n times (with $P^0 := \mathrm{Id}_{\mathcal{M}}$), we have $n+1$ different natural transformations $\partial^n_i : P^{n+1}X \to P^n X$ $(i = 0, \ldots, n)$, defined

(1.5)
$$\partial_i \ = \ \partial^n_i \ := \ P^i(\mathfrak{p}_{P^{n-i}X}) \ .$$

The natural transformations $\theta^{(i,j)}$ are required to satisfy the identities:

(1.6)
$$\partial^{n-1}_k \circ \theta^{(i,j)} = \begin{cases} \theta^{(i-1,j)} \circ (\partial^{i-1}_k \otimes \mathrm{Id}) & \text{if } 0 \leq k < i \\ \theta^{(i,j-1)} \circ (\mathrm{Id} \otimes \partial^{j-1}_{k-i}) & \text{if } i \leq k < n \end{cases}$$

for every $0 \leq k < i + j = n$.

1.7. *Remark.* The commutativity of (1.3) implies that the natural transformations of (1.5) satisfy the usual simplicial identities

(1.8)
$$\partial^{n-1}_i \circ \partial^n_j \ = \ \partial^{n-1}_{j-1} \circ \partial^n_i$$

for all $0 \leq i < j \leq n$.

1.9. **Paths and cubes.** The natural setting where such path categories arise is when a monoidal category $\mathcal{M}$ is also *simplicial*, in the sense of [Q, II, §1]. More specifically, we require the existence of an *unpointed path functor* $(-)^I : \mathcal{M} \to \mathcal{M}$ which behaves like a mapping space from the interval $[0, 1]$, so we have natural transformations

(a) $e^0, e^1 : X^I \to X$ (evaluation at the two endpoints),
(b) $s : X \to X^I$ with $e^0 s = e^1 s = \mathrm{Id}$ (the constant path), and
(c) $\widetilde{\theta}^L : X^I \otimes Y \to (X \otimes Y)^I$ and $\widetilde{\theta}^R : X \otimes Y^I \to (X \otimes Y)^I$ (paths in a product).

These make the following diagrams commute:

(1.10)
$$\begin{array}{ccc} X^I \otimes Y & \xrightarrow{\widetilde{\theta}^L} & (X \otimes Y)^I \\ {\scriptstyle s\otimes \mathrm{Id}}\uparrow\ \downarrow{\scriptstyle e^i_X \otimes \mathrm{Id}_Y} & & {\scriptstyle e^i_{X\otimes Y}}\downarrow\ \uparrow{\scriptstyle s} \\ X \otimes Y & \xrightarrow{=} & X \otimes Y \end{array} \qquad \begin{array}{ccc} X \otimes Y^I & \xrightarrow{\widetilde{\theta}^R} & (X \otimes Y)^I \\ {\scriptstyle \mathrm{Id}\otimes s}\uparrow\ \downarrow{\scriptstyle \mathrm{Id}_X \otimes e^i_Y} & & {\scriptstyle e^i_{X\otimes Y}}\downarrow\ \uparrow{\scriptstyle s} \\ X \otimes Y & \xrightarrow{=} & X \otimes Y \end{array}$$

for $i = 0, 1,$ as well as

$$\begin{array}{ccc} X^{I^2} & \xrightarrow{(e^i_X)^I} & X^I \\ {\scriptstyle e^j_{X^I}}\downarrow & & \downarrow{\scriptstyle e^j_X} \\ X^I & \xrightarrow[e^i_X]{} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} X^I \otimes Y^I & \xrightarrow{\widetilde{\theta}^R} & (X^I \otimes Y)^I \\ {\scriptstyle \widetilde{\theta}^L}\downarrow & & \downarrow{\scriptstyle (\widetilde{\theta}^L)^I} \\ (X \otimes Y^I)^I & \xrightarrow[(\widetilde{\theta}^R)^I]{} & (X \otimes Y)^{I^2} \end{array} \tag{1.11}$$

for $i, j \in \{0, 1\}$.

We may then define the required (pointed) path functor $P : \mathcal{M} \to \mathcal{M}$ by the functorial pullback diagram:

$$\begin{array}{ccc} PX & \longrightarrow & X^I \\ \downarrow & \boxed{\text{PB}} & \downarrow{\scriptstyle e^0} \\ * & \longrightarrow & X\ . \end{array} \tag{1.12}$$

The commutativity of the right hand square in (1.11) allows us to define either composite to be the natural transformation $\widehat{\theta}^{(1,1)} : X^I \otimes Y^I \to (X \otimes Y)^{I^2}$.

We see that $\widetilde{\theta}^L$ induces a natural transformation $\theta^L : PX \otimes Y \to P(X \otimes Y)$, and similarly $\theta^R : X \otimes PY \to P(X \otimes Y)$, making (1.2) commute.

Moreover, from (1.10) we see that (1.4) commutes, and that the natural transformations $\theta^{(i,j)}$ are associative and satisfy (1.6).

1.13. **Example.** The motivating example is provided by $\mathcal{M} = \mathsf{Top}_*$, with the monoidal structure given by the smash product $\otimes := \wedge$, and $X^I := \mathbf{map}_*(I, X)$ the mapping space out of the interval $I := \Delta[1]_+$. Thus PX is the usual pointed path space. Here $\mathbf{map}_*(X, Y)$ denotes the set $\mathrm{Hom}_{\mathsf{Top}_*}(X, Y)$ equipped with the compact-open topology.

1.14. **Example.** Similarly for $\mathcal{S}_*$, again with the smash product $\otimes := \wedge$ and $X^I := \mathrm{map}_*(\Delta[1]_+, X)$, where $\mathrm{map}_*(X, Y) \in \mathcal{S}_*$ denotes the simplicial mapping space with $\mathrm{map}_*(X, Y)_n := \mathrm{Hom}_{\mathcal{S}_*}(X \times \Delta[n]_+, Y)$.

When X is a Kan complex, we can use Kan's model for PX, where $(PX)_n := \mathrm{Ker}(d_1 d_2 \dots d_{n+1} : X_{n+1} \to X_0)$, and $\mathfrak{p}_X : PX \to X$ is d^i_0 in simplicial dimension i.

1.15. **Example.** Another variant is provided by a suitable category Sp of spectra with strictly associative smash product $\wedge$, such as the S-modules of [EKMM], the symmetric spectra of [HSS], and the orthogonal spectra of [MMSS]. One again has function spectra $\mathbf{map}_{\mathsf{Sp}}(X, Y)$, which can be used to define X^I and PX. The unit is the sphere spectrum S^0.

1.16. **Example.** For chain complexes of R-modules we have a monoidal structure with the tensor product $(\mathbf{A}_* \otimes \mathbf{B}_*)_n := \bigoplus_{i+j=n} A_i \otimes B_j$.

Recall that the *function complex* $\underline{\mathrm{Hom}}(\mathbf{A}_*, \mathbf{B}_*)$ is given by

$$\underline{\mathrm{Hom}}(\mathbf{A}_*, \mathbf{B}_*)_n \ := \ \prod_{i \in \mathbb{Z}} \mathrm{Hom}(A_i, B_{i+n})\ , \tag{1.17}$$

with $\partial_n((f_i)_{i\in\mathbb{Z}}) := (\partial^B_{i+n} f_i - (-1)^n f_{i-1} \partial^A_i)_{i\in\mathbb{Z}}$ for $(f_i \colon A_i \to B_{i+n})_{i\in\mathbb{Z}}$.

Thus for $\mathcal{M} = \mathsf{Ch}_R$ we may set $X^I := \underline{\mathbf{Hom}}(C_*(\Delta[1]; R), X)$, and see that $P\mathbf{A}_*$ has

$$(PA)_n = A_n \oplus A_{n+1} \quad \text{with} \quad \partial(a, a') = (\partial a, \partial a' + (-1)^{n+1} a) \ , \tag{1.18}$$

and $\mathfrak{p}_{\mathbf{A}_*}$ the projection.

1.19**. Cores and elements.** In any monoidal path category $\langle \mathcal{M}, \otimes, 1, (-)^I \rangle$ and for any $X \in \mathcal{M}$, we can think of $\mathrm{Hom}_{\mathcal{M}}(1, X)$ as the 'underlying set' of X, and think of a map $f : 1 \to X$ in $\mathcal{M}$ as an 'element' of X.

More generally, we may have a suitable monoidal subcategory $\mathcal{I}$ of $\mathcal{M}$, which we call a *core*, and define a *generalized element* of X to be any map $f : \alpha \to X$ in $\mathcal{M}$ with $\alpha \in \mathcal{I}$.

1.20. **Example.** We may always choose $\mathcal{I} = \{1\}$ to consist of the unit of $\mathcal{M}$ alone. However, in some cases other natural choices are possible:

(a) In the three examples of §1.13, §1.14, and §1.15, we can let $\mathcal{I}_S := \{S^n\}_{n=0}^{\infty}$ consist of all (non-negative dimensional) spheres – this is evidently closed under $\otimes = \wedge$.

(b) In the category of chain complexes over a ring R (§1.16), we let $\mathcal{I}_R := \{\widetilde{\mathbf{M}}(R, n)_*\}_{n \in \mathbb{Z}}$, where $\widetilde{\mathbf{M}}(R, n)_*$ is the Moore chain complex with $\widetilde{\mathbf{M}}(R, n)_i = R$ for $i = n$, and 0 otherwise. Again we see that $\widetilde{\mathbf{M}}(R, p)_* \otimes \widetilde{\mathbf{M}}(R, q)_* = \widetilde{\mathbf{M}}(R, p + q)_*$, so $\mathcal{I}_R$ is indeed a monoidal subcategory of $(\mathsf{Ch}_R, \otimes_R, \widetilde{\mathbf{M}}(R, 0)_*)$.

We see that a generalized element in a chain complex $\mathbf{A}_*$ is now a map $f : \widetilde{\mathbf{M}}(R, n)_* \to \mathbf{A}_*$ in Ch_R – that is, an n-cycle in $\mathbf{A}_*$.

(c) Other examples are also possible – for example, if $\mathcal{I}' := \{\mathbf{M}(\mathbb{Z}/p, n)\}_{n=1}^{\infty}$ is the collection of mod p Moore spaces, representing mod p homotopy groups (see [N]), then it is not itself a monoidal subcategory of $(\mathsf{Top}_*, \wedge, S^0)$, since it is not closed under smash products. However, when p is odd, the collection of finite wedges of such Moore spaces *is* monoidal, by [N, Corollary 6.6].

2. Higher order chain complexes

The structure defined in the previous section suffices to define higher order chain complexes, as in [BB]:

2.1**. Categories enriched in monoidal path categories.** Let $\mathcal{C}$ be a category enriched in a monoidal path category $\langle \mathcal{M}, \otimes, 1, P \rangle$, so that for any $a, b \in \mathrm{Obj}\ \mathcal{C}$ we have a *mapping object* $\mathbf{map}_{\mathcal{C}}(a, b)$ in $\mathcal{M}$, and for any $a, b, c \in \mathrm{Obj}\ \mathcal{C}$ we have a *composition map*

$$\mu = \mu_{a,b,c} \ : \ \mathbf{map}_{\mathcal{C}}(b, c) \ \otimes \ \mathbf{map}_{\mathcal{C}}(a, b) \ \longrightarrow \ \mathbf{map}_{\mathcal{C}}(a, c)$$

(written in the usual order for a composite), satisfying the standard associativity rules.

As in §1.19, we can think of a morphism $f : 1 \to \mathbf{map}_{\mathcal{C}}(a, b)$ in $\mathcal{M}$ as an 'element' of $\mathbf{map}_{\mathcal{C}}(a, b)$, or simply a *map* $f : a \to b$. In particular, we have 'identity maps' Id_a in $\mathbf{map}_{\mathcal{C}}(a, a)$ for each $a \in \mathrm{Obj}\ \mathcal{C}$, satisfying the usual unit rules.

In addition, a morphism $F : 1 \to P\mathbf{map}_{\mathcal{C}}(a, b)$ is called a *nullhomotopy* of $f := \mathfrak{p}_{\mathbf{map}_{\mathcal{C}}(a,b)} \circ F$. Higher order nullhomotopies are defined by maps $F : 1 \to P^i\mathbf{map}_{\mathcal{C}}(a, b)$.

The functoriality of P implies that we can also compose (higher order) nullhomotopies by means of the composite of

$$P^i\mathbf{map}_{\mathcal{C}}(b,c) \otimes P^j\mathbf{map}_{\mathcal{C}}(a,b) \xrightarrow{\theta^{(i,j)}} P^{i+j}[\mathbf{map}_{\mathcal{C}}(b,c) \otimes \mathbf{map}_{\mathcal{C}}(a,b)] \xrightarrow{P^{i+j}\mu} P^{i+j}\mathbf{map}_{\mathcal{C}}(a,c)\ , \tag{2.2}$$

which we denote by $\mu^{i,j} : P^i\mathbf{map}_{\mathcal{C}}(b,c) \otimes P^j\mathbf{map}_{\mathcal{C}}(a,b) \to P^{i+j}\mathbf{map}_{\mathcal{C}}(a,c)$. Again, the maps $\mu^{(-,-)}$ are associative.

For a general core $\mathcal{I} \subseteq \mathcal{M}$ (cf. §1.19), we have generalized elements given by maps $f : \alpha \to \mathbf{map}_{\mathcal{C}}(a,b)$ for $\alpha \in \mathcal{I}$. We use the fact that $\mathcal{I}$ is a monoidal subcategory to define the composite of $f : \alpha \to \mathbf{map}_{\mathcal{C}}(a,b)$ with $g : \beta \to \mathbf{map}_{\mathcal{C}}(b,c)$ $(\beta \in \mathcal{I})$ to be the composite in $\mathcal{M}$ of

$$\beta \otimes \alpha \xrightarrow{g\otimes f} \mathbf{map}_{\mathcal{C}}(b,c) \otimes \mathbf{map}_{\mathcal{C}}(a,b) \xrightarrow{\mu} \mathbf{map}_{\mathcal{C}}(a,c)\ , \tag{2.3}$$

and similarly for generalized (higher order) nullhomotopies.

