SUPERCONGRUENCES FOR SPORADIC SEQUENCES

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Dedicated to Frits Beukers on the occasion of his 60th birthday

ABSTRACT. We prove two-term supercongruences for generalizations of recently discovered sporadic sequences of Cooper. We also discuss recent progress and future directions concerning other types of supercongruences.

1. INTRODUCTION

The term *supercongruence* first appeared in Beukers' work [4] and was the subject of the Ph.D. thesis of Coster [13]. It refers to the fact that congruences of a certain type are stronger than those suggested by formal group theory. A motivating example in [4] and [13] is the Apéry numbers

$$(1) \qquad A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

which not only satisfy [17]

$$(2) \qquad A(mp) \equiv A(m) \pmod{p^3},$$

but the *two-term supercongruence* [13]

$$(3) \qquad A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}$$

for primes $p \geqslant 5$ and integers m, $r \geqslant 1$. In 1985, Beukers related these numbers to the p-th Fourier coefficient $a(p)$ of $\eta^4(2z)\eta^4(4z)$, where

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

is the Dedekind eta function and $q = e^{2\pi i z}$, with z in the upper half-plane. He proved that [5]

$$(4) \qquad A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p}$$

and then conjectured that (4) holds modulo p^2. In [2], Ahlgren and Ono proved this *modular supercongruence* using Gaussian hypergeometric series [21]. The techniques in [2] have been the basis for several recent results of this type (see [16], [25], [27], [29]–[32], [36]). Other types of supercongruences are

also of considerable interest. *Ramanujan-type supercongruences* are p-adic versions of formulas of Ramanujan which relate binomial sums to special values of the gamma function (or $1/\pi^a$, $a \geqslant 1$). For example, van Hamme [39] conjectured that for Ramanujan's formula

$$\sum_{k=0}^{\infty}(4k+1)\binom{-1/2}{k}^5 = \frac{2}{\Gamma(3/4)^4},$$

we have the p-adic analogue
(5)
$$\sum_{k=0}^{(p-1)/2}(4k+1)\binom{-1/2}{k}^5 \equiv \left\{ \begin{array}{ll} -\frac{p}{\Gamma_p(3/4)^4} & (\mathrm{mod}\ p^3), \quad \text{if } p \equiv 1 \pmod 4, \\ 0 & (\mathrm{mod}\ p^3), \quad \text{if } p \equiv 3 \pmod 4, \end{array} \right.$$

where $\Gamma_p(\cdot)$ is the p-adic gamma function. For a proof of (5), see [28]. For recent progress in this direction, see [11], [16], [22], [23], [26], [33] or [41]. Finally, *Atkin–Swinnerton-Dyer* supercongruences have been recently studied in [14], [15], [24] and [37].

In this paper, we consider the sequences of numbers given by

$$(6) \qquad s_7(n) = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}\binom{2k}{n}$$

as well as
(7)
$$s_{18}(n) = \sum_{k=0}^{[n/3]}(-1)^k\binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}\left[\binom{2n-3k-1}{n} + \binom{2n-3k}{n}\right],$$

with $s_{18}(0) = 1$. These "sporadic" sequences were recently discovered by Cooper [12] while performing a numerical search for sequences which appear as coefficients of series for $1/\pi$ and of series expansions in t of modular forms where t is a modular function. Here, the subscripts 7 and 18 are used in (6) and (7) as the associated modular function is of level 7 and 18, respectively (see Theorem 3.1 in [12]). In [12], Cooper searched for parameters (a, b, c, d) such that the recurrence relation

$$(8) \quad (n+1)^3 s(n+1) = (2n+1)(an^2 + an + b)s(n) - n(cn^2 + d)s(n-1),$$

with initial conditions $s(-1) = 0$, $s(0) = 1$, produces only integer values $s(n)$ for all $n \geqslant 0$. The tuple $(17, 5, 1, 0)$ corresponds to the Apéry numbers (1), while the tuples $(13, 4, -27, 3)$ and $(14, 6, -192, 12)$ correspond to the sequences $s_7(n)$ and $s_{18}(n)$, respectively. See [3] for the case $d = 0$. This search was motivated by Beukers' [6] and Zagier's [40] work on sequences $t(n)$ defined by

$$(9) \qquad (n+1)^2 t(n+1) = (an^2 + an + b)t(n) - cn^2 t(n-1),$$

with initial conditions $t(-1) = 0$, $t(0) = 1$, such that $t(n) \in \mathbb{Z}$ for all $n \geqslant 0$. Zagier's search yielded six sequences that are not either terminating,

polynomial, hypergeometric or Legendrian. These six sequences were called *sporadic*.

