CURVATURE ON THE INTEGERS, I

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ABSTRACT. Starting with a symmetric/antisymmetric matrix with integer co-efficients (which we view as an analogue of a metric/form on a principal bundle over the "manifold" $Spec\ \mathbb{Z}$) we introduce arithmetic analogues of Chern con-nections and their curvature (in which usual partial derivative operators act-ing on functions are replaced by Fermat quotient operators acting on integer numbers); curvature is introduced via the method of "analytic continuation between primes" [6]. We prove various non-vanishing, respectively vanishing results for curvature; morally, $Spec\ \mathbb{Z}$ will appear as "intrinsically curved." Along with [9, 10, 11], this theory can be viewed as taking first steps in devel-oping a "differential geometry of $Spec\ \mathbb{Z}$".

1. INTRODUCTION

In the present paper we will view $\mathbb{Z}$ as an analogue of a ring of "algebraic/analytic" functions on an infinite dimensional manifold in which the various "directions" cor-respond to the primes; C^{∞} objects in geometry will then roughly correspond to adelic objects in arithmetic. In the spirit of [3, 7, 6], the partial derivative opera-tors, acting on functions, will be replaced by Fermat quotient type operators (called *p-derivations*), acting on numbers. We will then want to develop an arithmetic ana-logue of connections and curvature on $Spec\ \mathbb{Z}$. In our paper the analogue of C^{∞} connections on $Spec\ \mathbb{Z}$ will be families (δ_p) indexed by rational primes p (belonging to a fixed, possibly infinite, set $\mathcal{V}$) where for each p, δ_p is a p-derivation on the p-adic completion of the ring $\mathcal{O}(GL_n)$ (where GL_n is the general linear group scheme over some ring of cyclotomic integers). Families (δ_p) as above will be referred to as *adelic connections.* One immediately encounters, however, the following difficulty: the operators δ_p, for various p's, do not act a priori on the same ring and, hence, one cannot directly consider their commutator; consequently the notion of curvature, which should correspond to such a family of commutators, seems problematic. We will overcome this problem in two ways: the first, to be explained in the present paper, is by implementing the technique of *analytic continuation between primes* introduced in [6]; the second way, to be explained in a sequel to this paper [12], is by *algebraization of Frobenius lifts by correspondences*, a method that has a "bi-rational/motivic" flavor. In both paradigms a notion of curvature can then be developed. Our main results, in this paper and its sequel [12], are theorems about the vanishing/non-vanishing of the curvatures of some remarkable adelic connec-tions, referred to as *Chern connections*, arising from [10]; these Chern connections are naturally attached to symmetric/antisymmetric matrices (equivalently to outer involutions of GL_n defining various forms of the classical groups SO_n and Sp_n), and are an arithmetic analogue of Chern connections [15, 18] attached to metrics/forms in differential geometry.

It is worth pointing out here that what we call *curvature* in our context is entirely different from what is called *p-curvature* in the arithmetic theory of differential equations that has been developed around the Grothendieck conjecture (cf., e.g., [17]); indeed our curvature here is about the "*p*-differentiation" of numbers with respect to primes p (in other words it is about d/dp) whereas the theory in [17] (and related papers) is about usual differentiation d/dt with respect to a variable t of power series in t with arithmetically interesting coefficients. In spite of these differences the two types of curvatures could interact; a model for such an interaction between d/dp and d/dt is in the papers [4, 5].

Along with [9, 10, 11], the present paper and its sequel [12] can be viewed as taking first steps in developing a "differential geometry of $Spec\ \mathbb{Z}$." The theory can be further developed by considering various (analogues of) contractions of curvature such as Ricci, mean, and scalar curvature; this will be done elsewhere. Another direction is the arithmetic analogue of the symplectic/Hamiltonian story introduced in [8]. The whole project is closely related to Borger's viewpoint on the field with one element [2]; cf. also the Introduction to [7].

Our paper is organized as follows. In section 2 we introduce our main arithmetic concepts preceded by a motivational discussion of the corresponding classical differential geometric concepts. In section 3 we state our main results. In section 4 we prove our results. In section 5 we add some remarks on the case when $\mathcal{V}$ consists of *one prime p only*; this case becomes, through the consideration of the p-derivations δ_p and "$\delta_{\overline{p}}$," analogous to the consideration of connections of vector (or principal) bundles on complex curves.

2. MAIN CONCEPTS

We start with a discussion of some classical concepts in differential geometry. We then proceed to introduce their arithmetic analogues.

2.1. **Classical differential geometry.** As a motivation for our arithmetic theory we "recall" some basic concepts from classical differential geometry. From a logical viewpoint this subsection will not play any role later. However our presentation of this classical material is not completely standard and the twist on the standard picture is meant to show the way to the arithmetic case.

2.1.1. *Connections in principal bundles.* Let A be a ring (commutative, with identity) and let $\delta_1, ..., \delta_m$ be commuting derivations

$$(2.1) \qquad\qquad \delta_i : A \to A.$$

Let B be an A-algebra. By a *connection* on B (equivalently, on the scheme $Spec\ B$) we will understand an m-tuple $\delta = (\delta_i)$ where for each i, $\delta_i : B \to B$ is a derivation lifting the corresponding derivation on A. We are especially interested in the following special case. Denote by $G = GL_n = Spec\ B$, with $B := A[x, \det(x)^{-1}]$,

the general linear group scheme over A, where $x = (x_{kl})$ is an $n \times n$ matrix of indeterminates. So $G(A) = GL_n(A)$ is the group of all invertible $n \times n$ matrices with entries in A and $\mathcal{O}(G) = A[x, \det(x)^{-1}]$. Denote by $\mathfrak{g} = Spec\ A[x]$ so $\mathfrak{g}(A)$ the additive group of all $n \times n$ matrices with entries in A; we view it both as associative algebra and as Lie algebra over A. By the above a *connection* on G is an m-tuple $\delta = (\delta_i)$, where

$$(2.2) \qquad\qquad\qquad \delta_i : \mathcal{O}(G) \to \mathcal{O}(G)$$

are derivations, extending the corresponding derivations on A. A connection $\delta = (\delta_i)$ on G will be called *right invariant* (or simply *invariant*) if for all i, $\delta_i x = A_i x$, where $A_i \in \mathfrak{g}(A)$. Here $\delta_i x$ is the matrix with entries $\delta_i x_{kl}$. More generally, in this paper, if u_{kl} are the entries of a matrix u with coefficients in a ring and f is a map of that ring into itself then we write $f(u)$ for the matrix $(f(u_{kl}))$ with entries $f(u)_{kl} = f(u_{kl})$.

2.1.2. *Curvature, 3-curvature, $(1,1)$-curvature.* For any connection $\delta = (\delta_i)$ on G one defines the *curvature* of δ as the family (φ_{ij}) where φ_{ij} are the A-derivations

$$(2.3) \qquad\qquad \varphi_{ij} := [\delta_i, \delta_j] := \delta_i \delta_j - \delta_j \delta_i : \mathcal{O}(G) \to \mathcal{O}(G).$$

If δ is invariant, with $\delta_i x = A_i x$, $A_i \in \mathfrak{g}(A)$, then

$$(2.4) \qquad\quad \varphi_{ij}(x) = F_{ij}x, \quad F_{ij} := \delta_i A_j - \delta_j A_i - [A_i, A_j] \in \mathfrak{g}(A),$$

where $[A_i, A_j] = A_i A_j - A_j A_i$ is the usual commutator of matrices. The matrix $F = (F_{ij})$ is also referred to, in this case, as the *curvature* of the invariant connection $\delta = (\delta_i)$.

For a connection $\delta = (\delta_i)$ one can define the 3-*curvature* of δ as the family (φ_{ijk}) of A-derivations

$$(2.5) \qquad\qquad \varphi_{ijk} := [\delta_i, [\delta_j, \delta_k]] = [\delta_i, \varphi_{jk}] : \mathcal{O}(G) \to \mathcal{O}(G).$$

For δ invariant as above, we have $\varphi_{ijk}(x) = (\delta_i F_{jk} - [A_i, F_{jk}])x$. In general, φ_{ijk} is antisymmetric in jk and satisfies the *Bianchi identity*: $\sum_{(ijk)} \varphi_{ijk} = 0$, where (ijk) indicates summation over the cyclic permutations of i, j, k. The 3-curvature plays a role in the Yang-Mills formalism [13].

