THE ASYMPTOTIC EXPANSION OF A HYPERGEOMETRIC SERIES COMING FROM MIRROR SYMMETRY

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Abstract. In this paper we give a description of the coefficients of the asymptotic expansion of the logarithmic derivative of a family of hypergeometric series. This family plays an important role in the computation of the reduced genus one Gromov-Witten invariants of projective hypersurfaces and the confirmation of Bershadsky, Cecotti, Ooguri, Vafa (BCOV) conjecture for genus one Gromov-Witten invariants of a generic quintic threefold by Zinger.

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1. Introduction

In [14] Zinger considered the hypergeometric series

$$\mathcal{F}(w,x)=\mathcal{F}_n(w,x)=\sum_{d=0}^{\infty}x^d\frac{\prod_{r=1}^{r=nd}(nw+r)}{\prod_{r=1}^{r=d}((w+r)^n-w^n)},\quad n\in\mathbb{N} \tag{1}$$

which is a deformation of the well-known hypergeometric series,

$$\mathcal{F}(x)=\mathcal{F}_n(x)=\sum_{d=0}^{\infty}\frac{(nd)!}{(d!)^n}x^d, \tag{2}$$

namely, the holomorphic solution of the Picard-Fuchs equation of a certain family of Calabi-Yau manifolds. The case $n=5$ is the most interesting one, because of its application in string theory, especially in the prediction of Gromov-Witten invariants (GW-invariants). The idea of computing GW-invariants using some explicit series goes back to [3]. In that paper the authors made a conjecture for computing the genus zero GW-invariant in terms of the Frobenius basis of the differential equation satisfied by $\mathcal{F}_5(x)$. This was proved independently by Givental [5], Lian, Lie and Yau [7]. The BCOV conjecture [2] is concerned with the genus one case which is confirmed by Zinger. To prove

2010 *Mathematics Subject Classification.* Primary 33C20, 32G34, Secondary 14J32, 14J33 .
Key words and phrases. hypergeometric series, deformation, asymptotic expansion, generating functions, elementary functions, Euler polynomials .

this conjecture Zinger first introduced a notion of reduced GW-invariants and found the following relation

$$N_{1,d} - N^0_{1,d} = \frac{1}{12} N_{0,d}, \tag{3}$$

where $N_{g,d}$ $(g = 0, 1)$ are the standard genus g GW-invariants of degree d and $N^0_{1,d}$ is the reduce one [13]. Then in [14] using deformation (1) he succeeded to compute $N^0_{1,d}$ for hypersurfaces. In particular for the quintic case the BCOV conjecture follows from (3). To compute $N^0_{1,d}$, Zinger needed the behavior of $\mathcal{F}(w, x)$ at $w = \infty$. In [12] the authors have shown that $\mathcal{F}(w, x)$ has a perturbative expansion

$$\mathcal{F}(w, x) \sim e^{\mu(x)w} \sum_{s=0}^{\infty} \Phi_s(x) w^{-s} \quad (w \to \infty), \tag{4}$$

or in other words for fixed x, $\log \mathcal{F}(w, x) = O(w)$. Thanks to this expansion they have shown that

$$\mu(x) = \int_0^x \frac{L(u) - 1}{u} du, \quad \Phi_0(x) = L(x), \tag{5}$$

where $L(x) = (1 - n^n x)^{-1/n}$. After this, with an ordinary differential equation they have given an inductive method to find all Φ_s, which according to Zinger may be needed for computing higher-genus GW-invariants of hypersurfaces.
Unfortunately this method is not useful in practice, because for each Φ_s, one has to solve a complicated non homogeneous differential equation and the complicity goes up with s. On the other hand the first few terms do not give any idea about the general behavior of Φ_s. But surprisingly when we take the logarithmic derivative of this perturbative expansion, we get a quite simpler formula which has been experimentally conjectured in [12]. The first goal of this article is to prove this conjecture and give a recursive formula for the coefficients. More precisely we prove the following theorem.

Theorem 1.1. *Let $\widetilde{\mathcal{F}}(w, x) = \sum_{s=0}^{\infty} \Phi_s(x) w^{-s}$, where Φ_s are defined in (4). Then*

(1) we have

$$x \frac{\partial}{\partial x} \log \widetilde{\mathcal{F}}(w, x) \sim \frac{1}{n} \sum_{s=0}^{\infty} \frac{P_s(n, X)}{(nwL)^s}, \tag{6}$$

where $X = (1 - n^n x)^{-1}, L = X^{1/n}$ and each $P_s(n, X) \in \mathbb{Q}[n, X]$ is a polynomial of degree $s + 1$ in X and $2s + 1$ in n.

(2) Let

$$\mathcal{P}(n, X, T) = \sum_{s=0}^{\infty} P_s(n, X) T^s, \tag{7}$$

and

$$\sum_{s=0}^{\infty} P_{s,i}(n, X) T^s := \Big(\mathcal{P}(n, X, T) + x\frac{\partial}{\partial x} - (X - 1)T\frac{\partial}{\partial T})\Big)^i (1). \tag{8}$$

Then for $s \geq 0$, $P_s(n,X)$ are given by the following two recursive equations.

$$P_{s,i+1}(n,X) = \left(n(X-1)X\frac{\partial}{\partial X} - s(X-1)\right)P_{s,i}(n,X) + \sum_{r=0}^{s} P_{r,i}(n,X)P_{s-r}(n,X), \quad i = 1,2,3,... \tag{9}$$

$$\sum_{r=1}^{s}\sum_{i=0}^{r} E_{r,i}(n,X)P_{s-r,i}(n,X) = 0, \quad s = 1,2,3,... \tag{10}$$

where $E_{r,i}(n,X) \in \mathbb{Q}[n,X]$ are known.

Remark. In the next section (see equations (19), (20) and (22)) we give a recursive equation for $E_{r,i}(n,X)$. Using this recursive one can find all $E_{k,i}(n,X)$. The table below shows few terms of $E_{r,i}$.

r	$E_{k,0}$	$E_{r,1}$	$E_{r,2}$
0	1	0	0
1	$-n(X-1)$	n	0
2	$n(n-1)\big(\frac{(n-2)(n-11)}{24}X-1\big)(X-1)$	$-\frac{3}{2}n^2(n-1)(X-1)$	$\frac{n(n-1)}{2}$

Then we study the coefficients of $P_s(n,X)$ with respect to n. Our motivation is the experimental computation and the conjecture which was given in [12] for the leading coefficient of $P_s(n,X)$. We state this in the following theorem.

Theorem 1.2. *Let*

$$\widehat{\mathcal{P}}(X,T) = \sum_{s=-1}^{\infty} \rho_s(X)T^s,$$

be the generating function of polynomials $\{\rho_s(X)\}_{s\geq 0}$, with an extra term $\rho_{-1}(X) = \log X$, where $\rho_s(X)$ for $s \geq 0$, is the leading coefficient of $P_s(n,X)$ with respect to n. Then we have

$$\widehat{\mathcal{P}}(X,T) = -\sum_{n=1}^{\infty} \frac{Uq^{n-\frac{1}{2}}}{1-Uq^{n-\frac{1}{2}}},$$

where $q = e^T, U = 1 - \frac{1}{X}$. Furthermore for each $s \geq -1$

$$\rho_s(X) = \alpha_{s+1}e_{s+1}(X), \tag{11}$$

where

$$\sum_{k\geq 0} \alpha_k t^k = \frac{t/2}{\sinh t/2} = 1 - \frac{1}{24}t^2 + \frac{7}{5760}t^4 - \frac{31}{967680}t^6 + \cdots \tag{12}$$

and $e_k(X)$ are defined inductively by

$$e_0(X) = \log X, \quad e_{k+1}(X) = X(X-1)\frac{\partial}{\partial X}e_k(X), \quad k \geq 0. \tag{13}$$

The method which we use to prove Theorem 1.2 theoretically can be applied for the rest of coefficients but in practice it is hopeless to find a concrete solution. Instead we show a description of the structure (see Section 5, Theorem 5.1). Roughly speaking we show that the image of the ℓ's top coefficient of $P_s(n,X)$ when s varies under a map (we call it the Euler map) belongs to the field of elementary functions.

The text is organized in the following way. In section 2 first we review those parts of the article [12] which we need in our proofs and then we prove Theorem 1.1. In section 3 we give a proof for Theorem 1.2. In Section 4 we introduce some algebraic formalism concerning Euler polynomials and Bernoulli numbers which will be needed later. Finally in Section 5 we prove the main theorem concerning the structure of the coefficients of $P_s(n, X)$.

2. Asymptotic expansion for $\mathcal{F}(w, x)$ and its logarithmic derivative

2.1. **Review on the computation of $\Phi_s(x)$.** In this part we review briefly the recursion for $\Phi_s(x)$ given in [12]. Denote $D = x\frac{d}{dx}$ and $D_w = D + w$. The function $\mathcal{F}(w, x)$ satisfies the ODE

$$\big(D_w^n - w^n - x\prod_{j=1}^{n}(nD_w + j)\big)\mathcal{F} = 0, \tag{14}$$

From the first equation of (5) we have $D_w e^{\mu w} = e^{\mu w}\widetilde{D}_w$, where $\widetilde{D}_w = D + Lw$. The function $\widetilde{\mathcal{F}}(w, x) = e^{-\mu(x)w}\mathcal{F}(w, x)$ satisfies the differential equation $\mathcal{L}\tilde{\mathcal{F}} = 0$, where $\mathcal{L}$ is the differential operator

$$\mathcal{L} = L^n\big(\tilde{D}_w^n - x\prod_{j=1}^{n}(n\widetilde{D}_w + j)\big) \tag{15}$$

$$\widetilde{D}_w^n - (Lw)^n - (L^n - 1)\sum_{r=1}^{n}\frac{S_r(n)}{n^r}\tilde{D}_w^{n-r}, \tag{16}$$

where $S_r(n)$ denotes the rth elementary symmetric function of $1, 2, \cdots, n$. By induction on k, the powers of $\widetilde{D}_w$ are given by

$$\widetilde{D}_w^k = \sum_{m=0}^{k}\widetilde{D}_w^m(1)D^{k-m}. \tag{17}$$

A second induction gives the formula

$$\widetilde{D}_w^m(1) = \sum_{j=0}^{m}\mathcal{H}_{m,j}(X)(Lw)^{m-j}, \tag{18}$$

where $X = L^n$ and

$$\mathcal{H}_{m,k}(n, X) = \sum_{j=1}^{k}\binom{m}{j+k}Q_{k,j}(n, X), \tag{19}$$

with $Q_{k,j} \in \mathbb{Z}[n^{-1}, X]$ defined inductively by

$$Q_{0,j} = \delta_{0,j}, \quad Q_{k,j} = (X - 1)\left(XQ'_{k-1,j} + \frac{jQ_{k-1,j} + (k + j - 1)Q_{k-1,j-1}}{n}\right) \quad k \geq 1. \tag{20}$$

Using (17),(18), we can expand $\mathcal{L}$ as $\mathcal{L} = \sum_{k=1}^{n}(Lw)^{n-k}\mathcal{L}_k$, with

$$\mathcal{L}_k = \sum_{i=0}^{k}\frac{E_{k,i}(n, X)}{n^{k-i}}D^i, \tag{21}$$

where

$$E_{k,i}(n,X) = \binom{n}{i}\mathcal{H}_{n-i,k-i}(n,X)\, n^{k-i}$$

$$- (X-1)\sum_{r=1}^{k-i}\binom{n-r}{i} S_r(n)\,\mathcal{H}_{n-i-r,k-i-r}(n,X)\, n^{k-i-r}, \tag{22}$$

Combining the differential equation $\mathcal{L}\widetilde{\mathcal{F}} = 0$, with the asymptotic expansion $\widetilde{\mathcal{F}}(w,x) = \sum_{s=0}^{\infty}\Phi_s(x)w^{-s}$, we obtain the following first order ODEs for Φ_s:

$$\mathcal{L}_1(\Phi_s) + \frac{1}{L}\mathcal{L}_2(\Phi_{s-1}) + \frac{1}{L^2}\mathcal{L}_3(\Phi_{s-2}) + \cdots + \frac{1}{L^{n-1}}\mathcal{L}_n(\Phi_{s-n+1}) = 0, \quad s \geq 0, \tag{23}$$

with the initial condition $\Phi_s(0) = \delta_{0,s}$.

In the case $s = 1$ we have $\mathcal{L}_1(\Phi_1) + \frac{1}{L}\mathcal{L}_2(\Phi_0) = 0$. From (21) and Table 1 we have

$$\mathcal{L}_1 = nD - (X-1) \tag{24}$$

$$\mathcal{L}_2 = \binom{n}{2}D^2 - \frac{3(n-1)}{2}(X-1)D + \frac{n-1}{n}\Big(\frac{(n-2)(n-11)}{24}X - 1\Big)(X-1), \tag{25}$$

It turns out

$$\Phi_1(x) = \frac{(n-2)(n+1)}{24n}(L(x) - L(x)^n),$$

$$\Phi_2(x) = \frac{(n-2)^2(n+1)^2}{2(24n)^2}(L - 2L^n + L^{2n-1}),$$

and in general $\Phi_s(x)$ for fixed s and n varying is an element of $\mathbb{Q}[n^{\pm 1}, L^{\pm 1}, X]$, where $X = L^n$ (See [12]).