Interestingly, Cooper conjectured the following congruences (see Conjecture 5.1 in [12]) which are reminiscent of (2).

Conjecture 1.1. *For any prime $p \geqslant 3$,*

$$(10) \qquad s_7(mp) \equiv s_7(m) \pmod{p^3}.$$

Likewise, for any prime p,

$$(11) \qquad s_{18}(mp) \equiv s_{18}(m) \pmod{p^2}.$$

The purpose of this paper is to exhibit that (10) and (11) are special cases of general two-term supercongruences. For integers A, B, C, let

$$(12) \qquad \mathcal{S}(n; A, B, C) = \sum_{k=0}^{n} \binom{n}{k}^A \binom{n+k}{k}^B \binom{2k}{n}^C.$$

Note that this family of sequences includes the Apéry numbers as well as the sequence $s_7(n)$.

Our main results are the following supercongruences, the first of which, in particular, generalizes the supercongruence (3) for the Apéry numbers.

Theorem 1.2. *Let $A \geqslant 2$ and $B, C \geqslant 0$ be integers. For any integers m, $r \geqslant 1$ and primes $p \geqslant 5$, we have*

$$(13) \qquad \mathcal{S}(mp^r; A, B, C) \equiv \mathcal{S}(mp^{r-1}; A, B, C) \pmod{p^{3r}}.$$

Theorem 1.3. *For any integers m, $r \geqslant 1$ and any primes p, we have*

$$(14) \qquad s_{18}(mp^r) \equiv s_{18}(mp^{r-1}) \pmod{p^{2r}}.$$

Note that by taking $(A, B, C) = (2, 1, 1)$ and $r = 1$ in Theorem 1.2, we prove (10) of Conjecture 1.1 for primes $p \geqslant 5$. Moreover, Theorem 1.2 shows a general supercongruence for the sporadic sequence

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n}^2,$$

which is case (ϵ) in [3] (see also Table 2 in Section 4). On the other hand, we remark that Theorem 1.3 is considerably simpler than Theorem 1.2 because it suffices to consider each summand of the sum (7), defining $s_{18}(n)$, individually. In both cases, our proof of the congruences of Conjecture 1.1 relies on the presence of the binomial sums (6) and (7) which were discovered by Zudilin.

Finally, we should mention that Cooper [12] conjectures congruences similar to (10) for $p = 2$, as well as a stronger version of (11) for $p = 2, 3$. These conjectures as well as (10) for $p = 3$ remain open. Based on numerical evidence, we actually conjecture that

$$(15) \qquad s_7(m2^r) \equiv s_7(m2^{r-1}) \pmod{2^{3r+2}}$$

for $m \geqslant 4$ and

(16) $s_7(m3^r) \equiv s_7(m3^{r-1}) \pmod{3^{3r}}$

for any positive integer m, as well as

(17) $s_{18}(m2^r) \equiv s_{18}(m2^{r-1}) \pmod{2^{2r+3}}$

for $m \geqslant 2$ and

(18) $s_{18}(m3^r) \equiv s_{18}(m3^{r-1}) \pmod{3^{3r-1}}$

for $m \geqslant 3$ (for $r = 1$ the congruence (18) empirically holds modulo 3^3). Slightly weaker congruences appear to hold in the cases when m is not large enough. We expect that these conjectures, which naturally generalize the ones from [12], can be established using the techniques we use in the case $p \geqslant 5$ when coupled with a careful and likely very technical analysis of the kind presented at the end of our proof of Theorem 1.3.

The remainder of the paper is organized as follows. Section 2 is devoted to the proofs of Theorems 1.2 and 1.3. We then indicate in Section 3 that these proofs readily generalize to other sequences of interest. In Section 4, we conclude with remarks concerning future directions. In particular, we discuss both proven and conjectural two-term supercongruences for all known sporadic sequences.

2. Proof of Theorems 1.2 and 1.3

Throughout this section, following [4], we let $\sum'$ denote the sum over indices not divisible by p.

We first recall the following version of Jacobsthal's binomial congruence [7]. For a proof, when $a, b \geqslant 0$, we refer to [18], [20], while the extension to negative integers is discussed in [38]. Similar congruences hold [18, 38] in the cases $p = 2$ and $p = 3$.