The above discussion has a "$(1,1)$ analogue" as follows. Assume that we have two invariant connections on G,

$$\delta = (\delta_i)_{1 \le i \le m}, \quad \bar{\delta} = (\bar{\delta}_i)_{1 \le i \le m}$$

for two m-tuples of derivations on A such that $\{\delta_1, ..., \delta_m, \bar{\delta}_1, ..., \bar{\delta}_m\}$ are pairwise commuting on A; we then say that $\bar{\delta}$ *commutes* with δ on A. We write $\delta_{\bar{i}} := \bar{\delta}_i$. We have at our disposal the curvature (F_{ij}) of δ and also the curvature $(F_{\bar{i}\bar{j}})$ of $\bar{\delta}$. On the other hand we can consider the commutators

$$\varphi_{i\bar{j}} := [\delta_i, \delta_{\bar{j}}] : \mathcal{O}(G) \to \mathcal{O}(G).$$

If $\delta_i x = A_i x$, $\delta_{\bar{i}} x = A_{\bar{i}} x$, then

$$\varphi_{i\bar{j}}(x) = F_{i\bar{j}}x, \quad F_{i\bar{j}} := \delta_i A_{\bar{j}} - \delta_{\bar{j}} A_i - [A_i, A_{\bar{j}}].$$

We call $(\varphi_{i\bar{j}})$, or $(F_{i\bar{j}})$, the $(1,1)$-*curvature* of δ with respect to $\bar{\delta}$.

2.1.3. *Chern connections.* Let $q \in G(A) = GL_n(A)$ be a symmetric, respectively antisymmetric matrix; hence $q^t = \pm q$, where the t superscript means transpose. Consider the maps $\mathcal{H}_q : G \to G$, $\mathcal{B}_q : G \times G \to G$ defined by

$$\mathcal{H}_q(x) = x^t q x, \quad \mathcal{B}_q(x, y) = x^t q y.$$

Consider the *trivial connection* $\delta_0 = (\delta_{0i})$ on G defined by $\delta_{0i} x = 0$. Then there is a unique invariant connection $\delta = (\delta_i)$ on G (which we refer to as the *Chern connection* on G attached to q) such that the following diagrams are commutative:

$$
\begin{array}{cccc}
\mathcal{O}(G) & \xleftarrow{\delta_i} & \mathcal{O}(G) & \\
(2.6) \quad \mathcal{H}_q \uparrow & & \uparrow \mathcal{H}_q & \\
\mathcal{O}(G) & \xleftarrow{\delta_{0i}} & \mathcal{O}(G) &
\end{array}
\qquad
\begin{array}{cccc}
\mathcal{O}(G) & \xleftarrow{\delta_i \otimes 1 + 1 \otimes \delta_{0i}} & \mathcal{O}(G) \otimes_A \mathcal{O}(G) \\
\delta_{0i} \otimes 1 + 1 \otimes \delta_i \uparrow & & \uparrow \mathcal{B}_q \\
\mathcal{O}(G) \otimes_A \mathcal{O}(G) & \xleftarrow{\mathcal{B}_q} & \mathcal{O}(G)
\end{array}
$$

In the above diagram we still denoted by $\mathcal{H}_q$ and $\mathcal{B}_q$ the ring homomorphisms induced by $\mathcal{H}_q$ and $\mathcal{B}_q$ respectively. To explain this and subsequent remarks it is convenient to switch to some classical notation as follows. Let $\delta_i x = A_i x$, set $\Gamma_i := -A_i^t$, let Γ_{ij}^k be the (j, k)-entry of Γ_i (the *Cristoffel symbols*), and set $\Gamma_{ijk} := \Gamma_{ij}^l q_{lk}$ (Einstein summation). Then the commutativity of the left diagram in 2.6 is equivalent to

$$(2.7) \qquad \qquad \delta_i q_{jk} = \Gamma_{ijk} \pm \Gamma_{ikj};$$

classically one refers to this condition as the *compatibility of the connection with the metric.* On the other hand the commutativity of the right diagram in 2.6 is equivalent to

$$(2.8) \qquad \qquad \Gamma_{ijk} = \pm \Gamma_{ikj}.$$

So, given q, equations 2.7 and 2.8 have a unique solution Γ_{ijk}, given by

$$(2.9) \qquad \qquad \Gamma_{ijk} = \frac{1}{2} \delta_i q_{jk},$$

equivalently by

$$(2.10) \qquad \qquad A_i = -\frac{1}{2} q^{-1} \delta_i q.$$

Our Chern connection δ attached to q is an analogue of hermitian Chern connections on hermitian vector bundles, cf. [15], p. 73., and also of the Duistermaat connections [14] in the real case. These analogies are explained in detail in [10]. For the Chern connection δ attached to q the curvature is given by

$$(2.11) \qquad F_{ij} = \frac{1}{4} \{ q^{-1}(\delta_i q) q^{-1}(\delta_j q) - q^{-1}(\delta_j q) q^{-1}(\delta_i q) \}.$$

Note that F_{ij} depends only of the first (rather than the first and second) order derivatives of q; cf. the discussion in [14]. On the other hand assume $\bar{\delta}$ is a connection commuting with δ on A such that $\bar{\delta}_i x = 0$; then the $(1, 1)$-curvature of δ with respect to $\bar{\delta}$ is given by

$$(2.12) \qquad \qquad F_{i\bar{j}} = \frac{1}{2} \delta_{\bar{j}} (q^{-1} \delta_i q).$$

In particular, if $n = 1$, $\rho_{i\bar{j}} =$ "$\frac{1}{2} \delta_i \delta_{\bar{j}} \log q$". Note that the $(1, 1)$-curvature depends on the first *and* second derivatives of q.

Remark 2.1. We would like to compare here our Chern connection with the Levi-Civita connection. Assume the notation above and assume, in addition, that $n = m$. (More invariantly we assume that a bijection is given between the index set for the derivations δ_i and the index set for the variables x_{kl}; fixing such a bijection can be viewed as an analogue of Cartan's *soldering* [19], p. 351.) One classically makes the following definition: the connection δ on G is *torsion free* if

$$(2.13) \qquad\qquad \Gamma_{ijk} = \Gamma_{jik}.$$

Notice the difference between the symmetry conditions 2.8 and 2.13. By the way the system consisting of 2.7, 2.8, 2.13, (with q given) has a solution Γ_{ijk} if and only if q satisfies the following equations:

$$(2.14) \qquad\qquad \delta_i q_{jk} = \delta_j q_{ik}.$$

If q satisfies the equations 2.14 we will say q is *Hessian*.

If $q^t = q$, there is a unique invariant connection δ satisfying the equations 2.7 and 2.13; it is given by

$$(2.15) \qquad\qquad \Gamma_{kij} = \frac{1}{2} \left(\delta_k q_{ij} + \delta_i q_{jk} - \delta_j q_{ki} \right)$$

and it is called the *Levi-Civita* connection attached to q. So the Levi-Civita connection and the Chern connection for a given $q = q^t$ coincide if and only if q is Hessian. There is a $(1,1)$ version of this in which the Hessian condition is replaced by the *Kähler condition*; we will not go into this here.

If $q^t = -q$, an invariant connection δ will be called *symplectic* if it satisfies equations 2.7 and 2.13. A q with $q^t = -q$ will be called *symplectic* if

$$(2.16) \qquad\qquad \delta_i q_{jk} + \delta_j q_{ki} + \delta_k q_{ij} = 0.$$

It is trivial to see that if δ is symplectic then q must be symplectic. Also, in case $q^t = -q$, conditions 2.7, 2.8, 2.13 imply

$$(2.17) \qquad\qquad \delta_k q_{ij} = 0, \quad \Gamma_{ijk} = 0, \quad \delta = \delta_0,$$

where we recall that $\delta_0 x = 0$. The first equality follows by combining 2.14, 2.16; the second follows from 2.9; the third follows from the second. So, for $q^t = -q$, we have that δ is both symplectic and Chern if and only if $\delta q_{jk} = 0$ and $\delta = \delta_0$.

The purpose of the above review is to point out that the Chern connection story will have an arithmetic analogue while the Levi-Civita story and the symplectic story, by themselves, will not have an arithmetic analogue in our paper. However the conjunction of the Chern and the Levi-Civita conditions (i.e. the Hessian/Kähler condition) *will* have an arithmetic analogue. Also the conjunction of the Chern and symplectic conditions will *also* have an arithmetic analogue.

2.2. **Arithmetic differential geometry.** Recall from [3] (cf. also [16]) that a *p-derivation* on a *p*-torsion free ring B is a set theoretic map $\delta_p : B \to B$, such that the map $\phi_p : B \to B$ defined by $\phi_p(b) := b^p + p\delta_p(b)$ is a ring homomorphism. Note that ϕ_p lifts the *p*-power Frobenius on B/pB; we will say that ϕ_p is the *lift of Frobenius* attached to δ_p. Also, for any ring B we denote by $B^{\widehat{p}}$ the *p*-adic completion of B, i.e., the projective limit of the rings $B/p^n B$.

