

KAZHDAN-LUSZTIG CONJECTURES AND SHADOWS OF HODGE THEORY

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ABSTRACT. We give an informal introduction to the authors' work on some conjectures of Kazhdan and Lusztig, building on work of Soergel and de Cataldo-Migliorini. This article is an expanded version of a lecture given by the second author at the Arbeitstagung in memory of Frederich Hirzebruch.

1. INTRODUCTION

It was a surprise and honour to be able to speak about our recent work at the Arbeitstagung in memory of Hirzebruch. These feelings are heightened by the fact that the decisive moments in the development of our joint work occurred at the Max-Planck-Institut in Bonn, which owes its very existence to Hirzebruch. In the following introduction we have tried to emphasize the aspects of our work which we believe Hirzebruch would have most enjoyed: compact Lie groups and the topology of their homogenous spaces; characteristic classes; Hodge theory; and more generally the remarkable topological properties of projective algebraic varieties.

Let G be a connected compact Lie group and T a maximal torus. A fundamental object in mathematics is the flag manifold G/T . We briefly recall Borel's beautiful and canonical description of its cohomology. Given a character $\lambda : T \rightarrow \mathbb{C}^*$ we can form the line bundle

$$L_\lambda := G \times_T \mathbb{C}$$

on G/T , defined as the quotient of $G \times \mathbb{C}$ by T -action given by $t \cdot (g, x) := (gt^{-1}, \lambda(t)x)$. Taking the Chern class of L_λ yields a homomorphism

$$X(T) \rightarrow H^2(G/T) : \lambda \mapsto c_1(L_\lambda).$$

from the lattice of characters to the second cohomology of G/T . If we identify $X(T) \otimes_{\mathbb{Z}} \mathbb{R} = (\text{Lie } T)^*$ via the differential and extend multiplicatively we get a morphism of graded algebras

$$R := S((\text{Lie } T)^*) \rightarrow H^\bullet(G/T; \mathbb{R}).$$

called the *Borel homomorphism*. (We let R denote the symmetric algebra on the dual of $\text{Lie } T$.) Borel showed that his homomorphism is surjective and identified its kernel with the ideal generated by W -invariant polynomials of positive degree. Here $W = N_G(T)/T$ denotes the Weyl group of G which acts on T by conjugation, hence on $\text{Lie } T$ and hence on R .

For example, let $G = U(n)$ be the unitary group, and T the subgroup of diagonal matrices ($\cong (S^1)^n$). Then the coordinate functions give an identification $R = \mathbb{R}[x_1, \dots, x_n]$, and W is the symmetric group on n -letters acting on R via permutation of variables. The Borel homomorphism gives an identification

$$\mathbb{R}[x_1, \dots, x_n] / \langle e_i \mid 1 \leq i \leq n \rangle = H^\bullet(G/T; \mathbb{R})$$

where e_i denotes the i^{th} elementary symmetric polynomial in x_1, \dots, x_n .

Let $G_{\mathbb{C}}$ denote the complexification of G and choose a Borel subgroup B containing the complexification of T . (For example if $G = U(n)$ then $G_{\mathbb{C}} = GL_n(\mathbb{C})$ and we could take B to be the subgroup of upper-triangular matrices.) A fundamental fact is that the natural map

$$G/T \rightarrow G_{\mathbb{C}}/B$$

is a diffeomorphism, and $G_{\mathbb{C}}/B$ is a projective algebraic variety.

For example, if $G = SU(2) \cong S^3$ then $G/T = S^2$ is the base of the Hopf fibration, and the above diffeomorphism is $S^2 \xrightarrow{\sim} \mathbb{P}^1\mathbb{C}$. More generally for $G = U(n)$ the above diffeomorphism can be seen as an instance of Gram-Schmidt orthogonalization. Fix a Hermitian form on \mathbb{C}^n . Then $G_{\mathbb{C}}/B$ parametrizes complete flags on \mathbb{C}^n , while G/T parametrizes collections of n ordered orthogonal complex lines. These spaces are clearly isomorphic.

The fact that $G/T = G_{\mathbb{C}}/B$ is a projective algebraic variety means that its cohomology satisfies a number of deep theorems from complex algebraic geometry. Set $H = H^{\bullet}(G_{\mathbb{C}}/B; \mathbb{R})$ and let N denote the complex dimension of $G_{\mathbb{C}}/B$. For us the following two results (the ‘‘shadows of Hodge theory’’ of the title) will be of fundamental importance.

Theorem 1.1 (Hard Lefschetz theorem). *Let $\lambda \in H^2$ denote the Chern class of an ample line bundle on $G_{\mathbb{C}}/B$ (i.e. $\lambda \in (\text{Lie } T)^*$ is a ‘dominant weight’, see (4.3)). Then for all $0 \leq i \leq N$ multiplication by λ^{N-i} gives an isomorphism:*

$$\lambda^{N-i} : H^i \xrightarrow{\sim} H^{2N-i}.$$

Because G/T is a compact manifold, Poincaré duality states that H^i and H^{2N-i} are non-degenerately paired by the Poincaré pairing $\langle -, - \rangle_{\text{Poinc}}$. On the other hand, after fixing λ as above the hard Lefschetz theorem gives us a way of identifying H^i and H^{2N-i} . The upshot is that for $0 \leq i \leq N$ we obtain a non-degenerate *Lefschetz form*:

$$\begin{aligned} H^i \times H^i &\rightarrow \mathbb{R} \\ (\alpha, \beta) &\mapsto \langle \alpha, \lambda^{N-i} \beta \rangle_{\text{Poinc}}. \end{aligned}$$

On the middle dimensional cohomology the Lefschetz form is just the Poincaré pairing. This is the only Lefschetz form which does not depend on the choice of ample class λ .

Theorem 1.2 (Hodge-Riemann bilinear relations). *For $0 \leq i \leq N$ the restriction of the Lefschetz form to $P^i := \ker(\lambda^{N-i+1}) \subset H^i$ is $(-1)^{i/2}$ -definite.*

Some comments are in order:

- (1) The odd cohomology of G/T vanishes as can be seen, for example, from the surjectivity of the Borel homomorphism. Hence the sign $(-1)^{i/2}$ makes sense.
- (2) For an arbitrary smooth projective algebraic variety the Hodge-Riemann bilinear relations are more complicated, involving the Hodge decomposition and a Hermitian form on the complex cohomology groups. However, the cohomology of the flag variety is always in (p, p) -type, so that we may use the simpler formulation above.

- (3) We will not make it explicit, but the Hodge-Riemann bilinear relations give formulas for the signatures of all Lefschetz forms in terms of the graded dimension of H .

We now come to the punchline of this survey. The hard Lefschetz theorem and Hodge-Riemann bilinear relations for $H^\bullet(G/B; \mathbb{R})$ are deep consequences of Hodge theory. On the other hand, we have seen that the Borel homomorphism gives us an elementary description of $H^\bullet(G/B; \mathbb{R})$ in terms of commutative algebra and invariant theory. Can one establish the hard Lefschetz theorem and Hodge-Riemann bilinear relations for $H^\bullet(G/B; \mathbb{R})$ algebraically? A crucial motivation for this question is the fact that $H^\bullet(G/B; \mathbb{R})$ has various algebraic cousins (described in §5) for which no geometric description is known. Remarkably, these cousins still satisfy analogs of Theorems 1.1 and 1.2. Establishing these Hodge-theoretic properties algebraically is the cornerstone of the authors' approach to conjectures of Kazhdan-Lusztig and Soergel.

