THE $SU(N)$ CASSON-LIN INVARIANTS FOR LINKS

HANS U. BODEN AND ERIC HARPER

ABSTRACT. We introduce the $SU(N)$ Casson-Lin invariants for links L in S^3 with more than one component. Writing $L = \ell_1 \cup \cdots \cup \ell_n$, we require as input an n-tuple $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ of *labels*, where a_j is associated with ℓ_j. The $SU(N)$ Casson-Lin invariant, denoted $h_{N,a}(L)$, gives an algebraic count of certain projective $SU(N)$ representations of the link group $\pi_1(S^3 \smallsetminus L)$, and the family $h_{N,a}$ of link invariants gives a natural extension of the $SU(2)$ Casson–Lin invariant, which was defined for knots by X.-S. Lin and for 2-component links by Harper and Saveliev. We compute the invariants for the Hopf link and more generally for chain links, and we show that, under mild conditions on the labels $(a_1, \ldots, a_n)$, the invariants $h_{N,a}(L)$ vanish whenever L is a split link.

INTRODUCTION

The goal of this paper is to construct $SU(N)$ Casson-Lin invariants $h_{N,a}(L)$ for oriented links L in S^3. These invariants are defined as a signed count of conjugacy classes of certain irreducible projective $SU(N)$ representations of $\pi_1(S^3 \smallsetminus L)$ with a non-trivial 2-cocycle. Given an oriented link L with n components, the 2-cocycle is determined by an n-tuple $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ of labels, and the choice of labels is made so that the resulting 2-cocycle is nontrivial. This is critical in what follows because it prohibits the existence of reducibles, see Proposition 2.2. We denote the resulting algebraic count as $h_{N,a}(L)$, and the following theorem is the main result of this paper.

Theorem. Suppose $L \subset S^3$ is an oriented n-component link with $n \geq 2$ and $a = (a_1, \ldots, a_n)$ is an allowable n-tuple of labels. Then the integer $h_{N,a}(L)$ is a well-defined invariant of L.

We briefly outline how the above theorem is established. By Alexander's Theorem [1], every link $L \subset S^3$ can be realized as the closure $L = \widehat{\sigma}$ for some braid $\sigma \in B_k$. The braid group B_k acts naturally on the free group F_k, and this induces an action

on the space of $SU(N)$ representations of F_k, which we denote as

$$R_k = \mathrm{Hom}\,(F_k, SU(N)) = SU(N) \times \cdots \times SU(N).$$

We extend this action to the wreath product $\mathbb{Z}_N \wr B_k$ as follows. Identifying $\mathbb{Z}_N$ with the center of $SU(N)$, then for $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k) \in (\mathbb{Z}_N)^k$ and $X = (X_1, \ldots, X_k) \in R_k$, we set $(\varepsilon, \sigma)(X) = (\varepsilon_1 \sigma(X)_1, \ldots, \varepsilon_k \sigma(X)_k)$. This extends the braid group action on R_k to an action of $\mathbb{Z}_N \wr B_k$, and in fact every fixed point $\mathrm{Fix}(\varepsilon\sigma) \subseteq R_k$ can be identified with a projective representation of the link group $G_L = \pi_1(S^3 \smallsetminus L)$.

The key result is Proposition 2.2, which shows that every element $X \in \mathrm{Fix}(\varepsilon\sigma)$ is irreducible. Consequently, writing $R_k^* \subset R_k$ for the subspace of irreducible $SU(N)$ representations, Proposition 2.2 implies that the graph $\Gamma_{\varepsilon\sigma}^*$ and the diagonal Δ_k^* intersect in a compact subset of $R_k^* \times R_k^*$. It follows that one can arrange transversality of the intersection $\Gamma_{\varepsilon\sigma}^* \cap \Delta_k^*$ by a compactly supported isotopy, and using natural orientations on the quotients $\widehat{\Gamma}_{\varepsilon\sigma} = \Gamma_{\varepsilon\sigma}^*/PU(N)$ and $\widehat{\Delta}_k = \Delta_k^*/PU(N)$, we define $h_{N,a}(\varepsilon\sigma)$ as the oriented intersection number of $\widehat{\Gamma}_{\varepsilon\sigma}$ and $\widehat{\Delta}_k$. Our main result is then established by showing that $h_{N,a}(\varepsilon\sigma)$ is independent of the choice of compatible k-tuple $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ (Proposition 3.4), and that it is invariant under the two Markov moves (Propositions 3.5 and 3.6). It follows that $h_{N,a}(L)$ gives a well-defined invariant of the link $L \subset S^3$.

One of the virtues of this approach is that it leads to a direct method for computing the invariants, and we illustrate this by computing $h_{N,a}(L)$ for the Hopf link and for chain links (Propositions 4.3 and 4.5) and by showing that the invariants vanish for split links (Proposition 4.6).

Gauge Theory. One motivation for defining link invariants in terms of the $SU(N)$ representations of the link group is that these representations can be identified with flat connections on a principal $SU(N)$ bundle over the link exterior, which allows for a gauge theoretic interpretation. This approach was originally used by Taubes to interpret Casson's invariant $\lambda(\Sigma)$ of homology 3-spheres Σ in terms of flat $SU(2)$ connections [17], and using similar ideas, Floer defined $\mathbb{Z}_8$-graded groups $HF_*(\Sigma)$ called the instanton Floer homology and whose Euler characteristic equals the Casson invariant [6].