2.2.1. *Adelic connections on GL_n.* Let now A be the ring $\mathbb{Z}[1/M, \zeta_N]$ where M is some even integer and ζ_N is a primitive N-th root of unity, $N \geq 1$. Let $G = GL_n = Spec\ A[x, \det(x)^{-1}]$ be the general linear group scheme over A, where $x = (x_{kl})$ is an $n \times n$ matrix of indeterminates. So $\mathcal{O}(G) = A[x \det(x)^{-1}]$. Finally let $\mathcal{V}$ be a (possibly infinite) set of primes in $\mathbb{Z}$ not dividing MN. We keep this notation throughout the paper. Note that for any p not dividing MN there is a unique p-derivation on A; indeed for any such p there is a unique ring endomorphism of A lifting the p-power Frobenius on A/pA; this endomorphism sends ζ_N into ζ_N^p.

Definition 2.2. An *adelic connection* on G is a family (δ_p), indexed by the primes p in $\mathcal{V}$ where, for each p, δ_p is a p-derivation on the p-adic completion $\mathcal{O}(G)^{\widehat{p}}$ of $\mathcal{O}(G)$.

The concept of adelic connection we just introduced, being adelic rather than global, should be viewed as an analogue of the differential geometric concept of C^∞, rather than analytic, connection in a principal bundle. In contrast to analytic connections which lead to trivial cohomology classes [1], C^∞ connections encode non-trivial cohomological information [18] and hence our adelic connections could be viewed as encoding some cohomology-like information on $Spec\ \mathbb{Z}$.

Given an adelic connection (δ_p) we can consider the attached family (ϕ_p) of lifts of Frobenius on the rings $\mathcal{O}(G)^{\widehat{p}}$; we shall still denote by $\phi_p : G^{\widehat{p}} \to G^{\widehat{p}}$ the induced morphism of p-adic formal schemes where $G^{\widehat{p}}$ is the p-adic completion of G.

Let T be the matrix $x - 1$ where 1 is the identity matrix.

Definition 2.3. An adelic connection $\delta = (\delta_p)$ on G *fixes* 1 if, for all p, $\phi_p : \mathcal{O}(G)^{\widehat{p}} \to \mathcal{O}(G)^{\widehat{p}}$ sends the ideal of 1 into itself. (Note that if this is the case then there is an induced homomorphism $\phi_p : A^{\widehat{p}}[[T]] \to A^{\widehat{p}}[[T]]$ for each p.) An adelic connection $\delta = (\delta_p)$ is said to be *global along* 1 if it fixes 1 and the induced homomorphisms $\phi_p : A^{\widehat{p}}[[T]] \to A^{\widehat{p}}[[T]]$ send $A[[T]]$ into $A[[T]]$. (We then still denote by $\phi_p : A[[T]] \to A[[T]]$ the induced homomorphisms.)

Morally this fits into the ideology of "analytic continuation between primes introduced in [6].

2.2.2. *Curvature, 3-curvature, $(1,1)$-curvature.* For adelic connections that are global along 1 one can introduce various concepts of curvature.

Definition 2.4. Let δ be an adelic connection on G that is global along 1, with associated homomorphisms $\phi_p : A[[T]] \to A[[T]]$. The *curvature* of δ is the family $\varphi = (\varphi_{pp'})$, indexed by pairs of primes p, p' in $\mathcal{V}$, where $\varphi_{pp'}$ are the $\mathbb{Z}$-module endomorphisms

$$(2.18) \qquad \varphi_{pp'} := \frac{1}{pp'}[\phi_p, \phi_{p'}] : A[[T]] \to A[[T]].$$

Here $[\alpha, \beta]$ denotes the usual commutator $\alpha \circ \beta - \beta \circ \alpha$ of two $\mathbb{Z}$-module endomorphisms α, β of $A[[T]]$. If $I \subset A[[T]]$ is the ideal generated by T then for any integer $\nu \geq 1$ we denote by

$$(2.19) \qquad \varphi_{pp'}^{(\nu)} : I/I^{\nu+1} \to I/I^{\nu+1}$$

the $\mathbb{Z}$-module maps induced by $\varphi_{pp'}$.

Definition 2.5. Let δ be an adelic connection on G that is global along 1, with associated homomorphisms $\phi_p : A[[T]] \to A[[T]]$. The 3-*curvature* of δ is the family $(\varphi_{pp'p''})$, indexed by triples of primes p, p', p'' in $\mathcal{V}$, where $\varphi_{pp'p''}$ are the $\mathbb{Z}$-module endomorphisms

$$\varphi_{pp'p''} := [\phi_p, \varphi_{p'p''}] = \frac{1}{p'p''}[\phi_p, [\phi_{p'}, \phi_{p''}]] : A[[T]] \to A[[T]].$$

Definition 2.6. Let δ and $\overline{\delta}$ be two adelic connections on G that are global along 1, with associated homomorphisms $\phi_p : A[[T]] \to A[[T]]$ and $\phi_{\overline{p}} : A[[T]] \to A[[T]]$, respectively. The $(1,1)$-*curvature* of δ with respect to $\overline{\delta}$ is the family $\varphi = (\varphi_{p\overline{p}'})$, indexed by pairs of primes p, p' in $\mathcal{V}$, where $\varphi_{p\overline{p}'}$ are the $\mathbb{Z}$-module endomorphisms

$$(2.20) \qquad \varphi_{p\overline{p}'} := \frac{1}{p\overline{p}'}[\phi_p, \phi_{\overline{p}'}] : A[[T]] \to A[[T]] \ \ \text{for } p \neq p' \text{ and}$$

$$(2.21) \qquad \varphi_{p\overline{p}} := \frac{1}{p}[\phi_p, \phi_{\overline{p}}] : A[[T]] \to A[[T]].$$

Again, if $I \subset A[[T]]$ is the ideal generated by T then for any integer $\nu \geq 1$ we denote by

$$(2.22) \qquad \varphi_{p\overline{p}'}^{(\nu)} : I/I^{\nu+1} \to I/I^{\nu+1}$$

the $\mathbb{Z}$-module maps induced by $\varphi_{p\overline{p}'}$. From now on, in this paper, we will take $\overline{\delta} = (\overline{\delta}_p) = (\delta_{\overline{p}})$ to be equal to the adelic connection $\delta_0 = (\delta_{0p})$, $\delta_{0p}x = 0$; this adelic connection is global along 1.

2.2.3. *Chern connections.* Next we would like to apply the concepts above to some basic adelic connections constructed in [10]. Let $q \in GL_n(A)$ with $q^t = \pm q$, where the t superscript means "transpose". Attached to q we have maps $\mathcal{H}_q : G \to G$ and $\mathcal{B}_q : G \times G \to G$ defined by

$$\mathcal{H}_q(x) = x^t q x, \quad \mathcal{B}_q(x, y) = x^t q y.$$

We continue to denote by $\mathcal{H}_q, \mathcal{B}_q$ the maps induced on the p-adic completions $G^{\widehat{p}}$ and $G^{\widehat{p}} \times G^{\widehat{p}}$. Consider the unique adelic connection $\delta_0 = (\delta_{0p})$ on G with $\delta_{0p}x = 0$. Denote by (ϕ_p) and (ϕ_{0p}) the families of lifts of Frobenius attached to an adelic connection δ and to δ_0 respectively. In particular $\phi_{0p}(x) = x^{(p)} := (x_{ij}^p)$. The following was proved in [10]:

Theorem 2.7. [10] *Let $q \in GL_n(A)$ be such that $q^t = \pm q$. Then there exists a unique adelic connection $\delta = (\delta_p)$ such that the following diagrams are commutative:*

$$(2.23) \qquad \begin{array}{ccc} G^{\widehat{p}} & \xrightarrow{\phi_p} & G^{\widehat{p}} \\ \mathcal{H}_q \downarrow & & \downarrow \mathcal{H}_q \\ G^{\widehat{p}} & \xrightarrow{\phi_{0p}} & G^{\widehat{p}} \end{array} \qquad \begin{array}{ccc} G^{\widehat{p}} & \xrightarrow{\phi_{0p} \times \phi_p} & G^{\widehat{p}} \times G^{\widehat{p}} \\ \phi_p \times \phi_{0p} \downarrow & & \downarrow \mathcal{B}_q \\ G^{\widehat{p}} \times G^{\widehat{p}} & \xrightarrow{\mathcal{B}_q} & G^{\widehat{p}} \end{array}$$

Definition 2.8. The adelic connection δ in Theorem 2.23 will be called the *Chern connection* (on $G = GL_n$) attached to q.

Our aim is to show that, for certain natural q's the Chern connection attached to q is global along 1 (hence has well defined curvatures); then we will state and prove vanishing and non-vanishing results for these curvatures.