Lemma 2.1. *For primes $p \geqslant 5$, integers a, b and integers $r, s \geqslant 1$,*

(19) $\binom{p^r a}{p^s b} / \binom{p^{r-1}a}{p^{s-1}b} \equiv 1 \pmod{p^{r+s+\min(r,s)}}.$

We will also make use, in the case $n = -2$, of the following simple congruences.

Lemma 2.2. *Let p be a prime and n an integer such that $n \not\equiv 0$ modulo $p - 1$. Then, for all integers $r \geqslant 0$,*

(20) $\displaystyle\sum_{k=1}^{p^r-1}{}' k^n \equiv 0 \pmod{p^r}.$

If, additionally, n is even, then, for primes $p \geqslant 5$,

(21) $\displaystyle\sum_{k=1}^{(p^r-1)/2}{}' \frac{1}{k^n} \equiv 0 \pmod{p^r}.$

Proof. Since $n \not\equiv 0$ modulo $p - 1$, we find an integer λ, not divisible by p, such that $\lambda^n \not\equiv 1$ modulo p. Then,

$$\lambda^n \sum_{k=1}^{p^r-1}{}' k^n = \sum_{k=1}^{p^r-1}{}' (\lambda k)^n \equiv \sum_{k=1}^{p^r-1}{}' k^n \pmod{p^r},$$

since the second and third sum run over the same residues modulo p^r. As λ^n is not divisible by p, the congruence (20) follows.

Congruence (21) follows since the sum in (21), modulo p^r, is exactly half of the sum in (20) if n is even. $\qquad\square$

Lemma 2.3. *For integers n, $k \geqslant 1$ and $A, B, C \geqslant 0$, define*

$$\mathcal{B}(n, k) = \mathcal{B}(n, k; A, B, C) = \binom{n}{k}^A \binom{n+k}{k}^B \binom{2k}{n}^C.$$

Then, for primes $p \geqslant 5$ and integers $A \geqslant 2$, $r, s \geqslant 1$ and $k \geqslant 0$ such that $p \nmid k$,

(22) $$\mathcal{B}(np^r, kp^s) \equiv \mathcal{B}(np^{r-1}, kp^{s-1}) \pmod{p^{3r}}.$$

Proof. By Jacobsthal's congruence (19), we have

$$\binom{np^r}{kp^s} \Big/ \binom{np^{r-1}}{kp^{s-1}} \equiv 1 \pmod{p^{r+s+\min(r,s)}}$$

as well as

$$\binom{np^r + kp^s}{np^r} \Big/ \binom{np^{r-1} + kp^{s-1}}{np^{r-1}} \equiv 1 \pmod{p^{r+2\min(r,s)}}$$

and

$$\binom{2kp^s}{np^r} \Big/ \binom{2kp^{s-1}}{np^{r-1}} \equiv 1 \pmod{p^{r+s+\min(r,s)}}.$$

Thus, if $s \geqslant r$ then congruence (22) follows immediately upon applying Jacobsthal's congruence to each binomial coefficient. On the other hand, suppose $s \leqslant r$. Then the same approach yields

(23) $$\mathcal{B}(np^r, kp^s) = \lambda \mathcal{B}(np^{r-1}, kp^{s-1})$$

with $\lambda \equiv 1$ modulo p^{r+2s}. Moreover, since $p \nmid k$, we have

$$\binom{np^r}{kp^s} \equiv \binom{np^{r-1}}{kp^{s-1}} \equiv 0 \pmod{p^{r-s}}.$$

As $A \geqslant 2$, it follows that $p^{2(r-s)}$ divides $\mathcal{B}(np^r, kp^s)$. Since $r + 2s + 2(r - s) = 3r$, congruence (22) follows from (23). $\qquad\square$

In the following, $[x]$ denotes the largest integer m such that $m \leqslant x$.