2.2. **Proof of Theorem 1.1.** Before giving the proof of Theorem 1.1 we will show how the recursive equations (9),(10) uniquely work. First for each $s \geq 1$ with equation (10) and given all $P_{s',i}(n,X)$ with $s' < s-1$ we find $P_{s-1}(n,X)$. Since $E_{s,i}(n,X)$ and $P_{s',i}(n,X)$ are polynomials, therefore $P_{s-1}(n,X)$ which is uniquely determined in this way will be a polynomial. With this information and equation (9) we find $P_{s-1,i}(n,X)$ for all $i > 1$. We show this procedure in some examples. From equation (10) we have

$$P_0 = -E_{1,0}.$$

Now with equation (9)

$$P_{0,2} = n(X-1)X\frac{d}{dX}P_0 + P_0^2, \quad P_{0,3} = (X-1)X\frac{d}{dX}P_{0,2} + P_0P_{0,2}.$$

Now equation (10) for $s = 2$ says

$$P_1 = -(E_{2,0} + E_{2,1}P_0 + E_{2,2}P_{0,2}),$$

so we can find P_1. Now using equation (9) for $s = 1$,

$$P_{1,2} = (X-1)(nX\frac{d}{dX} - 1)P_1 + 2P_0P_1.$$

Finally for $s = 3$:

$$P_2 = -(E_{3,0} + E_{3,1}P_0 + E_{3,2}P_{0,2} + E_{3,3}P_{0,3}$$
$$+E_{2,1}P_1 + E_{2,2}P_{1,2}),$$

and we can find P_2.

$$P_0(n,X) = X-1,$$
$$P_1(n,X) = -\frac{(n+1)(n-1)(n-2)}{24}(X-1)X,$$
$$P_2(n,X) = 0.$$

We define

$$\sum_{s=0}^{\infty}\Psi_{s,i}(n,x)\,T^s := \frac{\sum_{s=0}^{\infty} D^i\,\Phi_s(n,x)\,T^s}{\sum_{s=0}^{\infty}\Phi_s(n,x)\,T^s}, \tag{26}$$

where $D = x\frac{d}{dx}$. We notice that

$$\sum_{s=0}^{\infty}\Psi_s w^{-s} := \sum_{s=0}^{\infty}\Psi_{s,1}w^{-s} = x\frac{\partial}{\partial x}\log\tilde{\mathcal{F}}(w,x).$$

By differentiating from equation (26) we have

$$\sum_{s=0}^{\infty}D\Psi_{s,i}T^s = \frac{\sum_{s=0}^{\infty}D^{i+1}\Phi_sT^s}{\sum_{s=0}^{\infty}\Phi_sT^s} - \frac{\sum_{s=0}^{\infty}D^{i}\Phi_sT^s}{\sum_{s=0}^{\infty}\Phi_sT^s}.\frac{\sum_{s=0}^{\infty}D\Phi_sT^s}{\sum_{s=0}^{\infty}\Phi_sT^s},$$

or

$$\sum_{s=0}^{\infty}\Psi_{s,i+1}T^s = \sum_{s=0}^{\infty}D\Psi_{s,i}T^s + (\sum_{s=0}^{\infty}\Psi_{s,i}T^s)(\sum_{s=0}^{\infty}\Psi_sT^s). \tag{27}$$

We claim that

$$\sum_{r=0}^{s}\sum_{i=0}^{r}\frac{1}{n^{r-i}L^{r-1}}E_{r,i}\Psi_{s-r,i} = 0, \quad s = 1,2,3,... \tag{28}$$

For the moment let us assume that it is true and we show by induction on s,that

$$\Psi_{s,i} = \frac{P_{s,i}(n,X)}{n^{s+i}L^s}, \tag{29}$$

where $P_{s,i}(n,X) \in \mathbb{Q}[n,X]$ are given by the recursive equations (9),(10). For $s=0, i=1$ we have

$$\Psi_0 = \frac{D\Phi_0}{\Phi_0} = \frac{DL}{L} = Y = \frac{X-1}{n} = \frac{P_0}{n}.$$

Now if (28) is true for $s' \le s$ and $i' < i+1$, from equation (27) for $\Psi_{s,i+1}$ we have

$$\begin{aligned}\Psi_{s,i+1} &= D\Psi_{s,i} + \sum_{r=0}^{s}\Psi_{r,i}\Psi_{s-r,1}\\ &= D(\frac{P_{s,i}(n,X)}{n^{s+i}L^s}) + \sum_{r=0}^{s}\frac{P_{r,i}(n,X)}{n^{r+i}L^r}.\frac{P_{s-r,i}(n,X)}{n^{s-r+i}L^{s-r}}\\ &= \frac{nDP_{s,i} - s(X-1)P_{s,i}}{n^{s+i+1}L^s} + \sum_{r=0}^{s}\frac{P_{r,i}(n,X)P_{s-r,i}(n,X)}{n^{s+i+1}L^s}.\end{aligned}$$

We note that $D = x\frac{d}{dx} = X(X-1)\frac{d}{dX}$. Hence from equation (9) we obtain

$$\Psi_{s,i+1} = \frac{P_{s,i+1}(n,X)}{n^{s+i+1}L^s}.$$

Now coming back to equation(28) we get

$$\Psi_{s-1} = \frac{-1}{n^s L^{s-1}}(E_{s,0} + \sum_{r=2}^{s}\sum_{i=1}^{r} E_{r,i}P_{s-r,i}), \tag{30}$$

therefore from (10) we find

$$\Psi_{s-1} = \frac{P_{s-1}(n,X)}{n^s L^{s-1}},$$

which completes the induction step. The only thing is to prove the identity (28). We show this identity by induction on s.
For $s=1$ we have to check that

$$\frac{1}{n}E_{1,0} + E_{1,1}\Psi_0 = 0,$$

but $E_{1,0} = -nY$ and $E_{1,1} = n$. Therefore we have to show that $\Psi_0 = Y$. But by definition (26) for $i=1$ we have

$$\sum_{s=0}^{\infty}\Psi_s T^s = \frac{\sum_{s=0}^{\infty} D\Phi_s T^s}{\sum_{s=0}^{\infty}\Phi_s T^s} = Y + O(T). \tag{31}$$

Hence the identity is true for $s=1$. Now suppose that the identity is true for all $s' < s$, then from definition (26)

$$D\Phi_s = \sum_{i=0}^{s}\Phi_i\Psi_{s-i} = L\Psi_s + Y\Phi_s + \sum_{i=1}^{s-1}\Phi_i\Psi_{s-i}.$$

From (24) we have

$$\Psi_s = \frac{1}{L}(\frac{1}{n}\mathcal{L}_1(\Phi_s) - \sum_{j=1}^{s-1}\Phi_j\Psi_{s-j}). \tag{32}$$

From (23) we have

$$\mathcal{L}_1(\Phi_s) = -\sum_{r=2}^{s+1}\frac{1}{L^{r-1}}\mathcal{L}_r(\Phi_{s+1-r}) = -\sum_{r=2}^{s+1}\sum_{i=0}^{r}\frac{E_{r,i}}{n^{r-i-1}L^{r-1}}D^i(\Phi_{s-r+1}).$$

Plugging this into (32) we find

$$\Psi_s = -\sum_{r=2}^{s+1}\sum_{i=0}^{r}\frac{E_{r,i}}{n^{r-i}L^r}D^i(\Phi_{s-r+1}) - \frac{1}{L}\sum_{j=1}^{s-1}\Phi_j\Psi_{s-j}.$$

Now using the induction step for $\Psi_{s-j,1}$, $1 \le j \le s-1$, in the last equation we find

$$\begin{aligned}\Psi_s =& -\sum_{r=2}^{s+1}\sum_{i=0}^{r}\frac{E_{r,i}}{n^{r-i}L^r}D^i(\Phi_{s-r+1}) \\ &+ \sum_{j=1}^{s-1}\sum_{r=2}^{s-j+1}\sum_{i=1}^{r}\frac{\Phi_j E_{r,i}\Psi_{s-j+1-r,i}}{n^{r-i}L^r} + \sum_{j=1}^{s-1}\frac{\Phi_j E_{s-j+1,0}}{n^{s-j+1}L^{s-j+1}} \\ =& -\sum_{r=2}^{s+1}\sum_{i=1}^{r}\frac{E_{r,i}}{n^{r-i}L^r}D^i(\Phi_{s-r+1}) - \frac{E_{s+1,0}}{n^{s+1}L^s} \\ &+ \sum_{j=1}^{s-1}\sum_{r=2}^{s-j+1}\sum_{i=1}^{r}\frac{\Phi_j E_{r,i}\Psi_{s-j+1-r,i}}{n^{r-i}L^r}.\end{aligned} \tag{33}$$

For fixed $2 \leq r \leq s+1$ and $1 \leq i \leq r$, from the definition of $\Psi_{s,r}$ (equation (26)) we have

$$\Psi_{s-r+1,i} = \frac{1}{L}(D^i(\Phi_{s-r+1}) - \sum_{j=1}^{s-r+1} \Phi_j \Psi_{s-j+1-r,i}).$$

Applying this identity in (33) follows

$$\Psi_s = -\frac{E_{s+1,0}}{n^{s+1}L^s} - \sum_{r=2}^{s+1} \frac{E_{r,i}\Psi_{s-r+1,i}}{n^{r-i}L^{r-1}},$$

which completes the induction step.

For the degree of X, we see from the recursive equation of $\mathcal{H}_{m,j}$ by a simple induction that in this case the degree of X for $\mathcal{H}_{k,j}$ is j, so the degree of $E_{k,i}(n,X)$ will be $k-i$ and from the recursive equation (9), the result follows.

For the degree of n from (19) and (20) one can easily check that the degree of n in $E_{k,i}(n,X)$ is $2k-i$. From this and the recursive equation (10) we find the result.

3. Structure of the leading coefficient of $P_s(n,X)$

3.1. **Proof of Theorem 1.2.** In Theorem 1.1 we can consider $P_s(n,X)$ as a function of n and we can write

$$P_s(n,X) = \rho_s(X)n^{2s+1} + \cdots . \tag{34}$$

In this section we study the generating function of $\{\rho_s(X)\}_{s\geq 0}$, the leading coefficient of $P_s(n,X)$ and we give a complete description for it. By experiment we find

$$\rho_0(X) = \rho_2(X) = \rho_4(X) = 0,$$

and

$$\begin{aligned}
\rho_1(X) &= \frac{-1}{24}(X^2 - X), \\
\rho_3(X) &= \frac{7}{5760}(6X^4 - 12X^3 + 7X^2 - X), \\
\rho_5(X) &= -\frac{31}{967680}(120X^6 - 360X^5 + 390X^4 - 180X^3 + 31X^2 - X).
\end{aligned}$$

These results motivated the authors in [12] to guess that $\rho_s(X) = \alpha_{s+1}e_{s+1}(X)$, where α_k and $e_k(X)$ are defined in (12), (13). For $k \geq 1$, $e_k(X)$ are called Euler polynomials (do not confuse with the classical Euler polynomials, $E_k(x)$, which are given by the $\frac{2e^{tx}}{e^x+1} = \sum_{t=0}^{\infty} E_k(x)\frac{t^k}{k!}$). For example we have $e_2(X) = X^2 - X$, $e_3(X) = 2X^3 - 3X^2 + X$, and in general

$$e_k(X) = \sum_{i=1}^{k} (-1)^{k-i}(i-1)! \left\{ {k \atop i} \right\} X^i \quad \in \quad \mathbb{Z}[X], \tag{35}$$

where $\left\{ {k \atop l} \right\}$ is a Stirling number of the second kind.

Lemma 3.1. *Let $U := 1 - \frac{1}{X}$, then $e_k(X)$ is a power series in terms of U. More precisely*

$$e_k(X) = \sum_{d=1}^{\infty} d^{k-1}U^d, \quad k \geq 0 \tag{36}$$

Proof. Let us denote the right hand side of (36) by $\epsilon_k(U)$. Then we see that by definition

$$\epsilon_0(U) = -\log(1-U) = \log X = e_0(X), \quad \epsilon_{k+1}(U) = U\frac{\partial}{\partial U}\epsilon_k(U)$$

Now for $k \geq 1$, it suffices to note that $U\frac{\partial}{\partial U} = X(X-1)\frac{\partial}{\partial X}$. The result follows by a simple induction. □

In this section we prove Theorem 1.2. We give the proof in some steps.
Let $a_{k,i}(X)$ and $\rho_{s,i}(X)$ be the leading coefficients of $E_{k,i}(n, X)$ and $P_{s,i}(n, X)$ respectively. From recursive equations (9) and (10) we have

$$\sum_{r=1}^{s}\sum_{i=0}^{r} a_{r,i}(X)\rho_{s-r,i}(X) = 1, \tag{37}$$

$$\rho_{s,i+1}(X) = D\rho_{s,i}(X) + \sum_{r=0}^{s}\rho_{r,i}(X)\rho_{s-r}(X). \tag{38}$$

Lemma 3.2. *Let $k \geq i \geq 0$,*

$$a_{k,i}(X) = \lim_{n\to\infty}\frac{E_{k,i}(n, X)}{n^{2k-i}}, \tag{39}$$

be the leading term of $E_{k,i}(n, X)$. Then

$$a_{k,i}(X) = \frac{1}{i!}a_{k-i,0}(X).$$

Proof. Let $q_{k,j}(X)$ and $h_k(X)$ be the leading coefficients of $Q_{k,j}(n, X)$ and $\mathcal{H}_{n,k}(n, X)$ respectively (here k and j are fixed with $n \to \infty$).

$$Q_{k,j}(n, X) = q_{k,j}(X)\, n^{-j} + O(n^{-j-1}).$$

The first few terms are

$$\begin{aligned}
&q_{1,1}(X) = X - 1,\\
&q_{2,1}(X) = X(X-1), \quad q_{2,2}(X) = 3(X-1)^2,\\
&q_{3,1}(X) = X(X-1)(2X-1), \quad q_{3,2}(X) = 10X(X-1)^2, \quad q_{3,3}(X) = 15(X-1)^3.
\end{aligned}$$