3. Main results

Our first result is:

Theorem 3.1. *Let $q \in GL_n(A)$ be such that $q^t = \pm q$ and assume that all the entries of q are roots of unity or 0. Then:*

1) The Chern connection $\delta = (\delta_p)$ attached to q is global along 1.

2) If J is the ideal in $A[[T]]$ generated by the entries of the matrix $x^t q x - q$ and if (ϕ_p) are the lifts of Frobenius attached to (δ_p) then $\phi_p(J) \subset J$ for all p.

3) Assume $n = 2r$, $q = 1$, and $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a, b, c, d, $r \times r$ matrices. Let K be the ideal in $A[[T]]$ generated by the entries of the matrix $x^t x - 1$ and by the entries of the matrices $a - d$ and $b + c$. Let (ϕ_p) be the lifts of Frobenius attached to (δ_p). Then $\phi_p(K) \subset K$ for all p.

In particular, if all the entries of $q \in G(A)$ are roots of unity or 0, the curvature $(\varphi_{pp'})$, 3-curvature $(\varphi_{pp'p''})$, and $(1,1)$-curvature $(\varphi_{p\overline{p}'})$ of δ are defined.

Also we have induced $\mathbb{Z}$-module endomorphisms

$$\varphi_{pp'}^{SO} : A[[T]]/J \to A[[T]]/J;$$

the family $(\varphi_{pp'}^{SO})$ will be referred to as the *special orthogonal* curvature of the Chern connection attached to q. Of course, $A[[T]]/J$ is the ring of functions of the completion along 1 of the *special orthogonal subgroup scheme $SO(q)$* of $G = GL_n$ (defined as the identity component of the group defined by the equations $x^t q x = q$.)

If, in addition, $n = 2r$, $q = 1$, and if $(\delta_{\overline{p}})$ is the adelic connection with $\delta_{\overline{p}} x = 0$ then we have induced $\mathbb{Z}$-module homomorphisms

$$\varphi_{p\overline{p}'}^{U} : A[[T]]/K \to A[[T]]/K;$$

the family $(\varphi_{p\overline{p}'}^{U})$ will be referred to as the *unitary* $(1,1)$-curvature of the Chern connection attached to $q = 1$. Of course, $A[[T]]/K$ is the ring of functions of the completion along 1 of the *unitary subgroup scheme U_r^c* of $G = GL_{2r}$ defined by the equations $x^t x = 1$, $a = d$, $b = -c$. (This is, of course, an analogue of the classical complexified unitary group; see [10].)

Remark 3.2. The condition that the entries of a symmetric q be roots of unity or 0 can be arguably viewed as an analogue of the Chern+Levi-Civita=Hessian/Kähler condition in classical differential geometry. Also the condition that the entries of an antisymmetric q be roots of unity or 0 can be viewed as an analogue of the Chern+symplectic condition in the classical case.

Next let us say that a matrix $q \in GL_n(A)$ is *split* if it is one of the following:

$$(3.1) \qquad \begin{pmatrix} 0 & 1_r \\ -1_r & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1_r \\ 1_r & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1_r \\ 0 & 1_r & 0 \end{pmatrix},$$

where $n = 2r, 2r, 2r+1$ respectively and 1_r is the identity $r \times r$ matrix. By Theorem 3.1 the Chern connection attached to any split matrix is global along 1, hence has well defined curvature, 3-curvature, and $(1,1)$-curvature. Our next results are about the vanishing/non-vanishing of these curvatures.

In what follows $q \in G(A)$, $q^t = \pm q$.

For curvature we will prove:

Theorem 3.3. *Let q be split and let $(\varphi_{pp'})$ be the curvature of the Chern connection on G attached to q.*

1) Assume $n \geq 4$. Then for all $p \neq p'$ we have $\varphi_{pp'}^{(pp')} \neq 0$; in particular $\varphi_{pp'} \neq 0$.

2) Assume n even. Then for all p, p' we have $\varphi_{pp'}^{(2)} = 0$.

3) Assume $n = 2$ and $q^t = -q$. Then for all p, p' we have $\varphi_{pp'} = 0$.

4) Assume $n = 1$. Then for all p, p' we have $\varphi_{pp'} = 0$.

Morally what the above says is that for q split the curvature is non-vanishing for $n \geq 4$ (cf. assertion 1) but the non-vanishing of the curvature is somewhat subtle because it does not occur modulo I^3 (cf. assertion 2). For $q = -q^t$ in case $n = 2$ and for $q^t = q$ in case $n = 1$ the curvature vanishes (cf. assertions 3 and 4). Note that our Theorem says nothing about whether the curvature vanishes for $n = 2, 3$ in the case $q^t = q$. Indeed our method to prove assertion 1 of the above theorem, i.e. the fact that the lifts of Frobenius ϕ_p, corresponding to the Chern connection, do not commute on GL_n for $n \geq 4$ is to find a closed subgroup scheme $V \subset GL_n$ such that all ϕ_p send $V^{\widehat{p}}$ into itself and such that the restrictions of the ϕ_p's to $V^{\widehat{p}}$ extend to non-commuting endomorphisms of V. For n even V will be the (2-dimensional vector) group whose points are

$$
\begin{pmatrix}
1 & \ldots & 0 & 0 & \ldots & a \\
 & \ldots & & & \ldots & \\
0 & \ldots & 1 & b & \ldots & 0 \\
0 & \ldots & 0 & 1 & \ldots & 0 \\
 & \ldots & & & \ldots & \\
0 & \ldots & 0 & 0 & \ldots & 1
\end{pmatrix}
$$

where all the diagonal entries are 1 and all the entries except the diagonal entries and a, b are 0. For this subgroup to exist one needs $n \geq 4$. The same method will be used to prove Theorem 3.4 below; in that case the role of V will be played by a subgroup of $SO(q)$ isomorphic to the unipotent radical of GL_3.

For the special orthogonal curvature we will prove:

Theorem 3.4. *Let q be split and let $(\varphi_{pp'}^{SO})$ be the special orthogonal curvature of the Chern connection attached to q.*

1) Assume $n \geq 6$. Then for all $p \neq p'$ we have $\varphi_{pp'}^{SO} \neq 0$.

2) Assume $n = 2$. Then for all p, p' we have $\varphi_{pp'}^{SO} = 0$.

Note that the theorem says nothing about the cases $3 \leq n \leq 5$.

For the 3-curvature we will prove:

Theorem 3.5. *Let $q = q^t$ be split, let $n = 2r \geq 2$, and let $(\varphi_{pp'p''})$ be the 3-curvature of the Chern connection attached to q. Then $\varphi_{pp'p''} \neq 0$ for $p' \neq p''$.*

For the $(1,1)$-curvature, we will prove:

Theorem 3.6. *Let q be split, let $n = 2r \geq 2$ and let $(\varphi_{p\bar{p}'})$ be the $(1,1)$-curvature of the Chern connection attached to q. Then for all p, p' we have $\varphi_{p\bar{p}'}^{(1)} = 0$ and $\varphi_{p\bar{p}'}^{(2)} \neq 0$. In particular $\varphi_{p\bar{p}'} \neq 0$.*

Finally for the unitary $(1,1)$-curvature we will prove:

Theorem 3.7. *Let $n = 2r \geq 2$ and let $\varphi_{p\bar{p}'}^{U}$ be the unitary $(1,1)$-curvature of the Chern connection attached to attached to $q = 1$. Then $\varphi_{p\bar{p}'} \neq 0$ for all p, p'.*

4. Proofs

4.1. **Explicit formula.** Recall from [10] that for each p, the ϕ_p in Theorem 3.1 is induced by the unique ring homomorphism

$$\phi_p : \mathcal{O}(G)^{\widehat{p}} \to \mathcal{O}(G)^{\widehat{p}}, \quad x \mapsto \phi_p(x) = \Phi_p,$$

prolonging $\phi_p : A^{\widehat{p}} \to A^{\widehat{p}}$ where Φ_p is the $n \times n$ matrix with coefficients in $\mathcal{O}(G)^{\widehat{p}}$ defined by

$$(4.1) \qquad \Phi_p(x) = x^{(p)} \{ (x^{(p)t} \phi_p(q) x^{(p)})^{-1} (x^t q x)^{(p)} \}^{1/2}.$$

In this expression, the t exponent means *transpose*, $a^{(p)}$ for a matrix a is the matrix obtained from a by raising each entry to the pth power, and the $1/2$ power is computed using the usual matrix series

$$(4.2) \qquad (1 + u)^{1/2} = 1 + \sum_{i=1}^{\infty} \binom{1/2}{i} u^i$$

for u an $n \times n$ matrix of indeterminates. The formula is applied to

$$u = (x^{(p)t} \phi_p(q) x^{(p)})^{-1} (x^t q x)^{(p)} - 1,$$

which has coefficients in $p\mathcal{O}(G)$; this makes the series 4.2 p-adically convergent in view of the fact that $\binom{1/2}{i} \in \mathbb{Z}[1/2]$.

