

LOCAL ZETA FACTORS AND GEOMETRIES UNDER $\text{Spec } \mathbf{Z}$

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ABSTRACT. The first part of this note shows that the odd period polynomial of each Hecke cusp eigenform for full modular group produces via Rodriguez–Villegas transform ([Ro–V]) a polynomial satisfying the functional equation of zeta type and having nontrivial zeros only in the middle line of its critical strip. The second part discusses Chebyshev lambda–structure of the polynomial ring as Borger’s descent data to \mathbf{F}_1 and suggests its role in possible relation of $\Gamma_{\mathbf{R}}$ –factor to “real geometry over \mathbf{F}_1 (cf. also [CoCons2]).

Introduction

In his influential seminar talk [Se], Jean–Pierre Serre stated precise conjectures about the structure of local factors of zeta functions of algebraic varieties over arithmetic rings. In particular, he defined the local factors at complex, resp. real, archimedean completions of the base as multiplicative combinations of gamma functions involving Hodge numbers. (Of course, local factors at finite primes since Weil and Grothendieck were treated in terms of Galois representations on cohomology as characteristic polynomials of Frobenii.)

In my seminar talks [Ma1] dedicated to the geometry and arithmetics over Jacques Tits’ mythical “field with one element \mathbf{F}_1 ” I suggested the existence of respective local zetas “in characteristic one” and noticed that Riemann’s gamma–factor at the infinite prime looks like such a local factor in characteristic one of infinite–dimensional projective space $\mathbf{P}_{\mathbf{F}_1}^{\infty}$ appropriately regularized.

More precisely, in [Ma1] I defined the zeta function of $\mathbf{P}_{\mathbf{F}_1}^k$ as

$$(2\pi)^{-(k+1)} s(s-1) \dots (s-k). \quad (0.1)$$

On the other hand, Deninger ([De]) represented the basic Γ –factor at (complex) arithmetical infinity as the infinite determinant of complex Frobenius map and a regularized product

$$\Gamma_{\mathbf{C}}(s)^{-1} := \frac{(2\pi)^s}{\Gamma(s)} = \prod_{n \geq 0} \frac{s+n}{2\pi}. \quad (0.2)$$

Comparing (0.1) to (0.2), I suggested that this gamma-factor, with changed sign of s , might be imagined as the zeta-function of the infinite dimensional projective space over \mathbf{F}_1 . I did not discuss the problem of a similar interpretation of the real gamma-factor.

After 1992, there was a growing body of definitions and studies of \mathbf{F}_1 -geometries, cf. surveys and comprehensive bibliography in [Lo2], [Ma2]. In particular, Ch. Soulé in [So] put on a firm ground my heuristics about local zeta factors over \mathbf{F}_1 . In particular, natural factors of zetas of \mathbf{F}_1 -schemes turned out to be polynomials in s , satisfying a functional equation expressing their symmetry wrt a map $s \mapsto c - s$. In the main text, I will use for such polynomials a generic name “zeta polynomials”, complementing their description by the requirement that nontrivial zeros must lie on the vertical line at the middle of critical strip, cf. Theorem 1.3 below.

For other insights about \mathbf{F}_1 , see [KaS], [CoCons1] and the description of A. Smirnov’s work in [LeBr], and about Deninger’s program see [CoCons2].

However, the bridges between characteristics zero and one, and in particular the $\mathbf{P}_{\mathbf{F}_1}^\infty$ -heuristics about (0.2) still remain to a considerable degree elusive.

In this short note, I contribute additional strokes to this mystery.

In Section 1, I show that each cusp form f for $PGL(2, \mathbf{Z})$ which is eigenform for all Hecke operators, besides the usual p -factors of its Mellin transform, produces one more polynomial that looks like “local zeta factor in characteristic one”. This polynomial is obtained from the odd period polynomial of f in the same formal way as the Hilbert polynomial of a graded algebra is produced from its Poincaré series, see [Ro-V]. Formulas (1.7) and (1.8) below suggest that this formalism can be considered as a discrete version of the Mellin transform as well.

For analogies with zetas and geometric interpretations of the latter, cf. also [Go].