From equation (19) we have

$$\mathcal{H}_{n,k}(n, X) = \sum_{j=1}^{k}\binom{n}{k+j}Q_{k,j}(n, X) = \sum_{j=1}^{k}\frac{n^{k+j}}{(k+j)!}q_{k,j}(X)\, n^{-j} + O(n^{k-1}).$$

Hence

$$h_k(X) = \sum_{j=1}^{k}\frac{q_{k,j}(X)}{(k+j)!}. \tag{40}$$

We note that

$$S_r(n) = \frac{1}{2^r r!}\, n^{2r} + O(n^{2r-1}),$$

so from equation (22) we find

$$\begin{aligned}
a_{k,i}(X) &= \frac{h_{k-i}(X)}{i!} - (X-1)\sum_{r=1}^{k-i} \frac{h_{k-i-r}(X)}{2^r.r!\,i!} \\
&= \sum_{j=1}^{k-i} \frac{q_{k-i,j}(X)}{i!\,(k-i+j)!} - (X-1)\sum_{r=1}^{k-i}\sum_{j=1}^{k-i-r} \frac{q_{k-i-r,j}(X)}{2^r.r!\,i!\,(k-i-r+j)!} \\
&= \frac{1}{i!} a_{k-i,0}(X).
\end{aligned}$$

□

Now we set

$$\mathcal{A}_0(X,T) := \sum_{k=0}^{\infty} a_{k,0}(X)T^k, \tag{41}$$

and

$$R(X,T,Z) := \sum_{i=0}^{\infty} \mathcal{P}_i(X,T)\frac{Z^i}{i!}, \tag{42}$$

where

$$\mathcal{P}_i(X,T) := \sum_{s=0}^{\infty} \rho_{s,i}(X)T^s, \tag{43}$$

(note that $\mathcal{P}_1 = \mathcal{P}(X,T) = \sum_{s\geq 0} \rho_s(X)T^s$).
Hence we can rewrite equation (37) in the following compact form

$$\mathcal{A}_0(X,T)R(X,T,T) = 1. \tag{44}$$

Also from (38) we have

$$\mathcal{P}_i(X,T) = (D+\mathcal{P})^i(1) \quad i \geq 0, \tag{45}$$

where $D = (X-1)X\frac{\partial}{\partial X} = U\frac{\partial}{\partial U}$. Hence from equation (45) we have

$$\begin{aligned}
R(X,T,Z) &= \sum_{i=0}^{\infty} \mathcal{P}_i(X,T)\frac{Z^i}{i!} = \sum_{i=0}^{\infty} (D+\mathcal{P})^i(1)\frac{Z^i}{i!} \\
&= \exp\Big(\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}(X,T)\frac{Z^i}{i!}\Big) \\
&= \exp(\widetilde{\mathcal{P}}(Ue^Z,T) - \widetilde{\mathcal{P}}(U,T)),
\end{aligned} \tag{46}$$

where $\widetilde{\mathcal{P}}(U,T) = \int_0^U \mathcal{P}(\frac{1}{1-z},T)\frac{dz}{z}$.

The next step is to find a closed form for $\mathcal{A}_0(X,T)$ and $R(X,T,Z)$.

Proposition 3.1. *Let*

$$H(X,T,Z) = \sum_{j,k=0}^{\infty} q_{k,j}(X)\frac{T^{j+k}Z^j}{(j+k)!}. \tag{47}$$

be the generating function of $\{q_{k,j}(X)\}_{k,j\geq 0}$, where $q_{k,j}(X)$ is the leading coefficient of $Q_{k,j}(n,X)$.

i) We have

$$H(X,T,Z) = \exp(Zh(X,T)),$$

where

$$h(X,T) = \sum_{k=1}^{\infty} e_k(X)\frac{T^{k+1}}{(k+1)!} = Li_2(U) - Li_2(Ue^T) + T\log(1-U),$$

with $U = 1 - \frac{1}{X}$ *and*

$$Li_2(z) = \int_0^z -\frac{\log(1-u)}{u}du = \sum_{n=1}^{\infty}\frac{z^n}{n^2}$$

is the dilogarithm function.

ii) We have

$$\mathcal{A}_0(X,T) = H(X,T,T^{-1})\big(1-(X-1)(e^{T/2}-1)\big).$$

Proof. From (20) we find

$$q_{k,j}(X) = (X-1)Xq'_{k-1,j}(X) + (X-1)(j+k-1)q_{k-1,j-1}(X). \tag{48}$$

Hence $H(X,T,Z)$ satisfies the following homogeneous linear differential equation

$$[\frac{\partial}{\partial T} - X(X-1)\frac{\partial}{\partial X} - ZT(X-1)]H = 0. \tag{49}$$

We can write $H = \exp(h_0(X,T,Z))$ for some h_0, which satisfies

$$(\frac{\partial}{\partial T} - X(X-1)\frac{\partial}{\partial X})h_0 = ZT(X-1).$$

It follows that $h_0(X,T,Z) = Zh_1(X,T)$ and $h_1(X,T)$ satisfies

$$(\frac{\partial}{\partial T} - X(X-1)\frac{\partial}{\partial X})h_1 = T(X-1). \tag{50}$$

Now let $h_1(X,T) = \sum_{k=2}^{\infty}\widetilde{e}_k(X)\frac{T^k}{k!}$, then from (50) follows that

$$\widetilde{e}_2(X) = X-1, \quad \widetilde{e}_{k+1}(X) = X(X-1)\frac{\partial}{\partial X}\widetilde{e}_k(X) \quad k \geq 2,$$

which is exactly the definition of $e_k(X)$, and therefore $h_1(X,T) = h(X,T)$. The function $h(X,T)$ is obtained from

$$g(X,T) = \sum_{k=1}^{\infty} e_k(X)\frac{T^{k-1}}{(k-1)!},$$

by two times integrating respect to T. But by Lemma 3.1

$$\begin{aligned} g(X,T) &= \sum_{k=1}^{\infty}\epsilon_k(U)\frac{T^{k-1}}{(k-1)!} \\ &= \sum_{k=1}^{\infty} D^{k-1}\epsilon_1(U)\frac{T^{k-1}}{(k-1)!} \\ &= \frac{Ue^T}{1-Ue^T}. \end{aligned} \tag{51}$$

Therefore

$$h(X,T) = \int_0^T G(X,t)dt,$$

where

$$G(X,T)=\int_0^T g(X,t)dt=\int_0^T \frac{Ue^t}{1-Ue^t}dt=\log(1-U)-\log(1-Ue^T).$$

Finally

$$\begin{aligned} h(X,T) &= \int_0^T \log(1-Ue^t)dt+\int_0^T \log(1-U) \\ &= Li_2(U)-Li_2(Ue^T)+T\log(1-U). \end{aligned} \tag{52}$$

For the second part we have

$$\begin{aligned} \mathcal{A}_0(X,T) &= \sum_{j,k=0}^{\infty}\frac{q_{k,j}(X)}{(j+k)!}T^j-(X-1)\sum_{r=1}^{k}\sum_{j,k=0}^{\infty}\frac{q_{k-r,j}(X)}{2^r.r!\,(j+k-r)!}T^k \\ &= \sum_{j,k=0}^{\infty}\frac{q_{k,j}(X)}{(j+k)!}T^k-(X-1)\sum_{r=1}^{\infty}\frac{T^r}{2^r.r!}\sum_{j,k=0}^{\infty}\frac{q_{k,j}(X)}{(j+k)!}T^j \\ &= H(X,T,T^{-1})\big(1-(X-1)(e^{T/2}-1)\big). \end{aligned}$$

□

Lemma 3.3. *Let $S(X,T)\in Q[X][[T]]$, such that*

$$\sum_{i=1}^{\infty} D^{i-1}S(X,T)\frac{T^i}{i!}=0, \tag{53}$$

where $D=X(X-1)\frac{d}{dX}$. Then $S(X,T)$ is identically zero.

Proof. We have $D=U\frac{d}{dU}=\frac{d}{d\log V}$ where $U=e^V=1-\frac{1}{X}$. We set $\widetilde{S}(U,T)=S(X,T)$. Differentiating once more from equation (53), we get

$$\begin{aligned} 0 &= \sum_{i=1}^{\infty} D^iS(X,T)\frac{T^i}{i!} \\ &= \sum_{i=1}^{\infty}(U\frac{d}{dU})^i\widetilde{S}(U,T)\frac{T^i}{i!}=\widetilde{S}(Ue^T,T)-\widetilde{S}(U,T). \end{aligned}$$

It follows that $\widetilde{S}(Ue^T,T)=\widetilde{S}(U,T)$. Now let $\widetilde{S}(U,T)=\sum_{i=0}^{\infty}\widetilde{s}_i(U)T^i$, and k be the smallest indice such that $\widetilde{s}_k(U)\neq 0$. We have

$$\begin{aligned} 0=\widetilde{S}(Ue^T,T)-\widetilde{S}(U,T) &= T^k[\widetilde{s}_k(Ue^T)-\widetilde{s}_k(U)]+T^{k+1}O(T)+O(T^{k+2}) \\ &= T^k[\widetilde{s}'_k(U)T+O(T^2)]+O(T^{k+2})=T^{k+1}\widetilde{s}'_k(U)+O(T^{k+2}). \end{aligned}$$

Hence this implies that $\widetilde{s}_k(U)$ is constant. Substituting this into (53) we get

$$[\widetilde{s}_k(U)T^k+O(T^{k+1})]T+O(T^{k+1})\frac{T^2}{2!}+...=0$$

Hence $\widetilde{s}_k(U)\equiv 0$ and consequently $S(X,T)\equiv 0$. □

Now we are ready to proof Theorem 1.2.
From (44) we have

$$\log\mathcal{A}_0(X,T)+\log R(X,T,T)=0.$$

Hence by the second part of Proposition 3.1

$$\log H(X,T,T^{-1})+\log\big(1-(X-1)(e^{T/2}-1)\big)+\log R(X,T,T)=0. \tag{54}$$

We have

$$\log\bigl(1-(X-1)(e^{T/2}-1)\bigr)= -\log(1-U)+\log\bigl(1-Ue^{T/2}\bigr).$$

Plugging this in to (54) and using the first part of Proposition 3.1 and equation (46) we find

$$\widetilde{\mathcal{P}}(U,T)-\widetilde{\mathcal{P}}(Ue^{T},T)=\log\bigl(1-Ue^{T/2}\bigr)+\frac{1}{T}(Li_2(U)-Li_2(Ue^{T})). \tag{55}$$

Let

$$S(U,T)=-\widetilde{\mathcal{P}}(U,T)+\frac{1}{T}Li_2(U)+\sum_{k=1}^{\infty}\log\bigl(1-Ue^{(k-\frac{1}{2})T}\bigr). \tag{56}$$

From (55) we find that, $S(Ue^T,T)=S(U,T)$. Hence by Lemma 3.3 we have $S(U,T)=0$ and

$$\widetilde{\mathcal{P}}(U,T)=\frac{1}{T}Li_2(U)+\sum_{k=1}^{\infty}\log\bigl(1-Ue^{(k-\frac{1}{2})T}\bigr). \tag{57}$$

But by definition

$$\widetilde{\mathcal{P}}(U,T)=\int_0^U\mathcal{P}(\frac{1}{1-z},T)\frac{dz}{z}.$$

Hence Theorem 1.2 follows from (57) by derivative with respect to U and the fact that $\widehat{\mathcal{P}}=\mathcal{P}-\frac{1}{T}\log(1-U)$.

The only thing is to show that $\rho_s=\alpha_{s+1}e_{s+1}$. From the first part we have

$$\begin{aligned}\widehat{\mathcal{P}}(X,T)&=-\sum_{m=1}^{\infty}\Bigl(\sum_{\substack{n>0\\ odd}}q^{nm/2}\Bigr)U^m\\ &=\frac{1}{2}\sum_{m=1}^{\infty}\Bigl(\frac{1}{\sinh mT/2}\Bigr)U^m\\ &=\sum_{m=1}^{\infty}\Bigl(\sum_{k=0}^{\infty}\alpha_k(mT)^{k-1}\Bigr)U^m=\sum_{k=0}^{\infty}\alpha_kE_k(U)T^{k-1}.\end{aligned}$$

3.2. **Elliptic property.** The interesting point about Theorem 1.2 is that up to an elementary function and a shift $z\to z+\tau/2$, $\widehat{\mathcal{P}}(X,T)$ is quite similar to "half" of the Weirestrass ζ-function, i.e., $\zeta(\tau,z)=\frac{d}{dz}\log\theta(\tau,z)+\eta(1)z$, where

$$\theta(\tau,z)=\sum_{n\in\mathbb{Z}}(\frac{-4}{n})q^{n^2/8}y^{n/2}=q^{1/8}y^{1/2}\prod_{n=1}^{\infty}(1-q^n)(1-q^ny)(1-q^{n-1}y^{-1}),$$

is a theta function with $q=e^{2\pi i\tau},y=e^{2\pi iz}$ and $\eta:\Lambda_\tau\to\mathbb{C}$ the quasi-period homomorphism associated to $\Lambda_\tau=\mathbb{Z}+\tau\mathbb{Z}$.