4.2. **Proof of Theorem 3.1.** Since the entries of q are roots of unity or 0 we have $\phi_p(q) = q^{(p)}$. Set

$$v(T) := ((1 + T^t)^{(p)} \phi_p(q) (1 + T)^{(p)})^{-1} ((1 + T^t) q (1 + T))^{(p)}.$$

Now $u(T) := v(T) - 1$ has entries in

$$pA^{\widehat{p}}[[T]] \cap (T)A[[T]] = p(T)A[[T]].$$

Hence the matrix

$$\Phi_p^0(T) := \Phi_p(1 + T) - 1 = (1 + T)^{(p)} \cdot v(T)^{1/2} - 1,$$

which a priori has coefficients in $A^{\widehat{p}}[[T]]$, has actually coefficients in $A[[T]]$ and has zero constant term, which ends the proof of assertion 1 of the theorem.

To check assertion 2 note that the commutativity of the left diagram in 2.23 is equivalent to the equality

$$\Phi_p(x)^t \phi_p(q) \Phi_p(x) = (x^t q x)^{(p)}.$$

So we have

$$\phi_p(x^t q x - q) = \Phi_p(x)^t \phi_p(q) \Phi_p(x) - \phi_p(q) = (x^t q x)^{(p)} - q^{(p)}.$$

But now the entries of $(x^t q x)^{(p)} - q^{(p)}$ belong to the ideal generated by the entries of $x^t q x - q$ in $A[x] = A[T]$. We conclude that $\phi_p(J) \subset J$.

To check assertion 3 note that, by the above computation the entries of

$$\phi_p(x^t x - 1) = (x^t x)^{(p)} - 1$$

are in the ideal generated by the entries of $x^t x - 1$. It remains to show that Φ_p evaluated at the matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ has the form $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, i.e., it is in the centralizer of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which is clear from 4.1. $\qquad \square$

4.3. **Some remarkable polynomials.** In some of the next arguments certain polynomials will play a recurring role. Let v, w be two variables and let p, p', p'' be odd primes. Then consider the following two polynomials (one for the upper sign and one for the lower sign):

$$(4.3) \qquad f_p(v, w) := \frac{1}{2}(\pm(\pm v + w)^p + v^p \mp w^p) \in \mathbb{Z}[1/2][v, w].$$

Also consider the corresponding polynomials in $\mathbb{Z}[1/2][u, v]$:

$$(4.4) \qquad\qquad f_{pp'}(v, w) := f_{p'}(f_p(v, w), f_p(w, v)),$$

$$(4.5) \qquad\qquad g_{pp'}(v, w) := f_{pp'}(v, w) - f_{p'p}(v, w),$$

$$(4.6) \qquad f_{pp'p''}(v, w) := f_p(f_{p'}(f_{p''}(v, w), f_{p''}(w, v)), f_{p'}(f_{p''}(w, v), f_{p''}(v, w))),$$

$$(4.7) \qquad g_{pp'p''}(v, w) := f_{pp'p''}(v, w) - f_{p'pp''}(v, w) - f_{p''pp'}(v, w) + f_{p''p'p}(v, w).$$

Lemma 4.1. *For $p \neq p'$ we have $g_{pp'}(v, w) \neq 0$.*

Proof. We have:

$$f_{pp'}(v, w) := \tfrac{1}{2}(v \pm w)^{pp'}$$

$$(4.8) \qquad\qquad +\tfrac{1}{2}\left(\frac{\pm(\pm v + w)^p + v^p \mp w^p}{2}\right)^{p'}$$

$$\qquad\qquad \mp\tfrac{1}{2}\left(\frac{\pm(v \pm w)^p \mp v^p + w^p}{2}\right)^{p'}.$$

Consider the polynomial $f_{pp'}(v, 1) \in \mathbb{Z}[1/2][v]$. We have the following congruences mod (v^3) in $\mathbb{Z}[1/2][v]$:

$$f_{pp'}(v, 1) \equiv \tfrac{1}{2}\left(\pm 1 + pp'v \pm \frac{pp'(pp'-1)}{2}v^2\right)$$

$$\mp \tfrac{1}{2}\left(1 \pm \tfrac{p}{2}v + \frac{p(p-1)}{4}v^2\right)^{p'}$$

$$\equiv \tfrac{1}{2}\left(\pm 1 + pp'v \pm \frac{pp'(pp'-1)}{2}v^2\right)$$

$$\mp \tfrac{1}{2}\left(1 + p'(\pm\tfrac{p}{2}v + \frac{p(p-1)}{4}v^2) + \frac{p'(p'-1)}{2}(\pm\tfrac{p}{2}v)^2\right)$$

$$\equiv \frac{pp'}{4}v \pm \frac{pp'}{16}(3pp' - p - 2)v^2 \mod (v^3).$$

Similarly

$$f_{p'p}(v, 1) \equiv \frac{p'p}{4}v \pm \frac{p'p}{16}(3p'p - p' - 2)v^2 \mod (v^3)$$

hence we get

(4.9) $$g_{pp'}(v, 1) \equiv \pm \frac{pp'}{16}(p' - p)v^2 \mod (v^3).$$

This ends our proof.   □

In the next lemma we choose the upper sign in the definition of $f_p(v, w)$.

Lemma 4.2. $g_{pp'p''}(v, w) \neq 0$ for $p \neq p'$.

Proof. An immediate computation gives:

$$f_{pp'p''}(v, w) = \frac{1}{2}(v + w)^{pp'p''} + \frac{1}{2}h_{pp'p''}(v, w),$$

where

$$h_{pp'p''}(v, w) = \left(\frac{(v+w)^{p'p''}+f_{p''}(v,w)^{p'}-f_{p''}(w,v)^{p'}}{2} \right)^p$$

$$- \left(\frac{(v+w)^{p'p''}+f_{p''}(w,v)^{p'}-f_{p''}(v,w)^{p'}}{2} \right)^p.$$

Next, an easy computation as in Lemma 4.1 gives the following congruences mod v^3 in $\mathbb{Z}[1/2][v]$:

$$h_{pp'p''}(v, w) \equiv - \left(1 + \frac{3}{4}pp'p''v + \frac{pp'p''}{32}(p'p'' + 2p'' - 12 + 9pp'p'')v^2 \right) \mod (v^3).$$

Hence

$$
\begin{aligned}
2g_{pp'p''}(v, 1) &\equiv h_{pp'p''}(v, 1) - h_{p'pp''}(v, 1) - h_{p''pp'}(v, 1) + h_{p''p'p}(v, 1) \\
&\equiv -\tfrac{pp'p''}{32}\left((p'p'' + 2p'') - (pp'' + 2p'') - (pp' + 2p') + (p'p + 2p) \right)v^2 \\
&\equiv -\tfrac{pp'p''}{32}(p'' - 2)(p' - p)v^2 \mod (v^3).
\end{aligned}
$$

which ends our proof.   □

4.4. **Proof of assertion 1 in Theorem 3.3.** In what follows we assume $n = 2r$; the case $n = 2r + 1$ can be treated similarly. Let us make the convention that the upper sign in $\pm, \mp$ refers to the case $q^t = q$ and the lower sign refers to the case $q^t = -q$. Set $\phi_p(T) = \Phi_p^0$; so $\Phi_p^0 = \Phi_p^0(T)$ is a matrix with coefficients in $A[[T]]$, hence we may evaluate Φ_p^0 at various matrices. Let B be the $r \times r$ upper corner of the $2r \times 2r$ matrix T. By 4.1, evaluating Φ_p^0 at the matrix $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ gives

$$\Phi_p^0 \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & F_p(B) \\ 0 & 0 \end{pmatrix},$$

where

(4.10) $$F_p(B) := \frac{1}{2}(\pm(\pm B + B^t)^{(p)} + B^{(p)} \mp B^{(p)t}).$$

So $F_p(B)$ has coefficients in $A[B]$, hence the ring endomorphism

$$\phi_p : A[[B]] \to A[[B]], \quad B \mapsto \phi_p(B) = F_p(B)$$

sends $A[B]$ into itself. We have, of course, a commutative diagram

(4.11)
$$\begin{array}{ccc} A[[B]] & \xrightarrow{\phi_p} & A[[B]] \\ u \uparrow & & \uparrow u \\ A[[T]] & \xrightarrow{\phi_p} & A[[T]] \end{array}$$

where u is the surjective homomorphism sending T into $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$. In order to conclude the proof assume ϕ_p and $\phi_{p'}$ commute on $A[[T]]$ for some $p \neq p'$ and let us derive a contradiction. By 4.11 ϕ_p and $\phi_{p'}$ commute on $A[[B]]$. Clearly we have

$$F_p(B_{ij}) = F_p(B)_{ij} = \frac{1}{2}(\pm(\pm B_{ij} + B_{ji})^p + B_{ij}^p \mp B_{ji}^p) = f_p(B_{ij}, B_{ji}).$$