The structure of this (very informal) survey is as follows. In §2 we give a lightning introduction to intersection cohomology, which provides an improved cohomology theory for singular algebraic varieties. In §3 we discuss Schubert varieties, certain (usually singular) subvarieties of the flag variety which play an important role in representation theory. We also discuss Bott-Samelson resolutions of Schubert varieties. In §4 we discuss Soergel modules. The point is that one can give a purely algebraic/combinatorial description of the intersection cohomology of Schubert varieties, which only depends on the underlying Weyl group. In §5 we discuss Soergel modules for arbitrary Coxeter groups, which (currently) have no geometric interpretation. We also state our main theorem that these modules satisfy the “shadows of Hodge theory”. Finally, in §6 we discuss the amusing example of the coinvariant ring of a finite dihedral group.

2. INTERSECTION COHOMOLOGY AND THE DECOMPOSITION THEOREM

Poincaré duality, the hard Lefschetz theorem and Hodge-Riemann bilinear relations hold for the cohomology of any smooth projective variety. The statements of these results usually fail for singular varieties. However, in the 1970s Goresky and MacPherson invented intersection cohomology [GM80, GM83] and it was later proven that the analogues of these theorems hold for intersection cohomology. In this section we will try to give the vaguest of vague ideas as to what is going on, and hopefully convince the reader to go and read more. (The authors' favourite introduction to the theory is [dM09] whose emphasis agrees largely with that of this survey.¹ More information is contained in [Bor94, Rie04, Ara06] with the bible being [BBD82]. To stay motivated, Kleiman's excellent history of the subject [Kle07] is a must.)

Intersection cohomology associates to any complex variety X its “intersection cohomology groups” $IH^\bullet(X)$ (throughout this article we always take coefficients in \mathbb{R} , however there are versions of the theory with \mathbb{Q} and \mathbb{Z} -coefficients). Here are some basic properties of intersection cohomology:

- (1) $IH^\bullet(X)$ is a graded vector space, concentrated in degrees between 0 and $2N$, where N is the complex dimension of X ;
- (2) if X is smooth then $IH^\bullet(X) = H^\bullet(X)$;

¹Due, no doubt, to the influence which their work has had on the authors.

- (3) if X is projective then $IH^\bullet(X)$ is equipped with a non-degenerate Poincaré pairing $\langle -, - \rangle_{Poinc}$, which is the usual Poincaré pairing for X smooth.

However we caution the reader that:

- (1) the assignment $X \mapsto IH^\bullet(X)$ is not functorial: in general a morphism $f : X \rightarrow Y$ does not induce a pull-back map on intersection cohomology;
- (2) $IH^\bullet(X)$ is not a ring, but rather a module over the cohomology ring $H^\bullet(X)$.

(These two “failings” become less worrying when one interprets intersection cohomology in the language of constructible sheaves.) Finally, we come to the two key properties that will concern us in this article. We assume that X is a projective variety (not necessarily smooth):

- (1) multiplication by the first Chern class of an ample line bundle on $IH^\bullet(X)$ satisfies the hard Lefschetz theorem;
- (2) the groups $IH^\bullet(X)$ satisfy the Hodge-Riemann bilinear relations.

(To make sense of this second statement, one needs to know that $IH^\bullet(X)$ has a Hodge decomposition. This is true, but we will not discuss it. Below, we will only consider varieties whose Hodge decomposition only involves components of type (p, p) and so the naive formulation of the Hodge-Riemann bilinear relations in the form of Theorem 1.2 will be sufficient.)

Example 2.1. Consider the Grassmannian $Gr(2, 4)$ of planes in \mathbb{C}^4 . It is a smooth projective algebraic variety of complex dimension 4. Let $0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \mathbb{C}^3 \subset \mathbb{C}^4$ denote the standard coordinate flag on \mathbb{C}^4 . For any sequence of natural numbers $\underline{a} := (0 = a_0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 = 2)$ satisfying $a_i \leq a_{i+1} \leq a_i + 1$, consider the subvariety

$$C_{\underline{a}} := \{V \in Gr(2, 4) \mid \dim(V \cap \mathbb{C}^i) = a_i\}.$$

It is not difficult (by writing down charts for the Grassmannian) to see that each $C_{\underline{a}}$ is isomorphic to $\mathbb{C}^{d(\underline{a})}$ where $d(\underline{a}) = 7 - \sum_{i=0}^4 a_i$. Hence $Gr(2, 4)$ has a cell-decomposition with cells of real dimension 0, 2, 4, 4, 6, 8. The cohomology $H^\bullet(Gr(2, 4))$ is as follows:

0	1	2	3	4	5	6	7	8
\mathbb{R}	0	\mathbb{R}	0	\mathbb{R}^2	0	\mathbb{R}	0	\mathbb{R}

It is an easy exercise to use Schubert calculus (see e.g. [Hil82, III.3], which also discusses $Gr(2, 4)$ in more detail) to check the hard Lefschetz theorem and Hodge-Riemann bilinear relations by hand.

Now consider the subvariety

$$X := \{V \in Gr(2, 4) \mid \dim(V \cap \mathbb{C}^2) \geq 1\}.$$

Then X coincides with the closure of the cell $C_{0 \leq 0 \leq 1 \leq 1 \leq 2} \subset Gr(2, 4)$ (and thus is an example of a “Schubert variety”, as we will discuss in the next section). Hence X has real dimension 6 and has a cell-decomposition with cells of dimension (0, 2, 4, 4, 6). Its cohomology is as follows:

0	1	2	3	4	5	6
\mathbb{R}	0	\mathbb{R}	0	\mathbb{R}^2	0	\mathbb{R}

We conclude that X cannot satisfy Poincaré duality or the hard Lefschetz theorem. In particular X must be singular. In fact, X has a unique singular point $V_0 = \mathbb{C}^2$.

We will see below that the intersection cohomology $IH^\bullet(X)$ is as follows:

0	1	2	3	4	5	6
\mathbb{R}	0	\mathbb{R}^2	0	\mathbb{R}^2	0	\mathbb{R}

So in this example $IH^\bullet(X)$ seems to fit the bill (at least on the level of Betti numbers) of rescuing Poincaré duality and the hard Lefschetz theorem in a “minimal” way.

Probably the most fundamental theorem about intersection cohomology is the decomposition theorem. In its simplest form it says the following:

Theorem 2.2 (Decomposition theorem [BBD82, Sai89, dCM02, dCM05]). *Let $f : \tilde{X} \rightarrow X$ be a resolution, i.e. \tilde{X} is smooth and f is a projective birational morphism of algebraic varieties. Then $IH^\bullet(X)$ is a direct summand of $H^\bullet(\tilde{X})$, as modules over $H^\bullet(X)$.*

The decomposition theorem provides an invaluable tool for calculating intersection cohomology, which is otherwise a very difficult task.