Lemma 2.4. *For primes p, integers m and integers $k \geqslant 0$, $r \geqslant 1$,*

(24) $$\binom{mp^r - 1}{k}(-1)^k \equiv \binom{mp^{r-1} - 1}{[k/p]}(-1)^{[k/p]} \pmod{p^r}.$$

Proof. Following [4, Lemma 2], we split the defining product of the binomial coefficient, according to whether the index is divisible by p or not, to obtain

$$\binom{mp^r - 1}{k} = \prod_{j=1}^{k} \frac{mp^r - j}{j}$$

$$= \prod_{\substack{j=1 \\ p \nmid j}}^{k} \frac{mp^r - j}{j} \prod_{\lambda=1}^{[k/p]} \frac{mp^{r-1} - \lambda}{\lambda}$$

$$= \binom{mp^{r-1} - 1}{[k/p]} \prod_{\substack{j=1 \\ p \nmid j}}^{k} \frac{mp^r - j}{j}.$$

The claim follows upon reducing modulo p^r. □

Lemma 2.5. *For integers n, k, j and $A, B, C \geqslant 0$, define*

$$(25) \qquad \mathcal{C}(n, k, j) = \mathcal{C}(n, k, j; A, B, C) = \binom{n-1}{k}^A \binom{n+k}{k}^B \binom{j}{n}^C.$$

Then, for primes p and integers $n, k, j \geqslant 0$, $r \geqslant 1$,

$$\mathcal{C}(np^r, k, j) \equiv (-1)^{(k+[k/p])A} \mathcal{C}(np^{r-1}, [k/p], [j/p]) \pmod{p^r}.$$

Proof. Note that

$$(26) \qquad \binom{n+k}{k} = (-1)^k \binom{-n-1}{k}.$$

Using Lemma 2.4, we find that

$$(-1)^k \binom{-np^r - 1}{k} \equiv (-1)^{[k/p]} \binom{-np^{r-1} - 1}{[k/p]} \pmod{p^r}$$

or, equivalently,

$$(27) \qquad \binom{np^r + k}{k} \equiv \binom{np^{r-1} + [k/p]}{[k/p]} \pmod{p^r}.$$

In particular,

$$\binom{np^r + (j - np^r)}{j - np^r} \equiv \binom{np^{r-1} + [j/p] - np^{r-1}}{[j/p] - np^{r-1}} \pmod{p^r},$$

which is equivalent to

$$(28) \qquad \binom{j}{np^r} \equiv \binom{[j/p]}{np^{r-1}} \pmod{p^r}.$$

The proof thus follows upon combining (24), (27), (28). □

Proof of Theorem 1.2. Adapting the original approaches of [4] and [17], we split the binomial sum (12) as

$$\mathcal{S}(mp^r; A, B, C) = \sum_{s \geqslant 0} G_s(mp^r),$$

where

$$G_s(n) = \sum_k{}' \mathcal{B}(n, kp^s).$$

It follows from Lemma 2.3 that, for $s \geqslant 1$,

$$G_s(mp^r) \equiv G_{s-1}(mp^{r-1}) \pmod{p^{3r}}.$$

It therefore remains to show that

$$(29) \qquad G_0(mp^r) = \sum_k{}' \mathcal{B}(mp^r, k) \equiv 0 \pmod{p^{3r}}.$$

Note that

$$(30) \qquad \binom{mp^r}{k} = \frac{mp^r}{k}\binom{mp^r - 1}{k - 1}$$

is divisible by p^r if $p \nmid k$. Hence, if $A \geqslant 3$ then (29) is obviously true and (13) follows.

In the remainder, we consider the case $A = 2$. With (30) substituted into (29), we find that we need to show that

$$(31) \qquad \sum_k{}' \frac{1}{k^2}\binom{mp^r - 1}{k - 1}^2 \binom{mp^r + k}{k}^B \binom{2k}{mp^r}^C \equiv 0 \pmod{p^r}.$$

If $p \nmid k$ then $[(k-1)/p] = [k/p]$ so that, by Lemma 2.4,

$$(32) \qquad \binom{mp^r - 1}{k - 1}^2 \equiv \binom{mp^{r-1} - 1}{[k/p]}^2 \pmod{p^r}.$$

By (32) and Lemma 2.5 with $A = 0$, the left-hand side of (31) is congruent modulo p^r to

$$(33) \qquad \sum_k{}' \frac{1}{k^2}\binom{mp^{r-1} - 1}{[k/p]}^2 \binom{mp^{r-1} + [k/p]}{[k/p]}^B \binom{[2k/p]}{mp^{r-1}}^C.$$

Using the notation of (25) and (33), congruence (31) is equivalent to

$$(34) \qquad \sum_k{}' \frac{1}{k^2}\mathcal{C}(mp^{r-1}, [k/p], [2k/p]) \equiv 0 \pmod{p^r}.$$