Since $\phi_p \phi_{p'}(B) = F_{p'}(F_p(B))$ we have

$$\begin{aligned} \phi_p \phi_{p'}(B)_{ij} &= F_{p'}(F_p(B))_{ij} \\ \\ &= f_{p'}(F_p(B)_{ij}, F_p(B)_{ji}) \\ \\ &= f_{p'}(f_p(B_{ij}, B_{ji}), f_p(B_{ji}, B_{ij})) \\ \\ &= f_{pp'}(B_{ij}, B_{ji}). \end{aligned}$$

where $f_{pp'}$ is an in 4.4. Choose now two indices $i \neq j$ (this is possible because $n \geq 4$). Then we get $f_{pp'}(v, w) = f_{p'p}(v, w)$, i.e., $g_{pp'}(v, w) = 0$ which contradicts Lemma 4.1. $\qquad \square$

4.5. **Proof of Theorem 3.5.** In the notation of subsection 4.4 we have
$$\begin{aligned} p'p''\varphi_{pp'p''}(B) &= F_{p''}(F_{p'}(F_p(B))) - F_{p'}(F_{p''}(F_p(B))) \\ \\ &\quad - F_p(F_{p''}(F_{p'}(B))) + F_p(F_{p'}(F_{p''}(B))). \end{aligned}$$

Exactly as in that proof we have
$$F_{p''}(F_{p'}(F_p(B_{ij}))) = f_{p''p'p}(B_{ij}, B_{ji})$$

hence
$$p'p''\varphi_{pp'p''}(B_{ij}) = g_{p''p'p}(B_{ij}, B_{ji})$$
and, taking any two indices i, j with $i \neq j$ (which is possible because $n \geq 4$) we conclude by Lemma 4.2. $\qquad \square$

4.6. **Proof of Theorem 3.4.** For simplicity we only treat the case $n = 2r$; the case $n = 2r+1$ can be reduced to the case $n = 2r$. Set $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = T+1$. Let $H \subset G = GL_n$ be the subgroup scheme whose points are the specializations of the matrices $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. By the way, $H \cap SO(q)$ is the subgroup of H of all matrices in H such that $d = (a^t)^{-1}$. The embedding $H \subset G$ corresponds to the homomorphism sending a, d into themselves and b, c into 0. Set $\phi_p(x) = \Phi_p$. By 4.1 we have

$$\Phi_p \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} S_p(a, d) & 0 \\ 0 & S_p(d, a) \end{pmatrix},$$

where
$$S_p(a, d) := a^{(p)} \cdot \{(a^{(p)})^{-1}(d^{(p)t})^{-1}(d^t a)^{(p)}\}^{1/2}.$$

So $\phi_p : A[[T]] \to A[[T]]$ sends the ideal of H into itself. We also know that ϕ_p sends the ideal of $SO(q)$ into itself. Hence ϕ_p sends the ideal of $H \cap SO(q)$ into itself. So we have a commutative diagram

$$(4.12) \qquad \begin{array}{ccc} A[[a-1]] & \xrightarrow{\Sigma_p} & A[[a-1]] \\ u \uparrow & & \uparrow u \\ A[[T]]/J & \xrightarrow{\phi_p} & A[[T]]/J \end{array}$$

where

$$(4.13) \quad u(x) = \begin{pmatrix} a & 0 \\ 0 & (a^t)^{-1} \end{pmatrix}, \quad \Sigma_p(a) := S_p(a,(a^t)^{-1}) = a^{(p)} \cdot \{(a^{-1})^{(p)} a^{(p)}\}^{-1/2}.$$

To prove assertion 1 assume $n = 2r \geq 6$ and $\varphi_{pp'}^{SO} = 0$ for some $p \neq p'$ and seek a contradiction. We may assume $r = 3$. In view of diagram 4.12 we have

$$(4.14) \qquad \Sigma_{p'}\Sigma_p(a) = \Sigma_p\Sigma_{p'}(a).$$

A trivial computation using 4.13 yields

$$\Sigma_p \begin{pmatrix} 1 & u & v \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u^p & f_p(v,uw-v) \\ 0 & 1 & w^p \\ 0 & 0 & 1 \end{pmatrix};$$

from here a trivial (but somewhat "miraculous") computation yields:

$$\Sigma_p\Sigma_{p'} \begin{pmatrix} 1 & u & v \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u^{pp'} & f_{pp'}(v,uw-v) \\ 0 & 1 & w^{pp'} \\ 0 & 0 & 1 \end{pmatrix}.$$

In the above $f_p, f_{pp'}$ are the polynomial 4.3, 4.8 (with the choice of the upper sign). From 4.12 we get $f_{pp'} = f_{p'p}$ which contradicts Lemma 4.1; this ends our proof of assertion 1. Assertion 2 follows easily. $\qquad\square$

4.7. **Proof of assertion 2 in Theorem 2.23.** Let $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, B, C, D

are $r \times r$ matrices; the use of A here and the use of A as a name for the ring $\mathbb{Z}[1/M, \zeta_N]$ should not lead to any confusion. Denote by (T) the ideal generated by the entries of T. By our assumptions $q^t = \pm q$ and q is split. The upper sign in the proof below corresponds to the case $q^t = q$ and the lower sign corresponds to the case $q^t = -q$. Set $\phi_p(T) = \Phi_p^0$ and $\varphi_{pp'}(T) = \Phi_{pp'}^0$. So

$$\Phi_{pp'}^0(T) = \frac{1}{pp'}(\Phi_{p'}^0(\Phi_p^0(T)) - \Phi_p^0(\Phi_{p'}^0(T))).$$

We need to show that $\Phi_{pp'}^0 \equiv 0 \bmod (T)^3$.

We proceed by computing $\Phi_p(1+T)$ modulo $(T)^3$ using formula 4.1. Let $1 \leq i,j,k \leq r$ and set $\underline{i} = i + r$. We have

$$(1+T)^{(p)} = \begin{pmatrix} (1+T_{11})^p & T_{12}^p & \cdots \\ T_{21}^p & (1+T_{22})^p & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \equiv \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \bmod (T)^3,$$

where E_1 and E_2 are diagonal matrices with

$$(E_1)_{ii} = 1 + pT_{ii} + \begin{pmatrix} p \\ 2 \end{pmatrix} T_{ii}^2, \quad (E_2)_{ii} = 1 + pT_{\underline{ii}} + \begin{pmatrix} p \\ 2 \end{pmatrix} T_{\underline{ii}}^2.$$

Therefore,

$$(1 + T^t)^{(p)} q (1 + T)^{(p)} \equiv \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} q \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \mod(T)^3$$

$$= \begin{pmatrix} 0 & E_1 E_2 \\ \pm E_2 E_1 & 0 \end{pmatrix}$$

$$= G,$$

where the last equality defines G. Next, we have

$$((1 + T^t) q (1 + T))^{(p)} = \left(\begin{pmatrix} 1 + A^t & C^t \\ B^t & 1 + D^t \end{pmatrix} q \begin{pmatrix} 1 + A & B \\ C & 1 + D \end{pmatrix} \right)^{(p)}$$

$$= \begin{pmatrix} (C \pm C^t \pm C^t A + A^t C)^{(p)} & (1 + D + A^t + A^t D \pm C^t B)^{(p)} \\ (\pm 1 \pm A \pm D^t \pm D^t A + B^t C)^{(p)} & (\pm B + B^t + B^t D \pm D^t B)^{(p)} \end{pmatrix}$$

Write

(4.15) $$A_{ij} = T_{ij}, \quad B_{ij} = T_{i\underline{j}}, \quad C_{ij} = T_{\underline{i}j}, \quad D_{ij} = T_{\underline{i}\underline{j}},$$

hence

$$(A^t D)_{ij} = \sum_k T_{ki} T_{\underline{k}\underline{j}}, \qquad\qquad (D^t A)_{ij} = \sum_k T_{\underline{k}\underline{i}} T_{kj},$$

$$(B^t C)_{ij} = \sum_k T_{k\underline{i}} T_{\underline{k}j}, \qquad\qquad (C^t B)_{ij} = \sum_k T_{\underline{k}i} T_{k\underline{j}},$$

and define

(4.16) $$Q_i = \sum_k (T_{ki} T_{\underline{k}\underline{i}} \pm T_{\underline{k}i} T_{k\underline{i}}).$$