Example 2.3. In Example 2.1 we discussed the variety

$$X := \{V \in Gr(2, 4) \mid \dim(V \cap \mathbb{C}^2) \geq 1\}$$

which is projective with unique singular point $V_0 = \mathbb{C}^2$. Now X has a natural resolution $f : \tilde{X} \rightarrow X$ where

$$\tilde{X} = \{(V, W) \in Gr(2, 4) \times \mathbb{P}(\mathbb{C}^2) \mid W \subset V \cap \mathbb{C}^2\}$$

and $f(V, W) = V$. Clearly f is an isomorphism over $X \setminus \{V_0\}$ and has fibre $\mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2)$ over the singular point V_0 . Also, the projection $(V, W) \mapsto W$ realizes \tilde{X} as a \mathbb{P}^2 -bundle over \mathbb{P}^1 . In particular, \tilde{X} is smooth and its cohomology is as follows:

0	1	2	3	4	5	6
\mathbb{R}	0	\mathbb{R}^2	0	\mathbb{R}^2	0	\mathbb{R}

We conclude by the decomposition theorem that $IH^\bullet(X)$ is a summand of $H^\bullet(\tilde{X})$. In this case one has equality: $IH^\bullet(X) = H^\bullet(\tilde{X})$. One can see this directly as follows: first one checks that the pull-back map $H^i(X) \rightarrow H^i(\tilde{X})$ is injective. Now, because $IH^\bullet(X)$ is an $H^\bullet(X)$ -stable summand of $H^\bullet(\tilde{X})$ containing $\mathbb{R} = H^0(\tilde{X})$ we conclude that $IH^i(X) = H^i(\tilde{X})$ for $i \neq 2$. Finally, we must have $IH^2(X) = H^2(\tilde{X})$ because $IH^\bullet(X)$ satisfies Poincaré duality.

Let us now discuss the hard Lefschetz theorem and Hodge-Riemann bilinear relations for $IH^\bullet(X)$. Let λ be the class of an ample line bundle on X . Because $IH^\bullet(X) = H^\bullet(\tilde{X})$ in this example, the action of λ on $IH^\bullet(X)$ is identified with the action of $f^*\lambda$ on $H^\bullet(\tilde{X})$. We would like to know that $f^*\lambda$ acting on $H^\bullet(\tilde{X})$ satisfies the the hard Lefschetz theorem and Hodge-Riemann bilinear relations *even though $f^*\lambda$ is not an ample class on \tilde{X}* . This simple observation is the starting point for beautiful work of de Cataldo and Migliorini [dCM02, dCM05], who give a Hodge-theoretic proof of the decomposition theorem.

3. SCHUBERT VARIETIES AND BOTT-SAMELSON RESOLUTIONS

Recall our connected compact Lie group G , its complexification $G_{\mathbb{C}}$, the maximal torus $T \subset G$ and the Borel subgroup $T \subset B \subset G_{\mathbb{C}}$. To (G, T) we may associate a root system $\Phi \subset (\text{Lie } T)^*$. Our choice of Borel subgroup is equivalent to a choice of simple roots $\Delta \subset \Phi$. As we discussed in the introduction, the Weyl group $W = N_G(T)/T$ acts on $\text{Lie } T$ as a reflection group. The choice of simple roots $\Delta \subset \Phi$ gives a choice of *simple reflections* $S \subset W$. These simple reflections generate W and with respect to these generators W admits a *Coxeter presentation*:

$$W = \langle s \in S \mid s^2 = \text{id}, (st)^{m_{st}} = \text{id} \rangle$$

where $m_{st} \in \{2, 3, 4, 6\}$ can be read off the Dynkin diagram of G . Given $w \in W$ a *reduced expression* for w is an expression $w = s_1 \dots s_m$ with $s_i \in S$, having shortest length amongst all such expressions. The *length* $\ell(w)$ of w is the length of a reduced expression. The Weyl group W is finite, with a unique longest element w_0 .

From now on we will work with the flag variety $G_{\mathbb{C}}/B$ in its incarnation as a projective algebraic variety. It is an important fact (the ‘‘Bruhat decomposition’’) that B has finitely many orbits on $G_{\mathbb{C}}/B$ which are parametrized by the Weyl group W . In formulas we write:

$$G_{\mathbb{C}}/B = \bigsqcup_{w \in W} B \cdot wB/B$$

Each B -orbit $B \cdot wB/B$ is isomorphic to an affine space and its closure

$$X_w := \overline{B \cdot wB/B}$$

is a projective variety called a *Schubert variety*. It is of complex dimension $\ell(w)$. The two extreme cases are $X_{\text{id}} = B/B$, a point, and $X_{w_0} = G_{\mathbb{C}}/B$, the full flag variety.

More generally, given any subset $I \subset S$ we have a parabolic subgroup $B \subset P_I \subset G$ generated by B and (any choice of representatives of) the subset I . The quotient G/P_I is also a projective algebraic variety (called a *partial flag variety*) and the Bruhat decomposition takes the form

$$G/P_I := \bigsqcup_{w \in W^I} B \cdot wB/P_I$$

where W^I denotes a set of minimal length representatives for the cosets W/W_I . Again, the Schubert varieties are the closures $X_w^I := \overline{B \cdot wB/P_I} \subset G/P_I$, which are projective algebraic varieties of dimension $\ell(w)$.

Example 3.1. We discussed the more general setting of G/P_I to make contact with the Grassmannian in Example 2.1. Indeed, $Gr(2, 4) \cong GL_4(\mathbb{C})/P$ where P is the stabilizer of the fixed coordinate subspace $\mathbb{C}^2 \subset \mathbb{C}^4$. If B denotes the stabilizer of the coordinate flag $0 \subset \mathbb{C}^1 \subset \mathbb{C}^2 \subset \mathbb{C}^3 \subset \mathbb{C}^4$ (the upper triangular matrices) then the cells $C_{\underline{a}}$ of Example 2.1 are B -orbits on $Gr(2, 4)$. Hence our X is an example of a singular Schubert variety.

Schubert varieties are rarely smooth. We now discuss how to construct resolutions. We will focus on Schubert varieties in the full flag variety, although similar constructions work for Schubert varieties in partial flag varieties. Choose $w \in W$ and fix a reduced expression $w = s_1 s_2 \dots s_m$. For any $1 \leq i \leq m$ let us alter our

notation and write P_i for $P_{\{s_i\}} = \overline{Bs_iB}$, a (minimal) parabolic subgroup associated to the reflection s_i . Consider the space

$$BS(s_1, \dots, s_m) := P_1 \times^B P_2 \times^B \dots \times^B P_m/B.$$

The notation \times^B indicates that $BS(s_1, \dots, s_m)$ is the quotient of $P_1 \times P_2 \times \dots \times P_m$ by the action of B^m via

$$(b_1, b_2, \dots, b_m) \cdot (p_1, \dots, p_m) = (p_1 b_1^{-1}, b_1 p_2 b_2^{-1}, \dots, b_{m-1} p_m b_m^{-1}).$$

Then $BS(s_1, \dots, s_m)$ is a smooth projective *Bott-Samelson variety* and the multiplication map $P_1 \times \dots \times P_m \rightarrow G$ induces a morphism

$$f : BS(s_1, \dots, s_m) \rightarrow X_w$$

which is a resolution of X_w . (See [Dem74, Han73] and [Bri12, §2] for further discussion and applications of Bott-Samelson resolutions. The name Bott-Samelson resolution comes from [BS58] where related spaces are considered in the context of loop spaces of compact Lie groups.)

Example 3.2. If $G_{\mathbb{C}} = GL_n$, Bott-Samelson resolutions admit a more explicit description. Recall that GL_n/B is the variety of flags $V_{\bullet} = (0 = V_0 \subset V_1 \subset V_2 \subset \dots \subset V_n = \mathbb{C}^n)$ with $\dim V_i = i$. We identify W with the symmetric group S_n and S with the set of simple transpositions $\{s_i = (i, i+1) \mid 1 \leq i \leq n-1\}$. Given a reduced expression $s_{i_1} \dots s_{i_m}$ for $w \in W$ consider the variety $\widetilde{BS}(s_{i_1}, \dots, s_{i_m})$ of all m -tuples of flags $(V_{\bullet}^a)_{0 \leq a \leq m}$ such that:

- (1) V_{\bullet}^0 is the coordinate flag $V_{\bullet}^{\text{std}} = (0 \subset \mathbb{C}^1 \subset \dots \subset \mathbb{C}^n)$;
- (2) for all $1 \leq a \leq m$, $V_j^a = V_j^{a-1}$ for $j \neq i_a$.