In order to establish (34), we now show that

$$(35) \qquad \sum_k{}' \frac{1}{k^2}\mathcal{C}(mp^r, k, 2k) \equiv \sum_k{}' \frac{1}{k^2}\mathcal{C}(mp^{r-s}, [k/p^s], [2k/p^s]) \pmod{p^r}$$

for $s = 0, 1, \ldots, r$. The case $s = 0$ is trivial, while the case $s = 1$ follows from Lemma 2.5. If we now let $\{k : p^s\} := k - p^s[k/p^s]$, the remainder of k divided by p^s, then observe that

$$[2k/p^s] = 2[k/p^s] + \begin{cases} 1, & \text{if } \{k : p^s\} > p^s/2, \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$\sideset{}{'}\sum_{k} \frac{1}{k^2} \mathcal{C}(mp^{r-s}, [k/p^s], [2k/p^s]) = \sum_{n} \sideset{}{'}\sum_{[k/p^s]=n} \frac{1}{k^2} \mathcal{C}(mp^{r-s}, [k/p^s], [2k/p^s])$$

$$= \sum_{n} \mathcal{C}(mp^{r-s}, n, 2n) \sideset{}{'}\sum_{\substack{[k/p^s]=n \\ \{k:p^s\}<p^s/2}} \frac{1}{k^2}$$

(36)
$$+ \sum_{n} \mathcal{C}(mp^{r-s}, n, 2n+1) \sideset{}{'}\sum_{\substack{[k/p^s]=n \\ \{k:p^s\}>p^s/2}} \frac{1}{k^2}.$$

It follows from (21) of Lemma 2.2 that each of the inner sums in the last expression of (36) is divisible by p^s. Suppose that $s < r$. Thus, by (36) and Lemma 2.5, we now have

$$\sideset{}{'}\sum_{k} \frac{1}{k^2} \mathcal{C}(mp^{r-s}, [k/p^s], [2k/p^s])$$

$$\equiv \sum_{n} \mathcal{C}(mp^{r-s-1}, [n/p], [2n/p]) \sideset{}{'}\sum_{\substack{[k/p^s]=n \\ \{k:p^s\}<p^s/2}} \frac{1}{k^2}$$

$$+ \sum_{n} \mathcal{C}(mp^{r-s-1}, [n/p], [(2n+1)/p]) \sideset{}{'}\sum_{\substack{[k/p^s]=n \\ \{k:p^s\}>p^s/2}} \frac{1}{k^2} \pmod{p^r}$$

$$\equiv \sum_{n} \sideset{}{'}\sum_{\substack{[k/p^s]=n \\ \{k:p^s\}<p^s/2}} \frac{1}{k^2} \mathcal{C}(mp^{r-s-1}, [[k/p^s]/p], [[2k/p^s]/p])$$

$$+ \sum_{n} \sideset{}{'}\sum_{\substack{[k/p^s]=n \\ \{k:p^s\}>p^s/2}} \frac{1}{k^2} \mathcal{C}(mp^{r-s-1}, [[k/p^s]/p], [[2k/p^s]/p]) \pmod{p^r}$$

$$\equiv \sideset{}{'}\sum_{k} \frac{1}{k^2} \mathcal{C}(mp^{r-s-1}, [k/p^{s+1}], [2k/p^{s+1}]) \pmod{p^r}.$$

Hence (35) follows by induction on s. Moreover, the case $s = r$ in (36) shows that, for $s = r$ and hence all $s = 0, 1, \ldots, r$, the sums in (35) are divisible by p^r. In particular, the case $s = 1$ proves (34) and thus (13). $\qquad\square$

Proof of Theorem 1.3. As in [35], we use the identity

$$\binom{a-b}{c-d}\binom{b}{d} = \frac{\binom{a}{c}\binom{c}{d}\binom{a-c}{b-d}}{\binom{a}{b}},$$

to obtain that

$$\binom{2k}{k}\binom{2(n-k)}{n-k} = \frac{\binom{2n}{n}\binom{n}{k}^2}{\binom{2n}{2k}} = \binom{n}{k}\frac{(2k)!(2(n-k))!}{n!k!(n-k)!} = \binom{n}{k}S(n-k,k),$$

where $S(m,n)$ are the *super Catalan numbers*

$$S(m,n) = \frac{(2m)!(2n)!}{m!n!(m+n)!}.$$

We refer to [19] and the references therein for the history and properties of these numbers. Here, we only need that $S(n-k,k)$ is an integer; in fact, it is an even integer if $n \geqslant 1$.