If $w\in\Lambda$ and $\frac{w}{2}\notin\Lambda$, then

$$\eta(w)=2\,\zeta(\frac{1}{2}w;\tau). \tag{58}$$

Hence we have

$$\frac{1}{2\pi i}\zeta(\tau,z)=\frac{1}{2}+\frac{1}{2\pi i}\eta(1)z-\sum_{n=1}^{\infty}\Bigl(\frac{q^ny}{1-q^ny}-\frac{q^{n-1}y^{-1}}{1-q^{n-1}y^{-1}}\Bigr).$$

Using the above equation and (58) we find

$$\frac{1}{(2\pi i)^2}\eta(1) = -\frac{1}{12} + 2\sum_{n\geq 1}\frac{q^n}{(1-q^n)^2}.$$

Now by extending the recursive equation to $s \geq -1$ we can define

$$\widehat{R}(X,T,Z) = \sum_{i=0}^{\infty}\widehat{\mathcal{P}}_i(X,T)\frac{Z^i}{i!}, \tag{59}$$

where $\widehat{\mathcal{P}}_i(X,T) = \sum_{s=-1}^{\infty}\rho_{s,i}(X)T^s$. Then using equations (46) and (57) we find

$$\begin{aligned}
\widehat{R}(X,T,Z) &= \exp\big(\widetilde{\mathcal{P}}(Ue^Z,T) - \widetilde{\mathcal{P}}(U,T)\big)\\
&= \exp\big(\sum_{n\geq 1}\log(1-Ue^Ze^{(k-\frac{1}{2})T}) - \sum_{n\geq 1}\log(1-Ue^{(k-\frac{1}{2})T})\big)\\
&= \frac{\prod_{n\geq 1}(1-Ue^Ze^{(k-\frac{1}{2})T})}{\prod_{n\geq 1}(1-Ue^{(k-\frac{1}{2})T})} = \frac{(Uq^{\frac{-1}{2}}e^Z;q)_\infty}{(Uq^{\frac{-1}{2}};q)_\infty}, \qquad (60)
\end{aligned}$$

where $q = e^T$ and $(x;q)_\infty = \prod_{n\geq 1}(1-xq^n)$. We notice that by (59) and (60)

$$\widehat{\mathcal{P}}(X,T) = \frac{\partial\widehat{R}}{\partial Z}|_{Z=0} = -\sum_{k\geq 1}\frac{Uq^{k-\frac{1}{2}}}{1-Uq^{k-\frac{1}{2}}}.$$

We have also

$$U\frac{\partial}{\partial U}(\log\widehat{R}) = -\sum_{k\geq 1}\frac{Ue^Zq^{k-\frac{1}{2}}}{1-Ue^Zq^{k-\frac{1}{2}}} + \sum_{k\geq 1}\frac{Uq^{k-\frac{1}{2}}}{1-Uq^{k-\frac{1}{2}}}. \tag{61}$$

Hence up to a constant, at $Z_0 = -2\log U$,

$$U\frac{\partial}{\partial U}(\log\widehat{R})|_{Z=Z_0} = -\frac{1}{2\pi i}\zeta(\tau, z-\frac{\tau}{2}) - \frac{1}{2\pi i}\eta(1) - \frac{1}{2}.$$

4. The algebra of Euler polynomials and Stirling numbers

So far we have computed the leading coefficient of $P_s(n,X)$. The method which has been used, theoretically can be applied to the rest of the coefficients, but practically it is almost impossible because each time the computations become more and more complicated (for the second top coefficient see [8]). But this is not the end of the story. In Theorem 1.2 if we look at the leading coefficient and replace $e_s(X)$ by V^s we find

$$\sum_{s=1}^{\infty}\alpha_s V^s = \frac{V}{2\sinh V/2} - 1 \in \mathbb{Q}(V, e^{V/2})$$

which is an elementary function. The aim of the rest of this article is to prove such a statement for the ℓth top coefficient of $P_s(n,X)$, without giving a closed form for it. This will be done in Section 5. In this section we introduce some algebraic formalism concerning Euler polynomials and Stirling numbers which will be needed later and which seems of interest in itself.

4.1. **Euler multiplication and Euler map.** Euler polynomials defined in (35) with 1 form a basis of $\mathbb{Q}[X]$ and we can ask what the structure constants defined by $e_i(X)e_j(X) = \sum_k c_{ijk}\, e_k(X)$ are. For the classical Bernoulli polynomials and Euler polynomials this structure was given by N. Nielsen [9]. Similar formula holds for $e_k(X)$.

Proposition 4.1. *For $r, s \geq 1$ we have*

$$e_r(X).e_s(X) = \frac{(r-1)!(s-1)!}{(r+s-1)!} e_{r+s}(X)$$
$$+ \sum_{i=1}^{r+s-1} \frac{B_i}{i} \left[(-1)^{r-1}\binom{s-1}{r+s-i-1} + (-1)^{s-1}\binom{r-1}{r+s-i-1}\right] e_{r+s-i}(X),$$

where B_i is the i-th Bernoulli number.

Proof. Set $\widetilde{e}_r(X) = \dfrac{e_r(X)}{(r-1)!}$ and

$$\mathcal{E}(T) = \sum_{r=1}^{\infty} \widetilde{e}_r T^{r-1} = \frac{ue^T}{1-ue^T} = -\mathfrak{B}(v+T), \tag{62}$$

where $\mathfrak{B}(x) = \dfrac{e^x}{e^x-1} = \sum_{j=1}^{\infty} (-1)^j \dfrac{B_j}{j!}\, x^{j-1}$ and $u = e^v$. Rewriting Proposition 4.1 in a new form we find

$$\widetilde{e}_r\, \widetilde{e}_s = \widetilde{e}_{r+s} + (-1)^{r-1} \sum_{j=r}^{r+s-1} \frac{B_j\, \widetilde{e}_{r+s-j}}{j\,(r-1)!\,(j-r)!} + (-1)^{s-1} \sum_{j=s}^{r+s-1} \frac{B_j\, \widetilde{e}_{r+s-j}}{j\,(s-1)!\,(j-s)!}. \tag{63}$$

Hence we have

$$\mathcal{E}(T)\mathcal{E}(Z) = \sum_{r,s\geq 1} \widetilde{e}_r(X)\widetilde{e}_s(X) T^{r-1}Z^{s-1} = \sum_{r,s=1}^{\infty} \widetilde{e}_{r+s}(X) T^{r-1} Z^{s-1}$$
$$+ \sum_{j\geq r\geq 1} (-1)^{r-1} \frac{B_j T^{r-1} Z^{j-r}}{j\,(r-1)!\,(j-r)!} \left(\sum_{q\geq 1} \widetilde{e}_q(X)\, Z^{q-1}\right) + (T \leftrightarrow Z)$$
$$= \sum_{n=1}^{\infty} \widetilde{e}_n(X) \frac{T^{n-1} - Z^{n-1}}{T-Z}$$
$$+ \mathcal{E}(Z) \sum_{j\geq 1} \frac{(-1)^{j-1} B_j}{j!} (T-Z)^{j-1} + \mathcal{E}(T) \sum_{j\geq 1} \frac{(-1)^{j-1} B_j}{j!} (Z-T)^{j-1}$$
$$= \frac{\mathcal{E}(T) - \mathcal{E}(Z)}{T-Z} + \left(\mathfrak{B}(T-Z) - \frac{1}{Z-T}\right)\mathcal{E}(Z) + \left(\mathfrak{B}(Z-T) - \frac{1}{T-Z}\right)\mathcal{E}(T)$$
$$= \mathfrak{B}(T-Z)\mathcal{E}(Z) + \mathfrak{B}(Z-T)\mathcal{E}(T).$$

But $\mathfrak{B}(x) = \frac{1}{2}(1 + \coth(x/2))$, and $\mathfrak{B}(-x) = 1 - \mathfrak{B}(x)$. Therefore to prove the proposition one has to verify the following identity

$$\frac{1}{4}\left(1 + \coth(\frac{T+v}{2})\right)\left(1 + \coth(\frac{Z+v}{2})\right) =$$
$$= -\frac{1}{4}\left(1 + \coth(\frac{T-Z}{2})\right)\left(1 + \coth(\frac{Z+v}{2})\right) + (T \leftrightarrow Z),$$

which is straightforward. □

Theorem 4.1. *There is a commutative and associative action $*$ on $\mathbb{Q}[V]$, which is defined by any of the following three properties:*

- $e^{\alpha V} * e^{\beta V} = \dfrac{e^{\alpha V+\beta} - e^{\alpha+\beta V}}{e^{\alpha} - e^{\beta}}$.

  (Here we have to consider $e^V = \sum_{i=0}^{\infty} \frac{V^i}{i!}$ *and* $*$ *acts on each monomial and comparing the coefficient of* $\alpha^i\beta^j$ *in both sides gives the definition for* $V^i * V^j$*.)*
- *The map*
  $$\phi : \mathbb{Q}[V] \to (X-1)\mathbb{Q}[X] = \mathbb{Q}[e_1, e_2, \cdots],$$
  sending $V^i \to e_{i+1}$ $(i \geq 0)$ *is a ring isomorphism.*
- *The map* $*$ *is the composite map* $Mo\psi$*, where* ψ *defines the isomorphism* $\mathbb{Q}(V) \otimes \mathbb{Q}(V) \simeq \mathbb{Q}(V_1, V_2)$ *and the map*
  $$M : \mathbb{Q}[V_1, V_2] \to \mathbb{Q}[V]$$
  is defined by
  $$M(P(V_1, V_2)) =$$
  $$= -2P(0,0) + \int_0^V P(t, V-t)dt - \sum_{k=1}^{\infty}\Big[P(-k, V+k) + P(V+k, -k)\Big]. \tag{64}$$
  Here the infinite summation in (64) *is in the sense of 'zeta summation', i.e.*
  $$\sum_{k=1}^{\infty} k^{n-1} = \zeta(1-n) = \begin{cases} -\frac{B_n}{n} & n \geq 2, \\ -\frac{1}{2} & n = 1. \end{cases} \tag{65}$$

Proof. From the first definition the commutativity turns out from the fact that the right hand side of (4.1) is symmetric respect to α and β, which proves the commutativity. For associativity we have

$$\begin{aligned}
&(e^{\alpha V} * e^{\beta V}) * e^{\gamma V} \\
&= \frac{e^{\beta}}{e^{\alpha} - e^{\beta}}(e^{\alpha V} * e^{\gamma V}) - \frac{e^{\alpha}}{e^{\alpha} - e^{\beta}}(e^{\beta V} * e^{\gamma V}) \\
&= \frac{e^{\beta}}{e^{\alpha} - e^{\beta}}\Big(\frac{e^{\alpha V+\gamma} - e^{\alpha+\gamma V}}{e^{\alpha} - e^{\gamma}}\Big) - \frac{e^{\alpha}}{e^{\alpha} - e^{\beta}}\Big(\frac{e^{\beta V+\gamma} - e^{\beta+\gamma V}}{e^{\beta} - e^{\gamma}}\Big) \\
&= S(\alpha, \beta, \gamma) + S(\beta, \gamma, \alpha) + S(\gamma, \alpha, \beta)
\end{aligned}$$

where

$$S(\alpha, \beta, \gamma) = \frac{1}{(e^{\alpha-\beta} - 1)(e^{\alpha-\gamma} - 1)} e^{\alpha V}.$$

This proves the associativity.

Now for equivalency, for $r + s \geq 1$, by definition of M, we have

$$\begin{aligned}
M(V_1^{r-1}V_2^{s-1}) &= \int_0^V t^{r-1}(V-t)^{s-1}\,dt \\
&+ \sum_{k=1}^{\infty}(-k)^{r-1}(V+k)^{s-1} + \sum_{k=1}^{\infty}(-k)^{s-1}(V+k)^{r-1} \\
&= \frac{(r-1)!(s-1)!}{(r+s-1)!}V^{r+s-1} \\
&+ \sum_{j=0}^{s}(-1)^{r-1}\frac{B_{j+r}}{j+r}\binom{s}{j}V^{s-j-1} + (r \leftrightarrow s),
\end{aligned}$$

which exactly by Proposition 4.1 equals the inverse image of $\phi^{-1}(e_r e_s)$. Finally from the above equation we have

$$
\begin{aligned}
M\left(\frac{(\alpha V_1)^{r-1}}{(r-1)!}\frac{(\beta V_2)^{s-1}}{(s-1)!}\right) &= \alpha^{r-1}\beta^{s-1}\frac{V^{r+s-1}}{(r+s-1)!} \\
&\quad + \sum_{j=r}^{r+s-1}\frac{B_j}{j!}\binom{j-1}{r-1}(-\alpha)^{r-1}\beta^{j-r}\frac{(\beta V)^{s+r-j-1}}{(s+r-j-1)!} + (\alpha\leftrightarrow\beta). \qquad (66)
\end{aligned}
$$

Summing over all $r, s \geq 1$ we find

$$
\begin{aligned}
M(e^{\alpha V_1+\beta V_2}) &= \frac{e^{\alpha V}-e^{\beta V}}{\alpha-\beta} + e^{\beta V}\sum_{j=1}^{\infty}\frac{B_j}{j!}(\beta-\alpha)^{j-1} + e^{\alpha V}\sum_{j=1}^{\infty}\frac{B_j}{j!}(\alpha-\beta)^{j-1} \\
&= \frac{e^{\alpha V}-e^{\beta V}}{\alpha-\beta} + e^{\beta V}\Big(\frac{1}{e^{\beta-\alpha}-1}-\frac{1}{\beta-\alpha}\Big) + e^{\alpha V}\Big(\frac{1}{e^{\alpha-\beta}-1}-\frac{1}{\alpha-\beta}\Big) \\
&= \frac{e^{\alpha V+\beta}-e^{\alpha+\beta V}}{e^{\alpha}-e^{\beta}} = e^{\alpha V} * e^{\beta V}.
\end{aligned}
$$

□

Definition. We define the Euler map

$$\Phi_d : \mathbb{Q}[V_1, \cdots, V_d] \to \mathbb{Q}[X]\,[T]$$

on the basis $\{V_1^{i_1}\cdots V_d^{i_d}\}$ by

$$
\Phi_d(V_1^{i_1}\cdots V_d^{i_d}) = \begin{cases} e_{i_1}(X)\cdots e_{i_d}(X)T^{i_1+\cdots i_d} & \text{if} \quad i_1,\cdots,i_d \geq 1, \\ 0 & \text{otherwise,} \end{cases} \qquad (67)
$$

where $e_i(X)$ $(i \geq 1)$ are Euler polynomials and this map is extended by linearity.