That is, $\widetilde{BS}(s_{i_1}, \dots, s_{i_m})$ is the variety of sequences of $m+1$ flags which begin at the coordinate flag, and where, in passing from the $(j-1)^{\text{st}}$ to the j^{th} step, we are only allowed to change the i_j^{th} dimensional subspace.

Let $p_0 = 1$. Then the map

$$(p_1, \dots, p_m) \mapsto (p_0 \dots p_a V_{\bullet}^{\text{std}})_{a=0}^m$$

gives an isomorphism $BS(s_1, \dots, s_m) \rightarrow \widetilde{BS}(s_1, \dots, s_m)$. Under this isomorphism the map f becomes the projection to the final flag: $f((V_{\bullet}^a)_{a=1}^m) = V_{\bullet}^m$.

4. SOERTEL MODULES AND INTERSECTION COHOMOLOGY

In a landmark paper [Soe90], Soergel explained how to calculate the intersection cohomology of Schubert varieties in a purely algebraic way. Though much less explicit, one way of viewing this result is as a generalization of Borel's description of the cohomology of the flag variety.

The idea is as follows. In the last section we discussed the Bott-Samelson resolutions of Schubert varieties

$$f : BS(s_1, \dots, s_m) \rightarrow X_w \subset G_{\mathbb{C}}/B$$

where $w = s_1 \dots s_m$ is a reduced expression for w . By the decomposition theorem $IH^{\bullet}(X_w)$, the intersection cohomology of the Schubert variety $X_w \subset G_{\mathbb{C}}/B$, is a summand of $H^{\bullet}(BS(s_1, \dots, s_m))$. Moreover, we have pull-back maps

$$H^{\bullet}(G_{\mathbb{C}}/B) \rightarrow H^{\bullet}(X_w) \rightarrow H^{\bullet}(BS(s_1, \dots, s_m))$$

and $IH^{\bullet}(X_w)$ is even a summand of $H^{\bullet}(BS(s_1, \dots, s_m))$ as an $H^{\bullet}(G_{\mathbb{C}}/B)$ -module. (The surjectivity of the restriction map $H^{\bullet}(G_{\mathbb{C}}/B) \rightarrow H^{\bullet}(X_w)$ follows because both

spaces have compatible cell-decompositions.) Remarkably, this algebraic structure already determines the summand $IH^\bullet(X_w)$ (see [Soe90, Erweiterungssatz]):

Theorem 4.1 (Soergel). *Let $w = s_1 \dots s_m$ denote a reduced expression for w as above. Consider $H^\bullet(BS(s_1, \dots, s_m))$ as a $H^\bullet(G_{\mathbb{C}}/B)$ -module. Then $IH(X_w)$ may be described as the indecomposable graded $H^\bullet(G_{\mathbb{C}}/B)$ -module direct summand with non-trivial degree zero part.*

A word of caution: The realization of $IH^\bullet(X_w)$ inside $H^\bullet(BS(s_1, \dots, s_m))$ is not canonical in general. We can certainly decompose $H^\bullet(BS(s_1, \dots, s_m))$ into graded indecomposable $H^\bullet(G_{\mathbb{C}}/B)$ -modules. Although this decomposition is not canonical, the Krull-Schmidt theorem ensures that the isomorphism type and multiplicities of indecomposable summands do not depend on the chosen decomposition. The above theorem states that, for any such decomposition, the unique indecomposable module with non-trivial degree zero part is isomorphic to $IH^\bullet(X_w)$ (as an $H^\bullet(G_{\mathbb{C}}/B)$ -module).

We now explain (following Soergel) how one may give an algebraic description of all players in the above theorem. Recall that $R = S((\text{Lie } T)^*)$ denotes the symmetric algebra on the dual of $\text{Lie } T$, graded so that $(\text{Lie } T)^*$ has degree 2. The Weyl group W acts on R , and for any simple reflection $s \in S$ we denote by R^s the invariants under s . It is not difficult to see that R is a free graded module of rank 2 over R^s with basis $\{1, \alpha_s\}$, where α_s is the simple root associated to $s \in S$. (In essence this is the high-school fact that any polynomial can be written as the sum of its even and odd parts.)

The starting point is the following observation:

Proposition 4.2 (Soergel). *One has an isomorphism of graded algebras*

$$H^\bullet(BS(s_1, \dots, s_m)) = R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \dots R \otimes_{R^{s_m}} R \otimes_R \mathbb{R}$$

where the final term is an R -algebra via $\mathbb{R} \cong R/R^{>0}$.

For example, for any $s \in S$ we have $BS(s) = P_s/B \cong \mathbb{P}^1$ and $R \otimes_{R^s} R \otimes_R \mathbb{R} = R \otimes_{R^s} \mathbb{R}$ is 2-dimensional, with graded basis $\{1 \otimes 1, \alpha_s \otimes 1\}$ of degrees 0 and 2. More generally, one can show that

$$R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \dots R \otimes_{R^{s_m}} R \otimes_R \mathbb{R} = R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \dots R \otimes_{R^{s_m}} \mathbb{R}$$

has graded basis $\alpha_{s_1}^{\varepsilon_1} \otimes \alpha_{s_2}^{\varepsilon_2} \otimes \dots \otimes \alpha_{s_m}^{\varepsilon_m} \otimes 1$ where $(\varepsilon_a)_{a=1}^m$ is any tuple of zeroes and ones. In particular, its Poincaré polynomial is $(1 + q^2)^m$.

Recall that in the introduction we described the Borel isomorphism:

$$H^\bullet(G/B) \cong R/(R_+^W).$$

Notice that left multiplication by any invariant polynomial of positive degree acts as zero on

$$H^\bullet(BS(s_1, \dots, s_m)) = R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \dots R \otimes_{R^{s_m}} R \otimes_R \mathbb{R}.$$

We conclude that $R \otimes_{R^{s_1}} \dots R \otimes_{R^{s_m}} R \otimes_R \mathbb{R}$ is a module over $R/(R_+^W)$. Geometrically, this corresponds to the pull-back map on cohomology

$$H^\bullet(G_{\mathbb{C}}/B) \rightarrow H^\bullet(BS(s_1, \dots, s_m))$$

discussed above.

We can now reformulate Theorem 4.1 algebraically as follows:

Theorem 4.3 (Soergel [Soe90]). *Let D_w be any indecomposable $R/(R_+^W)$ -module direct summand of*

$$H^\bullet(BS(s_1, \dots, s_m)) = R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \dots R \otimes_{R^{s_m}} R \otimes_R \mathbb{R}$$

containing the element $1 \otimes 1 \otimes \dots \otimes 1$, where $w = s_1 \dots s_m$ is a reduced expression for w . Then D_w is well-defined up to isomorphism (i.e. does not depend on the choice of reduced expression) and $D_w \cong IH^\bullet(X_w)$.

The modules $\{D_w \mid w \in W\}$ are the (indecomposable) *Soergel modules*.