Denote the summand of (7) by $\mathcal{D}(n,k)$, that is

$$\mathcal{D}(n,k) = (-1)^k \binom{n}{k}^2 S(n-k,k) \left[\binom{2n-3k-1}{n} + \binom{2n-3k}{n}\right],$$

so that

$$s_{18}(mp^r) = \sum_{s\geqslant 0}\sideset{}{'}\sum_{k} \mathcal{D}(mp^r, kp^s).$$

In analogy with Lemma 2.3, we claim that, for primes $p \geqslant 5$ and integers $r, s \geqslant 1$,

(37) $$\mathcal{D}(mp^r, kp^s) \equiv \mathcal{D}(mp^{r-1}, kp^{s-1}) \pmod{p^{3r}}.$$

A direct application of Lemma 2.1 shows that

$$\binom{2mp^r - 3kp^s}{mp^r} \bigg/ \binom{2mp^{r-1} - 3kp^{s-1}}{mp^{r-1}} \equiv 1 \pmod{p^{r+2\min(r,s)}}$$

as well as

$$S(mp^r - kp^s, kp^s)/S(mp^{r-1} - kp^{s-1}, kp^{s-1}) \equiv 1 \pmod{p^{r+2\min(r,s)}}.$$

On the other hand, for all integers a and integers $b \geqslant 0$,

(38) $$\binom{a+b}{b} = (-1)^b \binom{-a-1}{b},$$

so that

$$\binom{2n-3k-1}{n} = (-1)^n \binom{3k-n}{n}.$$

Hence, Lemma 2.1 implies that

$$\binom{2mp^r - 3kp^s - 1}{mp^r} \bigg/ \binom{2mp^{r-1} - 3kp^{s-1} - 1}{mp^{r-1}} \equiv 1 \pmod{p^{r+2\min(r,s)}}.$$

Proceeding as in the proof of Lemma 2.3, we therefore obtain (37). The congruence (14) then follows if we can prove that, for integers k such that $p \nmid k$,

$$\mathcal{D}(mp^r, k) \equiv 0 \pmod{p^{2r}}.$$

This is an immediate consequence of the fact that $\mathcal{D}(n, k)$ is divisible by $\binom{n}{k}^2$.

Finally, let us briefly indicate how to obtain the corresponding congruences in the case that $p = 2$ or $p = 3$. For $p = 3$, congruence (19) of Lemma 2.1 only holds modulo $p^{r+s+\min(r,s)-1}$. The same arguments as above then show that congruence (37) holds modulo p^{3r-1}, and hence modulo p^{2r}, for $p = 3$. The case $p = 2$ requires some more attention. In that case, the counterpart of congruence (19) is

$$\binom{2^r a}{2^s b} / \binom{2^{r-1} a}{2^{s-1} b} \equiv \varepsilon \pmod{2^{r+s+\min(r,s)-2}}$$

with $\varepsilon = -1$, if $2^{r-1}a \equiv 0$, $2^{s-1}b \equiv 1$ modulo 2, and $\varepsilon = 1$ otherwise. Hence, applying the same arguments as for $p > 2$ shows that

$$\mathcal{D}(2^r m, 2^s k) = \lambda \mathcal{D}(2^{r-1} m, 2^{s-1} k)$$

with $\lambda \equiv \pm 1$ modulo $2^{r+2\min(r,s)-2}$ and, hence, $\lambda \equiv 1$ modulo 2. Moreover, both sides are divisible by $2^{2\max(r-s,0)+1}$ because $\binom{2^r m}{2^s k}$ is divisible by 2^{r-s}, if $r \geqslant s$, and the super Catalan numbers $S(n-k, k)$ are even when $n \geqslant 1$. In the cases $r = 1$ or $s = 1$, this suffices to conclude that (37) holds modulo p^{2r} for $p = 2$. On the other hand, if $r \geqslant 2$ and $s \geqslant 2$, then going through the above computations reveals that $\lambda \equiv 1$ modulo $2^{r+2\min(r,s)-2}$, and hence modulo $2^{r+\min(r,s)}$. Together with the divisibility of $\mathcal{D}(2^r m, 2^s k)$ by 2^{r-s}, if $r \geqslant s$, we again find that (37) holds modulo p^{2r} for $p = 2$. $\square$

3. COMMENTS ON DIRECT GENERALIZATIONS

The approaches of the proof of Theorems 1.2 and 1.3, which are based on [17] and [4], generalize easily to other sequences.