From (1.6) we see that:

$$\partial_k^{n-1} \circ \mu^{i,j} = \begin{cases} \mu^{i-1,j} \circ (\partial_k^{i-1} \otimes \mathrm{Id}) & \text{if } 0 \leq k < i \\ \mu^{i,j-1} \circ (\mathrm{Id} \otimes \partial_{k-i}^{j-1}) & \text{if } i \leq k < i+j \end{cases} \tag{2.4}$$

for every $0 \leq k < i+j = n$.

2.5. *Remark.* If the path structure P comes from a unpointed path structure $(-)^I$ as in §1.9, a morphism $F : 1 \to \mathbf{map}_{\mathcal{C}}(a,b)^I$ in $\mathcal{M}$ is called a *homotopy* $F : f_0 \sim f_1$ between $f_0 := e^0_{\mathbf{map}_{\mathcal{C}}} \circ F$ and $f_1 := e^1_{\mathbf{map}_{\mathcal{C}}} \circ F$.

Higher order homotopies are defined by maps $F : 1 \to \mathbf{map}_{\mathcal{C}}{}^{I^i}(a,b)$, and the functoriality of $(-)^I$ implies that we can compose (higher order) homotopies by means of the composite of

$$\mathbf{map}_{\mathcal{C}}(b,c)^{I^i} \otimes \mathbf{map}_{\mathcal{C}}(a,b)^{I^j} \xrightarrow{\widehat{\theta}^{(i,j)}} [\mathbf{map}_{\mathcal{C}}(b,c) \otimes \mathbf{map}_{\mathcal{C}}(a,b)]^{I^{i+j}} \xrightarrow{\mu^{I^{i+j}}} \mathbf{map}_{\mathcal{C}}(a,c)^{I^{i+j}}\ ,$$

which we denote by $\widetilde{\mu}^{i,j} : \mathbf{map}_{\mathcal{C}}(b,c)^{I^i} \otimes \mathbf{map}_{\mathcal{C}}(a,b)^{I^j} \to (\mathbf{map}_{\mathcal{C}}(a,c))^{I^{i+j}}$. These induce the maps $\mu^{i,j}$, as in §1.9.

2.6. **Definition.** Assume given a monoidal path category $\langle \mathcal{M}, \otimes, 1, P\rangle$ with core $\mathcal{I}$ in $\mathcal{M}$ (cf. §1.19), and choose an ordered set $\Gamma = (\gamma_1, \ldots, \gamma_N)$ of N core elements.

An *n-th order chain complex* $\mathcal{K} = \langle K, \{\{F^k_{(i)}\}_{i=k+1}^N\}_{k=0}^n\rangle$ *over* $\mathcal{M}$ *(for* Γ*) of length* $N \geq n+2$ consists of:

(a) A category K enriched over $\mathcal{M}$, with $\mathrm{Obj}\,(K) = \{a_0, \ldots, a_N\}$ and

$$\mathbf{map}_K(a_i,\, a_j) \;=\; \begin{cases} 1 \amalg * & \text{if } i = j \\ * & \text{if } i < j\ . \end{cases} \tag{2.7}$$

K will be called the *underlying category* of the n-th order chain complex $\mathcal{K}$.

(b) For each $0 \leq k \leq n$ and $i = k+1, \ldots N$, generalized elements

$$F^k_{(i)} : \gamma_{i-k} \otimes \ldots \otimes \gamma_i \to P^k\mathbf{map}_K(a_i,\, a_{i-k-1})$$

such that

$$\partial_t \circ F^k_{(i)} \;=\; \mu^{k-t-1,t}(F^{k-t-1}_{(i-t-1)} \otimes F^t_{(i)}) \tag{2.8}$$

for all $0 \leq t < k$.

When $N=n+2$, we simply call $\mathcal{K}$ an *n-th order chain complex.*

2.9. *Remark.* Typically we are given a fixed category $\mathcal{C}$ enriched in a monoidal path category $\langle\mathcal{M},\otimes,1,P\rangle$, and the underlying category K for a higher order chain complex $\mathcal{K}$ will simply be a finite subcategory of $\mathcal{C}$ (usually not full, because of condition (2.7)). Such a $\mathcal{K}$ will be called an *n-th order chain complex in* $\mathcal{C}$.

2.10. **Definition.** Given an n-th order chain complex $\mathcal{K}=\langle K,\{\{F^k_{(i)}\}^N_{i=k+1}\}^n_{k=0}\rangle$ over $\mathcal{M}$ (for Γ) of length N, and an enriched functor $\phi:K\to L$ over $\mathcal{M}$ (which we may assume to be the identity on objects, with L also satisfying (2.7)), the *induced* n-th order chain complex $\mathcal{L}=\langle L,\{\{G^k_{(i)}\}^N_{i=k+1}\}^n_{k=0}\rangle$ over $\mathcal{M}$ (for the same Γ) is defined by setting

$$G^k_{(i)}:=\phi(F^k_{(i)}):\gamma_{i-k}\otimes\ldots\otimes\gamma_i\to P^k\mathbf{map}_L(a_i,\,a_{i-k-1})$$

for all $0\le k\le n$ and $k<i\le N$.

2.11. *Remark.* Note that we do *not* assume that we have n-th order nullhomotopies $F^n_{(i)}\in P^n\mathbf{map}_K(a_i,\,a_{i-n-1})$ (for $i>n$) satisfying (2.8).

However, from (2.8) and (2.4) we see that:

$$\partial_s\circ\partial_t\circ F^k_{(i)} \;=\; \mu^{k-t-2,t}(\mu^{k-s-t-2,s}(F^{k-s-t-2}_{(i-s-t-2)}\otimes F^s_{(i-t-1)})\otimes F^t_{(i)})$$

if $s+t<k-1$, and

$$\partial_s\circ\partial_t\circ F^k_{(i)} \;=\; \mu^{k-t-1,t-1}(F^{k-t-1}_{(i-t-1)}\otimes\mu^{k-s-2,s+t-k+1}(F^{k-s-2}_{(i-s-t+k-2)}\otimes F^{s+t-k+1}_{(i)}))$$

if $k-1\le s+t$. Thus from the simplicial identity $\partial_s\circ\partial_t=\partial_{t-1}\circ\partial_s$ for $0\le s<t$ we deduce that the maps $\{F^k_{(i)}\}$ must satisfy:

(2.12)

$$\mu(F^r_{(i-s-t-2)}\otimes F^s_{(i-t-1)}\otimes F^t_{(i)})\begin{cases}\mu(F^{r+1}_{(i-s-t-3)}\otimes F^{t-1}_{(i-s-1)}\otimes F^s_{(i)}) & \text{if } s<t\\ \mu(F^t_{(i-r-s-2)}\otimes F^s_{(i-r-1)}\otimes F^r_{(i)}) & \text{if } s\ge r \text{ and } t=0\\ \mu(F^{s+1}_{(i-r-t-3)}\otimes F^r_{(i-t-2)}\otimes F^{t-1}_{(i)}) & \text{if } s\ge r \text{ and } t>0\ ,\end{cases}$$

where we have simplified the notation using the associativity of μ.

2.13**. A cubical description.** Higher order chain complexes were originally defined in [BB, §4] in terms of a cubical enrichment, which is well suited to describing higher homotopies. In general, for an $(n-1)$-st order chain complex

$$a_{n+1}\xrightarrow{F^0_{(n+1)}}a_n\xrightarrow{F^0_{(n)}}a_{n-2}\to\ldots\to a_1\xrightarrow{F^0_{(1)}}a_0\ , \tag{2.14}$$

we may describe the choices of higher homotopies $F^k_{(i)}$ succinctly by arranging them as the collection of all the cubical faces in the boundary of I^{n+2} containing a fixed vertex (which is indexed by $F^0_{(1)}\otimes F^0_{(2)}\otimes\ldots F^0_{(n)}\otimes F^0_{(n+1)}$).

The k-faces are indexed by

$$F^{k_1}_{(i_1)}\otimes\ldots\otimes F^{k_r}_{(i_r)}\;\in\;P^{k_1}\mathbf{map}_K(a_{i_1},\,a_0)\otimes\ldots\otimes P^{k_r}\mathbf{map}_K(a_{n+1},\,a_{n-k_r})\ , \tag{2.15}$$

with $\sum_{j=1}^r k_j=k$, $i_j=\sum_{t=1}^{j-1}(k_t+1)$, and $r=n-k+1$ (so $i_1=k_1+1$ and $i_r=n+1$).

By intersecting the corner of ∂I^{n+2} with a transverse hyperplane in $\mathbb{R}^{n+1}$ we obtain an $(n+1)$-simplex σ, whose n-faces correspond to the $(n+1)$-facets of the corner, and so on. More precisely, the cone on this simplex (with cone point the chosen vertex v of I^{n+2}) is homeomorphic to I^{n+2}, with each $(n+1)$-face of the cone obtained from an $(n+1)$-facet τ of the corner by identifying the n-corner opposite v in τ to a single n-simplex in the base of the cone. See Figure 2.16.

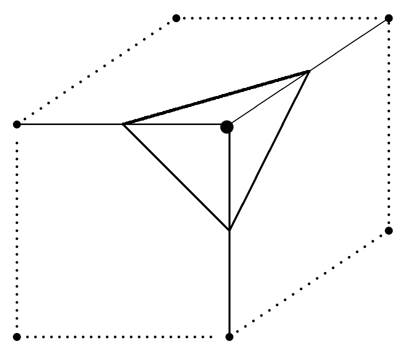
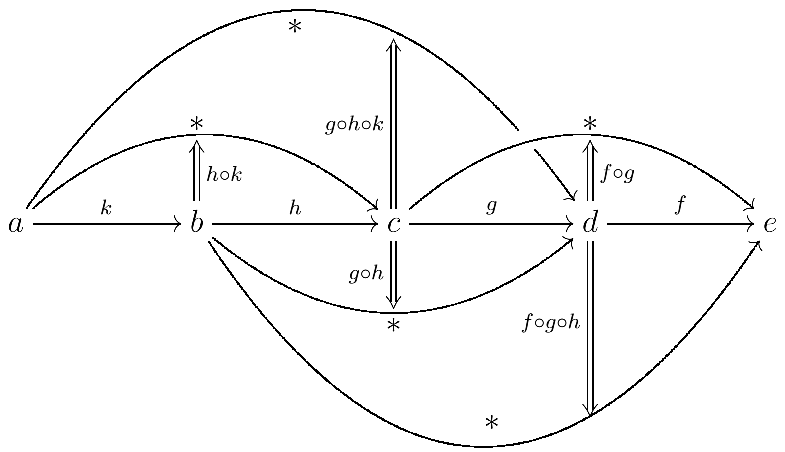


FIGURE 2.16. Corner of 3-cube and transverse 2-simplex

This explains why the maps $\partial_i^n : P^{n+1}X \to P^n X$ of §1.1, which relate the various $\otimes$-composites appearing as facets of ∂I^{n+1}, satisfy simplicial, rather than cubical, identities.

2.17. **Example.** Consider a second order chain complex

(2.18)



in Top_*, say, in which we have $n+1=4$ composable maps: $F^0_{(1)} = f$, $F^0_{(2)} = g$, and so on, with all adjacent composites nullhomotopic.

In this case we may choose nullhomotopies as indicated, namely: $F^1_{(2)} = f \circ g$ in $\mathrm{map}_*(c,e)^{I^1}$ (with $e^0(f \circ g) = *$ and $e^1(f \circ g) = fg$), $F^1_{(3)} = g \circ h$ in $\mathrm{map}_*(b,d)^{I^1}$, and $F^1_{(4)} = h \circ k$ in $\mathrm{map}_*(a,c)^{I^1}$ – so that in fact $f \circ g$ is in the pointed path space $P\,\mathrm{map}_*(c,e)$. Similarly, $F^2_{(4)} = f \circ g \circ h$ is a homotopy of nullhomotopies between $h^*(f \circ g)$ and $f^*(g \circ h)$.

The more suggestive notation $f \circ g$, and so on, is motivated by the cubical Boardman-Vogt W-construction of [BV, §3], as explained in [BB, §5]: we think a k-th order homotopy as a k-cube in the appropriate mapping spaces.

If we apply the usual composition map

$$\mu : \ \mathrm{map}_*(c,d) \ \otimes \ \mathrm{map}_*(a,c)^{I^1} \ \to \ \mathrm{map}_*(a,d)^{I^1}$$

to $g \otimes h \circ k$, we obtain a nullhomotopy of ghk, and similarly for $g \circ h \otimes k$ in $\mathrm{map}_*(b,d)^{I^1} \otimes \mathrm{map}_*(a,b)$. Thus we may ask if these two nullhomotopies are themselves homotopic (relative to ghk): if so, we have a 2-cube $g \circ h \circ k$ in $\mathrm{map}_*(a,d)^{I^2}$, which in fact lies in $P^2\,\mathrm{map}_*(a,d)$. The "formal" post-composition with $f \in \mathrm{map}_*(d,e)$ yields $f \otimes g \circ h \circ k$ in $\mathrm{map}_*(d,e) \otimes P^2\,\mathrm{map}_*(a,d)$. Together with the other two formal composites $f \circ g \circ h \otimes k$ in $P^2\,\mathrm{map}_*(b,e) \otimes \mathrm{map}_*(a,b)$ and $f \circ g \otimes h \circ k$ in $P\,\mathrm{map}_*(c,e) \otimes P\,\mathrm{map}_*(a,c)$, it fits into the corner of the 3-cube described in Figure 2.19 (where we use both notations $F^2_{(2)} = f \circ g$, and so on, to label facets).