Example 4.4. We consider the case of $G = GL_3(\mathbb{C})$ in which case

$$W = S_3 = \{\text{id}, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$$

(we use the conventions of Example 3.2). In this case it turns out that all Schubert varieties are smooth. Also, if $\ell(w) \leq 2$ then any Bott-Samelson resolution is an isomorphism. We conclude

$$\begin{aligned} D_{\text{id}} &= \mathbb{R} \\ D_{s_1} &= H^\bullet(BS(s_1)) = R \otimes_{R^{s_1}} \mathbb{R} & D_{s_2} &= R \otimes_{R^{s_2}} \mathbb{R} \\ D_{s_1s_2} &= H^\bullet(BS(s_1, s_2)) = R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \mathbb{R} & D_{s_2s_1} &= R \otimes_{R^{s_2}} R \otimes_{R^{s_1}} \mathbb{R} \end{aligned}$$

(A pleasant exercise for the reader is to verify that in all these examples above D_x is a cyclic (hence indecomposable) module over R . This is not usually the case, and is related to the (rational) smoothness of the Schubert varieties in question.)

The element $w_0 = s_1s_2s_1$ is more interesting. In this case the Bott-Samelson resolution

$$BS(s_1, s_2, s_1) \rightarrow X_{w_0} = G/B$$

is not an isomorphism. As previously discussed, the Poincaré polynomial of

$$(4.1) \quad H^\bullet(BS(s_1, s_2, s_1)) = R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} R \otimes_{R^{s_1}} \mathbb{R}$$

is $(1 + q^2)^3$ whereas the Poincaré polynomial of

$$(4.2) \quad IH^\bullet(X_{w_0}) = H^\bullet(G/B) = R/(R_+^W)$$

is $(1 + q^2)(1 + q^2 + q^4)$. In this case the reader may verify that (4.2) is a summand of (4.1). In fact one has an isomorphism of graded $R/(R_+^W)$ -modules:

$$R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} R \otimes_{R^{s_1}} \mathbb{R} = R/(R_+^W) \oplus (R \otimes_{R^s} \mathbb{R}(-2)).$$

Here $R \otimes_{R^s} \mathbb{R}(-2)$ denotes the shift of $R \otimes_{R^s} \mathbb{R}$ in the grading such that its generator $1 \otimes 1$ occurs in degree 2. This extra summand can be embedded into (4.1) via the map which sends

$$f \otimes 1 \mapsto f \otimes \alpha_{s_2} \otimes 1 \otimes 1 + f \otimes 1 \otimes \alpha_{s_2} \otimes 1$$

for $f \in R$.

Example 4.5. If w_0 denotes the longest element of W then $X_{w_0} = G_{\mathbb{C}}/B$, the (smooth) flag variety of G . In particular

$$IH^\bullet(X_{w_0}) = H^\bullet(G_{\mathbb{C}}/B) = R/(R_+^W)$$

by the Borel isomorphism. Theorem 4.3 asserts that $R/(R_+^W)$ occurs as a direct summand of

$$R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \dots \otimes_{R^{s_m}} \mathbb{R}$$

for any reduced expression $w_0 = s_1 \dots s_m$. This is by no means obvious! We have seen an instance of this in the previous example.

Remark 4.6. In this section we could have worked in the category of graded R -modules, rather than the category of graded $R/(R_+^W)$ -modules, and it would change nothing. All the R -modules in question will factor through $R/(R_+^W)$. In the next section, we will work with R -modules instead.

We now discuss hard Lefschetz and the Hodge–Riemann bilinear relations. Recall that our Borel subgroup $B \subset G_{\mathbb{C}}$ determines a set of simple roots $\Delta \subset \Phi \subset (\text{Lie } T)^*$ and simple coroots $\Delta^\vee \subset \Phi^\vee \subset \text{Lie } T$. Under the isomorphism

$$H^2(G_{\mathbb{C}}/B) \cong (\text{Lie } T)^*$$

the ample cone (i.e. the $\mathbb{R}_{>0}$ -stable subset of $H^2(G_{\mathbb{C}}/B)$ generated by Chern classes of ample line bundles on $G_{\mathbb{C}}/B$) is the cone of dominant weights for $\text{Lie } T$:

$$(4.3) \quad (\text{Lie } T)_+^* := \{\lambda \in (\text{Lie } T)^* \mid \langle \lambda, \alpha^\vee \rangle > 0 \text{ for all } \alpha^\vee \in \Delta^\vee\}.$$

The hard Lefschetz theorem then asserts that left multiplication by any $\lambda \in (\text{Lie } T)_+^*$ satisfies the hard Lefschetz theorem on $D_w = IH^\bullet(X_w)$. That is, for all $i \geq 0$, multiplication by λ^i induces an isomorphism

$$\lambda^i : D_w^{\ell(w)-i} \xrightarrow{\sim} D_w^{\ell(w)+i}.$$

To discuss the Hodge–Riemann relations we need to make the Poincaré pairing $\langle -, - \rangle_{\text{Poinc}}$ explicit for D_w . We first discuss the Poincaré form on $H^\bullet(BS(s_1, \dots, s_m))$. Recall that for any oriented manifold M the Poincaré form in de Rham cohomology is given by

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \beta.$$

We imitate this algebraically as follows. By the discussion after Proposition 4.2, the degree $2m$ component of

$$H^\bullet(BS(s_1, \dots, s_m)) = R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \dots R \otimes_{R^{s_m}} \mathbb{R}$$

is one-dimensional and is spanned by the vector $c_{\text{top}} := \alpha_{s_1} \otimes \alpha_{s_2} \otimes \dots \otimes \alpha_{s_m} \otimes 1$. We can define a bilinear form $\langle -, - \rangle$ on $R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \dots R \otimes_{R^{s_m}} \mathbb{R}$ via

$$\langle f, g \rangle = \text{Tr}(fg)$$

where fg denotes the term-wise multiplication, and Tr is the functional which returns the coefficient of c_{top} . Then $\langle -, - \rangle$ is a non-degenerate symmetric form which agrees up to a positive scalar with the intersection form on $H^\bullet(BS(s_1, \dots, s_m))$.

Now recall that D_w is obtained as summand of $R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \dots R \otimes_{R^{s_m}} \mathbb{R}$, for a reduced expression of w . Fixing such an inclusion we obtain a form on D_w via restriction of the form $\langle -, - \rangle$. In fact, this form is well-defined (i.e. depends neither on the choice of reduced expression nor embedding) up to a positive scalar. One can show that this form agrees with the Poincaré pairing on $D_w = IH^\bullet(X_w)$ up to a positive scalar. The Hodge–Riemann bilinear relations then hold for D_w with respect to this form and left multiplication by any $\lambda \in (\text{Lie } T)_+^*$.

5. SOERGEL MODULES FOR ARBITRARY COXETER SYSTEMS

Now let (W, S) denote an arbitrary Coxeter system. That is, W is a group with a distinguished set of generators S and a presentation

$$W = \langle s \in S \mid (st)^{m_{st}} = \text{id} \rangle$$

such that $m_{ss} = 1$ and $m_{st} = m_{ts} \in \{2, 3, 4, \dots, \infty\}$ for all $s \neq t$. (We interpret $(st)^\infty = \text{id}$ as there being no relation). As we discussed above, the Weyl groups of compact Lie groups are Coxeter groups. In the 1930's Coxeter proved that the finite reflection groups are exactly the finite Coxeter groups, and achieved in this way a classification. As well as the finite reflection groups arising in Lie theory (of types A, \dots, G) one has the symmetries of the regular n -gon (a dihedral group of type $I_2(n)$) for $n \neq 3, 4, 6$, the symmetries of the icosahedron (a group of type H_3) and the symmetries of a regular polytope in \mathbb{R}^4 with 600 sides (a group of type H_4).