We notice that Φ_d is not injective for $d > 1$. From Lemma 3.1 (the alternative definition for Euler polynomials) for every $h \in \mathbb{Q}[V_1, \cdots, V_d]$ we can represent Φ_d as follows:

$$
\Phi_d\big(V_1\cdots V_d h(V_1,\cdots,V_d)\big) = \sum_{m_1,\cdots,m_d\geq 1} h(m_1T,\cdots,m_dT)T^dU^{m_1+\cdots m_d}. \qquad (68)
$$

Proposition 4.2. *Let $E_d = V_1\cdots V_d\mathbb{Q}[V_1,\cdots,V_d]$. Then there is an associative and commutative multiplication $\star$ on E_1 such that For $d > 1$, the map Φ_d on E_d is the composite of*

$$E_d \simeq E_1^{\otimes d} \xrightarrow{\star} E_1 \xrightarrow{\Phi_1} \mathbb{Q}[X][T].$$

Proof. Since multiplication by V gives an isomorphism $V\mathbb{Q}[V] \simeq \mathbb{Q}[V]$, hence we can define $\star$ on $V\mathbb{Q}[V]$ as $V^i \star V^j = V.(V^{i-1} * V^{j-1})$, where $*$ is already constructed in Theorem 4.1. The statement follows. □

Now by linearity one can extend the map Φ_d to $\mathbb{Q}[[V_1,\cdots,V_d]]$, the completion of $\mathbb{Q}[V_1,\cdots,V_d]$. We have the following lemma.

Lemma 4.1. *Let $\mathbf{E}_d = V_1\cdots V_d\mathbb{Q}[[V_1,\cdots,V_d]]$. Then the following two diagrams are commutative:*

$$
\begin{array}{ccc}
\mathbf{E_d} & \xrightarrow{\Phi_d} & \mathbb{Q}[X]\,[[T]] \\
{\scriptstyle (V_1+\cdots+V_d).}\big\downarrow & & \big\downarrow{\scriptstyle TX(X-1)\frac{\partial}{\partial X}} \\
\mathbf{E_d} & \xrightarrow{\Phi_d} & \mathbb{Q}[X]\,[[T]]
\end{array}
$$

and

$$\begin{array}{ccc} \mathbf{E_d} & \xrightarrow{\Phi_d} & \mathbb{Q}[X]\,[[T]] \\ {\scriptstyle V_1\frac{\partial}{\partial V_1}+\cdots+V_d\frac{\partial}{\partial V_d}}\big\downarrow & & \big\downarrow{\scriptstyle T\frac{\partial}{\partial T}} \\ \mathbf{E_d} & \xrightarrow{\Phi_d} & \mathbb{Q}[X]\,[[T]] \end{array}$$

Proof. We recall that $D = X(X-1)\frac{\partial}{\partial X}$ and $D(e_i) = e_{i+1}$, so the map $\Phi_1 : V_1\mathbb{Q}[[V_1]] \to \mathbb{Q}[X][[T]]$ satisfies

$$\Phi_1(V_1 f(V_1)) = TD\Phi_1(f(V_1)),$$

for any function $f(V_1) \in V_1\mathbb{Q}[[V_1]]$. Now by extending to d variables we find that

$$\Phi_d\big((V_1 + \cdots + V_d)f(\mathbf{V})\big) = TD\Phi_d(f(\mathbf{V})),$$

for any function $f(\mathbf{V}) \in V_1 \cdots V_d\mathbb{Q}[[\mathbf{V}]]$, where $\mathbf{V} = (V_1, \cdots, V_d)$.
One can also see this from (68). We have

$$\begin{aligned} TD\Phi_d(V_1 \cdots V_d h(\mathbf{V})) &= \sum (m_1 + \cdots m_d) h(m_1 V_1, \cdots, m_d V_d) U^{m_1 + \cdots m_d} \\ &= \Phi_d\big((V_1 + \cdots + V_d) V_1 \cdots V_d h(\mathbf{V})\big). \end{aligned}$$

The commutativity of the second diagram is obvious by definition. □

4.2. **Review of Stirling Numbers.** The number of permutations of n symbols which have exactly m cycles is called a Stirling number of the first kind and equals $\left[{n \atop m}\right]$, where $\left[{n \atop m}\right]$ given by the following generating functions:

$$x(x-1)\cdots(x-n+1) = \sum_{m=0}^{n} (-1)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix} x^m, \tag{69}$$

$$\frac{\log(1+y)^m}{m!} = \sum_{n=m}^{\infty} (-1)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix} \frac{y^n}{n!}. \tag{70}$$

For example

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!, \quad \begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)!\,(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}).$$
$$\begin{bmatrix} n \\ n-1 \end{bmatrix} = \binom{n}{2}, \quad \begin{bmatrix} n \\ n-2 \end{bmatrix} = \frac{3n-1}{4}\binom{n}{3}, \quad \begin{bmatrix} n \\ n-3 \end{bmatrix} = \binom{n}{2}\binom{n}{4}.$$

The number of ways of partitioning a set of n elements into m non-empty subsets is called a Stirling number of the second kind and denoted by $\left\{{n \atop m}\right\}$. We have the following generating functions for them:

$$x^n = \sum_{m=0}^{n} \left\{ {n \atop m} \right\} (x)_m, \tag{71}$$

$$\frac{(e^y-1)^m}{m!} = \sum_{n=m}^{\infty} \left\{ {n \atop m} \right\} \frac{y^n}{n!}, \tag{72}$$

$$\frac{z^m}{(1-z)\cdots(1-mz)} = \sum_{n=m}^{\infty} \left\{ {n \atop m} \right\} z^n, \tag{73}$$

where $(x)_m = x(x-1)\cdots(x-m+1)$ is the Pochhammer symbol. For example

$$\left\{ n \atop 1 \right\} = 1, \quad \left\{ n \atop 2 \right\} = 2^{n-1} - 1, \quad \left\{ n \atop 3 \right\} = \frac{1}{2}(3^{n-1}+1) - 2^{n-1}.$$
$$\left\{ n \atop n \right\} = 1, \quad \left\{ n \atop n-1 \right\} = \binom{n}{2}, \quad \left\{ n \atop n-2 \right\} = \binom{n}{2}\frac{3n^2-11n+10}{12}.$$

Stirling numbers, like binomial coefficients, can be defined by recursive equations:

$$\left[n+1 \atop m \right] = n\left[n \atop m \right] + \left[n \atop m-1 \right], \tag{74}$$

$$\left\{ n+1 \atop m \right\} = m\left\{ n \atop m \right\} + \left\{ n \atop m-1 \right\}. \tag{75}$$

One of the advantage of these definitions is that they hold for all integers n, m, and we have the following duality law discovered by D. Knuth

$$\left\{ n \atop m \right\} = \left[-m \atop -n \right]. \tag{76}$$

Now let $S_p(r,n)$ (where r is omitted if it equals 0), be the pth elementary symmetric function of $r, r+1, \cdots, n$. For $r=0$, from equation (69) we have $S_p(n-1) = \left[n \atop n-p \right]$. The following lemma gives a formula for $S_p(r,m)$ in terms of Stirling numbers.

Lemma 4.2. *We have for all $p, r \geq 0$,*

$$S_p(r, n-1) = \sum_{v=0}^{p} (-1)^v \left\{ v+r-1 \atop r-1 \right\} \left[n \atop n-p+v \right]. \tag{77}$$

Proof. From (69) follows

$$(x)_{r-1}(x-r)\cdots(x-n+1) = \sum_{m=0}^{n} (-1)^{n-m} \left[n \atop m \right] x^m.$$

Now by definition of $S_p(r,n)$ and from (73) we find

$$\begin{aligned}
&\sum_{p=0}^{n-1} (-1)^p S_p(r,n-1) x^{n-p-r} = (x-r)\cdots(x-n+1) \\
&= (x)_{r-1}^{-1} \sum_{m=0}^{n} (-1)^{n-m} \left[n \atop m \right] x^m = \sum_{v=0}^{\infty} \left\{ v+r-1 \atop r-1 \right\} x^{-v-r} \sum_{m=0}^{n} (-1)^{n-m} \left[n \atop m \right] x^m \\
&= \sum_{m=0}^{n} \sum_{v=0}^{\infty} (-1)^{n-m} \left\{ v+r-1 \atop r-1 \right\} \left[n \atop m \right] x^{m-v-r}.
\end{aligned}$$

Setting $m = n-p+v$ in the right hand side of the above equation we get the result. □

Lemma 4.3. *We have*

i) When m varies, for fixed p, $\left[m \atop m-p \right]$ is a polynomial of degree $2p$ of m. More precisely we have

$$\left[m \atop m-p \right] = \sum_{k=p}^{2p} c_{p,k}\,(m)_k, \quad p \geq 0, \tag{78}$$

for some rational numbers $c_{p,k}$ given by the formula

$$c_{p,k} = \sum_{j=1}^{k} \frac{(-1)^{k-j+p}}{j!\,(k-j)!} \begin{bmatrix} j \\ j-p \end{bmatrix} \tag{79}$$

ii) when r, m vary, for fixed p, the coefficient of m^{2r-p} in $\left[\begin{smallmatrix} m+1 \\ m+1-r \end{smallmatrix}\right]$ is $\dfrac{\sigma_p(r)}{2^r r!}$, where $\sigma_p(r)$ is a polynomial of degree $2p$.

iii) We have for all $p, r, i, t, t', t'' \geq 0$

$$\frac{\left[\begin{smallmatrix} m \\ m-p \end{smallmatrix}\right]}{i!(m-r-i)!} = \sum_{t=p}^{2p} \sum_{t'=0}^{t} \sum_{t''=0}^{t-t'} \frac{c_{p,t} \binom{t}{t',t''} (r)_{t'}}{(i-t'')!\,(m-i-r-t+t'+t'')!}, \tag{80}$$

Proof. We set

$$\begin{bmatrix} m \\ m-p \end{bmatrix} = \sum_{k \geq 0} c_{p,k}\,(m)_k,$$

for unknown $c_{p,k}$. First we want to show that $c_{p,k}$ is zero for $k \notin \{p, \cdots, 2p\}$. We set $C(x, y) = e^{-y}(1 + xy)^{1/x}$ and we claim that the coefficient of $x^p y^k$ in $C(x, y)$ is $(-1)^p c_{p,k}$. From (69), (70) we have

$$\begin{aligned} (1+xy)^{1/x} &= \sum_{m \geq 0} (x^{-1})_m \frac{x^m y^m}{m!} = \sum_{m \geq p \geq 0} (-1)^p \begin{bmatrix} m \\ m-p \end{bmatrix} \frac{x^p y^m}{m!} \\ &= \sum_{m,p,k \geq 0} (-1)^p k! \binom{m}{k} c_{p,k}\, x^{m-p} \frac{y^m}{m!} \\ &= \sum_{p,k \geq 0} (-1)^p c_{p,k}\, x^p \sum_{m \geq k} \frac{y^m}{(m-k)!} \\ &= e^y \sum_{p,k \geq 0} (-1)^p c_{p,k}\, x^p y^k. \end{aligned}$$

Hence we get

$$C(x,y) = \sum_{p,k \geq 0} (-1)^p c_{p,k}\, x^p y^k = e^{-\frac{xy^2}{2} u(xy)} = \sum_{d=0}^{\infty} \left(\frac{-xy^2}{2} \right)^d \frac{u(xy)^d}{d!}, \tag{81}$$

where

$$u(z) = -\frac{2}{z^2}(\log(z+1) - z) = 1 - \frac{2}{3}z + \frac{2}{4}z^2 - \frac{2}{5}z^3 + \cdots$$

Since the power of y in the right hand side of (81) is strictly bigger than the power of x (except $d = 0$), the left hand side is so and therefore $k \geq p + 1$. Moreover a general term in the right hand side is $x^{d+r} y^{2d+r}$, consequently in the left hand side $2p \geq k$.
Now the coefficient of $x^p y^k$ in $C(x, y)$ in one hand is $(-1)^p c_{p,k}$ and on the other hand by definition is $\sum_{j=1}^{k} \dfrac{(-1)^{k-j}}{j!(k-j)!} \left[\begin{smallmatrix} j \\ j-p \end{smallmatrix}\right]$, which gives the formula for $c_{p,k}$.
For the second part we write

$$\begin{bmatrix} m+1 \\ m-r+1 \end{bmatrix} = \frac{1}{2^r r!} \sum_{k=0}^{2r-1} \sigma_k(r) m^{2r-k},$$

and we want to show that for fixed p, $\sigma_p(r)$ is a polynomial of degree 2ℓ. Using the recursive equation (74)

$$\begin{bmatrix} m+1 \\ m-r+1 \end{bmatrix} - \begin{bmatrix} m \\ m-r \end{bmatrix} = m \begin{bmatrix} m \\ m-r+1 \end{bmatrix}, \tag{82}$$

the coefficient of m^{2r-p} in both sides gives us the following identity

$$\sigma_p(r) = \sum_{k=0}^{p} (-1)^{p-k} \sigma_k(r) \binom{2r-k}{p-k} + 2r \sum_{k=0}^{p-1} (-1)^{p-k-1} \sigma_k(r-1) \binom{2r-2-k}{p-k-1}, \tag{83}$$

or

$$\begin{aligned} 2r\, \sigma_{p-1}(r-1) - (2r-p+1)\, \sigma_{p-1}(r) = &\sum_{k=0}^{p-2} (-1)^{p-k} \sigma_k(r) \binom{2r-k}{p-k} \\ &+ 2r \sum_{k=0}^{p-2} (-1)^{p-k-1} \sigma_k(r-1) \binom{2r-2-k}{p-k-1}. \end{aligned}$$