Let H be the diagonal matrix with

$$H_{ii} = 1 + p(T_{ii} + T_{\underline{i}\underline{i}} + Q_i) + \binom{p}{2}(T_{ii}^2 + T_{\underline{i}\underline{i}}^2 + 2 T_{ii} T_{\underline{i}\underline{i}}).$$

Then we have

$$((1 + T^t) q (1 + T))^{(p)} \equiv \begin{pmatrix} 0 & H \\ \pm H & 0 \end{pmatrix} = F \mod(T)^3,$$

where the last equality is the definition of F. One trivially checks that

$$G^{-1} F = \begin{pmatrix} E_1^{-1} E_2^{-1} H & 0 \\ 0 & E_2^{-1} E_1^{-1} H \end{pmatrix} \equiv 1 \mod (T)^2.$$

By the binomial expansion

$$(1 + (G^{-1} F - 1))^{1/2} \equiv 1 + \frac{1}{2}(G^{-1} F - 1) \mod(T)^3.$$

So

$$\Phi_p(1 + T) \equiv (1 + T)^{(p)} + \frac{1}{2}(1 + T)^{(p)} \frac{1}{2}(G^{-1} F - 1)$$

$$\equiv \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \begin{pmatrix} E_1^{-1} E_2^{-1} H - 1 & 0 \\ 0 & E_2^{-1} E_1^{-1} H - 1 \end{pmatrix}$$

$$\equiv \frac{1}{2} \begin{pmatrix} E_2^{-1} H + E_1 & 0 \\ 0 & E_1^{-1} H + E_2 \end{pmatrix} \mod (T)^3.$$

Set $X = E_2^{-1}H + E_1$ and $Y = E_1^{-1}H + E_2$ and note that X and Y are diagonal matrices. Then

$$(4.17) \qquad \Phi_p(1+T) \equiv \frac{1}{2}\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \mod (T)^3.$$

On the other hand

$$
\begin{aligned}
X_{ii} &\equiv (1 - pT_{\underline{ii}} - \tbinom{p}{2}T_{\underline{ii}}^2 + p^2 T_{\underline{ii}}^2)\times \\
&\quad \times (1 + p(T_{ii} + T_{\underline{ii}} + Q_i) + \tbinom{p}{2}(T_{ii}^2 + T_{\underline{ii}}^2 + 2T_{ii}T_{\underline{ii}})) \\
&\quad + (1 + pT_{ii} + \tbinom{p}{2}T_{ii}^2) \mod (T)^3 \\
&\equiv 2 + 2pT_{ii} - pT_{ii}T_{\underline{ii}} + 2\tbinom{p}{2}T_{ii}^2 + pQ_i \mod (T)^3,
\end{aligned}
$$

and similarly

$$(4.19) \qquad Y_{ii} \equiv 2 + 2pT_{\underline{ii}} - pT_{ii}T_{\underline{ii}} + 2\binom{p}{2}T_{\underline{ii}}^2 + pQ_i \mod (T)^3.$$

By the way equations 4.17, 4.18, and 4.19 provide a formula for $\Phi_p(1+T) \mod (T)^3$ and, in particular, one gets that $\Phi_p(1+T)$ is congruent modulo $(T)^3$ to a diagonal matrix.

Now, in order to compute $\Phi_{pp'}^0 \mod (T)^3$ set $Z_p = \Phi_p(1+T) = \Phi_p^0(T)+1$. Since Z_p is congruent mod $(T)^3$ to a diagonal matrix and since for any $n \times n$ diagonal matrix Δ and any $n \times n$ matrix Γ we have $(\Gamma\Delta)^{(p')} = \Gamma^{(p')}\Delta^{(p')}$, the following holds:

$$
\begin{aligned}
\Phi_{p'}^0(\Phi_p^0(T)) &= Z_p^{(p')}\left[\left(Z_p^{t(p')}qZ_p^{(p')}\right)^{-1}(Z_p^t q Z_p)^{(p')}\right]^{\frac{1}{2}} - 1 \\
&\equiv Z_p^{(p')}\left[\left((Z_p^t q Z_p)^{(p')}\right)^{-1}(Z_p^t q Z_p)^{(p')}\right]^{\frac{1}{2}} - 1 \mod (T)^3 \\
&= Z_p^{(p')} - 1.
\end{aligned}
$$

By 4.18 we get

$$(4.20) \quad (\tfrac{1}{2}X_{ii})^{p'} \equiv 1 + pp'T_{ii} - \frac{pp'}{2}T_{ii}T_{\underline{ii}} + \frac{pp'}{2}Q_i + \frac{pp'(pp'-1)}{2}T_{ii}^2 \mod (T)^3,$$

and similarly, by 4.19, we get

$$(4.21) \quad (\tfrac{1}{2}Y_{ii})^{p'} \equiv 1 + pp'T_{\underline{ii}} - \frac{pp'}{2}T_{ii}T_{\underline{ii}} + \frac{pp'}{2}Q_i + \frac{pp'(pp'-1)}{2}T_{\underline{ii}}^2 \mod (T)^3.$$

Consequently

$$Z_p^{(p')} \equiv Z_{p'}^{(p)} \mod (T)^3,$$

hence $\Phi_{pp'}^0 \equiv 0 \mod (T)^3$. $\qquad \square$

Remark 4.3. As already noticed in the proof above, Φ_p^0 is congruent mod $(T)^3$ to a diagonal matrix. Note, on the other hand, that Φ_p^0 is *not* congruent mod $(T)^4$ to a diagonal matrix (e.g. for $p = 3$). By the way, modulo $(T)^2$, the situation is simpler; we have

$$\Phi_p^0 \equiv \mathrm{diag}(pT_{11}, ..., pT_{nn}) \mod (T)^2.$$

4.8. **Proof of Theorem 3.6.** Recall that by our assumptions $n = 2r \geq 2$ and $q^t = \pm q$ is split. We let $\Phi^0_{p\overline{p}'ij} := \varphi_{p\overline{p}'}(T_{ij})$ be the entries of the matrix $\Phi^0_{p\overline{p}'} := \varphi_{p\overline{p}'}(T)$ where $i, j = 1, ..., n = 2r$. As before, we set $\underline{i} = i + r$ for $1 \leq i \leq r$. Also we use the notation in 4.15 and 4.16. Our theorem will be proved if we prove the following congruences:

1) $\Phi^0_{p\overline{p}'ii} \equiv \Phi^0_{p\overline{p}'\underline{i}\underline{i}} \equiv \frac{1}{2}(Q_i - T_{ii}T_{\underline{i}\underline{i}}) \bmod (T)^3$ for $1 \leq i \leq r$ and $p \neq p'$.

2) $\Phi^0_{p\overline{p}ii} \equiv \Phi^0_{p\overline{p}\underline{i}\underline{i}} \equiv \frac{p}{2}(Q_i - T_{ii}T_{\underline{i}\underline{i}}) \bmod (T)^3$ for $1 \leq i \leq r$.

3) $\Phi^0_{p\overline{p}'ij} \equiv 0 \bmod (T)^3$ for $1 \leq i, j \leq n = 2r$, $i \neq j$.

To check these we use the notation in subsection 4.7. Set

$$Z_{\overline{p}'} := \Phi_{\overline{p}'}(1 + T) = \Phi^0_{\overline{p}'}(T) + 1 = (1 + T)^{(p')}.$$

Then exactly as in subsection 4.7 we have

$$\Phi^0_{\overline{p}'}(\Phi^0_p(T)) = Z^{(p')}_p - 1,$$

$$\Phi^0_p(\Phi^0_{\overline{p}'}(T)) = Z^{(p)}_{\overline{p}'} - 1.$$

Now $Z^{(p')}_p \bmod (T)^3$ was computed in 4.20 and 4.21 while $Z^{(p)}_{\overline{p}'} \bmod (T)^3$ is, of course, trivial to compute; then assertions 1, 2, 3 above follow. $\square$

Since assertion 4 in Theorem 2.23 trivially follows from the formula 4.1 all we need to conclude the proof of Theorem 2.23 is:

4.9. **Proof of assertion 3 in Theorem 3.3.** We need to show that ϕ_p and $\phi_{p'}$ commute on $A[[T]]$ for $q^t = -q$, q split. In this case formula 4.1 yields

$$\Phi_p(x) = \lambda_p(x) \cdot x^{(p)}, \qquad \lambda_p(x) = \left(\frac{\det(x^{(p)})}{(\det(x))^p}\right)^{-1/2}.$$