It was realized later (by Coxeter, Tits, ...) that Coxeter groups form an interesting class of groups whether or not they are finite. They encompass groups generated by affine reflections in euclidean space (affine Weyl groups), certain hyperbolic reflection groups etc. One can treat these groups in a uniform way thanks to the existence of their *geometric representation*. Let $\mathfrak{h} = \bigoplus_{s \in S} \mathbb{R}\alpha_s^\vee$ for formal symbols α_s^\vee , and define a form on \mathfrak{h} via

$$(\alpha_s^\vee, \alpha_t^\vee) = -\cos(\pi/m_{st}).$$

Although this form is positive definite if and only if W is finite, one can still imagine that each α_s^\vee has length 1 and the angle between α_s^\vee and α_t^\vee for $s \neq t$ is $(m_{st} - 1)\pi/m_{st}$. It is not difficult to verify (see [Bou68, V.4.1] or [Hum90, 5.3]) that the assignment

$$s(v) := v - 2(v, \alpha_s^\vee)\alpha_s^\vee$$

defines a representation of W on \mathfrak{h} . In fact it is faithful ([Bou68, V.4.4.2] or [Hum90, Corollary 5.4])

If W happens to be the Weyl group of our $T \subset G$ from the introduction then (by rescaling the coroots so that they all have length 1 with respect to a W -invariant form) one may construct a W -equivariant isomorphism

$$\text{Lie } T \cong \mathfrak{h}.$$

Hence one can think of this setup as providing the action of W on the Lie algebra of a maximal torus, even though the corresponding Lie group might not exist!

The main point of the previous section is that one may describe the intersection cohomology, Poincaré pairing and ample cone entirely algebraically, using only \mathfrak{h} , its basis and its W -action. That is, let us (re)define $R = S(\mathfrak{h}^*)$ to be the symmetric algebra on \mathfrak{h}^* (alias the regular functions on \mathfrak{h}), graded with $\deg \mathfrak{h}^* = 2$. Then W acts on R via graded algebra automorphisms. Imitating the constructions of the previous section one obtains graded R -modules D_w (well-defined up to isomorphism), the only difference being that we work in the category of R -modules rather than $R/(R_+^W)$ -modules.² We call the modules D_w the (indecomposable) *Soergel*

²Although all the R -modules will factor through $R/(R_+^W)$, we prefer the ring R for philosophical reasons. When W is infinite, the ring $R/(R_+^W)$ is infinite-dimensional, as R^W has the “wrong” transcendence degree, and the Chevalley theorem does not hold. The ring R behaves in a uniform way for all Coxeter groups, while the quotient ring $R/(R_+^W)$ does not.

modules. As in the Weyl group case, the modules D_w are finite dimensional over \mathbb{R} and are equipped with non-degenerate ‘‘Poincaré pairings’’:

$$\langle -, - \rangle : D_w^i \times D_w^{2\ell(w)-i} \rightarrow \mathbb{R}.$$

Our main theorem is that these modules D_w ‘‘look like the intersection cohomology of a Schubert variety’’. Consider the ‘‘ample cone’’:

$$\mathfrak{h}_+^* := \{ \lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_s^\vee \rangle > 0 \text{ for all } s \in S \}.$$

Theorem 5.1 ([EW12]). *For any $w \in W$, let D_w be as above.*

- (1) (*Hard Lefschetz theorem*) *For any $i \leq \ell(w)$, left multiplication by λ^i for any $\lambda \in \mathfrak{h}_+^*$ gives an isomorphism*

$$\lambda^{\ell(w)-i} : D_w^i \xrightarrow{\sim} D_w^{2\ell(w)-i}$$

- (2) (*Hodge-Riemann bilinear relations*) *For any $i \leq \ell(w)$ and $\lambda \in \mathfrak{h}_+^*$ the restriction of the form*

$$(f, g) := \langle f, \lambda^{\ell(w)-i} g \rangle$$

on D_w^i to $P^i = \ker \lambda^{\ell(w)-i+1} \subset D_w^i$ is $(-1)^{i/2}$ -definite.

Some remarks:

- (1) The graded modules D_w are zero in odd-degree (as is immediate from their definition as a summand of $R \otimes_{R^{s_1}} \cdots \otimes_{R^{s_m}} \mathbb{R}$) and so the sign $(-1)^{i/2}$ makes sense.
- (2) The motivation behind establishing the above theorem is a conjecture made by Soergel in [Soe07, Vermutung 1.13]. In fact, the above theorem forms part of a complicated inductive proof of Soergel’s conjecture. Soergel was led to his conjecture as an algebraic means of understanding the Kazhdan-Lusztig basis of the Hecke algebra and the Kazhdan-Lusztig conjecture on characters of simple highest weight modules over complex semi-simple Lie algebras. The definition of the Kazhdan-Lusztig basis and the statement of the Kazhdan-Lusztig conjecture is ‘‘elementary’’ but, prior to the above results, needed powerful tools from algebraic geometry (e.g. Deligne’s proof of the Weil conjectures) for its resolution. Because of this reliance on algebraic geometry, these methods break down for arbitrary Coxeter systems, for which no flag variety exists. In some sense the above theorem is interesting because it provides a ‘‘geometry’’ for Kazhdan-Lusztig theory for Coxeter groups which do not come from Lie groups or generalizations (affine, Kac-Moody, ...) thereof. This was Soergel’s aim in formulating his conjecture.
- (3) Our proof is inspired by the beautiful work of de Cataldo and Migliorini [dCM02, dCM05], which proves the decomposition theorem using only classical Hodge theory.
- (4) The idea of considering the ‘‘intersection cohomology’’ of a Schubert variety associated to any element in a Coxeter group has also been pursued by Dyer [Dye95, Dye09] and Fiebig [Fie08]. There is also a closely related theory non-rational polytopes (where the associated toric variety is missing) [BL03, Kar04, BF07].

- (5) In Example 4.5 we saw that if W is a Weyl group then an important example of a Soergel module is

$$D_{w_0} \cong R/(R_+^W).$$

In fact this isomorphism holds for any finite Coxeter group W with longest element w_0 . The “coinvariant”³ algebra $R/(R_+^W)$ has been studied by many authors from many points of view. However even in this basic example it seems to be difficult to check the hard Lefschetz theorem or Hodge-Riemann bilinear relations directly. In the next section we will do this by hand when W is a dihedral group.

- (6) In [EW12] we work with \mathfrak{h} a slightly larger representation containing the geometric representation. We do this for technical reasons (to ensure that the category of Soergel bimodules is well-behaved). However, one can deduce Theorem 5.1 from the results of [EW12]. The idea of using the results for the slightly larger representation to deduce results for the geometric representation goes back to Libedinsky [Lib08].
- (7) (For the experts.) In [EW12] we prove the results above for certain R -modules \overline{B}_w , whose definition differs subtly from that of D_w . However, given that \overline{B}_w is indecomposable as an R -module, one can show easily that \overline{B}_w and D_w are isomorphic. This will be explained elsewhere.

6. THE FLAG VARIETY OF A DIHEDRAL GROUP

In this final section we amuse ourselves with the coinvariant ring of a finite dihedral group. We check the hard Lefschetz property and Hodge-Riemann bilinear relations directly.