In Section 2, I suggest how an expected gamma-bridge between characteristics zero and one could take into account the fact that in Serre’s picture gamma-factors corresponding to real and complex infinite arithmetic primes are different. To this end, I appeal to J. Borger’s identification of lambda-structures on schemes with descent data to \mathbf{F}_1 ([Bo], [LeBr]), and to the idea that Habiro rings are lifts to $\text{Spec } \mathbf{Z}$ of “rings of analytic functions” in characteristic one suggested in [Ma2]. Then it turns out that two different lambda-structures on the polynomial ring, the toric one and the Chebyshev one, faithfully reflect the difference between complex and real analytic geometry in characteristic one.

Notice that lambda-structures naturally appear in several contexts, related to zetas: see e. g. [CoCons2], [Na] and [Ra]. It would be interesting to include Borger's philosophy into these contexts as well.

1. Zeta polynomials from cusp forms

1.1. Period polynomials and period functions. Here we are considering modular forms with respect to $PSL(2, \mathbf{Z})$, k is a positive even weight; $w := k - 2$; S_k denotes the space of cusp forms, M_k is the space all modular forms of weight k .

Period polynomials for cusp forms are defined by:

$$r_f(z) := \int_0^{i\infty} f(\tau)(\tau - z)^{k-2} d\tau, \quad r_f^\pm(z) := \frac{r_f(z) \pm r_f(-z)}{2}.$$

The following more general formula is valid also for Eisenstein series: if $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z} \in M_k$, define its *Eichler integral* by

$$\mathcal{E}_f(z) := \int_z^{i\infty} (f(\tau) - a_0)(\tau - z)^{k-2} d\tau = -\frac{(k-2)!}{(2\pi i)^{k-1}} \sum_{n=1}^{\infty} \frac{a_n}{n^{k-1}} e^{2\pi n z}$$

and then define *period function* by

$$r_f(z) := \mathcal{E}_f(z) - z^{k-2} \mathcal{E}_f(-1/z), \quad r_f^\pm(z) := \frac{r_f(z) \pm r_f(-z)}{2}.$$

If f is not cusp form, then $r_f(z) \in z^{-1} \mathbf{C}[z]$.

1.2. Spaces of period functions. If $g \in PSL(2, \mathbf{Z})$, $g(z) = \frac{az+b}{cz+d}$, the right action $|_w$ of g on the space V_w of polynomials r of degree $\leq w$ is defined by

$$(r|_w g)(z) := (cz+d)^w r(g(z)).$$

Let $S(z) = -1/z$, $U(z) = 1 - 1/z$ and

$$Y_w := \{ r \in V_w \mid r|_w(1+S) = r|_w(1+U+U^2) = 0 \}. \quad (1.1)$$

For $f \in S_k$, we have $r_f(z) \in Y_w$. Let Y_w^\pm mean the respective subspaces of even/odd polynomials.

It is well known (Eichler–Shimura) that the map $r^- : f \mapsto r_f^-(z)$ defines an isomorphism $S_k \rightarrow Y_w^-$, whereas r^+ defines an embedding of codimension one $S_k \rightarrow Y_w^+$.

Recently it was proved ([ConFaIm]) that if $f \in S_k$ is a Hecke eigenform, then

$$U_f(z) := \frac{r_f^-(z)}{z(z^2 - 4)(z^2 - 1/4)(z^2 - 1)^2} \quad (1.2)$$

is a polynomial without real zeros whose complex zeros all lie on the unit circle. Clearly, its degree is $e := w - 10$.

1.3. Theorem. *Fix an integer $d > e = w - 10$ and put*

$$P_f(z) := \frac{U_f(z)}{(1 - z)^d}. \quad (1.3)$$

There exists a polynomial $H_f(x) \in \mathbf{C}[x]$ of degree $d - 1$ such that

$$P_f(z) = \sum_{n=0}^{\infty} H_f(n) z^n$$

for $|z| < 1$. This polynomial satisfies the functional equation

$$H_f(x) = (-1)^{d-1} H(-d + e - x) \quad (1.4)$$

and it vanishes at $x = -1, \dots, -d + e + 1$. All its remaining zeros lie on the vertical line $\operatorname{Re} x = -(d - e - 1)/2$.