For $p=1$ we find $\sigma_0(r) = 1$ and by an induction we find the result.
For the third part using the identity

$$(1+x)^m = (1+x)^r (1+x)^i (1+x)^{m-i-r}, \tag{84}$$

the power of x^t in both sides gives the following equation

$$\binom{m}{t} = \sum_{t', t'' \geq 0} \binom{r}{t'} \binom{i}{t''} \binom{m-i-r}{t-t'-t''}.$$

This equation together with the first part prove the third part. □

5. Structure of the coefficients of $P_s(n, X)$ as a function of n

In this section we prove our main theorem. It says that for each fixed $\ell \geq 0$ the ℓth top coefficient of $P_s(n, X)$ with respect to n is a finite sum of the image of some elementary functions under the Euler map $\Phi_d : \mathbb{Q}[[V_1, \cdots, V_d]] \longrightarrow \mathbb{Q}[X][[T]]$. To state the theorem we introduce some notations. Let K_d be the following ring

$$K_d := \mathbb{Q}(V_1, \cdots, V_d, e^{V_1/2}, \cdots, e^{V_d/2}) \cap \mathbb{Q}[[V_1, \cdots, V_d]]. \tag{85}$$

Then we define

$$\mathcal{K}_d := \Phi_d(K_d) \otimes \mathbb{Q}[T, T^{-1}] \subset \mathbb{Q}[X][T^{-1}, T]]$$

and we set $\mathcal{K} = \sum_{d \geq 1} \mathcal{K}_d$.

Lemma 5.1. *The space $\mathcal{K} \subset \mathbb{Q}[X][[T]]$ is closed under multiplication and differentiation.*

Proof. The multiplication follows from

$$\begin{aligned} &\Phi_d\big(F(V_1, \cdots, V_d) T^i\big) . \Phi_{d'}\big(G(V_1, \cdots, V_{d'}) T^j\big) = \\ &= \Phi_{d+d'}\big(F(V_1, \cdots, V_d) G(V_{d+1}, \cdots, V_{d+d'}) T^{i+j}\big). \end{aligned}$$

The differentiation with respect to T follows from the second diagram of Lemma 4.1 and the fact that K_d is closed under $\sum_i V_i \frac{\partial}{\partial V_i}$. □

Corollary 5.1. *Suppose $a, b \in \mathbb{Q}[X][[T]]$, such that the functions $\frac{a}{b}$ and $\frac{d}{dT}(\log b) = \frac{b'}{b}$ are elements of $\mathcal{K}$. Then $\frac{a^{(k)}}{b} \in \mathcal{K}$ for all $k \geq 0$.*

Proof. This follows by induction on k, because

$$\frac{a^{(k+1)}}{b} = (\frac{a^{(k)}}{b})' + \frac{a^{(k)}}{b}.\frac{b'}{b} \in \mathcal{K}.$$

□

Theorem 5.1. *Denote by $\rho_s^{(\ell)}(X)$ the coefficient of $n^{2s+1-\ell}$ in $P_s(n,X)$ and*

$$\mathcal{P}^{(\ell)}(X,T) = \sum_{s=0}^{\infty} \rho_s^{(\ell)}(X) T^s$$

the corresponding generating function. Then we have

$$\mathcal{P}^{(0)}(X,T) \in \mathcal{K}_1, \quad \mathcal{P}^{(\ell)}(X,T) \in \sum_{d=1}^{2\ell} \mathcal{K}_d \qquad (\ell \geq 1).$$

Remark. The proof is constructive. Indeed we show that there exist effectively computable functions

$$\Pi^{(\ell,d)}(V_1, \cdots, V_d, T) \in K_d \otimes \mathbb{Q}[T, T^{-1}],$$

such that

$$\mathcal{P}^{(\ell)}(X,T) = \sum_{d=1}^{2\ell} \Phi_d(\Pi^{(\ell,d)}). \tag{86}$$

Examples. The case $\ell = 0$ is essentially the content of section 3. Indeed from Theorem 1.2 we have

$$\mathcal{P}^{(0)}(X,T) = \mathcal{P}(X,T) = \sum_{s=1}^{\infty} \alpha_s e_s(X) T^{s-1} = T^{-1}\Phi_1\big(S(V)\big) \in \mathcal{K}_1.$$

where $S(V) = \frac{V/2}{\sinh(V/2)}$.
Similarly the case $\ell = 1$ (the second top coefficient) follows from what we did in [8] (Section 5, Theorem 5.1.1). We have

$$\mathcal{P}^{(1)}(X,T) = \Phi_2(\Pi^{(1,2)}) + \Phi_1(\Pi^{(1,1)}),$$

where

$$\Pi^{(1,2)} = T^{-1} S(V_1) S(V_2) S(V_1+V_2) \cosh(\frac{V_1 - V_2}{2}),$$
$$\Pi^{(1,1)} = T^{-1}\big(V + \frac{V^2}{4}\big) S(V) - T^{-1}\frac{V^2}{2} S'(V) - \frac{V}{6} S(V).$$

5.1. **Statements of auxiliary results.** To prove Theorem 5.1 we will need some propositions and a theorem that we state in this section. The proofs will be given in the next section. Recall the definition of $P_{s,i}(n,X)$ from (8). We can write

$$P_{s,i}(n,X) = \sum_{\ell=0}^{2s+1} \rho_{s,i}^{(\ell)}(X) n^{2s+i-\ell}, \quad (s \geq 0, i \geq 1).$$

For $i \geq 1$ we set

$$\mathcal{P}_i^{(\ell)}(X,T) := \sum_{s=0}^{\infty} \rho_{s,i}^{(\ell)}(X) T^s, \quad \mathcal{P}_0^{(\ell)} = \delta_{0\ell}. \tag{87}$$

We define also

$$R_\ell(X,T,Z) := \sum_{i=0}^{\infty} \mathcal{P}_i^{(\ell)}(X,T) \frac{Z^i}{i!}, \quad \ell \geq 0. \tag{88}$$

Proposition 5.1. *For $\ell \geq 1$ we have*

$$\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}^{(\ell)}(X,T)\frac{T^i}{i!} \in \sum_{d=1}^{2\ell}\mathcal{K}_d.$$

Proposition 5.2. *Let*

$$\mathcal{H}(X,T,Z,W) = \sum_{j,k\geq 0} Q_{k,j}(\frac{1}{W},X)\frac{T^{j+k}}{(j+k)!}(Z/W)^k, \quad W = \frac{1}{n},$$

where $Q_{k,j}(n,X)$ is defined as in (20)*. Then $\mathcal{H} = \exp(Z\hbar(X,T,W))$, where*

$$\hbar(X,T,W) = \frac{1}{W}\int_0^T \left[\left(\frac{1-U}{1-Ue^t}\right)^W -1\right] dt. \tag{89}$$

Remark. Define coefficients $q_{k,j}^{(\ell)}(X)$ by the expansion

$$Q_{k,j}(n,X) = \sum_{\ell=0}^{j} q_{k,j}^{(\ell)}(X)n^{-j-\ell}, \quad (n\to\infty), \tag{90}$$

and let $H_\ell(X,T,Z)$ be the coefficient of W^ℓ in $\mathcal{H}$. Then by definition of $\mathcal{H}$ we have

$$H_\ell(X,T,Z) = \sum_{j,k\geq 0} q_{k,j}^{(\ell)}(X)\frac{T^{j+k}Z^j}{(j+k)!}. \tag{91}$$

We note that in the previous notation in Section 3 $H_0 = H$.

Now let

$$\lambda_{p,v}(T) = \sum_{r=1}^{\infty}\frac{\sigma_p(r)\left\{{v+r-1 \atop r-1}\right\}}{2^r r!}T^r, \qquad p,v\geq 0. \tag{92}$$

where $\dfrac{\sigma_p(r)}{2^r r!}$ is the coefficient of m^{2r-p} in $\left[{m+1 \atop m-r+1}\right]$ (see Lemma 4.3).
For example we have

$$\lambda_{0,0}(T) = \sum_{r\geq 1}\frac{T^r}{2^r r!} = e^{T/2}-1,$$

$$\lambda_{0,1}(T) = \sum_{r\geq 2}\frac{r(r-1)}{2^{r+1}r!}T^r = \frac{T^2}{8}e^{T/2},$$

$$\lambda_{1,0}(T) = \sum_{r\geq 1}\frac{r(5-2r)}{3\cdot 2^r r!}T^r = (\frac{T}{2}-\frac{T^2}{6})e^{T/2}.$$

We notice that for $p+v\geq 1$, $e^{-T/2}\lambda_{p,v}(T)$ is a polynomial of degree $2(p+v)$. In fact for $p\geq 1$ $\sigma_p(r)\in r\mathbb{Q}[r]$ of degree $2p$ and $\left\{{v+r-1 \atop r-1}\right\}\in\mathbb{Q}[r]$ is a polynomial of degree $2v$ (see Lemma 4.3, equations (83), (74),(75) and the Knuth duality formula (76)). Therefore we can write

$$\lambda_{p,v}(T) = \sum_{i=1}^{p+v} a_i D^i(e^{T/2}) \in e^{T/2}\mathbb{Q}[T], \quad a_i\in\mathbb{Q}.$$

Now set

$$\Lambda(T,W) := \sum_{p,v\geq 0}(-1)^v\lambda_{p,v}(T)W^{v+p} = (e^{T/2}-1) + (\frac{T}{2}-\frac{7}{24}T^2)e^{T/2}W + \cdots . \tag{93}$$

Then we have the following theorem, in which Q is considered as known and we are trying to find $\mathfrak{R}$.

Theorem 5.2. *Let* $\mathfrak{R}(X,T,Z,W) := \sum_{\ell=0}^{\infty} R_\ell(X,T,Z)W^\ell$, *with* R_ℓ *as in* (88). *Set*

$$Q(X,T,Z,W) = \mathcal{H}(X,T,Z,W)\big(1-(X-1)\Lambda(T,W)\big).$$

Then

$$C(W,T\frac{\partial}{\partial T})\left(\mathfrak{R}(X,Z_1,T,W)Q(X,T,Z_2,W)\right)\Big|_{Z_1=T,\, Z_2=T^{-1}} = 1, \tag{94}$$

where

$$C(x,y) = e^{-y}(1+xy)^{1/x} = 1 - x\frac{y^2}{2} + x^2(\frac{y^3}{3}+\frac{y^4}{8}) + \cdots \in \mathbb{Q}[[x,y]].$$

Remark. In applying the formula we have to consider $(T\frac{\partial}{\partial T})^k$ as $T^k\frac{\partial^k}{\partial T^k}$.

From Proposition 5.2 and Theorem 5.2 one can compute inductively all $R_\ell(X,T,Z)$ only on the diagonal $Z=T$. We explain later how from this we can obtain $\mathcal{P}^{(\ell)}(X,T)$. In sections 3 we did this for $\ell=0$. In the following example we illustrate the theorem by verifying the case $\ell=0$ again. We show also how to obtain the case $\ell=1$.

Example. For $\ell=0$ the constant term with respect to W in the left hand side of (94) is $R(X,T,T)H(X,T,T^{-1})(X-(X-1)e^{T/2})$ (recall that $R=R_0, H=H_0$), so from Theorem 5.2 we have

$$R(X,T,T)H(X,T,T^{-1})(X-(X-1)e^{T/2}) = 1,$$

which is exactly equation (54) for computing the leading coefficient.
For $\ell=1$ we need to find the coefficient of W in the left hand side of (94). We have

$$\begin{aligned}&\big(1-\frac{T^2}{2}W\frac{\partial^2}{\partial T^2}\big)\left(((1-(X-1)\lambda_{0,0})(HR+HR_1+RH_1)\right)\Big|_{Z_1=T,\, Z_2=T^{-1}}\\ &\quad + (X-1)(\lambda_{0,1}-\lambda_{1,0})HRW\Big)\Big|_{Z_1=T,\, Z_2=T^{-1}} = 1.\end{aligned}$$

From this by knowing H, H_1, R and $\lambda_{0,0}, \lambda_{0,1}, \lambda_{1,0}$, one can get R_1.

5.2. **Proof of Theorem 5.1.** In this subsection and the next we prove all propositions and theorems of Section 5. It is organized as follows. First we assume Proposition 5.1 and we prove Theorem 5.1. Next we prove Proposition 5.1 using Proposition 5.2 and Theorem 5.2. Finally we prove Proposition 5.2. The next subsection is devoted to the proof of Theorem 5.2.