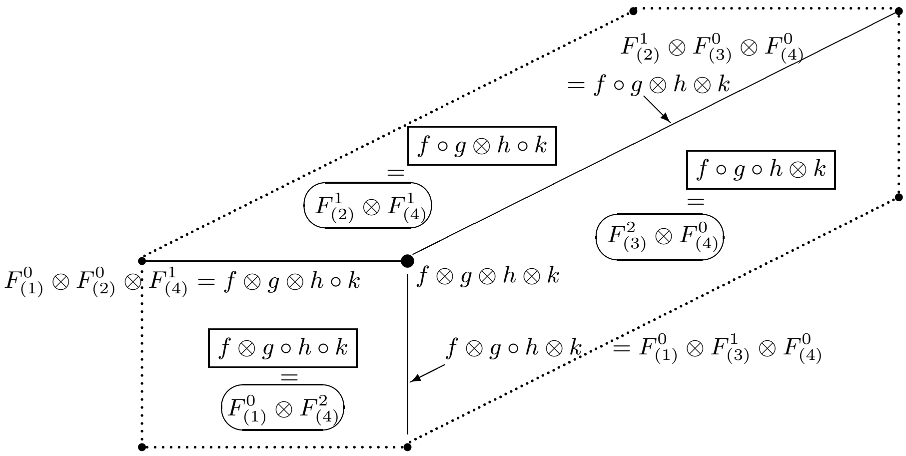


FIGURE 2.19. The cubical corner

All vertices but the central one represent the zero map, and the dotted edges represent the trivial nullhomotopy of the zero map (and similarly for the invisible facets of the cube, representing the trivial second-order homotopy of the trivial nullhomotopy).

2.20. *Remark.* The cubical formalism may be used to describe the iterated path complex $P^n\mathbf{A}_*$ in the category of chain complexes (see §1.16):

We may use the conventions of §2.13 to identify the k-faces of the corner of an n-cube I^n (adjacent to a fixed vertex v), for $0 \leq k \leq n$, with the $(k-1)$-dimensional faces $\sigma^k_{(i)}$ of the standard $(n-1)$-simplex $\Delta[n-1]$ for $0 \leq i \leq \binom{n}{k} - 1$ (see Figure 2.16). Thus I^n itself is labelled $\sigma^n_{(0)}$ (corresponding to $\Delta[n-1]$), with the n $(n-1)$-facets of I^n adjacent to v labelled $\sigma^{n-1}_{(0)} = d_0\sigma^n_{(0)}$, $\sigma^{n-1}_{(1)} = d_1\sigma^n_{(0)}$, and so on. The vertex v is labelled $\sigma^0_{(0)}$ (not corresponding to any real face of $\Delta[n-1]$).

Then

$$(P^nA)_j \;=\; \bigoplus_{0\leq k\leq n}\;\bigoplus_{0\leq i<\binom{n}{k}} A^{[\sigma^k_{(i)}]}_{j+k}\,, \tag{2.21}$$

with the differential $\partial^{P^nA}\colon (P^nA)_j \to (P^nA)_{j-1}$ sending $a \in A^{[\sigma^k_{(i)}]}_{j+k}$ to $\partial^A(a)$ in the summand $A^{[\sigma^k_{(i)}]}_{j+k-1}$ of $(P^nA)_{j-1}$, and to $(-1)^{n+k+t}a$ in the summand $A^{[d_t\sigma^k_{(i)}]}_{j+k-1}$.

The structure maps $\partial^n_i : P^n\mathbf{A}_* \to P^{n-1}\mathbf{A}_*$ are given by the projections onto the summands labelled by the i-th simplicial facet of $\Delta[n]$ and its simplicial faces, for $0 \leq i \leq n-1$.

2.22. **Example.** The double path complex $P^2\mathbf{A}_*$ is given by

$$(P^2A)_j = A_j \oplus A_{j+1} \oplus A_{j+1} \oplus A_{j+2}\ , \tag{2.23}$$

with

$$\partial(x,a,a',y) = (\partial x, \partial a + (-1)^{j+1}x, \partial a' + (-1)^{j+1}x, \partial y + (-1)^j(a-a'))\ . \tag{2.24}$$

2.25. **Example.** Similarly, $(P^3A)_j$ is given by

$$A_j^{[\sigma^0_{(0)}]} \oplus A_{j+1}^{[\sigma^1_{(0)}]} \oplus A_{j+1}^{[\sigma^1_{(1)}]} \oplus A_{j+1}^{[\sigma^1_{(2)}]} \oplus A_{j+2}^{[\sigma^2_{(0)}]} \oplus A_{j+2}^{[\sigma^2_{(1)}]} \oplus A_{j+2}^{[\sigma^1_{(2)}]} \oplus A_{j+3}^{[\sigma^3_{(0)}]}$$

and

$$\begin{aligned}\partial(a,b_0,b_1,b_2,c_0,c_1,c_2,d) = (&\partial a, \partial b_0 - \tau x, \partial b_1 - \tau x, \partial b_2 - \tau x,\\ &\partial c_0 + \tau(b_1-b_0), \partial c_1 + \tau(b_2-b_0), \partial c_2 + \tau(b_2-b_1), \partial d - \tau(c_2-c_1+c_1))\end{aligned}$$

for $\tau = (-1)^j$.

3. Higher Toda brackets

We now show how one may define the higher Toda bracket corresponding to a higher order chain complex. First, we need to define the object housing it:

3.1. **Definition.** In any monoidal path category $\langle \mathcal{M}, \otimes, 1, (-)^I\rangle$ we define the (modified) *n-fold loop functor* $\widetilde{\Omega}^n : \mathcal{M} \to \mathcal{M}$ to be the limit:

$$\widetilde{\Omega}^n X \ := \ \lim_{1\leq k\leq n} \ P^k X \tag{3.2}$$

where the limit is taken all the natural maps $\partial^k_i : P^kX \to P^{k-1}X$ of §1.1. By §2.13, we may think of this as a diagram indexed by the dual of the standard n-simplex.

The simplicial identities (1.8) imply that there is a natural map

$$\widetilde{\sigma}^n_X \ : \ P^{n+1}X \ \to \ \widetilde{\Omega}^n X\ , \tag{3.3}$$

which composes with the structure maps $\pi_t : \widetilde{\Omega}^n X \to P^nX$ for the limit to yield the face maps $\partial_t : P^{n+1}X \to P^nX$ $(i = 0, \ldots, n)$, since $\widetilde{\Omega}^n X$ is the n-th matching object for the restricted augmented simplicial object $P^\bullet X$ (cf. [Hi, §16.3.7]).

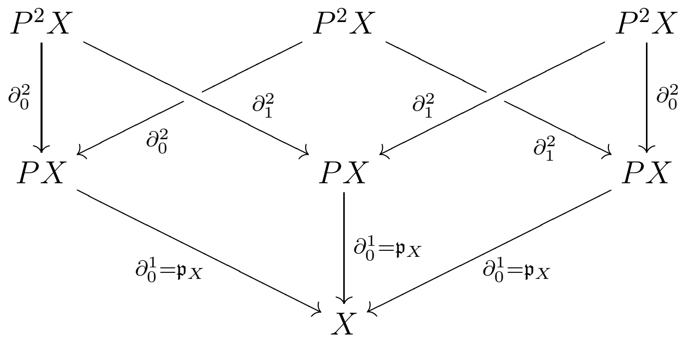
For $n = 0$ we set $\widetilde{\Omega}^0 X := X$.

3.4. **Example.** By §2.13, we may think of (3.2) as the limit of a diagram indexed by the dual of the standard n-simplex. Thus $\widetilde{\Omega}^1 X$ is the pullback in:

$$\begin{array}{ccc} \widetilde{\Omega}^1 X & \longrightarrow & PX \\ \Big\downarrow & \boxed{\text{PB}} & \Big\downarrow \mathfrak{p}_X \\ PX & \xrightarrow{\ \mathfrak{p}_X\ } & X, \end{array} \tag{3.5}$$

indexed by the inclusion of the two vertices into $\Delta[1]$, while $\widetilde{\Omega}^2 X$ is the limit of the diagram:

(3.6)



3.7. **Definition.** Let $\mathcal{K}$ be an $(n-1)$-st order chain complex (of length $n+1$) enriched in a monoidal path category $\langle \mathcal{M}, \otimes, 1, (-)^I \rangle$ (for a set $\Gamma = (\gamma_1, \dots, \gamma_{n+1})$ of core elements), as in §2.6. If we apply the iterated composition map to each k-face of the form (2.15), we obtain an 'element'

$$\mu(F_{(i_1)}^{k_1} \otimes \dots \otimes F_{(i_r)}^{k_r}) \; : \; \gamma_1 \otimes \dots \otimes \gamma_{n+1} \; \to \; P^k \mathbf{map}_K(a_{n+1},\, a_0) \tag{3.8}$$

(using the associativity of μ),

From (2.8) and (2.4) we see that these elements (3.8) are compatible under the face maps $\partial_t : P^k \mathbf{map}_K(a_{n+1},\, a_0) \to P^{k-1} \mathbf{map}_K(a_{n+1},\, a_0)$, so that they fit together to define an element

$$\langle \mathcal{K} \rangle \; : \; \gamma_1 \otimes \dots \otimes \gamma_{n+1} \; \to \; \widetilde{\Omega}^{n-1} \mathbf{map}_K(a_{n+1},\, a_0) \tag{3.9}$$

which we call the *value of the n-th order Toda bracket* associated to the chain complex $\mathcal{K}$.

If $\langle \mathcal{K} \rangle$ lifts along the map $\widetilde{\sigma}_X^{n-1} : P^n X \to \widetilde{\Omega}^{n-1} X$ of (3.3), we say that this value of the Toda bracket *vanishes*.

3.10. *Remark.* Given an $(n-1)$-st order chain complex $\mathcal{K} = \langle K, \{\{F_{(i)}^k\}_{i=k+1}^{n+1}\}_{k=0}^{n-1} \rangle$ over $\mathcal{M}$ (for Γ), any enriched functor $\phi : K \to L$ over $\mathcal{M}$ as in §2.10 takes $\langle \mathcal{K} \rangle$ to

$$\langle \mathcal{L} \rangle \; : \; \gamma_1 \otimes \dots \otimes \gamma_{n+1} \; \to \; \widetilde{\Omega}^{n-1} \mathbf{map}_L(a_{n+1},\, a_0)$$

where $\mathcal{L}$ is the $(n-1)$-st order chain complex induced by ϕ, by functoriality of the limits in $\mathcal{M}$.

3.11**. Massey products.** Massey products (and their higher order versions) also fit into our setting, although they cannot be defined as ordinary Toda brackets in a model category. This is because a (unital associative) differential graded algebra $\mathbf{A}_*$ over a commutative ground ring R can be thought of as a category $\mathcal{C}$ with a single object ξ enriched in $(\mathsf{Ch}_R, \otimes_R, \widetilde{\mathbf{M}}(R, 0)_*)$, with $\mathrm{Hom}_{\mathcal{C}}(\xi, \xi) := \mathbf{A}_*$.

In this context we choose the core of Ch_R to be $\mathcal{I}_R$ as in §1.20(b). Thus an $(n-1)$-st order chain complex in $\mathbf{A}_*$ consists of:

(a) The sequence of objects – necessarily $a_i = \xi$ for all i.
(b) A sequence of generalized maps $F_{(i)}^0 : \widetilde{\mathbf{M}}(R, m_i)_* \to \mathrm{Hom}_{\mathcal{C}}(\xi, \xi)$ for $i = 1, \dots, n+1$, which may be identified with an m_i-cycle $H_i^0 \in Z_{m_i} \mathbf{A}_*$ (see §1.20(b)).

(c) A sequence of generalized nullhomotopies $F^1_{(i)} \in P\mathbf{A}_*$ $(i = 2, \ldots, n+1)$, with $\mathfrak{p}_{\mathbf{A}_*}(F^1_{(i)}) = \mu(f_{i-1} \otimes f_i)$. From the description in §1.16 we see that $F^1_{(i)}$ is completely determined by an element $H^1_i \in A_{m_i+m_{i-1}+1}$ with $d(H^1_i) = H^0_{i-1} \cdot H^0_i$ (where d is the differential and $\cdot$ is the multiplication in $\mathbf{A}_*$).

(d) From §2.22 we see that a 'second-order nullhomotopy' $F^2_{(i)} \in P^2\mathbf{A}_*$ $(i = 3, \ldots, n+1)$, which is a $(j+2)$-cycle for $j := m_i+m_{i-1}+m_{i-2}$, is determined uniquely by the element $H^2_i \in A_j$ (the last summand in (2.23)). From the last term in (2.24) we see that $F^2_{(i)}$ being a cycle means that

$$d(H^2_i) \;=\; (-1)^{j+1}\,(H^0_{i-2} \cdot H^1_i - H^1_{i-1} \cdot H^0_i)\;.$$

(e) In general, for each $1 \leq k < n$ and $i = k+1, \ldots n+1$, we have a (generalized) $F^k_{(i)} \in P^k\mathbf{A}_*$ which is a $(j+k)$-cycle for $j := \sum_{t=i-k}^{i} m_t$, with

$$\partial_t \circ F^k_{(i)} \;=\; F^{k-t-1}_{(i-t-1)} \cdot F^t_{(i)}\;, \tag{3.12}$$

and from the description in §2.20 we see that again $F^k_{(i)}$ is completely determined by the component H^k_i in the summand $A^{[\sigma^k_{(i)}]}_{j+k}$, with

$$d(H^k_i) \;=\; (-1)^{k+j+1}\sum_{t=0}^{k-1}(-1)^t H^t_{i-k+t} \cdot H^{k-t-1}_i\;.$$

Thus by Definition 3.7 we see that the value of the $(n+1)$-st order Toda bracket associated to this $(n-1)$-st order chain complex in $\mathbf{A}_*$ is the element in $\widetilde{\Omega}^{n-1}\mathbf{A}_* = \lim_{1\leq k<n} P^k\mathbf{A}_*$ determined by the coherent choice of elements

$$H^t_{i-k+t} \cdot H^{n-t}_i \;\in A_{j+n} \qquad \text{for}\;\; t = 1, \ldots, n\;, \tag{3.13}$$

where $j := \sum_{t=1}^{n} m_t$.