Proof. This is a direct application of the Proposition in sec. 3 of [Ro–V] (due in more general form to Popoviciu), and of its Corollary. One condition for applicability of this Proposition is ensured by the theorem about zeros of (1.2) from [CoFaIm]. We have only to check the functional equation (9) from this Proposition, i. e. the identity

$$P_f(1/z) = (-1)^d z^{d-e} P_f(z). \quad (1.5)$$

Rewriting (1.5) as

$$\frac{U_f(1/z)}{(1 - 1/z)^d} = (-1)^d z^{d-e} \frac{U_f(z)}{(1 - z)^d}$$

one sees that it is equivalent to

$$U_f(1/z) = z^{-e}U_f(z)$$

that is, in view of (1.2),

$$\frac{r_f^-(1/z)}{z^{-1}(z^{-2}-4)(z^{-2}-1/4)(z^{-2}-1)^2} = z^{10-w} \frac{r_f^-(z)}{z(z^2-4)(z^2-1/4)(z^2-1)^2}. \quad (1.6)$$

Now, from $r|_w(1+S) = 0$ it follows that $r_f^-(1/z) = -r_f^-(-1/z) = z^{-w}r_f^-(z)$. Inserting this into (1.6), we finally get (1.5).

1.4. Remarks. a). In [ER], it was proved that all zeros of the full period polynomial of a Hecke cusp form lie on the unit circle. Similarly, all zeros of $zr_f(z)$ for Eisenstein Hecke series lie on the unit circle.

However, I was unable to fit these cases into the framework of the Rodriguez–Villegas construction, because the analog of the functional equation (1.5) seemingly fails for the complete period polynomial.

b). I use the generic catchword "zeta polynomials" for polynomials of one variable satisfying a version of functional equation such as (1.4) and "Riemann conjecture". In [Ro–V], it was in particular proved that Hilbert polynomials of certain graded rings are zeta polynomials. Golyshev ([Go]) considered rings of homogeneous functions on Fano and Calabi–Yau varieties with respect to anticanonical or related projective embeddings and found interesting geometric correlates of these results.

Moreover, comparing the formula

$$H_f(n) = \frac{1}{2\pi i} \int_{\gamma} P_f(z) z^{-(n+1)} dz \quad (1.7)$$

(where γ is a small contour around zero) with the Mellin transform

$$Z_f(s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^{i\infty} f(z) \left(\frac{z}{i}\right)^{s-1} d\left(\frac{z}{i}\right) \quad (1.8)$$

one sees a considerable formal analogy: morally, H_f is "discrete Mellin transform of P_f ".

In particular, the argument n of H_f corresponds to the classical $-s$: this is consistent with observations in [Ro–V] and [Go].

However, finding an appropriate geometric living space for zeta polynomials H_f associated with Hecke cusp forms seemingly requires more general realm of “geometries under $\text{Spec } \mathbf{Z}$ ”. The problem is that in most versions of \mathbf{F}_1 -geometries those zeta polynomials that appear as zeta functions of motives over \mathbf{F}_{1^n} have only integer zeros: cf. e. g. [Lo1]. To the contrary, our H_f seem to come from some non-Tate motives and geometric objects lying below $\text{Spec } \mathbf{Z}$ but not over \mathbf{F}_1 . I expect that they arise from the levels below $\text{Spec } \mathbf{Z}$ to which such moduli stacks as $\overline{M}_{1,n}$ can be descended.

c) Notice finally that period polynomials appear also in the studies of Galois action on the Grothendieck–Teichmüller groupoid: see [Sch], [Hai] and [Po] and references therein about their role in Hodge realisations. One can guess that a special role of period polynomials of Hecke eigenforms will become clearer in the light of étale setting.