Proof of Theorem 5.1. Proposition 5.1 says that

$$\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}^{(\ell)}\frac{T^i}{i!} = \sum_{d=1}^{2\ell}\Phi_d(\widehat{\Pi}^{(\ell,d)}(V_1,\cdots,V_d,T)), \quad \ell \geq 1$$

where $\widehat{\Pi}^{(\ell,d)} \in K_d \otimes \mathbb{Q}[T,T^{-1}]$.
Now we set $\Pi^{(\ell,d)}(\mathbf{V},T) = T^{-1}\widehat{\Pi}^{(\ell,d)}(\mathbf{V},T)\frac{V_1+\cdots+V_d}{\exp(V_1+\cdots+V_d)-1}$, and we claim that

$$\mathcal{P}^{(\ell)}(X,T) = \sum_{d=1}^{2\ell}\Phi_d(\Pi^{(\ell,d)}(V_1,\cdots,V_d,T)) \in \sum_{d=1}^{2\ell}\mathcal{K}_d.$$

First of all from Lemma 4.1, we have the following commutative diagram:

$$\begin{array}{ccc} K_d & \xrightarrow{\Phi_d} & \mathbb{Q}[X]\,[[T]] \\ {\scriptstyle T\frac{\exp(V_1+\cdots+V_d)-1}{V_1+\cdots V_d}}\Big\downarrow & & \Big\downarrow {\scriptstyle \sum_{i=1}^{\infty}\frac{T^i}{i!}D^{i-1}} \\ K_d & \xrightarrow{\Phi_d} & \mathbb{Q}[X]\,[[T]] \end{array}$$

This diagram implies that for $\mathcal{Q}(X,T) := \sum_{d\geq 1} \Phi_d(\Pi^{(\ell,d)})$, we have

$$\sum_{i=1}^{\infty} D^{i-1}\mathcal{Q}(X,T)\frac{T^i}{i!} = \sum_{i=1}^{\infty} D^{i-1}\mathcal{P}^{(\ell)}(X,T)\frac{T^i}{i!}.$$

From Lemma 3.3 we conclude $\mathcal{P}^{(\ell)}(X,T) = \mathcal{Q}(X,T)$ and the proof of the theorem is complete. $\square$

Proof of Proposition 5.1. To prove the statement, we prove (assuming Theorem 5.2 and Proposition 5.2) the following two statements:

A. We show that $\dfrac{R_\ell(X,T,T)}{R(X,T,T)} \in \sum_{d=1}^{2\ell} \mathcal{K}_d$.

B. We prove

$$\left(\frac{R_\ell(X,T,Z)}{R(X,T,Z)} - \sum_{i=1}^{\infty} D^{i-1}\mathcal{P}^{(\ell)}(X,T)\frac{Z^i}{i!}\right)\Bigg|_{Z=T} \in \sum_{d=1}^{2\ell-1} \mathcal{K}_d.$$

These together imply the statement.
Now we start with Statement **A**. First we show that the left hand side belongs to $\mathcal{K}$ and then we give the upper bound.
From Theorem 5.2 and by expanding the compact form of equation (94), for $\ell \geq 1$ and looking for the coefficient of W^ℓ in both sides, we find

$$\begin{aligned}
&(X-(X-1)e^{T/2})H(X,T,T^{-1})R_\ell(X,T,T) = \\
&= (X-(X-1)e^{T/2}) \sum_{t,p,i,j\geq 0} \sum_{\substack{i+j=\ell-p \\ i<\ell}} c_{p,t}\, T^t \frac{\partial^t}{\partial T^t}\Big(R_i(X,Z_1,T)H_j(X,T,Z_2)\Big)\Big|_{\substack{Z_1=T, \\ Z_2=T^{-1}}} \\
&- (X-1)\sum \sum_{\substack{i+j+k=\ell-p \\ i<\ell}} c_{p-v,t}\, T^t \frac{\partial^t}{\partial T^t}\Big(R_i(X,Z_1,T)H_j(X,T,Z_2)\lambda_{k,v}(T)\Big)\Big|_{\substack{Z_1=T, \\ Z_2=T^{-1}}},
\end{aligned} \tag{95}$$

where the first sum runs over $t,p,v,i,j,k \geq 0$.
From Lemma 5.1, $\mathcal{K}$ is a ring, hence from the above equation to show that $\dfrac{R_\ell(X,T,T)}{R(X,T,T)} \in \mathcal{K}$, it is enough to prove that for $i < \ell$ and $j,t,k \geq 0$

$$\frac{R_i^{(t)}(X,Z,T)}{R(X,Z,T)}\Bigg|_{Z=T}, \quad \frac{H_j^{(t)}(X,T,Z)}{H(X,T,Z)}\Bigg|_{Z=T^{-1}}, \quad \frac{(X-1)\lambda_{k,v}^{(t)}(T)}{X-(X-1)e^{T/2}} \in \mathcal{K}. \tag{96}$$

To verify (96), we see that $\frac{R'}{R}$, $\frac{H'}{H}$ and $\frac{(X-1)e^{T/2}}{X-(X-1)e^{T/2}}$ are elements of $\mathcal{K}_1 \subset \mathcal{K}$. In fact from (46) we have

$$\frac{R'(X,Z,T)}{R(X,Z,T)}\Big|_{Z=T} = \sum_{i=1}^{\infty} D^{i-1}\mathcal{P}\frac{T^{i-1}}{(i-1)!} = \sum_{i,s=1}^{\infty} \alpha_{s+1}\, e_{s+i}\frac{T^{s+i-1}}{(i-1)!}$$
$$= T^{-1}\Phi_1\big(\sum_{i,s\geq 1} \alpha_{s+1}\frac{V^{s+i}}{(i-1)!}\big) \in T^{-1}\Phi_1(K_1) \subset \mathcal{K}_1.$$

From the first part of Proposition 3.1 we have

$$\frac{H'(X,T,Z)}{H(X,T,Z)}\Big|_{Z=T^{-1}} = T^{-1}h'(X,T)$$
$$= T^{-1}\sum_{k\geq 1} e_k(X)\frac{T^k}{k!} = T^{-1}\Phi_1\big(\sum_{k\geq 1}\frac{V^k}{k!}\big) \in \mathcal{K}_1,$$

and finally $\frac{(X-1)e^{T/2}}{X-(X-1)e^{T/2}} = \sum_{k\geq 1} e_k(X)\frac{T^{k-1}}{2^k(k-1)!} \in \mathcal{K}_1$. Hence from Corollary 5.1 it is enough to verify (96) only for $t=0$. But in that case, the first part of (96) is true by induction (since $i<\ell$) and the last one follows from the fact that $\lambda_{k,v}(T) \in \mathbb{Q}[T]e^{T/2}$, so $\dfrac{(X-1)\lambda_{k,v}(T)}{X-(X-1)e^{T/2}} \in \mathcal{K}$.

For $\dfrac{H_j}{H}$, from Proposition 5.2 we have

$$\frac{H_j(X,T,Z)}{H(X,T,Z)}|_{Z=T^{-1}} = \frac{\partial^j \exp\big(Z\hbar(X,T,W) - Zh(X,T)\big)}{\partial W^j}\Big|_{W=0,\, Z=T^{-1}}$$

(recall that by Proposition 52, $H(X,T,Z) = \exp(Zh(X,T))$, or $h(X,T) = \hbar(X,T,0)$). But we have

$$\frac{\partial^j \exp(\hbar(W) - h)}{\partial W^j}\,\Big|_{W=0} \in \mathbb{Q}[\partial\hbar, \partial^2\hbar, \cdots, \partial^j\hbar]\,\Big|_{W=0},$$

where ∂ denotes $\frac{\partial}{\partial W}$. From Theorem 5.2

$$\frac{\partial^j\hbar}{\partial W^j}|_{W=0} = \int_0^T \log^{j+1}\Big(\frac{1-U}{1-Ue^t}\Big)dt = \int_0^T\Big(\sum_{k=1}^{\infty} e_k(X)\frac{t^k}{k!}\Big)^{j+1}dt$$
$$= \int_0^T \sum_{k_1,\cdots,k_{j+1}\geq 1} e_{k_1}\cdots e_{k_{j+1}}\frac{t^{k_1+\cdots k_{j+1}}}{k_1!\cdots k_{j+1}!}\,dt$$
$$= T\Phi_{j+1}\big(\sum_{k_1,\cdots,k_{j+1}\geq 1}\frac{V_1^{k_1}\cdots V_{j+1}^{k_{j+1}}}{k_1!\cdots k_{j+1}!\,(k_1+\cdots k_{j+1}+1)}\big).$$

But

$$\sum_{k_1,\cdots,k_{i+1}\geq 1}\frac{V_1^{k_1}\cdots V_{j+1}^{k_{j+1}}}{k_1!\cdots k_{j+1}!\,(k_1+\cdots k_{j+1}+1)}$$
$$= \int_0^1 \big(e^{V_1T}-1\big)\cdots\big(e^{V_{j+1}T}-1\big)\,dT = \sum_{\mathbf{s}}\frac{\exp(V_{\mathbf{s}})-1}{V_{\mathbf{s}}} \in K_{j+1},$$

where $\mathbf{s}$ runs over all subsets of $\{1, \cdots, j+1\}$ and $V_{\mathbf{s}} = \sum_{p \in \mathbf{s}} V_p$. As a consequence $\dfrac{H_j(X,T,Z)}{H(X,T,Z)} \big|_{Z=T^{-1}} \in \mathcal{K}$, or more precisely

$$\frac{H_j(X,T,Z)}{H(X,T,Z)} \big|_{Z=T^{-1}} \in \sum_{d=1}^{j+1} \mathcal{K}_d, \quad j \geq 1. \tag{97}$$

Denote r_ℓ the upper bound for the sum of the right hand side of Claim **A**. We look at the equation (95). This maximum is obtained in the right hand side of (95), when $p = 0$ and for $j \geq 1$ we find that

$$r_\ell = \max\{r_{\ell-j} + j + 1 | j = 1, \cdots, \ell\},$$

which implies that $r_\ell = 2\ell$ and the proof of the Statement **A** is complete.
Now we prove Statement **B**. From the recursive equation (9) we have

$$\rho_{s,i+1}^{(\ell)}(X) = D\rho_{s,i}^{(\ell)}(X) - s(X-1)\rho_{s,i}^{(\ell-1)}(X) + \sum_{k=0}^{\ell} \sum_{r=0}^{s} \rho_{r,i}^{(k)}(X)\rho_{s-r}^{(\ell-k)}(X). \tag{98}$$

By (87) this is equivalent to

$$\begin{aligned} \mathcal{P}_{i+1}^{(\ell)}(X,T) &= D\mathcal{P}_i^{(\ell)}(X,T) - (X-1)\Theta\mathcal{P}_i^{(\ell-1)}(X,T) \\ &+ \sum_{k=0}^{\ell} \mathcal{P}_i^{(k)}(X,T)\mathcal{P}^{(\ell-k)}(X,T), \end{aligned} \tag{99}$$

where $\Theta = T\frac{\partial}{\partial T}$. Then by (88) we find

$$\begin{aligned} \frac{d}{dZ} R_\ell(X,T,Z) &= DR_\ell(X,T,Z) - (X-1)T\frac{d}{dT}R_{\ell-1}(X,T,Z) \\ &+ \sum_{k=0}^{\ell} \mathcal{P}^{(\ell-k)}(X,T)R_k(X,T,Z). \end{aligned}$$

Hence if we set $\mathfrak{P}(X,T,W) := \sum_{\ell=0}^{\infty} \mathcal{P}^{(\ell)}(X,T)W^\ell$ we have

$$\left(\frac{d}{dZ} - D - (X-1)W\Theta\right)\mathfrak{R}(X,T,Z,W) = \mathfrak{P}(X,T,W)\mathfrak{R}(X,T,Z,W). \tag{100}$$

Therefore $\mathfrak{R} = \exp(F)$ for some F which satisfies

$$\left(\frac{d}{dZ} - D - (X-1)W\Theta\right)F(X,T,Z,W) = \mathfrak{P}(X,T,W),$$

with the right hand side independent of Z. We write

$$F(X,T,Z,W) = \sum_{k=0}^{\infty} F_k(X,T,Z)W^k.$$

Then we have

$$(\frac{d}{dZ} - D)F_0 = \mathcal{P}^{(0)} = \mathcal{P}, \tag{101}$$

$$(\frac{d}{dZ} - D)F_k = \mathcal{P}^{(k)} + (X-1)\Theta F_{k-1} \quad (k \geq 1). \tag{102}$$

Equation (101) can easily be solved. We note that $\mathfrak{R}(X,T,0,0)=1$, so $F_0(X,T,0)=0$, and we have

$$F_0(X,T,Z)=\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}(X,T)\frac{Z^i}{i!}, \tag{103}$$

and for (102) using the identity

$$\binom{i}{j_1,\cdots,j_s}=\binom{i-1}{j_1-1,j_2,\cdots,j_s}+\cdots+\binom{i-1}{j_1,\cdots,j_{s-1},j_s-1}+\binom{i-1}{j_1,\cdots,j_s},$$

one can check directly that the solution of (102) is given by

$$\begin{aligned} F_k(X,T,Z) &= \sum_{i=1}^{\infty} D^{i-1}\mathcal{P}^{(k)}(X,T)\frac{Z^i}{i!}\\ &+\sum_{i=1}^{\infty}\sum_{s=1}^{k}\sum_{j_1,\cdots,j_s\geq 1}\frac{1}{s!}\binom{i-1}{j_1,\cdots,j_s}e_{j_1}\cdots e_{j_s}D^{i-\mathbf{j}-1}\Theta^s\mathcal{P}^{(k-s)}\frac{Z^i}{i!}, \end{aligned}\tag{104}$$

where $\mathbf{j}=j_1+\cdots+j_s$.
Since $k<\ell$ we have by induction

$$\mathcal{P}^{(k)}(X,T,T)=\sum_{d=1}^{2k}\Phi_d(\Pi^{(k,d)}),\quad \Pi^{(k,d)}\in K_d\otimes\mathbb{Q}[T,T^{-1}]. \tag{105}$$