4. Higher Toda brackets in model categories

In order to *define* the values of higher Toda brackets, all we need is a category enriched in a monoidal path category $\mathcal{M}$. However, in applications we want to use such Toda brackets, either as obstructions to rectifying diagrams, or as invariants used in computations (e.g., of differentials in spectral sequence). For this we need to make an additional

4.1. **Definition.** A *path model category* is a pointed monoidal model category $\langle\mathcal{M}, \otimes, 1\rangle$ in the sense of [Ho, Ch. 4] which satisfies the conditions of either of [BM, Theorem 1.9,Theorem 1.10], and which is also a simplicial model category as in [Q, II, §2], equipped with a core $\mathcal{I}$ (cf. §1.19) consisting of cofibrant objects, and a natural transformation

$$\zeta_{X,Y,K} : X^K \otimes Y^K \to (X \otimes Y)^K \tag{4.2}$$

(natural in $X, Y \in \mathcal{M}$ and $K \in \mathcal{S}$).

4.3. *Remark.* By [Ho, Proposition 4.2.19], a path model category actually has a $\mathcal{S}_*$-model category structure – that is, we have functors $(-)^K : \mathcal{M} \to \mathcal{M}$ and $(-)\otimes K : \mathcal{M} \to \mathcal{M}$ for every *pointed* simplicial set $K \in \mathcal{S}_*$, satisfying the usual axioms.

4.4. **Examples.** In practice we shall be interested only in the following examples:

(a) The monoidal structure on Top is cartesian, so we actually have a natural homeomorphism $\tilde{\zeta} : X^K \times Y^K \xrightarrow{\cong} (X \times Y)^K$. It is readily verified that in the pointed version $\langle \mathsf{Top}_*, \wedge, S^0 \rangle$ of §1.13, the map $\tilde{\zeta}$ induces $\zeta : X^K \wedge Y^K \to (X \wedge Y)^K$.

(b) The monoidal structure on $\mathcal{S}$ is also cartesian, so in the pointed version $\langle \mathcal{S}_*, \wedge, S^0 \rangle$ of §1.14 we also have an induced map as in (4.2).

(c) If we use symmetric spectra as our model for Sp (cf. §1.15) we see that the spectrum X^K is defined levelwise, so we have (4.2) as for Top_*.

(d) In the category $\langle \mathsf{Ch}_R, \otimes, \widetilde{\mathbf{M}}(R, 0)_* \rangle$ of chain complexes of R-modules (§1.16), the monoidal structure is not cartesian, but the simplicial structure is defined by setting ${\mathbf{A}_*}^K := \underline{\mathrm{Hom}}(C_*K, \mathbf{A}_*)$ (where C_*K is the simplicial chain complex of $K \in \mathcal{S}$). The natural transformation (4.2) is induced by the diagonal $\Delta : K \to K \times K$ in $\mathcal{S}$.

Note that all of these satisfy the hypotheses of one of [BM, Theorem 1.9,Theorem 1.10], by [BM, §1.8] and [Lu, Proposition A.3.2.4-A.3.2.24], so they are in fact path model categories.

4.5. *Remark.* In this case the simplicial structure defines the functor $(-)^I : \mathcal{M} \to \mathcal{M}$, with $X^I := X^{\Delta[1]}$ (cf. [Q, II, §1]), and $PX \hookrightarrow X^{\Delta[1]}$ is defined by the pullback (1.12). We can therefore identify $P^k X$ for each $k \geq 0$ with the subobject of $X^{[0,1]^k}$ consisting of all maps of the k-cube sending the corner opposite a fixed vertex to the basepoint (see Figure 2.19).

Thus $\widetilde{\Omega}^n X$ is a subobject of $\lim_k \ \mathrm{map}_*([0,1]^k, X)$, which by adjunction may be identified with $X^{\mathrm{colim}_k [0,1]^k}$. Thus $\widetilde{\Omega}^n X$ itself is just $\mathrm{map}_*(\mathrm{colim}\ \widetilde{[0,1]^k},\ X)$, where the colimit is now taken over all proper faces of $[0,1]^{n+1}$, and we identify the corner opposite our chosen vertex of $[0,1]^{n+1}$ to a point. This colimit is homeomorphic to an n-sphere, so $\widetilde{\Omega}^n X$ is homotopy equivalent to the n-fold loop space $\Omega^n X$, defined as usual by iterating the functor $\Omega : \mathcal{M} \to \mathcal{M}$ given by the pullback

$$\begin{array}{ccc} \Omega X & \longrightarrow & PX \\ \downarrow & \boxed{\mathrm{PB}} & \downarrow{\scriptstyle \mathfrak{p}_X} \\ * & \longrightarrow & X. \end{array} \tag{4.6}$$

4.7. *Remark.* In any path model category $\mathcal{M}$, for any fibrant object X we have an equivalence relation $\sim$ on the set of morphisms $\mathrm{Hom}_{\mathcal{M}}(1, X)$ (cf. §1.19), given by:

$$f \sim g \quad \Leftrightarrow \quad \exists F : 1 \to X^I \ \text{ such that } \ e^0 \circ F = f \ \text{ and } \ e^1 \circ F = g \ .$$

We then define the (pointed) *set of components* $\pi_0 X$ to be the set of equivalence classes in $\mathrm{Hom}_{\mathcal{M}}(1, X)$ under $\sim$.

Now let $\mathcal{C}$ be a category enriched in $\mathcal{M}$, and assume the mapping objects $\mathbf{map}_{\mathcal{C}}(a, b)$ are fibrant (e.g., if all objects in $\mathcal{M}$ are fibrant, as in Top_*), If we denote $\pi_0 \mathbf{map}_{\mathcal{C}}(a, b)$ simply by $[a, b]$, from §2.1 we see that μ induces an associative composition on $[-, -]$, so that this serves as the set of morphisms in the *homotopy category* $\mathrm{ho}\,\mathcal{C}$ of the $\mathcal{M}$-enriched category $\mathcal{C}$ (with the same objects as $\mathcal{C}$).

4.8. **Definition.** More generally, if $\mathcal{I}$ is the core of a path model category $\mathcal{M}$, for any core element γ (which is cofibrant by Definition 4.1) the simplicial enrichment $\mathrm{map}_{\mathcal{M}}$ in $\mathcal{M}$ allows us to identify $[\gamma, X]$ with $\pi_0\,\mathrm{map}_{\mathcal{M}}(\gamma, X)$ (see [Q, II, 2.6]).

Thus if $\mathcal{C}$ is enriched in $\mathcal{M}$, we may set

$$[a,b]_\gamma \ := \ \pi_0 \operatorname{map}_{\mathcal{M}}(\gamma, \mathbf{map}_{\mathcal{C}}(a,b)) \ .$$

for any $a,b \in \mathcal{C}$ and $\gamma \in \mathcal{I}$.

Note that for any $\gamma, \delta \in \mathcal{I}$ and $i \geq 0$, the bifunctor $\otimes$, the map $\zeta_{X,Y,\Delta[i]}$ of (4.2) for $X := \mathbf{map}_{\mathcal{C}}(b,c)$ and $Y := \mathbf{map}_{\mathcal{C}}(a,b)$, and the composition $\mu : X \otimes Y \to Z$ (for $Z := \mathbf{map}_{\mathcal{C}}(a,c)$) induce natural maps of sets

$$\begin{aligned}(\operatorname{map}_{\mathcal{M}}(\gamma, X) \times \operatorname{map}_{\mathcal{M}}(\delta, Y))_i \ &= \ \operatorname{Hom}_{\mathcal{M}}(\gamma, X^{\Delta[i]}) \times \operatorname{Hom}_{\mathcal{M}}(\delta, Y^{\Delta[i]}) \xrightarrow{\otimes_*} \\ \operatorname{Hom}_{\mathcal{M}}(\gamma \otimes \delta, X^{\Delta[i]} \otimes Y^{\Delta[i]}) &\xrightarrow{\zeta} \operatorname{Hom}_{\mathcal{M}}(\gamma \otimes \delta, (X \otimes Y)^{\Delta[i]}) \xrightarrow{(\mu^{\Delta[i]})_*} \\ &\operatorname{Hom}_{\mathcal{M}}(\gamma \otimes \delta, Z^{\Delta[i]}) \ = \ (\operatorname{map}_{\mathcal{M}}(\gamma \otimes \delta, Z))_i\end{aligned}$$

and thus a composition map $\nu : \operatorname{map}_{\mathcal{M}}(\gamma, X) \times \operatorname{map}_{\mathcal{M}}(\delta, Y) \to \operatorname{map}_{\mathcal{M}}(\gamma \otimes \delta, Z)$ in $\mathcal{S}$. Thus induces an associative composition map

$$\nu_* : [b,c]_\gamma \ \times \ [a,b]_\delta \ \to \ [a,c]_{\gamma \otimes \delta} \ . \tag{4.9}$$

Thus we have an $\mathcal{I}$-graded category denoted by $\operatorname{ho}^{\mathcal{I}} \mathcal{C}$, called the *$\mathcal{I}$-homotopy category* of $\mathcal{C}$.

4.10. **Definition.** Assume given a path model category $\mathcal{M}$ with core $\mathcal{I}$. We say that a category K enriched in $\mathcal{M}$ is *fibrant* if $\mathbf{map}_K(a,b)$ is fibrant in $\mathcal{M}$ for any $a,b \in K$. Note that since each $\gamma \in \mathcal{I}$ is cofibrant, this implies that $\operatorname{map}_{\mathcal{M}}(\gamma, \mathbf{map}_K(a,b))$ is a fibrant simplicial set, by SM7.

An enriched functor $\phi : K \to L$ between categories K and L enriched in $\mathcal{M}$ is a *Dwyer-Kan equivalence* if

(a) For all $a,b \in \mathcal{C}$, $\phi : \mathbf{map}_K(a,b) \to \mathbf{map}_L(\phi(a), \phi(b))$ is a weak equivalence in $\mathcal{M}$.
(b) The induced functor $\phi_* : \operatorname{ho}^{\mathcal{I}} K \to \operatorname{ho}^{\mathcal{I}} L$ is an equivalence of $\mathcal{I}$-graded categories.

See [SS], and compare [BM].

We say that such a Dwyer-Kan equivalence is a *trivial fibration* if each $\phi : \mathbf{map}_K(a,b) \to \mathbf{map}_L(\phi(a), \phi(b))$ is a fibration in $\mathcal{M}$.

By Definition 4.1 and [BM, Theorem 1.9-1.10] we have:

4.11. **Theorem.** *There is a canonical model category structure on the category* $\mathcal{M}$-Cat *of small categories enriched in any path model category* $\mathcal{M}$*, in which the trivial fibrations and fibrant categories are defined object-wise, and the weak equivalences are the Dwyer-Kan equivalences.*

4.12. **Definition.** Let $\mathcal{M}$ be a path model category with core $\mathcal{I}$, and let $\mathcal{K}^{(0)} = \langle K, \{F^0_{(i)}\}_{i=1}^{n+1}\} \rangle$ be a fixed fibrant 0-th order chain complex of length $n+1$ over $\mathcal{M}$ for $\Gamma \subseteq \mathcal{I}$. We define $\mathcal{L}_{\mathcal{K}^{(0)}}$ to be the collection of all possible fibrant $(n-1)$-st order chain complexes $\mathcal{K}$ (of length $n+1$) extending $\mathcal{K}^{(0)}$.

Each $\mathcal{K} \in \mathcal{L}_{\mathcal{K}^{(0)}}$ has a value $\langle \mathcal{K} \rangle : \gamma_1 \otimes \ldots \otimes \gamma_{n+1} \to \widetilde{\Omega}^{n-1} \mathbf{map}_K(a_{n+1}, \, a_0)$, as in (3.9), which we may identify with a 0-simplex in the corresponding simplicial mapping space

$$\langle \mathcal{K} \rangle \in \operatorname{map}_{\mathcal{M}}(\gamma_1 \otimes \ldots \otimes \gamma_{n+1}, \ \widetilde{\Omega}^{n-1} \mathbf{map}_K(a_{n+1}, \, a_0))_0 \ . \tag{4.13}$$

By Remark 4.5 $\widetilde{\Omega}^{n-1}\mathbf{map}_K(a_{n+1},\, a_0)$ is weakly equivalent to the $(n-1)$-fold loop space on the mapping space $\mathrm{map}_{\mathcal{C}}(a_{n+1},\, a_0)$ in $\mathcal{M}$ (cf. [Q, I, §2]). Moreover, we have a natural isomorphism

$$\mathrm{map}_{\mathcal{M}}(Y, X^L) \xrightarrow{\cong} \mathrm{map}_{\mathcal{S}}(L, \mathrm{map}_{\mathcal{M}}(Y, X)) \tag{4.14}$$

for any $X, Y \in \mathcal{M}$ and $L \in \mathcal{S}$ any finite simplicial set, by [Q, II, §1]), so we may identify the path component $[\langle\mathcal{K}\rangle]$ of this 0-simplex with the corresponding element in

$$\begin{aligned}\pi_0\,\mathrm{map}_{\mathcal{M}}(\gamma_1\otimes\ldots\otimes\gamma_{n+1},\ \Omega^{n-1}\mathbf{map}_K(a_{n+1},\, a_0))\\ \cong \pi_0\Omega^{n-1}\,\mathrm{map}_{\mathcal{M}}(\gamma_1\otimes\ldots\otimes\gamma_{n+1},\ \mathbf{map}_K(a_{n+1},\, a_0))\\ \cong \pi_{n-1}\,\mathrm{map}_{\mathcal{M}}(\gamma_1\otimes\ldots\otimes\gamma_{n+1},\ \mathbf{map}_K(a_{n+1},\, a_0))\end{aligned}$$

We call the set

$$\langle\langle\mathcal{K}^{(0)}\rangle\rangle \ := \ \{[\langle\mathcal{K}\rangle] \in \pi_{n-1}\,\mathrm{map}_{\mathcal{M}}(\gamma_1\otimes\ldots\otimes\gamma_{n+1},\ \mathbf{map}_K(a_{n+1},\, a_0)) \ : \ \mathcal{K}\in\mathcal{L}_{\mathcal{K}^{(0)}}$$

the *n-th order Toda bracket for* $\mathcal{K}^{(0)}$. We say that it *vanishes* if $0 \in \langle\langle\mathcal{K}^{(0)}\rangle\rangle$.