Since $x^{(pp')} = x^{(p)(p')} = x^{(p')(p)}$ and

$$\phi_p(T) = \lambda_p(1 + T) \cdot (1 + T)^{(p)} - 1$$

we get:

$$
\begin{aligned}
\phi_p(\phi_{p'}(T)) &= \lambda_{p'}(\lambda_p(x) \cdot x^{(p)}) \cdot (\lambda_p(x) \cdot x^{(p)})^{(p')} - 1 \\[2mm]
&= \lambda_{p'}(x^{(p)}) \cdot (\lambda_p(x))^{p'} \cdot x^{(pp')} - 1 \\[2mm]
&= \left(\frac{\det(x^{(pp')})}{\det(x^{(p)})^{p'}}\right)^{-1/2} \cdot \left(\frac{\det(x^{(p)})^{p'}}{(\det(x))^{pp'}}\right)^{-1/2} \cdot x^{(pp')} - 1 \\[2mm]
&= \left(\frac{\det(x^{(pp')})}{(\det(x))^{pp'}}\right)^{-1/2} \cdot x^{(pp')} - 1 \\[2mm]
&= \phi_{p'}(\phi_p(T)),
\end{aligned}
$$

which proved the desired commutation. $\square$

To conclude the proofs of our theorems we need:

Proof of Theorem 3.7. We may assume $M = 2$ and $N = 1$ so our ground ring is $\mathbb{Z}[1/M, \zeta_N] = \mathbb{Z}[1/2]$. Again, write $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so $a = 1 + A$,

$b = B$, $c = C$, $d = 1 + D$. Denote by GL_r^c the subgroup of GL_{2r} defined by the equations $a = d$, $b = -c$. So U_r^c is defined in GL_r^c by the equations $aa^t + bb^t = 1$, $ba^t = ab^t$, equivalently by the equations:

$$F := A + A^t + AA^t + BB^t, \quad G := (B + BA^t) - (B^t + AB^t).$$

Similarly $GL_1^c = Spec \, \mathbb{Z}[1/2][\alpha, \beta, \frac{1}{(1+\alpha)^2+\beta^2}]$, with α, β two variables, and U_1^c is defined in GL_1^c by the equation

$$f := 2\alpha + \alpha^2 + \beta^2.$$

Consider the embedding of GL_1^c into GL_r^c given by the homomorphism between their coordinate rings

$$u : \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto \begin{pmatrix} \alpha \cdot 1_r & \beta \cdot 1_r \\ -\beta \cdot 1_r & \alpha \cdot 1_r \end{pmatrix}.$$

The ideal of the image of GL_1^c via this embedding is easily seen to be sent into itself by the lifts of Frobenius ϕ_p on $(GL_r^c)^{\widehat{p}}$; hence the ideal of U_1^c is sent into itself by the lifts of Frobenius ϕ_p on $(U_r^c)^{\widehat{p}}$. So we have a commutative diagram

(4.22)
$$\begin{array}{ccc}
\mathbb{Z}[1/2][[\alpha, \beta]]/(f) & \xrightarrow{\phi_p} & \mathbb{Z}[1/2][[\alpha, \beta]]/(f) \\
u \uparrow & & \uparrow u \\
\mathbb{Z}[1/2][[A, B]]/(F, G) & \xrightarrow{\phi_p} & \mathbb{Z}[1/2][[A, B]]/(F, G) \\
v \uparrow & & \uparrow v \\
\mathbb{Z}[1/2][[T]] & \xrightarrow{\phi_p} & \mathbb{Z}[1/2][[T]]
\end{array}$$

where v sends A into A, B into B, C into $-B$ and D into A. An easy computation yields

$$\phi_p \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} = \begin{pmatrix} \alpha_p & \beta_p \\ -\beta_p & \alpha_p \end{pmatrix}$$

where

$$\alpha_p = (1+\alpha)^p K_p(\alpha, \beta) - 1, \quad \beta_p = \beta^p K_p(\alpha, \beta),$$

$$K_p(\alpha, \beta) := \left(\frac{((1+\alpha)^2 + \beta^2)^p}{(1+\alpha)^{2p} + \beta^{2p}} \right)^{1/2}.$$

On the other hand we have

$$\phi_{\overline{p}'} \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} = \begin{pmatrix} (1+\alpha)^{p'} - 1 & \beta^{p'} \\ -\beta^{p'} & (1+\alpha)^{p'} \end{pmatrix}.$$

Assume $\varphi_{p\overline{p}'}^U = 0$ and seek a contradiction. The equality $\varphi_{p\overline{p}'}^U = 0$ implies that ϕ_p and $\phi_{\overline{p}'}$ commute on $\mathbb{Z}[1/2][[\alpha, \beta]]/(f)$ which immediately implies

$$K_p(\alpha, \beta)^{p'} = K_p((1+\alpha)^{p'} - 1, \beta^{p'}).$$

Squaring the latter equation and using the relation $\beta^2 = -2\alpha - \alpha^2$, we get the following equality of polynomials in α:

$$(1+\alpha)^{2pp'} - (2\alpha + \alpha^2)^{pp'} = ((1+\alpha)^{2p} - (2\alpha + \alpha^2)^p)^{p'} \cdot ((1+\alpha)^{2p'} - (2\alpha + \alpha^2)^{p'})^p.$$

Picking out the coefficients of α we get $2pp' = 4pp'$, a contradiction; this ends the proof. $\qquad\square$

5. The case of one prime

So far there was no restriction on the cardinality of our "index set" of primes $\mathcal{V}$. Interestingly, restricting to the case when $\mathcal{V}$ has one element, is still interesting and actually allows one to introduce $(1,1)$-curvature for connections that are not necessarily global along the identity. We give in what follows some details.

As in the rest of the paper we let $A = \mathbb{Z}[1/M, \zeta_N]$ with M even. Consider the set $\mathcal{V} = \{p\}$ consisting of one prime p only, such that p does not divide MN. In this case an adelic connection is simply a p-derivation $\delta_p : \mathcal{O}(G)^{\widehat{p}} \to \mathcal{O}(G)^{\widehat{p}}$. Now let us consider two adelic connections δ_p and $\overline{\delta}_p = \delta_{\overline{p}}$ (not necessarily global along 1). To these one can attach the $(1,1)$-*curvature*:

$$(5.1) \qquad \varphi_{p\overline{p}} := \frac{1}{p}[\phi_p, \phi_{\overline{p}}] : \mathcal{O}(G)^{\widehat{p}} \to \mathcal{O}(G)^{\widehat{p}}.$$

In case δ and $\overline{\delta}$ are global along 1, 5.1 is compatible (induces) 2.21. Note that $\Phi_{p\overline{p}} := \phi_p(x)$ is an $n \times n$ matrix with coefficients in $\mathcal{O}(G) = A[x, \det(x)^{-1}]^{\widehat{p}}$ hence the value, $\Phi_{p\overline{p}}(1)$, of $\Phi_{p\overline{p}}$ at $x = 1$ is an $n \times n$ matrix with entries in $A^{\widehat{p}}$. Under the assumptions above we have:

Theorem 5.1. *Let $q \in GL_n(A)$, $q^t = \pm q$. Let $\delta = (\delta_p)$ be the Chern connection attached to q and let $\overline{\delta} = (\delta_{\overline{p}})$ be the adelic connection with $\delta_{\overline{p}} x = 0$; here both adelic connections are indexed by the set $\{p\}$ with one prime p. Let $\varphi_{p\overline{p}}$ be the $(1,1)$-curvature of δ with respect to $\overline{\delta}$ and let $\Phi_{p\overline{p}} := \varphi_{p\overline{p}}(x)$. Then:*

1) $\Phi_{p\overline{p}}(1) = -\delta_{\overline{p}}\left(\left(1 + p(q^{(p)})^{-1}\delta_p q\right)^{-1/2}\right).$

2) If $n = 1$ then $\Phi_{p\overline{p}} = \Phi_{p\overline{p}}(1) \cdot x^{p^2}$.

3) $\Phi_{p\overline{p}}(1) = 0$ if and only if each entry of q is either 0 or a root of unity. Moreover, if $n = 1$, then $\varphi_{p\overline{p}} = 0$ if and only if q is a root of unity.

Proof. The formulae in assertions 1 and 2 follows from a direct computation using 4.1. Assertion 3 follows from assertions 1 and 2 plus the following two facts: i) if $\delta_p a = 0$ or $\delta_{\overline{p}} a = 0$ for some $a \in A$ then a is a root of unity or 0; and ii) if $1 + pa$ is a root of unity or 0 for some $a \in A$ then $a = 0$. $\square$

Remark 5.2. Note the analogy between the formula in assertion 1 of Theorem 5.1 and formula 2.12.

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