From Lemma 4.1 and (105) we have

$$\begin{aligned} &e_{j_1}\cdots e_{j_s}D^{i-\mathbf{j}-1}\Theta^s\mathcal{P}^{(k-s)}(X,T)T^{i-1}=\\ &=\sum_{d=1}^{2(k-s)}\sum_{s=1}^{k}\Phi_{d+s}\Big(V_{d+1}^{j_1}\cdots V_{d+s}^{j_s}\big(V_1+\cdots+V_d\big)^{i-\mathbf{j}-1}\big(\mathbf{V}\frac{\partial}{\partial\mathbf{V}}+\Theta\big)^s\Pi^{(k-s,d)}\Big), \end{aligned}$$

where $\mathbf{V}\frac{\partial}{\partial\mathbf{V}}=V_1\frac{\partial}{\partial V_1}+\cdots V_d\frac{\partial}{\partial V_d}$. Hence from (104) and the definition of Φ_d we have

$$\begin{aligned} F_k(X,T,T) &= \sum_{i=1}^{\infty} D^{i-1}\mathcal{P}^{(k)}\frac{T^i}{i!}\\ &+T\sum_{s=1}^{k}\sum_{d=1}^{2(k-s)}\Phi_{d+s}\Big(\frac{\exp(V_1+\cdots+V_{d+s})-1}{s!\big(V_1+\cdots+V_d\big)}\big(\mathbf{V}\frac{\partial}{\partial\mathbf{V}}+\Theta\big)^s\Pi^{(k-s,d)}\Big). \end{aligned}$$

Hence it follows for $k<\ell$, that $F_k(X,T,T)\in\mathcal{K}$ and the upper bound is $2k-1$ and consequently by Lemma 5.1, $\mathbb{Q}[F_1,\cdots,F_{\ell-1}]\subset\mathcal{K}$.
For $k=\ell$, the same argument gives

$$[F_\ell(X,T,Z)-\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}^{(\ell)}(X,T)\frac{Z^i}{i!}]\Big|_{Z=T}\in\mathcal{K}. \tag{106}$$

with the upper bound $d=2\ell-1$. Hence Claim **B** is equivalent to

$$\frac{R_\ell(X,T,T)}{R(X,T,T)}-F_\ell(X,T,T)\in\mathcal{K}$$

with the upper bound $2\ell-1$. But we have

$$\frac{R_\ell(X,T,T)}{R(X,T,T)}=\frac{\partial^\ell\mathfrak{R}/R}{\partial W^\ell}\Big|_{W=0}=\frac{\partial^\ell\exp(F-F_0)}{\partial W^\ell}\Big|_{W=0}=F_\ell+G,$$

where $G \in \mathbb{Q}[F_1, \cdots, F_{\ell-1}] \subset \mathcal{K}$ with upper bound $2\ell - 1$. This completes the proof of Proposition 5.1. $\square$

Proof of Proposition 5.2. From equation (20), $\mathcal{H}$ satisfies the following homogeneous linear differential equation

$$\left[\frac{\partial}{\partial T} - X(X-1)\frac{\partial}{\partial X} - ZT(X-1) - W(X-1)Z\frac{\partial}{\partial Z}\right]\mathcal{H} = 0. \tag{107}$$

It follows that $\mathcal{H}(X,T,Z,W) = \exp(Z\hbar(X,T,W))$, for some $\hbar$ which satisfies the following differential equation

$$\frac{\partial}{\partial T}\hbar - X(X-1)\frac{\partial}{\partial X}\hbar - W(X-1)\hbar = T(X-1).$$

Then we write $\hbar(X,T,W) = \sum_{i=0}^{\infty} h_i(X,W)\frac{T^i}{i!}$. But by definition of $\mathcal{H}$ we have $\mathcal{H} = 1 + \frac{X-1}{2}T^2 Z + \cdots$ which follows $h_0 = h_1 = 0$ and from the above equation it turns out

$$h_2 = X - 1 = e_1, \quad h_{i+1} = (D + We_1)h_{i-1}, \quad i \geq 2,$$

or $h_i = \frac{1}{W}(D + We_1)^i(1)$, for $i \geq 1$. Finally we have

$$\begin{aligned} \hbar' &= \frac{1}{W}\sum_{i=1}^{\infty}\big(D + We_1\big)^i(1)\frac{T^i}{i!} = \frac{1}{W}\left[\exp\big(W\sum_{i=1}^{\infty} e_i\frac{T^i}{i!}\big) - 1\right] \\ &= \frac{1}{W}\left[\left(\frac{1-U}{1-Ue^T}\right)^W - 1\right]. \end{aligned} \tag{108}$$

Therefore (89) follows. $\square$

5.3. **Proof of Theorem 5.2.** In this final section we prove Theorem 5.2. The recursive equation (10), which we repeat for convenience, says

$$\sum_{k=1}^{s}\sum_{i=0}^{k} E_{k,i}(n,X)P_{s-k,i}(n,X) = 0, \tag{109}$$

where $E_{k,i}(n,X)$ is defined in (22).
We define coefficients $a_{k,j}^{(2k-\ell)}(X)$ by the expansion

$$E_{k,i}(n,X) = \sum_{\ell=0}^{2k-i-1} a_{k,i}^{(2k-\ell)}(X)\, n^{2k-i-\ell}, \quad (n \to \infty),$$

The coefficient of $n^{2s-\ell}$ in the left hand side of (109) is

$$\sum_{k=1}^{s}\sum_{p=0}^{\ell}\sum_{i=0}^{k} a_{k,i}^{(2k-\ell+p)}(X)\rho_{s-k,i}^{(2s-2k-p)}(X). \tag{110}$$

Hence by definition of $\mathcal{P}_i^{(\ell)}$ we have

$$\sum_{\ell=0}^{\infty}\sum_{p=0}^{\ell}\sum_{i=0}^{\infty}\left(\sum_{k=i}^{\infty} a_{k,i}^{(2k-\ell+p)}(X)T^k\right)\mathcal{P}_i^{(p)}(X,T) = 1. \tag{111}$$

We show that for fixed i

$$\sum_{\ell,k=0}^{\infty} a_{k,i}^{(2k-\ell)}(X)T^k W^\ell = C(W, T\frac{\partial}{\partial T})\bigg(\big(1 - (X-1)\Lambda(T,W)\big)\mathcal{H}\,\frac{T^i}{i!}\bigg)\bigg|_{Z=T^{-1}}. \tag{112}$$

Note that one can consider this case as an special case of Theorem 5.2 when all $\rho_{s,i}^{(\ell)} = 1$ $(s, i, \ell \geq 0)$.
To do this we need to write $a_{k,i}^{(2k-p)}$ as a sum of terms which depend on $k-i$ and i. The function $E_{k,i}(n, X)$ in equation (22) depends on n through the quantities $\binom{n-r}{i}$, $S_r(n)$ and $\mathcal{H}_{n-i-r,k-i-r}(n, X)$ $(0 \leq r \leq k-i)$. We expand the first of these by

$$\binom{n-r}{i} = \frac{1}{i!}\sum_{p=0}^{i}(-1)^p S_p(r, r+i-1)\, n^{i-1},$$

where coefficients can be expressed in terms of Stirling numbers of the first and second kind by (77). Using (19) and equation (90) we expand $\mathcal{H}_{m,k}$ in terms of $q_{k,j}(X)$. Finally $S_r(n)$ by the second part of Lemma 4.3 have the following expansion

$$S_r(n) = \sum_{p=0}^{2r-1}\frac{\sigma_p(r)}{2^r r!}n^{2r-p} \quad r \geq 1.$$

Therefore from (22) we have

$$\begin{aligned}
a_{k,i}^{(2k-\ell)}(X) &= \sum_{j=1}^{k-i}\sum_{p=0}^{\ell}(-1)^p\frac{S_p(k+j-1)q_{k-i,j}^{(\ell-p)}(X)}{i!\,(k-i+j)!}\\
&- (X-1)\sum_{r=1}^{k-i}\sum_{j=1}^{k-i-r}\sum_{p+p'+p''=\ell}(-1)^p\frac{\sigma_{p'}(r)}{2^r r!}\,\frac{S_p(r, k+j-1)}{i!\,(k-i-r+j)!}q_{k-i-r,j}^{(p'')}(X)\\
&= \sum_{j=1}^{k-i}\sum_{p=0}^{\ell}(-1)^p\begin{bmatrix}k+j\\k+j-p\end{bmatrix}\frac{q_{k-i,j}^{(\ell-p)}(X)}{i!\,(k-i+j)!}\\
&- (X-1)\sum_{r=1}^{k-i}\sum_{j=1}^{k-i-r}\sum_{p+p'+p''=\ell}\sum_{v=0}^{p}(-1)^{v+p}\frac{\sigma_{p'}(r)\left\{{v+r-1 \atop r-1}\right\}}{2^r r!}\,\frac{\left[{k+j \atop k+j-p+v}\right]}{i!\,(k-i-r+j)!}q_{k-i-r,j}^{(p'')}(X).
\end{aligned} \tag{113}$$

Expanding $\left[{k+j \atop k+j-p+v}\right]$ by (80) and forming a generating function we can write this as

$$\begin{aligned}
&\sum_{k=0}^{\infty}a_{k,i}^{(2k-\ell)}(X)T^k =\\
&\sum_{k,j,p,t,t'\geq 0}(-1)^p\binom{t}{t'}\frac{c_{p,t}\,q_{k-i,j}^{(\ell-p)}(X)T^k}{(i-t')!\,(k+j-i-t+t')!}\\
&- (X-1)\sum_{k,j,p,p',v,t,t',t''\geq 0}(-1)^p\binom{t}{t',t''}\frac{c_{p-v,t}\,\lambda_{p',v}^{(t')}(T)\,q_{k-i,j}^{(\ell-p-p')}(X)T^{k+t'}}{(i-t'')!\,(k+j-i-t+t'+t'')!}\\
=&\sum_{p=0}^{\ell}\sum_{t,t'\geq 0}(-1)^p\binom{t}{t'}c_{p,t}\,H_{\ell-p}^{(t-t')}(X,T,Z)\frac{T^{i+t-t'}}{(i-t')!}\Big|_{Z=T^{-1}}\\
&- (X-1)\sum_{p+p'=\ell}\sum_{v,t,t',t''\geq 0}(-1)^p\binom{t}{t',t''}c_{p-v,t}\,\lambda_{p',v}^{(t)}(T)\,H_{\ell-p-p'}^{(t-t'-t'')}(X,T,Z)\frac{T^{i+t-t''}}{(i-t'')!}\Big|_{Z=T^{-1}},
\end{aligned}$$

where $\lambda_{p,v}$ and H_ℓ are as in (92), (91) respectively.
Now multiplying by W^ℓ and summing over $\ell \geq 0$ and using the Leibniz rule we find

$$\sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty} a_{k,i}^{(2k-\ell)}(X)T^kW^\ell$$
$$= \sum_{\ell=0}^{\infty}\sum_{p=0}^{\ell}\sum_{t=0}^{2p}(-1)^p c_{p,t}\,\frac{\partial^t}{\partial T^t}\big(H_{\ell-p}(X,T,Z)\frac{T^i}{i!}\big)W^\ell T^t\,\Big|_{Z=T^{-1}}$$
$$(114)\qquad - (X-1)\sum_{\ell=0}^{\infty}\sum_{p=0}^{\ell}\sum_{v=0}^{p}\sum_{t=0}^{2p}(-1)^p c_{p-v,t}\,\frac{\partial^t}{\partial T^t}\left(H_{\ell-p-p'}(X,T,Z)\lambda_{p',v}(T)\frac{T^i}{i!}\right)W^\ell T^t\Big|_{Z=T^{-1}}.$$

We note that except the term $\frac{T^i}{i!}$ the other terms are independent of i. From definitions of $\mathcal{H}$ and Λ we get

$$\sum_{\ell,k=0}^{\infty} a_{k,i}^{(2k-\ell)}(X)T^kW^\ell = \sum_{\ell=0}^{\infty}\sum_{p=0}^{\ell}\sum_{t=0}^{2p}(-1)^p c_{p,t}\,W^pT^t\,\frac{\partial^t}{\partial T^t}\big(\mathcal{H}\,\frac{T^i}{i!}\big)\Big|_{Z=T^{-1}}$$
$$- (X-1)\sum_{\ell=0}^{\infty}\sum_{p=0}^{\ell}\sum_{t=0}^{2p}(-1)^p c_{p,t}\,W^pT^t\,\frac{\partial^t}{\partial T^t}\left(\Lambda\,\mathcal{H}\,\frac{T^i}{i!}\right)\Big|_{Z=T^{-1}}.$$

Hence equation (112) follows from the definition of $C(x,y)$.
Now in order to prove the statement of the theorem we multiply equation (111) by W^ℓ and sum over all $\ell \geq 0$. For fixed i we replace inside the parenthesis of (111) with the right hand side of (112) . Also replacing T by Z_1 in $\mathcal{P}_i(X,T)$ and passing through the parenthesis and then summing over all $i \geq 0$ the statement of theorem immediately follows from the definition of $\mathfrak{R}$.

Acknowledgment This paper is based on the main part of my PhD thesis. Here I would like to express my gratitude to my advisor Don Zagier for introducing the problem and scientific advise. As a PhD student I was financially supported by the Max Planck institute for mathematics in Bonn. Currently I am a post doc at IMPA Rio de Janeiro. I would like to thank these institutes for their hospitality and support. I would also like to thank H. Movasati for reading the draft and made useful suggestions.

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