Of course, $\langle\langle\mathcal{K}^{(0)}\rangle\rangle$ may be empty (if there are no $(n-1)$-st order chain complexes $\mathcal{K}$ extending $\mathcal{K}^{(0)}$). It vanishes if and only if there is an n-th order chain complex extending $\mathcal{K}^{(0)}$.

4.15. *Remark.* When K is a higher chain complex in $\mathcal{C} = \mathcal{M}$ in a monoidal path category enriched over itself (e.g., for $\mathcal{M} = \mathsf{Top}_*$ or $\mathcal{S}_*$), the homotopy class $[\langle K\rangle]$ may be thought of as an element in the group

$$[\mathbf{\Sigma}^{n-1}\gamma_1\otimes\ldots\otimes\gamma_{n+1}\otimes a_{n+1},\ a_0]_*$$

Moreover, $[\langle K\rangle]$ vanishes if and only if it represents the zero element in this group.

4.16. **Lemma.** *If* $\mathcal{M}$ *is a simplicial model category and* $f : X \to Y$ *is a (trivial) fibration between fibrant objects in* $\mathcal{M}$*, then the induced maps* $P^k f : P^k X \to P^k Y$ *and* $\widetilde{\Omega}^k f : \widetilde{\Omega}^k X \to \widetilde{\Omega}^k Y$ *are (trivial) fibrations for all* $k \geq 1$. *Furthermore, if* $f : X \to Y$ *is a weak equivalence between fibrant and cofibrant objects in* $\mathcal{M}$*, so are* $P^k f : P^k X \to P^k Y$ *and* $\widetilde{\Omega}^k f : \widetilde{\Omega}^k X \to \widetilde{\Omega}^k Y$.

Proof. Using Axiom SM7 for the simplicial model category $\mathcal{M}$, the natural isomorphism (4.14), and SM7 for $\mathcal{S}$ itself (cf. [Q, II, §1-3]), we see that

(a) Any (trivial) cofibration $i : K \hookrightarrow L$ in $\mathcal{S}$ induces a (trivial) fibration $i^* : X^L \twoheadrightarrow X^K$, as long as $X \in \mathcal{M}$ is fibrant.
(b) Any (trivial) fibration $f : X \to Y$ in $\mathcal{M}$ induces a (trivial) fibration $f_* : X^K \twoheadrightarrow Y^K$ for any (necessarily cofibrant) $K \in \mathcal{S}$.

In particular, let C^n_+ denote the sub-simplicial set of the cube boundary ∂I^n consisting of all facets adjacent to a fixed corner v (i.e., the cubical star of v in ∂I^n), with ∂C^n its boundary (the cubical link of v). The cofibration $i : \partial C^n_+ \hookrightarrow C^n_+$ makes $i^\# : X^{C^n_+} \to X^{\partial C^n_+}$ a fibration in $\mathcal{M}$, by (a).

In particular, the pullback square

$$\begin{array}{ccc} \widetilde{\Omega}^{n-1}X & \longrightarrow & X^{C^n_+} \\ \downarrow & \boxed{\text{PB}} & \downarrow\!\!\downarrow i^\# \\ * & \longrightarrow & X^{\partial C^n_+} \end{array} \tag{4.17}$$

defining $\widetilde{\Omega}^{n-1}X$ (see §3.1 and compare §2.13) is a homotopy pullback (see [Mat]).

Thus if $f : X \to Y$ is a (trivial) fibration in $\mathcal{M}$, then the induced map $\widetilde{\Omega}^{n-1}f : \widetilde{\Omega}^{n-1}X \to \widetilde{\Omega}^{n-1}Y$ is a (trivial) fibration, by (b).

Similarly, if we consider the (pointed) cofibration sequence in $\mathcal{S}_*$:

$$S^0 = \{0, *\} \hookrightarrow \Delta[1]_+ = [0,1] \cup \{*\} \twoheadrightarrow \Delta[1] = [0,1]$$

(with $*$ as basepoint in the first two, and 0 as the basepoint in the cofiber), we see from the corresponding fibration sequence in $\mathcal{M}$:

$$PX = X^{\Delta[1]} \hookrightarrow X^I \twoheadrightarrow X^{S^0} = X$$

that if $f : X \to Y$ is a (trivial) fibration in $\mathcal{M}$, so is $Pf : PX \to PY$, by (b) again (see §4.3 above). □

4.18. **Lemma.** *If X is a fibrant object in a simplicial model category $\mathcal{M}$, then for each $n \geq 0$ the map $\widetilde{\sigma}^n_X : P^{n+1}X \to \widetilde{\Omega}^n X$ of (3.3) is a fibration.*

Note that for $n = 0$, $\widetilde{\Omega}^0 X = X$ and $\widetilde{\sigma}^0_X$ is simply $\mathfrak{p}_X : PX \to X$.

Proof. If we consider the map of cofibration sequences (pushouts to $*$) in $\mathcal{S}$:

$$(4.19)\qquad \begin{array}{ccccc} \partial C^n_+ & \hookrightarrow & C^n_+ & \longrightarrow & C^n_+/\partial C^n_+ \\ \downarrow & & \downarrow & & \downarrow \\ C^n_- & \hookrightarrow & I^n & \longrightarrow & I^n/\partial C^n_- \end{array}$$

we see that the natural map $C^n_+/\partial C^n_+ \to I^n/\partial C^n_-$ is an inclusion (cofibration) in $\mathcal{S}_*$, so the natural map it induces – namely, $\widetilde{\sigma}^n_X :: P^{n+1}X \to \widetilde{\Omega}^n X$ – is a fibration by (b) above.

For $n = 0$ this follows directly because $\mathfrak{p}_X$ is a pullback in the following diagram:

$$(4.20)\qquad \begin{array}{ccc} PX & \longrightarrow & X^I \\ {\scriptstyle \mathfrak{p}_X}\downarrow & \boxed{\text{PB}} & \downarrow{\scriptstyle e^0 \top e^1} \\ X & \xrightarrow{\ \mathrm{Id} \top *\ } & X \times X \end{array}$$

where $e^0 \top e^1$ is a fibration since it is induced by the cofibration $\{0,1\} \hookrightarrow \Delta[1]$ in $\mathcal{S}$. □

4.21. **Theorem.** *Let $\mathcal{M}$ be a path model category with core $\mathcal{I}$, and let $\mathcal{K}^{(0)} = \langle K, \{F^0_{(i)}\}_{i=1}^{n+1}\rangle$ and $\mathcal{L}^{(0)} = \langle L, \{G^0_{(i)}\}_{i=1}^{n+1}\rangle$ be 0-th order chain complexes of length $n+1$ over $\mathcal{M}$ (for the same $\Gamma \subseteq \mathcal{I}$) with K and L fibrant, and let $\phi^{(0)} : \mathcal{K}^{(0)} \to \mathcal{L}^{(0)}$ be a map of 0-th order chain complexes which is a Dwyer-Kan equivalence. Then the resulting equivalence of categories $\phi_* : \mathrm{ho}^{\mathcal{I}} K \to \mathrm{ho}^{\mathcal{I}} L$ induces a bijection between $\langle\langle \mathcal{K}^{(0)} \rangle\rangle$ and $\langle\langle \mathcal{L}^{(0)} \rangle\rangle$.*

Proof. We assume for simplicity that ϕ is the identity on objects, so we may identify both $\pi_0 \operatorname{map}_{\mathcal{M}}(\gamma, \mathbf{map}_K(a, a'))$ and $\pi_0 \operatorname{map}_{\mathcal{M}}(\gamma, \mathbf{map}_L(a, a'))$ as $[a, a']_\gamma$. Similarly we may identify the groups $\pi_* \operatorname{map}_{\mathcal{M}}(\gamma, \mathbf{map}_K(a, a'))$ and $\pi_* \operatorname{map}_{\mathcal{M}}(\gamma, \mathbf{map}_L(a, a'))$.

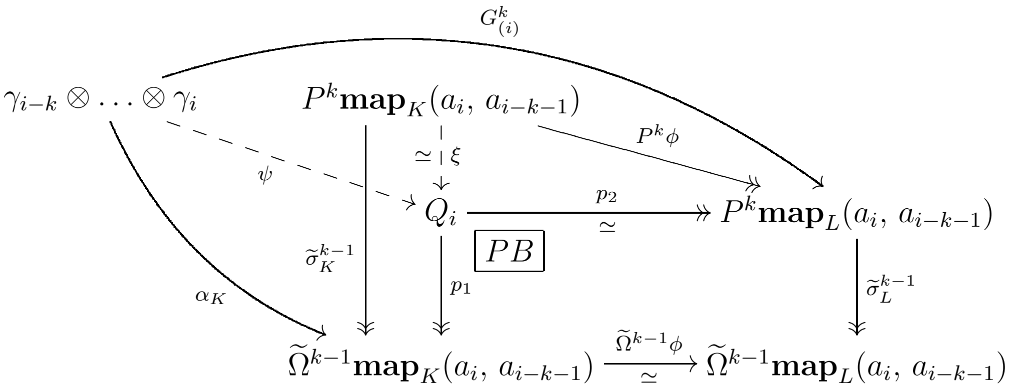
Given an $(n-1)$-st order chain complex $\mathcal{K}$ extending $\mathcal{K}^{(0)}$, ϕ induces an $(n-1)$-st order chain complex $\mathcal{L}$ extending $\mathcal{L}^{(0)}$, as in §2.10, and takes the value $\langle\mathcal{K}\rangle \subset [a_{n+1},\, a_0]_{\gamma_1\otimes\ldots\otimes\gamma_{n+1}}$ to $\langle\mathcal{L}\rangle$.

(a) First assume that $\phi^{(0)} : \mathcal{K}^{(0)} \to \mathcal{L}^{(0)}$ is a trivial fibration.

To show that the above correspondence is a bijection, let $\mathcal{L}$ be an $(n-1)$-st order chain complex extending $\mathcal{L}^{(0)}$. We show by induction on $k \geq 0$ that we have an k-th order chain complex $\mathcal{K}^{(k)}$ extending $\mathcal{K}^{(0)}$, where $\phi_*\mathcal{K}^{(k)}$ agrees with $\mathcal{L}$ to k-th order (by assumption this holds for $k=0$).

In the induction step, we have a $(k-1)$-st order chain complex $\mathcal{K}^{(k-1)}$ such that $\phi_*\mathcal{K}^{(k-1)}$ agrees with $\mathcal{L}$ to $(k-1)$-st order, which we wish to extend to $\mathcal{K}^{(k)}$. Thus we have a commuting diagram

(4.22)



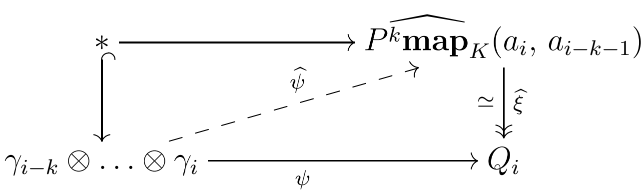
in which Q_i is the pullback as indicated, and $\alpha_K : \gamma_{i-k}\otimes\ldots\otimes\gamma_i \to \widetilde{\Omega}^{k-1}\mathbf{map}_K(a_i,\, a_{i-k-1})$ into the limit is induced by the maps $F^{k-1}_{(t)}$ $(t=0,\ldots,k-1)$.

Here p_2 is a trivial fibration and p_1 is a fibration by base change (using Lemmas 4.16 and 4.18). The maps $\psi : \gamma_{i-1}\otimes\gamma_i \to Q_i$ and $\xi : P^k\mathbf{map}_K(a_i, a_{i-k-1}) \to Q_i$ exist by the universal property, and ξ is a weak equivalence by the 2 out of 3 property. Factor ξ as

$$P^k\mathbf{map}_K(a_i, a_{i-k-1}) \xrightarrow{j} \widehat{P^k\mathbf{map}}_K(a_i, a_{i-k-1}) \xrightarrow{\widehat{\xi}} Q_i\ ,$$

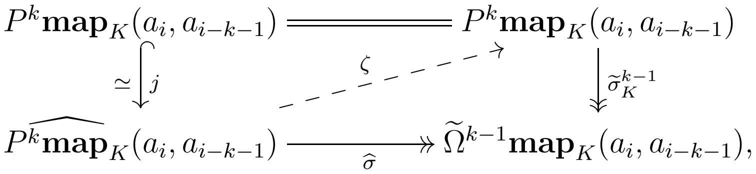
where j a trivial cofibration and $\widehat{\xi}$ is a trivial fibration. Since $\gamma_{i-k}\otimes\ldots\otimes\gamma_i \in \mathcal{I}$ is cofibrant, we have a lifting as indicated in the solid commuting square:

(4.23)



Since j is a trivial cofibration and $\widetilde{\sigma}^k_X$ is a fibration (for $X := \widetilde{\Omega}^{k-1}\mathbf{map}_K(a_i, a_{i-k-1})$) by Lemma 4.18, we have a lift ζ as indicated in:

(4.24)



for $\widehat{\sigma} := p_1 \circ \widehat{\xi}$. Thus if we set $F^k_{(i)} : \gamma_{i-k} \otimes \ldots \otimes \gamma_i \to P^k \mathbf{map}_K(a_i, a_{i-k-1})$ equal to $\zeta \circ \widehat{\psi}$, we see that

$$\pi_t \circ \widetilde{\sigma}^{k-1}_K \circ F^k_{(i)} \;=\; \partial_t \circ F^k_{(i)} \;=\; \mu^{k-t-1,t}(F^{k-t-1}_{(i-t-1)} \otimes F^t_{(i)})$$

(see §3.1 and (2.8)) for all $0 \leq t < k$, and $\phi \circ F^k_{(i)} = G^k_{(i)}$.