Example 3.1. For instance, consider the sequence

$$(39) \qquad Z(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n} \right),$$

which is case (η) in [3] (see also Table 2 in Section 4). We claim that the proof of Theorem 1.3 naturally extends to show that, for primes $p \geqslant 5$,

$$(40) \qquad\qquad Z(mp^r) \equiv Z(mp^{r-1}) \pmod{p^{3r}}.$$

Similar to the proof of Theorem 1.3, write

$$Z(mp^r) = \sum_{s \geqslant 0} {\sum_k}' \mathcal{A}(mp^r, kp^s),$$

where

$$\mathcal{A}(n,k) = (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n} \right).$$

As in the proof of Theorem 1.3, we find that for primes $p \geqslant 5$,

$$\mathcal{A}(mp^r, kp^s) \equiv \mathcal{A}(mp^{r-1}, kp^{s-1}) \pmod{p^{3r}}.$$

On the other hand, the presence of $\binom{n}{k}^3$ shows that, for $p \nmid k$,

$$\mathcal{A}(mp^r, k) \equiv 0 \pmod{p^{3r}}.$$

Combining these two congruences, we conclude that the supercongruence (40) indeed holds.

Example 3.2. The proof of Theorem 1.3 directly generalizes to supercongruences for the following family of sequences which includes $s_{18}(n)$ as the case $(A, B, C, D, E) = (1, 1, 1, 1, 1)$. For nonnegative integers A, B, C, D, E, define $\mathcal{T}(n; A, B, C, D, E)$ to be the sequence

$$\sum_{k=0}^{[n/3]} (-1)^k \binom{n}{k}^A \binom{2k}{k}^B \binom{2(n-k)}{n-k}^C \left[\binom{2n-3k-1}{n}^D + \binom{2n-3k}{n}^E \right].$$

If $A \geqslant 1$, $B \geqslant 1$ and $C \geqslant 1$, then

$$\mathcal{T}(mp^r; A, B, C, D, E) \equiv \mathcal{T}(mp^{r-1}; A, B, C, D, E) \pmod{p^{2r}}$$

for all primes p. More generally, we have, for instance, that if $A \geqslant 2$, $B \geqslant 1$ and $C \geqslant 1$, then

$$\mathcal{T}(mp^r; A, B, C, D, E) \equiv \mathcal{T}(mp^{r-1}; A, B, C, D, E) \pmod{p^{3r}}$$

for all primes $p \geqslant 5$.

Example 3.3. In [8] and [9], the sequence

$$u(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n}{k}$$

is studied. In particular, it is proved that $u(mp) \equiv u(m)$ modulo p^3 for all primes $p \geqslant 5$. Following the approach of Theorem 1.2, we obtain that, more generally,

$$u(mp^r) \equiv u(mp^{r-1}) \pmod{p^{3r}}$$

for all primes $p \geqslant 5$. Let a, b be nonnegative integers. It is shown in [8] that the more general sequences

$$u_{a,b}^{\varepsilon}(n) = \sum_{k=0}^{n} (-1)^{\varepsilon k} \binom{n}{k}^a \binom{2n}{k}^b$$

satisfy, for primes $p \geqslant 5$, the congruence

$$u_{a,b}^{\varepsilon}(p) \equiv u_{a,b}^{\varepsilon}(1) \pmod{p^3}$$

unless $(\varepsilon, a, b) = (0, 0, 1)$ or $(0, 1, 0)$. Again, we can generalize this congruence by using the approach of Theorem 1.2 to show that, for primes $p \geqslant 5$,

$$u_{a,b}^{\varepsilon}(mp^r) \equiv u_{a,b}(mp^{r-1}) \pmod{p^{3r}}$$

provided that $a + b \geqslant 2$.