2. Habiro Lambda–Rings

2.1. Habiro rings. The Habiro ring \mathcal{H} of one variable over \mathbf{Z} is defined as the projective limit of quotient rings $\mathbf{Z}[q]/(f(q))$ where $f(q)$ runs over the multiplicative set of monic polynomials whose all roots are roots of unity. This ring was introduced and studied in [Hab], and in [Ma2] it was suggested to consider it as “*the ring of analytic functions on \mathbf{G}_m lifted from \mathbf{F}_1 .*” In fact, $\mathbf{Z}[q]$ is naturally embedded into the Habiro completion \mathcal{H} , and q becomes invertible there, so that \mathcal{H} can be also defined as a completion of $\mathbf{Z}[q, q^{-1}]$. One can extend this definition to the case of several invertible variables that is, functions on tori.

2.2. Lambda–rings. J. Borger developed in [Bo] the idea to interpret Grothendieck lambda–structures on schemes as general descent data to \mathbf{F}_1 . It is therefore natural to expect that the Habiro ring admits a natural lambda–structure.

Here we will be concerned only with commutative rings A flat over \mathbf{Z} in which case a lambda–structure can be considered simply as a system of commuting lifts of Frobenii: ring homomorphisms $\psi^p : A \rightarrow A$ for each prime p such that $\psi^p(x) \equiv x^p \pmod{pA}$ for all $x \in A$ and $\psi^{p_1}\psi^{p_2} = \psi^{p_2}\psi^{p_1}$. In particular, we can define $\psi^k : A \rightarrow A$ for all positive integers k by multiplicativity.

The most natural lambda–structure on $\mathbf{Z}[q]$ and $\mathbf{Z}[q, q^{-1}]$ is determined by $\psi^k(q) = q^k$, and since it is compatible with the projective limit over the system of

cyclotomic polynomials in q , it is inherited by the Habiro ring. We will call this structure *toric one*.

However, the polynomial ring $\mathbf{Z}[r]$ admits one more lambda-structure discovered by Clauwens ([Cl]). In this structure,

$$\psi^k(r) := T_k(r)$$

where T_k is the k -th Chebyshev polynomial. Our next result describes a subring $\mathcal{H}_0 \subset \mathcal{H}$ which can be endowed with Chebyshev lambda-structure.

2.3. Proposition. (i) Consider in the Habiro ring \mathcal{H} the subring \mathcal{H}_0 defined as the completion of the polynomial subring $\mathbf{Z}[r]$, where

$$r := 1 + q + \sum_{n=1}^{\infty} q^n \cdot (1 - q) \dots (1 - q^n). \quad (2.1)$$

This subring is invariant with respect to the standard lambda-structure ψ^k , which induces on this subring, in terms of the coordinate r , the Chebyshev lambda-structure.

(ii) \mathcal{H}_0 is strictly smaller than \mathcal{H} .

Proof. (i) In \mathcal{H} , we have the convergent expression for q^{-1} (see [Hab], Prop. 7.1):

$$q^{-1} = 1 + \sum_{n=1}^{\infty} q^n \cdot (1 - q) \dots (1 - q^n)$$

Hence $r = q + q^{-1}$. Moreover, using one of the definitions of Chebyshev polynomials, we see that

$$\psi^k(r) = q^k + q^{-k} = T_k(q + q^{-1}) = T_k(r).$$

(ii) In order to see that \mathcal{H}_0 is strictly smaller than \mathcal{H} , we can use the following result due to Habiro. Any element of \mathcal{H} determines a function on the set of roots of unity μ_{∞} with values in $\mathbf{Z}[\mu_{\infty}]$, and the resulting map

$$\mathcal{H} \rightarrow \text{Map}(\mu_{\infty}, \mathbf{Z}[\mu_{\infty}])$$

is an embedding (see [Hab]). The element q corresponds to the tautological map $\mu_{\infty} \rightarrow \mathbf{Z}[\mu_{\infty}]$.

Then all elements of $\mathbf{Z}[r]$ become functions invariant under the involution $\zeta \rightarrow \zeta^{-1}$ of μ_∞ and their values are as well invariant. This property holds after the completion. Hence $q \notin \mathcal{H}_0$.

Notice that for each complex embedding of μ_∞ and any $\eta \in \mu_\infty$, $\eta + \eta^{-1}$ is real. This is why we referred to “real analytic geometry over \mathbf{F}_1 .”

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