Thus by induction we see that any $(n-1)$-st order chain complex $\mathcal{L}^{(n-1)}$ extending $\mathcal{L}^{(0)}$ lifts along ϕ to $\mathcal{K}^{(n-1)}$, so that ϕ_* is surjective.

On the other hand, since ϕ is a trivial fibration in $\mathrm{map}_{\mathcal{M}}$, in particular $\widetilde{\Omega}^{n-1}\phi : \widetilde{\Omega}^{n-1}\mathbf{map}_K(a_{n+1}, a_0) \to \widetilde{\Omega}^{n-1}\mathbf{map}_L(a_{n+1}, a_0)$ is a trivial fibration in $\mathcal{M}$, so it induces an isomorphism

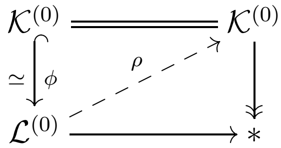
$$\pi_{n-1}\,\mathrm{map}_{\mathcal{M}}(\gamma_1 \ldots \gamma_{n+1}, \mathbf{map}_K(a_{n+1}, a_0)) \xrightarrow{\cong} \pi_{n-1}\,\mathrm{map}_{\mathcal{M}}(\gamma_1 \ldots \gamma_{n+1}, \mathbf{map}_L(a_{n+1}, a_0))$$

by SM7. Thus if $[\langle \phi_* \mathcal{K} \rangle] = [\langle \phi_* \mathcal{K}' \rangle]$ in $\pi_{n-1}\,\mathrm{map}_{\mathcal{M}}(\gamma_1 \ldots \gamma_{n+1},\ \mathbf{map}_L(a_{n+1},\ a_0))$, then $[\langle \mathcal{K} \rangle] = [\langle \mathcal{K}' \rangle]$ in $\pi_{n-1}\,\mathrm{map}_{\mathcal{M}}(\gamma_1 \ldots \gamma_{n+1},\ \mathbf{map}_K(a_{n+1},\ a_0))$.

We can see directly that $\langle \mathcal{L}^{(n-1)} \rangle$ vanishes if and only if it lifts to $F^n_{(n+1)} : \gamma_0 \otimes \ldots \otimes \gamma_{n+1} \to P^{n+1}\mathbf{map}_K(a_{n+1}, a_0)$, this happens if and only if the corresponding value $\langle \mathcal{K}^{(n-1)} \rangle$ vanishes, too.

(b) Now assume that $\phi^{(0)} : \mathcal{K}^{(0)} \to \mathcal{L}^{(0)}$ is an arbitrary weak equivalence, but that $\mathcal{K}^{(0)}$ and $\mathcal{L}^{(0)}$ are both fibrant and cofibrant. Factoring $\phi^{(0)}$ as a trivial cofibration followed by a trivial fibration, by (a) it suffices to assume that $\phi^{(0)}$ is a trivial cofibration. This implies that we have a lifting as indicated in the diagram of $\mathcal{M}$-categories

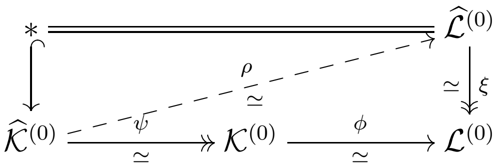
(4.25)



using Theorem 4.11. Thus by [Ho, Proposition 1.2.8], ϕ is a homotopy equivalence (with strict left inverse ρ). Therefore, if $H : \mathcal{L}^{(0)} \to (\mathcal{L}^{(0)})^{\mathcal{I}}$ is a right homotopy $\phi \circ \rho \sim \mathrm{Id}$ into a path object for $\mathcal{L}^{(0)}$ in $\mathcal{M}$-Cat (cf. [Q, I, §1]), the two trivial fibrations $d_0, d_1 : (\mathcal{L}^{(0)})^{\mathcal{I}} \twoheadrightarrow \mathcal{L}^{(0)}$ induce the required bijection by (a).

(c) Finally, if $\phi : \mathcal{K}^{(0)} \to \mathcal{L}^{(0)}$ is any Dwyer-Kan equivalence, with cofibrant replacements $\psi : \widehat{\mathcal{K}}^{(0)} \twoheadrightarrow \mathcal{K}^{(0)}$ and $\xi : \widehat{\mathcal{L}}^{(0)} \twoheadrightarrow \mathcal{L}^{(0)}$ in $\mathcal{M}$-Cat (so both ψ and ξ are trivial fibrations), we have a lifting

(4.26)



where ρ is a Dwyer-Kan equivalence between fibrant and cofibrant $\mathcal{M}$-categories, so it induces a bijection as required by (b), while ψ and ξ are trivial fibrations in $\mathcal{M}$-Cat, so they induce the required bijections by (a). Since the lower right quadrangle in (4.26) commutes, ϕ also induces a bijection as required. $\square$

4.27. **Definition.** Given a path model category $\mathcal{M}$ with core $\mathcal{I}$, let $\mathcal{C}$ be a (small) subcategory of $\mathcal{M}$-Cat consisting of fibrant 0-th order chain complexes of length

$N=n+1$ for $\Gamma\subseteq\mathcal{I}$. If $\sim$ is the equivalence relation on $\mathcal{C}$ generated by Dwyer-Kan equivalences, let $\operatorname{Ho}^{\Gamma}\mathcal{C}:=\mathcal{C}/\sim$. An equivalence class in $\operatorname{Ho}^{\Gamma}\mathcal{C}$ will be called a *homotopy chain complex* for Γ.

4.28. **Example.** Our motivating example is when $\mathcal{C}$ is an $\mathcal{M}$-subcategory of a model category $\mathcal{C}'$, whose weak equivalences $f:X\to Y$ between fibrant objects are maps inducing an isomorphism in $f_*:\pi_*\operatorname{map}_{\mathcal{M}}(\gamma,\mathbf{map}_{\mathcal{C}'}(Z,X))\to\pi_*\operatorname{map}_{\mathcal{M}}(\gamma,\mathbf{map}_{\mathcal{C}'}(Z,Y))$ for every cofibrant $Z\in\mathcal{C}'$ and every $\gamma\in\mathcal{I}$. Examples include those of §§1.13-1.16 with $\mathcal{I}$ as in §1.20.

In this case a homotopy chain complex Λ of length $n+1$ in $\operatorname{Ho}^{\Gamma}\mathcal{C}$ is represented by a sequence of elements

$$\varphi_i\in[a_i,\,a_{i-1}]_{\gamma_i}\;\cong\;\pi_0\operatorname{map}_{\mathcal{M}}(\gamma_i,\,\mathbf{map}_{\mathcal{C}}(a_i,\,a_{i-1}))\qquad(i=1,\ldots,n+1)\tag{4.29}$$

such that

$$\nu_*(\varphi_{i-1},\varphi_i)=0\quad\text{in}\quad[a_i,\,a_{i-2}]_{\gamma_{i-1}\otimes\gamma_i}\qquad(i=2,\ldots n+1)\;,$$

in the notation of §4.8.

In particular, when $\mathcal{I}=\{1\}$, Λ may be described by a diagram:

$$a_{n+1}\xrightarrow{\varphi_{n+1}}a_n\xrightarrow{\varphi_n}a_{n-2}\to\ldots\to a_1\xrightarrow{\varphi_1}a_0\;,\tag{4.30}$$

in $\operatorname{ho}\mathcal{C}'$ such that $\varphi_{i-1}\circ\varphi_i=0$ for $i=2,\ldots n+1$.

However, in the context of Massey products (cf. §3.11), we do not have such a model category $\mathcal{C}'$ available. In this case, we let $\mathcal{C}$ be a set of DGAs over R with a given homology algebra, $\Gamma=\mathcal{I}_R$ as in §1.20(b), and a homotopy chain complex Λ in $\operatorname{Ho}^{\Gamma}\mathcal{C}$ is a quasi-isomorphism class of DGAs in $\mathcal{C}$.

4.31. **Definition.** Given a path model category $\mathcal{M}$ with core $\mathcal{I}$, a category $\mathcal{C}$ as in §4.27 for $\Gamma\subseteq\mathcal{I}$, and a homotopy chain complex Λ of length $n+1$ for Γ, the corresponding *n-th order Toda bracket* $\langle\langle\Lambda\rangle\rangle$ is defined to be $\langle\langle\mathcal{K}^{(0)}\rangle\rangle\subseteq\pi_{n-1}\operatorname{map}_{\mathcal{M}}(\gamma_1\otimes\ldots\otimes\gamma_{n+1},\;\mathbf{map}_K(a_{n+1},\,a_0))$ for some representative $\mathcal{K}^{(0)}$ of Λ.

4.32. *Remark.* By Theorem 4.21, $\langle\langle\Lambda\rangle\rangle$ is well-defined.

4.33**. Massey products in DGAs.** Since Ch_R is a model category, we can consider higher Toda brackets for a differential graded algebra $\mathbf{A}_*$, as in §3.11 (we think of $\mathbf{A}_*$ as a chain complex, rather than a cochain complex, but since we allow arbitrary $\mathbb{Z}$-grading, this is no restriction).

A chain complex Λ of length $n+1$ in $\operatorname{ho}\mathbf{A}_*$ consists of a sequence $(\gamma_i)_{i=1}^{n+1}$ of homology classes in $H_*\mathbf{A}_*$, with $\gamma_i\cdot\gamma_{i+1}=0$ for $i=1,\ldots,n$. If we choose an n-th order chain complex (that is, a DGA $\mathbf{A}_*$) realizing Λ, as above, we obtain the element given by (3.13) in $\widetilde{\Omega}^{n-1}\mathbf{A}_*$. However, because we are working over Mod_R we can define the identification $\widetilde{\Omega}^{n-1}\mathbf{A}_*\cong\Omega^{n-1}\mathbf{A}_*$ using the Dold-Kan equivalence (essentially, by the homotopy addition theorem – cf. [Mu]), and thus obtain the value

$$\sum_{t=1}^{n}\,(-1)^t\;H^t_{i-k+t}\cdot H^{n-t}_i\;\in A_{j+n}\tag{4.34}$$

in $\Omega^{n-1}\mathbf{A}_*$, which is readily seen to be a $(j+n-1)$–cycle for $j:=\sum_{t=1}^n m_t$.

By comparing this formula with the classical definition of the higher Massey product (see, e.g., [Ta, (V.4)]), we find:

4.35. **Proposition.** *The higher Toda brackets in a differential graded algebra* $\mathbf{A}_*$ *are identical with the usual higher Massey products.*

5. Toda brackets for chain complexes

We now study Toda brackets in the category $\mathsf{Ch}_R^{\geqslant 0}$ of non-negatively graded chain complexes over a hereditary ring R, such as $\mathbb{Z}$. It turns out that in this case even ordinary Toda brackets have a finer "homological" structure, which we describe.

5.1**. Chain complexes over hereditary rings.** Since R is hereditary, if $Q_0(G)$ is a functorial free cover of an R-module G, we have a projective presentation

$$0 \to Q_1(G) \xrightarrow{\alpha^G} Q_0(G) \xrightarrow{r} G \to 0 \ ,$$

where $Q_1(G) := \mathrm{Ker}(r)$.

We then define the n-th *Moore complex* $\mathbf{M}(G,n)_*$ for an R-module G to be the chain complex with $(M(G,n))_{n+1} := Q_1(G)$, $(M(G,n))_n := Q_0(G)$, and 0 otherwise, with $\partial_{n+1} = \alpha^G$. This yields a functor $\widehat{C}_*\colon \mathsf{grMod}_R^{\geqslant 0} \to \mathsf{Ch}_R^{\geqslant 0}$ with

$$\widehat{C}_*(\mathcal{E}_*) \ := \ \bigoplus_{n\geq 0} \mathbf{M}(E_n, n)_* \ . \tag{5.2}$$

Recall that $\mathsf{Ch}_R^{\geqslant 0}$ has a model structure in which quasi-isomorphisms are the weak equivalences, and a chain complexes is cofibrant if and only if it is projective in each dimension (see [Ho, §2.3]). Because R is hereditary, any $\mathbf{A}_* \in \mathsf{Ch}_R^{\geqslant 0}$ is uniquely determined up to weak equivalence by the graded R-module $H_*\mathbf{A}_*$ (cf. [D1, Theorem 3.4]).

Therefore, if we enrich $\mathsf{grMod}_R^{\geqslant 0}$ over Ch_R by setting

$$\underline{\mathbf{Hom}}(\mathcal{E}_*, \mathcal{F}_*) \ := \ \underline{\mathrm{Hom}}(\widehat{C}_*(\mathcal{E}_*), \widehat{C}_*(\mathcal{F}_*))$$

(see §1.16), $\widehat{C}_*$ becomes an enriched embedding, and in fact:

5.3. **Lemma.** *The functor* $\widehat{C}_*\colon \mathsf{grMod}_R^{\geqslant 0} \to \mathsf{Ch}_R^{\geqslant 0}$ *is a Dwyer-Kan equivalence over* Ch_R*.*

Since the right-hand side of (5.2) is a coproduct, we see that $\underline{\mathbf{Hom}}(\mathcal{E}_*, \mathcal{F}_*)$ naturally splits as a product

$$\prod_{n\geq 0} \big(\underline{\mathrm{Hom}}(\mathbf{M}(E_n,n)_*,\, \mathbf{M}(F_n,n)_*) \times \underline{\mathrm{Hom}}(\mathbf{M}(E_n,n)_*,\, \mathbf{M}(F_{n+1},n+1)_*)\big) \times P, \tag{5.4}$$

where P is a product of similar terms, but with $H_0 P = 0$. Moreover, since

$$[\mathbf{M}(E,n)_*,\, \mathbf{M}(F,n)_*] \cong \mathrm{Hom}_R(E,\, F) \text{ and } [\mathbf{M}(E,n)_*,\, \mathbf{M}(F,n+1)_*] \cong \mathrm{Ext}_R(E,\, F), \tag{5.5}$$

we see that (5.4) is an enriched version of the Universal Coefficient Theorem for chain complexes, stating that for chain complexes over a hereditary ring R there is a (split) short exact sequence:

$$0 \to \prod_{n>0} \mathrm{Ext}_R(H_{n-1}\mathbf{A}_*, H_n\mathbf{B}_*) \to [\mathbf{A}_*,\, \mathbf{B}_*] \to \prod_{n\geq 0} \mathrm{Hom}_R(H_n\mathbf{A}_*, H_n\mathbf{B}_*) \to 0$$

(cf. [D2, Corollary 10.13]). Note that in our version for $\mathsf{grMod}_R^{\geqslant 0}$, the splitting is natural!