4. Concluding remarks

There are several directions for future work. First, we have numerically checked that each of the six sporadic examples of Zagier (labelled **A**, **B**, **C**, **D**, **E**, **F** in [40]) and the six sporadic examples in [3] (labelled (δ), (η), (α), (ϵ), (ζ), (γ)) satisfies a two-term supercongruence. Precisely, cases **A** and **D** are modulo p^{3r} while cases **B**, **C**, **E**, **F** are modulo p^{2r}. All six cases from [3] are modulo p^{3r}. Cases **A** and **D** have been proven by Coster [13] and cases **C** and **E** were settled by the first two authors in [34] and [35]. Cases (α) and (γ) have been proven in [35] and [13], respectively. As mentioned in the introduction, the proof of Theorem 1.2 implies case (ϵ). On the other hand, case (η) is proven in Example 3.1 of Section 3. For a discussion concerning connections between the six sporadic examples in [3] and [40], see Theorem 4.1 in [3] or Theorem 3.5 and Tables 1 and 2 in [10].

The information on supercongruences for Apéry-like numbers is summarized in Tables 1 and 2. The value for k indicates that the sequence $A(n)$ (at least conjecturally in cases **B**, **F**, (δ) and (ζ)) satisfies the supercongruence

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{kr}}$$

for primes $p \geqslant 5$. In the cases where this congruence has been proven, a reference is indicated in the final column. It would be of interest to prove cases **B**, **F**, (δ) and (ζ) and, more generally, provide a framework for all two-term supercongruences.

Secondly, for each of the 15 sporadic cases in Tables 1 and 2, one could ask if there exists a modular form $f(z)$ whose p-th Fourier coefficient satisfies a modular supercongruence. This is true for cases **D** [1] and (γ) [2].

Finally, it also appears that all known Ramanujan-type series for $1/\pi^a$, $a \geqslant 1$, have a p-adic analogue which satisfies a Ramanujan-type supercongruence. Different techniques have been employed as there is currently no general explanation for this occurrence. For example, the first author and McCarthy [28] utilized Gaussian hypergeometric series and Whipple's transformation to prove (5). Zudilin [41] proved several Ramanujan-type supercongruences using the Wilf–Zeilberger method while Long [26] used a combination of combinatorial identities, p-adic analysis and transformations and "strange" evaluations of ordinary hypergeometric series due to Whipple, Gessel and Gosper.

(a,b,c)	[40]	[3]	$A(n)$	k	
$(7,2,-8)$	**A**	(a)	$\displaystyle\sum_k \binom{n}{k}^3$	3	[13]
$(11,3,-1)$	**D**	(b)	$\displaystyle\sum_k \binom{n}{k}^2\binom{n+k}{n}$	3	[13]
$(10,3,9)$	**C**	(c)	$\displaystyle\sum_k \binom{n}{k}^2\binom{2k}{k}$	2	[34]
$(12,4,32)$	**E**	(d)	$\displaystyle\sum_k \binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}$	2	[35]
$(9,3,27)$	**B**	(f)	$\displaystyle\sum_k (-1)^k 3^{n-3k}\binom{n}{3k}\frac{(3k)!}{k!^3}$	2	
$(17,6,72)$	**F**	(g)	$\displaystyle\sum_{k,l}(-1)^k 8^{n-k}\binom{n}{k}\binom{k}{l}^3$	2	

TABLE 1. Sporadic sequences from [40] for (9)

(a,b,c,d)	[3], [12]	$A(n)$	k	
$(7,3,81,0)$	(δ)	$\displaystyle\sum_k (-1)^k 3^{n-3k}\binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3}$	3	
$(11,5,125,0)$	(η)	defined in (39)	3	(40)
$(10,4,64,0)$	(α)	$\displaystyle\sum_k \binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$	3	[35]
$(12,4,16,0)$	(ϵ)	$\displaystyle\sum_k \binom{n}{k}^2\binom{2k}{n}^2$	3	(13)
$(9,3,-27,0)$	(ζ)	$\displaystyle\sum_{k,l} \binom{n}{k}^2\binom{n}{l}\binom{k}{l}\binom{k+l}{n}$	3	
$(17,5,1,0)$	(γ)	$\displaystyle\sum_k \binom{n}{k}^2\binom{n+k}{n}^2$	3	[13]
$(6,2,-64,4)$	s_{10}	$\displaystyle\sum_k \binom{n}{k}^4$	3	[13]
$(13,4,-27,3)$	s_7	$\displaystyle\sum_k \binom{n}{k}^2\binom{n+k}{k}\binom{2k}{n}$	3	(13)
$(14,6,-192,12)$	s_{18}	defined in (7)	2	(14)

TABLE 2. Sporadic sequences from [3] and [12] for (8)

wonderful working conditions. Finally, we thank the referee for their helpful comments and suggestions.

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