6.1. Gauß’s q -numbers. We start by recalling Gauß’s q -numbers. By definition

$$[n] := q^{-n+1} + q^{-n+3} + \cdots + q^{n-3} + q^{n-1} = \frac{q^n - q^{-n}}{q - q^{-1}} \in \mathbb{Z}[q^{\pm 1}].$$

Many identities between numbers can be lifted to identities between q -numbers. We will need

$$(6.1) \quad [2][n] = [n+1] + [n-1]$$

$$(6.2) \quad [n]^2 = [2n-1] + [2n-3] + \cdots + [1].$$

$$(6.3) \quad [n][n+1] = [2n] + [2n-2] + \cdots + [2].$$

For the representation theorist, $[n]$ is the character of the simple $\mathfrak{sl}_2(\mathbb{C})$ -module of dimension n , and the relations above are instances of the Clebsch-Gordan formula.

If $\zeta = e^{2\pi i/2m} \in \mathbb{C}$ then we can specialize $q = \zeta$ to obtain algebraic integers $[n]_\zeta \in \mathbb{R}$. Because $\zeta^m = -1$ we have

$$(6.4) \quad [m]_\zeta = 0, \quad [i]_\zeta = [m-i]_\zeta, \quad [i+m]_\zeta = -[i]_\zeta.$$

Because ζ^n has positive imaginary part for $n < m$, it is clear that

$$(6.5) \quad [n]_\zeta \text{ is positive for } 0 < n < m.$$

We use this positivity in a crucial way below. Had we foolishly chosen ζ to be a primitive $2m^{\text{th}}$ root of unity with non-maximal real part, (6.5) would fail.

³W. Soergel pointed out that this is a bad name, as it has nothing whatsoever to do with coinvariants.

6.2. The reflection representation of a dihedral group. Now let W be a finite dihedral group of order $2m$. That is $S = \{s_1, s_2\}$ and

$$W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^m = \text{id} \rangle.$$

Let $\mathfrak{h} = \mathbb{R}\alpha_1^\vee \oplus \mathbb{R}\alpha_2^\vee$ be the geometric representation of (W, S) , as in §5. Because W is finite the form $(-, -)$ on \mathfrak{h} is non-degenerate. We define simple roots $\alpha_1, \alpha_2 \in \mathfrak{h}^*$ by $\alpha_1 = 2(\alpha_1^\vee, -)$ and $\alpha_2 = 2(\alpha_2^\vee, -)$. Then the ‘‘Cartan’’ matrix is

$$(6.6) \quad (\langle \alpha_i^\vee, \alpha_j \rangle)_{i,j \in \{1,2\}} = \begin{pmatrix} 2 & -\varphi \\ -\varphi & 2 \end{pmatrix}$$

where $\varphi = 2 \cos(\pi/m)$. Note that $\varphi = \zeta + \zeta^{-1}$ where $\zeta = e^{2\pi i/2m} \in \mathbb{C}$. Hence $\varphi = [2]_\zeta$ in the notation of the previous section. In particular it is an algebraic integer.

Example 6.1. Throughout we will use the first non-Weyl-group case $m = 5$ to illustrate what is going on. In this case $[2]_\zeta = [3]_\zeta$ and the relation $[2]^2 = [3] + [1]$ gives $\varphi^2 = \varphi + 1$. Thus φ is the golden ratio.

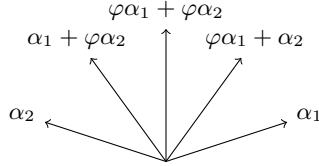
For all $v \in \mathfrak{h}^*$ we have

$$s_1(v) = v - \langle v, \alpha_1^\vee \rangle \alpha_1 \quad \text{and} \quad s_2(v) = v - \langle v, \alpha_2^\vee \rangle \alpha_2.$$

It is a pleasant exercise for the reader to verify that the set $\Phi = W \cdot \{\alpha_1, \alpha_2\}$ gives something like a root system in \mathfrak{h}^* . We have $\Phi = \Phi^+ \cup -\Phi^+$ where

$$(6.7) \quad \Phi^+ = \{[i]_\zeta \alpha_1 + [i-1]_\zeta \alpha_2 \mid 1 \leq i \leq m\}.$$

Example 6.2. For $m = 5$ one can picture the ‘‘positive roots’’ Φ^+ as follows:



Let $T := \bigcup wSw^{-1}$. Then T are precisely the elements of W which act as reflections on \mathfrak{h} (and \mathfrak{h}^*). One has a bijection

$$T \xrightarrow{\sim} \Phi^+ : t \mapsto \alpha_t$$

such that $t(\alpha_t) = -\alpha_t$ for all $t \in T$.

6.3. Schubert calculus. In the following we describe Schubert calculus for the coinvariant ring. Most of what we say here is valid for any finite Coxeter group. A good reference for the unproved statements below is [Hil82].

Let R denote the symmetric algebra on \mathfrak{h}^* and H the coinvariant algebra

$$H := R/(R_+^W).$$

For each $s \in S$ consider the divided difference operator

$$\partial_s(f) = \frac{f - s(f)}{\alpha_s}.$$

Then ∂_s preserves R and decreases degrees by 2. Given $x \in W$ we define

$$\partial_x = \partial_{s_1} \dots \partial_{s_m}$$

where $x = s_1 \dots s_m$ is a reduced expression for x . The operators ∂_s satisfy the braid relations, and therefore ∂_x is well-defined. The operators ∂_x kill invariant polynomials and hence commute with multiplication by invariants. In particular they preserve the ideal (R_+^W) and induce operators on H .

Let $\pi := \prod_{\alpha \in \Phi^+} \alpha$ denote the product of the positive roots. For any $x \in W$ define $Y_x \in H$ as the image of $\partial_x(\pi)$ in H . Because π has degree $2\ell(w_0)$, Y_x has degree $\deg Y_x = 2(\ell(w_0) - \ell(x))$.

Theorem 6.3. *The elements $\{Y_x \mid x \in W\}$ give a basis for H .*

This basis is called the *Schubert basis*. When W is a Weyl group each Y_x maps under the Borel isomorphism to the fundamental class of a Schubert variety [BGG73].

We can define a bilinear form $\langle -, - \rangle$ on H as follows:

$$\langle f, g \rangle := \frac{1}{2m} \partial_{w_0}(fg)$$

Then for all $x, z \in W$ one has:

$$(6.8) \quad \langle Y_x, Y_z \rangle = \delta_{w_0, x^{-1}z}.$$

In particular $\langle -, - \rangle$ is a non-degenerate form on H .

The following ‘‘Chevalley’’ formula describes the action of an element $f \in \mathfrak{h}^*$ in the basis $\{Y_x\}$:

$$(6.9) \quad f \cdot Y_x = \sum_{\substack{t \in T \\ \ell(tx) = \ell(x) - 1}} \langle f, \alpha_t^\vee \rangle Y_{tx}$$

Example 6.4. Figure 1 depicts the case $m = 5$. Each edge is labelled with the coroot which, when paired against f , gives the scalar coefficient that describes the action of f . Using (6.7) the reader can guess what the picture looks like for general m .