5.6. **Notation.** From (5.4) we see that there are two kinds of indecomposable maps of chain complexes (and their nullhomotopies) (see (5.7)):

(a) 'Hom-type' maps $H(f) : \mathbf{M}(E,n)_* \to \mathbf{M}(F,n)_*$, determined by

$$f_n^{10} \colon Q_0(E) \to Q_0(F) \qquad \text{and} \quad f_n^{11} \colon Q_1(E) \to Q_1(F)\ .$$

A nullhomotopy $H(S) : H(f) \sim 0$ is given by $S_n^{01} \colon Q_0(E) \to Q_1(F)$, the factorization of f_n^{00} through $Q_1(F) \hookrightarrow Q_0(F)$. If it exists, it is unique.

(b) 'Ext-type' maps

$$E(f) : \mathbf{M}(E,n)_* \to \mathbf{M}(F',n+1)_*\ ,$$

determined by $f_n^{01} \colon Q_1(E) \to Q_0(F')$. A nullhomotopy $E(S) : E(f) \sim 0$ is given by $S_n^{00} \colon Q_0(E) \to Q_0(F')$ and $S_n^{11} \colon Q_1(E) \to Q_1(F')$.

$$\text{(5.7)} \qquad \mathbf{H}(\mathbf{f}) : \quad \begin{array}{ccc} Q_1(E_n) & \xrightarrow{f_n^{11}} & Q_1(F_n) \\ \downarrow & \overset{S_n^{01}}{\nearrow} & \downarrow \\ Q_0(E_n) & \xrightarrow[f_n^{00}]{} & Q_0(F_n) \end{array} \qquad \mathbf{E}(\mathbf{f}) : \quad \begin{array}{ccc} & & Q_1(F_{n+1}) \\ & \overset{S_n^{11}}{\nearrow} & \downarrow \\ Q_1(E_n) & \xrightarrow{f_n^{01}} & Q_0(F_{n+1}) \\ \downarrow & \underset{S_n^{00}}{\nearrow} & \\ Q_0(E_n) & & \end{array}$$

5.8. **Secondary chain complexes in $\mathsf{grMod}_R^{\geqslant 0}$.** In light of the above discussion, we see that any secondary chain complex

$$\text{(5.9)} \qquad \widehat{C}_*(\mathcal{E}_*) \xrightarrow{f} \widehat{C}_*(\mathcal{F}_*) \xrightarrow{g} \widehat{C}_*(\mathcal{G}_*) \xrightarrow{h} \widehat{C}_*(\mathcal{H}_*)$$

in the Ch_R-enriched category $\mathsf{grMod}_R^{\geqslant 0}$ is a direct sum of secondary chain complexes of one of the following four elementary forms:

$$\text{(5.10)} \qquad \begin{aligned} &\mathbf{M}(E_n,n)_* \xrightarrow{H(f)\top E(f)} \mathbf{M}(F_n,n)_* \oplus \mathbf{M}(F_{n+1},n+1)_* \xrightarrow{H(g)\perp E(g)} \\ &\qquad \mathbf{M}(G_{n+1},n+1)_* \xrightarrow{H(h)} \mathbf{M}(H_{n+1},n+1)_*\ , \end{aligned}$$

$$\text{(5.11)} \qquad \begin{aligned} &\mathbf{M}(E_n,n)_* \xrightarrow{H(f)\top E(f)} \mathbf{M}(F_n,n)_* \oplus \mathbf{M}(F_{n+1},n+1)_* \xrightarrow{H(g)\perp E(g)} \\ &\qquad \mathbf{M}(G_{n+1},n+1)_* \xrightarrow{E(h)} \mathbf{M}(H_{n+2},n+2)_*\ , \end{aligned}$$

$$\text{(5.12)} \qquad \begin{aligned} &\mathbf{M}(E_n,n)_* \xrightarrow{H(f)} \mathbf{M}(F_n,n)_* \xrightarrow{H(g)\top E(g)} \\ &\qquad \mathbf{M}(G_n,n)_* \oplus \mathbf{M}(G_{n+1},n+1)_* \xrightarrow{H(h)\perp E(h)} \mathbf{M}(H_{n+1},n+1)_* \end{aligned}$$

$$\text{(5.13)} \qquad \begin{aligned} &\mathbf{M}(E_n,n)_* \xrightarrow{E(f)} \mathbf{M}(F_{n+1},n+1)_* \xrightarrow{H(g)\top E(g)} \\ &\qquad \mathbf{M}(G_{n+1},n+1)_* \oplus \mathbf{M}(G_{n+2},n+2)_* \xrightarrow{H(h)\perp E(h)} \mathbf{M}(H_{n+2},n+2)_* \end{aligned}$$

Two additional hypothetical forms, namely:

(i) $\mathbf{M}(E_n, n)_* \xrightarrow{H(f)} \mathbf{M}(F_n, n)_* \xrightarrow{H(g)} \mathbf{M}(G_n, n)_* \xrightarrow{H(h)} \mathbf{M}(H_n, n)_*$

(ii) $\mathbf{M}(E_n, n)_* \xrightarrow{E(f)} \mathbf{M}(F_{n+1}, n+1)_* \xrightarrow{E(g)} \mathbf{M}(G_{n+2}, n+2)_* \xrightarrow{E(h)} \mathbf{M}(H_{n+3}, n+3)_*$

in fact are irrelevant to Toda brackets, for dimensional reasons.

Moreover, the four elementary secondary chain complexes may or may not split further into one of the following six *atomic* forms:

(a) $\mathbf{M}(E_n, n)_* \xrightarrow{H(f)} \mathbf{M}(F_n, n)_* \xrightarrow{H(g)} \mathbf{M}(G_n, n)_* \xrightarrow{E(h)} \mathbf{M}(H_{n+1}, n+1)_*$ and two similar cases with a single E-term;

(b) $\mathbf{M}(E_n, n)_* \xrightarrow{H(f)} \mathbf{M}(F_n, n)_* \xrightarrow{E(g)} \mathbf{M}(G_{n+1}, n+1)_* \xrightarrow{E(h)} \mathbf{M}(H_{n+2}, n+2)_*$ and two similar cases with a single H-term.

5.14**. Secondary Toda brackets in $\mathsf{grMod}_R^{\geqslant 0}$.** By Definition 4.31, a secondary Toda bracket in the Ch_R-enriched category $\mathsf{grMod}_R^{\geqslant 0}$ is associated to a homotopy chain complex Λ of length 3 in $\operatorname{ho}\mathsf{grMod}_R^{\geqslant 0}$ as in (4.30). This means that we replace the actual chain maps in each of the twelve examples of §5.8 by their homotopy classes: that is, elements in $\operatorname{Hom}_R(E, F)$ or $\operatorname{Ext}_R(E, F')$, respectively.

The compositions $\operatorname{Hom}(E, F) \otimes \operatorname{Ext}(F, G) \to \operatorname{Ext}(E, G)$ $\operatorname{Ext}(E, F) \otimes \operatorname{Hom}(F, G) \to \operatorname{Ext}(E, G)$ simply define the functoriality of Ext, while $\operatorname{Ext}(E, F) \otimes \operatorname{Ext}(F, G) \to \operatorname{Ext}(E, G)$ vanishes for dimension reasons. Nevertheless, the associated Toda bracket may be non-trivial.

Note that in this case, as in the original construction of Toda in [To2] (see also [Sp1]), the subset $\langle\langle\Lambda\rangle\rangle$ of $[\mathbf{\Sigma}\mathcal{E}_*, \mathcal{H}_*]$ is actually a double coset of the group

$$(\Sigma f)^\sharp[\mathbf{\Sigma}\mathcal{F}_*, \mathcal{H}_*] + h_\sharp[\mathbf{\Sigma}\mathcal{E}_*, \mathcal{G}_*] ,$$

so we can think of $\langle\langle\Lambda\rangle\rangle$, which we usually denote simply by $\langle h, g, f\rangle$, as taking value in the quotient abelian group

$$\langle h, g, f\rangle \in (\Sigma f)^\sharp[\mathbf{\Sigma}\mathcal{F}_*, \mathcal{H}_*] \setminus [\mathbf{\Sigma}\mathcal{E}_*, \mathcal{H}_*] \;/\; h_\sharp[\mathbf{\Sigma}\mathcal{E}_*, \mathcal{G}_*] . \tag{5.15}$$

Thus the elementary examples of §5.8 may be interpreted as secondary operations in Ext_R, defined under certain vanishing assumptions, and with an explicit indeterminacy (which may be less than that indicated in (5.15) in any specific case).

For example, in (5.11) (case (e) above), the operation is defined for elements in the pullback of

$$\begin{array}{c}
\operatorname{Hom}(E_n, F_n) \otimes \operatorname{Ext}(F_n, G_{n+1}) \otimes \operatorname{Ext}(G_{n+1}, H_{n+2}) \\
\qquad\qquad \searrow^{\text{comp} \otimes \text{Id}} \\
\qquad\qquad\qquad\qquad \operatorname{Ext}(E_n, G_{n+1}) \otimes \operatorname{Ext}(G_{n+1}, H_{n+2}) \\
\qquad\qquad \nearrow_{\text{comp} \otimes \text{Id}} \\
\operatorname{Ext}(E_n, F_{n+1}) \otimes \operatorname{Hom}(F_{n+1}, G_{n+1}) \otimes \operatorname{Ext}(G_{n+1}, H_{n+2})
\end{array}$$

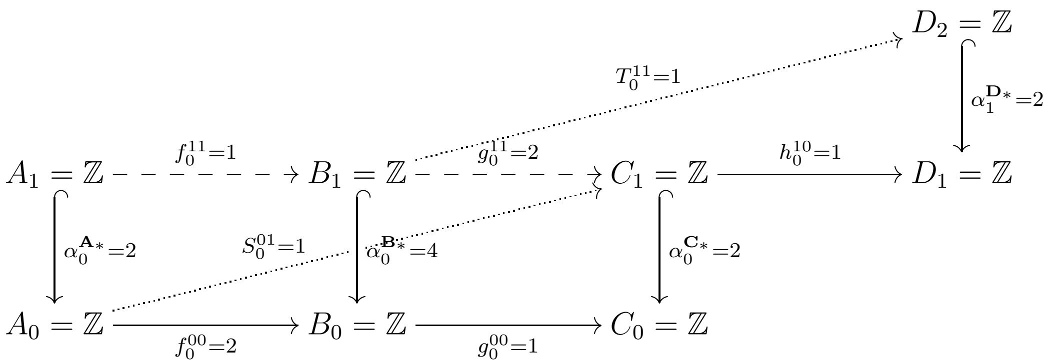
and takes value in the quotient group $\operatorname{Ext}(E_n, H_{n+1}) \;/\; h_\sharp \operatorname{Hom}(E_n, G_n)$, where $h_\sharp \operatorname{Hom}(E_n, G_n)$ refers to the image of the given element $h \in \operatorname{Ext}(G_n, H_{n+1})$ under precomposition with all elements of $\operatorname{Hom}(E_n, G_n)$.

It turns out that cases (a) and (d) are trivial for dimension reasons, but we shall now provide examples of non-triviality for four of the remaining cases.

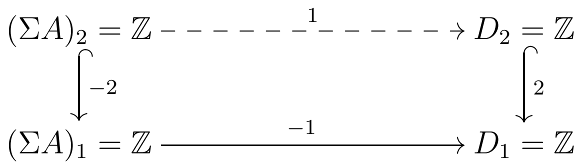
5.16. **Example.** Consider the homotopy chain complex Λ in $\mathrm{ho}\,\mathsf{grMod}_R^{\geqslant 0}$ given by $E_0 = \mathbb{Z}/2$, $F_0 = \mathbb{Z}/4$, $G_0 = \mathbb{Z}/2$, and $H_1 = \mathbb{Z}/2$, with the corresponding maps

$$\begin{aligned} f = 2 \in \mathbb{Z}/2 &= \mathrm{Hom}(E_0,\ F_0) = \mathit{Hom}(\mathbb{Z}/2,\ \mathbb{Z}/4) \\ g = 1 \in \mathbb{Z}/2 &= \mathrm{Hom}(F_0,\ G_0) = \mathrm{Hom}(\mathbb{Z}/4, \mathbb{Z}/2) \\ h = 2 \in \mathbb{Z}/2 &= \mathrm{Ext}(F_0,\ H_1) = \mathrm{Ext}(\mathbb{Z}/2, \mathbb{Z}/2)\ . \end{aligned}$$

By Remark 4.32, we may choose any cofibrant chain complexes in $\mathsf{Ch}_{\mathbb{Z}}$ to realize Λ, not necessarily the functorial versions $\widehat{C}_*(\mathcal{E}_*)$, and so on. In our case we shall use the following minimal secondary chain complex:



The Toda bracket is given by:



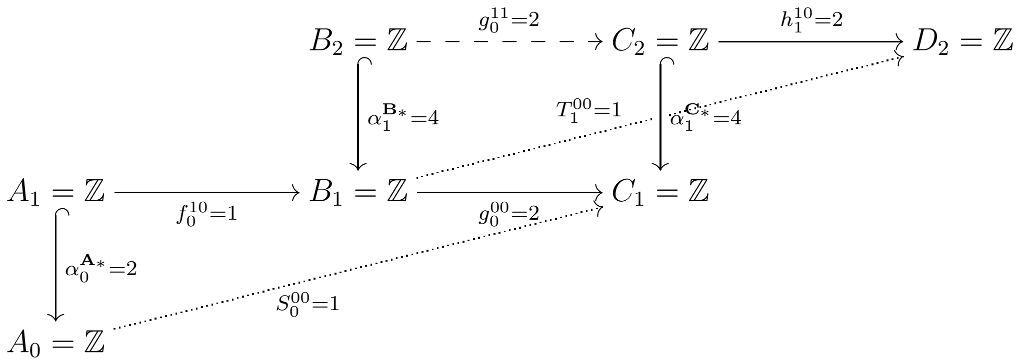
The indeterminacy is given by

$$\begin{aligned} &(\Sigma f)^\sharp[\boldsymbol{\Sigma}\mathcal{F}_*,\ \mathcal{H}_*] + h_\sharp[\boldsymbol{\Sigma}\mathcal{E}_*,\ \mathcal{G}_*] \\ = {} &\Sigma f^\sharp\,\mathrm{Hom}(F_0, H_1) + h_\sharp\,\mathrm{Hom}(E_0, G_1) = 2\cdot(\mathbb{Z}/2) + 0 = 0\ . \end{aligned}$$

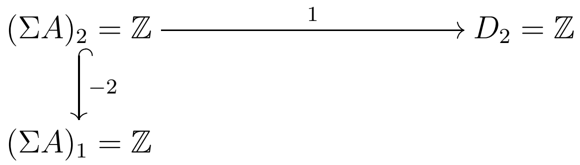
Hence the Toda bracket $\langle h, g, f\rangle$ does not vanish.

5.17. **Example.** Consider the homotopy chain complex in $\mathrm{ho}\,\mathsf{grMod}_R^{\geqslant 0}$ given by $E_0 = \mathbb{Z}/2$, $F_1 = \mathbb{Z}/4$, $G_1 = \mathbb{Z}/4$, and $H_2 = \mathbb{Z}$, with the corresponding maps $f = 1 \in \mathbb{Z}/2 = \mathrm{Ext}(\mathbb{Z}/2, \mathbb{Z}/4)$, $g = 2 \in \mathbb{Z}/4 = \mathrm{Hom}(\mathbb{Z}/4, \mathbb{Z}/4)$, and $h = 2 \in \mathbb{Z}/4 = \mathrm{Ext}(\mathbb{Z}/4, \mathbb{Z})$.

We choose the following associated secondary chain complex:



The Toda bracket is represented as follows:



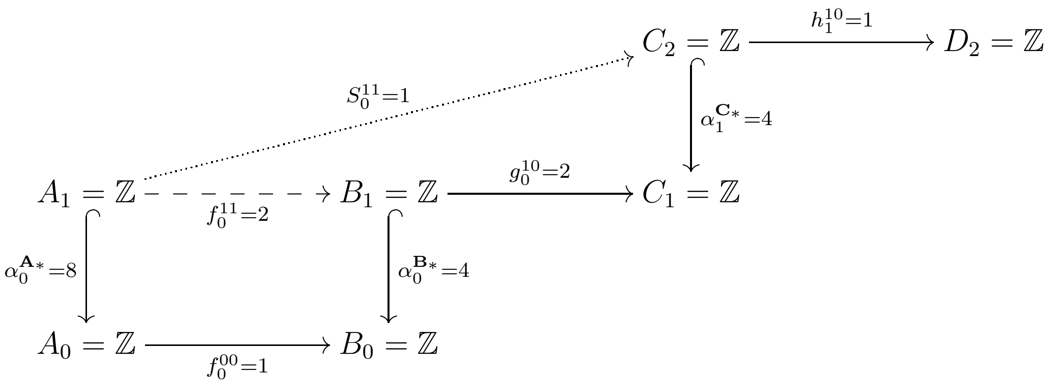
which is a generator of $\operatorname{Ext}(\mathbb{Z}/2,\mathbb{Z}) = \mathbb{Z}/2$. The indeterminacy is

$$(\Sigma f)^{\sharp}[\boldsymbol{\Sigma}\mathcal{F}_*,\ \mathcal{H}_*] + h_{\sharp}[\boldsymbol{\Sigma}\mathcal{E}_*,\ \mathcal{G}_*]$$
$$= \Sigma f^{\sharp}\operatorname{Hom}(F_0,H_1) + h_{\sharp}\operatorname{Hom}(E_0,G_1) \;=\; 1\cdot 0 + 2\cdot(\mathbb{Z}/2) \;=\; 0\ .$$

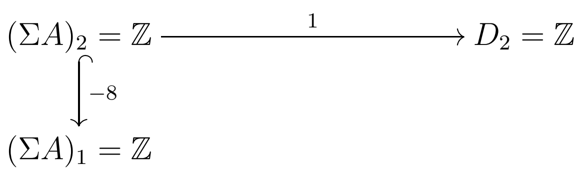
Hence the Toda bracket $\langle h,g,f\rangle$ does not vanish.

5.18. **Example.** Consider the homotopy chain complex in $\operatorname{ho}\mathsf{grMod}_R^{\geqslant 0}$ given by $E_0 = \mathbb{Z}/8$, $F_1 = \mathbb{Z}/4$, $G_1 = \mathbb{Z}/4$, and $H_2 = \mathbb{Z}$, with the corresponding maps $f = 1 \in \mathbb{Z}/4 = \operatorname{Hom}(\mathbb{Z}/8,\mathbb{Z}/4)$, $g = 2 \in \mathbb{Z}/4 = \operatorname{Ext}(\mathbb{Z}/4,\mathbb{Z}/4)$, and $h = 1 \in \mathbb{Z}/4 = \operatorname{Ext}(\mathbb{Z}/4,\mathbb{Z})$.

We may choose the following associated secondary chain complex:



The Toda bracket is given by:



which is a generator of $\operatorname{Ext}(\mathbb{Z}/8,\mathbb{Z}) = \mathbb{Z}/8$. The indeterminacy is

$$(\Sigma f)^{\sharp}[\boldsymbol{\Sigma}\mathcal{F}_*,\ \mathcal{H}_*] + h_{\sharp}[\boldsymbol{\Sigma}\mathcal{E}_*,\ \mathcal{G}_*] \;=\; f^{\sharp}\operatorname{Ext}(F_0,H_2) + h_{\sharp}\operatorname{Hom}(E_0,G_1)\ .$$

A generator of $f^\sharp\,\mathrm{Ext}(F_0,H_2)=1\cdot\mathrm{Ext}(\mathbb{Z}/4,\mathbb{Z})=\mathbb{Z}/4$ in $\mathrm{Ext}(E_0,H_2)$ is given by

$$\begin{array}{ccccc}
(\Sigma A)_2=\mathbb{Z} & \overset{2}{\dashrightarrow} & (\Sigma B)_2=\mathbb{Z} & \xrightarrow{\;1\;} & D_2=\mathbb{Z}\\
\big\downarrow{\scriptstyle -8} & & \big\downarrow{\scriptstyle -4} & & \\
(\Sigma A)_1=\mathbb{Z} & \xrightarrow{\;1\;} & (\Sigma B)_1=\mathbb{Z} & &
\end{array}$$

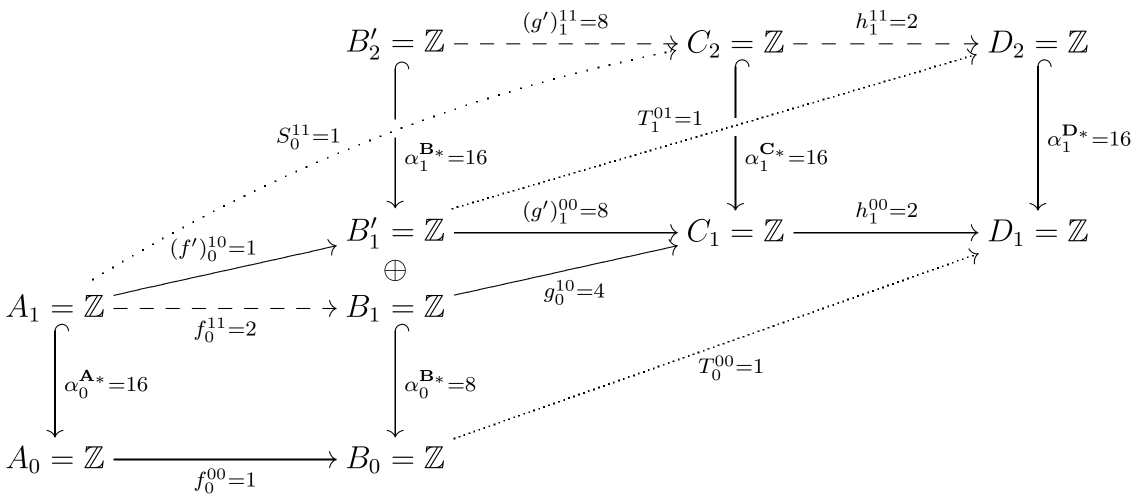
while a generator of $h_\sharp\,\mathrm{Hom}(E_0,G_1)=1\cdot\mathrm{Hom}(\mathbb{Z}/8,\mathbb{Z}/4)=\mathbb{Z}/4$ in $\mathrm{Ext}(E_0,H_2)$ is given by

$$\begin{array}{ccc}
(\Sigma A)_2=\mathbb{Z} & \xrightarrow{\;2\;} & D_2=\mathbb{Z}\\
\big\downarrow{\scriptstyle -8} & & \\
(\Sigma A)_1=\mathbb{Z} & &
\end{array}$$

so the total indeterminacy is the subgroup $\mathbb{Z}/4\subseteq\mathbb{Z}/8=\mathrm{Ext}(\mathbb{Z}/8,\mathbb{Z})=\mathrm{Ext}(E_0,H_2)$. Since the Toda bracket $\langle h,g,f\rangle$ is represented by a generator of this $\mathbb{Z}/8$, it does not vanish.

5.19. **Example.** Consider the homotopy chain complex in $\mathrm{ho}\,\mathsf{grMod}_R^{\geqslant 0}$ given by $E_0=\mathbb{Z}/16$, $F_0=\mathbb{Z}/8$, $F_1=\mathbb{Z}/16$, $G_1=\mathbb{Z}/16$, and $H_2=\mathbb{Z}/16$, with the corresponding maps $f=1\in\mathbb{Z}/8=\mathrm{Hom}(E_0,F_0)$, $f'\in\mathbb{Z}/16=\mathrm{Ext}(E_0,F_1)$, $g=4\in\mathbb{Z}/8=\mathrm{Ext}(F_0,G_1)$, $g'\in\mathbb{Z}/16=\mathrm{Hom}(F_1,G_1)$ and $h=2\in\mathbb{Z}/16=\mathrm{Hom}(G_1,H_1)$.

We may choose the following associated secondary chain complex:



The Toda bracket is given by:

$$\begin{array}{ccc}
(\Sigma A)_2=\mathbb{Z} & \overset{h^{11}_1\circ S^{11}_0-T^{01}_1\circ(f')^{10}_0=1}{\dashrightarrow} & D_2=\mathbb{Z}\\
\big\downarrow{\scriptstyle -16} & & \big\downarrow{\scriptstyle 16}\\
(\Sigma A)_1=\mathbb{Z} & \xrightarrow{\;-f^{00}_0\circ T^{00}_0=-1\;} & D_1=\mathbb{Z}
\end{array}$$

which is a generator of $\mathrm{Hom}(\mathbb{Z}/16,\mathbb{Z}/16)=\mathbb{Z}/16$. The indeterminacy is

$$(\Sigma f)^\sharp[\mathbf{\Sigma}\mathcal{F}_*,\,\mathcal{H}_*]+h_\sharp[\mathbf{\Sigma}\mathcal{E}_*,\,\mathcal{G}_*]\;=\;f^\sharp\,\mathrm{Hom}(F_0,H_1)+h_\sharp\,\mathrm{Hom}(E_0,G_1)\ .$$

A generator of $f^\sharp \operatorname{Hom}(F_0, H_1) = 1 \cdot \operatorname{Hom}(\mathbb{Z}/8, \mathbb{Z}/16) = \mathbb{Z}/8$ in $\operatorname{Hom}(E_0, H_1)$ is given by

$$\begin{array}{ccccc}
(\Sigma A)_2 = \mathbb{Z} & \overset{f_0^{11}=2}{\dashrightarrow} & (\Sigma B)_2 = \mathbb{Z} & \overset{-1}{\dashrightarrow} & D_2 = \mathbb{Z} \\
\big\downarrow {\scriptstyle -16} & & \big\downarrow {\scriptstyle -8} & & \big\downarrow {\scriptstyle 16} \\
(\Sigma A)_1 = \mathbb{Z} & \xrightarrow{f_0^{00}=1} & (\Sigma B)_1 = \mathbb{Z} & \xrightarrow{2} & D_1 = \mathbb{Z}
\end{array}$$

while a generator of $h_\sharp \operatorname{Hom}(E_0, G_1) = 2 \cdot \operatorname{Hom}(\mathbb{Z}/16, \mathbb{Z}/16) = \mathbb{Z}/8$ in $\operatorname{Hom}(E_0, H_1)$ is given by

$$\begin{array}{ccc}
(\Sigma A)_2 = \mathbb{Z} & \overset{-2}{\dashrightarrow} & D_2 = \mathbb{Z} \\
\big\downarrow {\scriptstyle -16} & & \big\downarrow {\scriptstyle 16} \\
(\Sigma A)_1 = \mathbb{Z} & \xrightarrow{2} & D_1 = \mathbb{Z}
\end{array}$$

so the Toda bracket $\langle h, g, f \rangle$ does not vanish.

5.20. *Remark.* See [Ba2, §6.12] for a calculation relating Toda brackets in topology with a certain operation in homological algebra.

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