Proposition 6.5. *Suppose that $\lambda \in \mathfrak{h}^*$ is such that $\langle \alpha_i^\vee, \lambda \rangle > 0$ for $i = 1, 2$. Then multiplication by λ on H satisfies the hard Lefschetz theorem, and the Hodge-Riemann bilinear relations hold.*

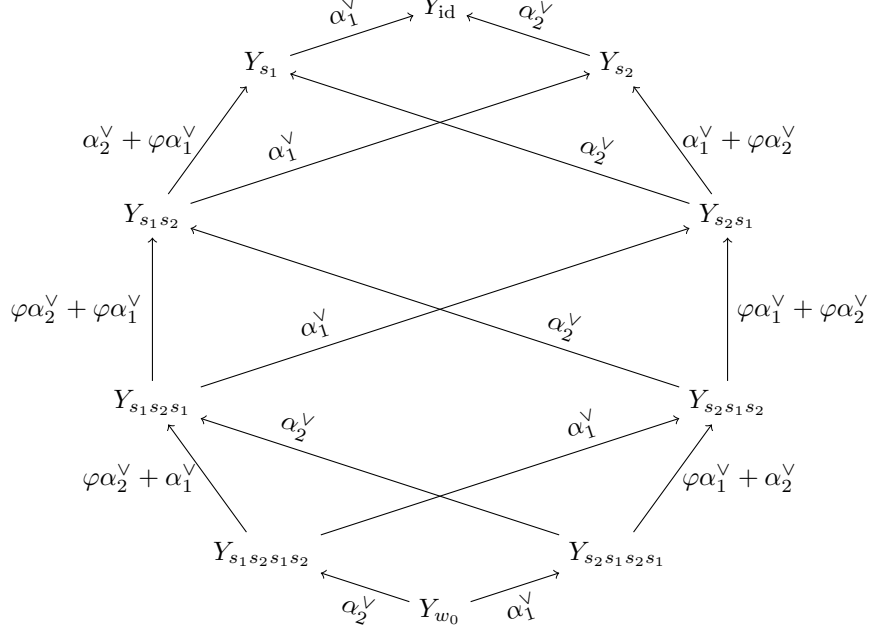
Proof. It is immediate from (6.9) that if λ is as in the proposition and if $x \neq \text{id}$ then λY_x is a sum of various Y_z with strictly positive coefficients (two terms occur if $\ell(x) < m - 1$ and one term occurs if $\ell(x) = m - 1$). Hence $\lambda^m Y_{w_0}$ is a strictly positive constant times Y_{id} . In particular $\lambda^m : H^0 = \mathbb{R}Y_{w_0} \rightarrow H^{2m} = \mathbb{R}Y_{\text{id}}$ is an isomorphism. By (6.8) we have

$$\langle Y_{w_0}, \lambda^m Y_{w_0} \rangle > 0$$

and hence the Lefschetz form is positive definite on H^0 .

We now fix $1 \leq i < m - 1$ and consider multiplication by $f \in \mathfrak{h}^*$ as a map $H^{2i} \rightarrow H^{2i+2}$. The following diagram depicts the effect in the Schubert basis:

$$(6.10) \quad \begin{array}{ccc} & Y_a & Y_b \\ & \swarrow & \searrow \\ [i]_\zeta \alpha_1^\vee + [i+1]_\zeta \alpha_2^\vee & \alpha_1^\vee & \alpha_2^\vee \\ & \swarrow & \searrow \\ Y_{s_1 b} & & Y_{s_2 a} \\ & \swarrow & \searrow \\ & [i+1]_\zeta \alpha_1^\vee + [i]_\zeta \alpha_2^\vee & \end{array}$$

FIGURE 1. The Chevalley formula for the dihedral group with $m = 5$:

where a and b (resp. s_2a and s_1b) are the unique elements of length $\ell(w_0) - i - 1$ (resp. $\ell(w_0) - i$). Remember that α_i^\vee here represents the scalar $\langle f, \alpha_i^\vee \rangle$. We now calculate the determinant:

$$\begin{aligned} & \det \begin{pmatrix} [i]_\zeta \alpha_1^\vee + [i+1]_\zeta \alpha_2^\vee & \alpha_2^\vee \\ \alpha_1^\vee & [i+1]_\zeta \alpha_1^\vee + [i]_\zeta \alpha_2^\vee \end{pmatrix} = \\ &= [i]_\zeta [i+1]_\zeta (\alpha_1^\vee)^2 + ([i]_\zeta^2 + [i+1]_\zeta^2 - 1) \alpha_1^\vee \alpha_2^\vee + [i]_\zeta [i+1]_\zeta (\alpha_2^\vee)^2 \\ &= [i]_\zeta [i+1]_\zeta (\alpha_1^\vee)^2 + [2]_\zeta [i]_\zeta [i+1]_\zeta \alpha_1^\vee \alpha_2^\vee + [i]_\zeta [i+1]_\zeta (\alpha_2^\vee)^2 \end{aligned}$$

(using (6.1), (6.2) and (6.3)). All q -numbers appearing here are positive by (6.5).

If λ is as in the proposition, then the determinant of multiplication by λ is positive. So λ gives an isomorphism $H^{2i} \xrightarrow{\sim} H^{2i+2}$ for each $1 \leq i \leq m-2$, and λ^{m-2} gives an isomorphism $H^2 \xrightarrow{\sim} H^{2m-2}$. Therefore the hard Lefschetz theorem holds for λ , with primitive classes occurring only in degrees 0 and 2.

It remains to check the Hodge-Riemann bilinear relations. We have already seen that the Lefschetz form on H^0 is positive definite. We need to know that the restriction of the Lefschetz form on H^2 to $\ker \lambda^{m-1}$ is negative definite. Now $\langle \lambda Y_{w_0}, \lambda Y_{w_0} \rangle = \langle Y_{w_0}, Y_{w_0} \rangle > 0$, and if $\gamma \in H^2$ denotes a generator for $\ker \lambda^{m-1}$ then $\langle \lambda Y_{w_0}, \gamma \rangle = \langle \lambda Y_{w_0}, \lambda^{m-2} \gamma \rangle = \langle Y_{w_0}, \lambda^{m-1} \gamma \rangle = 0$. Hence the Hodge-Riemann relations hold if and only if the signature of the Lefschetz form on H^2 is zero.

From the definition of the Lefschetz form, it is immediate that $\lambda : H^{2i} \rightarrow H^{2i+2}$ is an isometry with respect to the Lefschetz forms, so long as $2 \leq 2i \leq m-2$. Thus when m is even (resp. odd) it is enough to show that the signature of the Lefschetz form is zero on H^m (resp. H^{m-1}).

Suppose m is even. The Lefschetz form on the middle dimension H^m is the same as the pairing. By (6.8) this form has Gram matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which has signature 0.

Suppose $m = 2k+1$ is odd; we check the signature of the Lefschetz form on H^{m-1} . We are reduced to studying (6.10) with $\ell(a) = \ell(b) = k$ and $\ell(s_2a) = \ell(s_1b) = k+1$. We see by (6.8) that Y_{s_1b}, Y_{s_2a} is a basis dual to Y_b, Y_a . We get that the Lefschetz form on H^{m-1} is given by

$$\begin{pmatrix} \alpha_1^\vee & [k+1]_\zeta \alpha_1^\vee + [k]_\zeta \alpha_2^\vee \\ [k]_\zeta \alpha_1^\vee + [k+1]_\zeta \alpha_2^\vee & \alpha_2^\vee \end{pmatrix},$$

and $[k] = [k+1]$ is positive. For any λ as in the proposition, this is a symmetric matrix with strictly positive entries and negative determinant (by our calculation above). Hence its signature is zero and the Hodge-Riemann relations are satisfied as claimed. \square

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