The Casson-Lin invariants can also be interpreted gauge theoretically, as we now explain. The Casson-Lin invariant was originally defined in [14] by X.-S. Lin, who described an invariant $h(K)$ of knots $K \subset S^3$ as an algebraic count of conjugacy classes of *tracefree* irreducible $SU(2)$ representations of $\pi_1(S^3 \smallsetminus K)$ and proved that $h(K) = \mathrm{sign}(K)/2$, half the signature of K. In [9], C. Herald used gauge theory to define an extended Casson-Lin invariant $h_\alpha(K)$ for knots $K \subset \Sigma^3$ in homology 3-spheres which allows for more general meridional trace conditions, and he generalized Lin's formula by showing that $h_\alpha(K) = \mathrm{sign}_\alpha(K)/2$, half the Tristram-Levine α-twisted signature of K. (Similar results were obtained by M. Heusener and J. Kroll in [10].) O. Collin and B. Steer then used moduli spaces of orbifold connections to

define an associated Floer homology theory for knots whose Euler characteristic equals $h_\alpha(K)$ in [4], and P. Kronheimer and T. Mrowka further developed the instanton Floer homology theory of knots in [12], and they used it to prove a strong nontriviality result for Khovanov homology in [13].

The second author and N. Saveliev used projective $SU(2)$ representations to extend the Casson-Lin invariant to 2-component links L in S^3 in [7], and they showed that $h(L) = \pm lk(\ell_1, \ell_2)$, the linking number of $L = \ell_1 \cup \ell_2$. They gave a gauge theoretic description of the invariant $h(L)$ in [8], where they also described Floer homology groups with Euler characteristic equal to $h(L)$.

In view of all of these results, it is natural to ask whether the $SU(N)$ Casson-Lin invariants introduced here can also be interpreted gauge theoretically. We plan to address this question in a future article using moduli spaces of projective $SU(N)$ representations (cf. [16]). We hope to use this approach to extend the Casson-Lin invariants $h_{N,a}(L)$ to links $L \subset \Sigma^3$ in homology 3-spheres and to describe corresponding Floer homology groups. In particular, we expect this approach will help clarify the relationship between the invariants $h_{N,a}(L)$ studied here and the $SU(N)$ instanton Floer groups constructed by Kronheimer and Mrowka in [12]. It is possible that this approach will also shed light on other interesting questions, such as whether and how the Casson-Lin invariants are related to classical link invariants, such as the higher linking numbers.

We give a brief outline of the contents of this paper. In Section 1, we introduce the notation for braids $\sigma \in B_k$, links $L \subset S^3$, and $SU(N)$ representations that is used throughout the article. In Section 2, we introduce allowable labels $(a_1, \ldots, a_n)$ for a given n-component link $L \subset S^3$, and compatible k-tuple $(\varepsilon_1, \ldots, \varepsilon_k)$ for a braid $\sigma \in B_k$ with closure L. We also introduce projective $SU(N)$ representations of the link group G_L and establish irreducibility of elements of $\text{Fix}(\varepsilon\sigma)$. In Section 3, we define the invariant $h_{N,a}(L)$ as an oriented intersection number and prove it is independent of the various choices involved. In Section 4, we calculate the invariants $h_{N,a}(L)$ for the Hopf link and the n-component chain link, and we prove a general vanishing result for the invariants for split links.

1. Braids and representations

In this section, we introduce the results for braids, links, and $SU(N)$ representations that will be used in throughout the article.

1.1. **The braid group.** We denote by B_k the group of geometric braids on k-strands with standard generators $\sigma_1, \ldots, \sigma_{k-1}$ and relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|j - i| > 1$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. The generators are depicted in Figure 1 below, and note that in this paper we follow Convention 1.13 of [11].
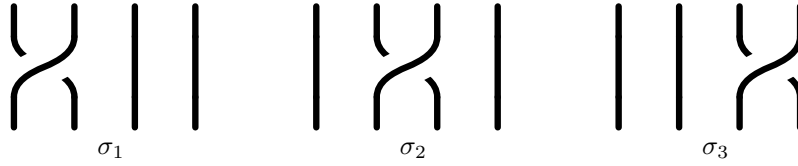
FIGURE 1. The three generators of B_4

Each braid $\sigma \in B_k$ determines a permutation, and the resulting map $B_k \to S_k$, which sends the generator σ_i to the transposition $\bar{\sigma}_i = (i, i+1)$, is a surjection. Given $\sigma \in B_k$, we let $\bar{\sigma} \in S_k$ denote the corresponding permutation. Under this map, the symmetric group S_k acts on the set $\{1, \ldots, k\}$ on the right. For $j \in \{1, \ldots, k\}$, we write $(i)^{\bar{\sigma}}$ for the image of i under $\bar{\sigma} \in S_k$.

Let F_k be the free group with free generating set $x_1, \ldots, x_k$. There is a natural right action of the braid group B_k on F_k defined by setting $\sigma_i \colon F_k \to F_k$ to be the map

$$\begin{aligned} x_i &\mapsto x_{i+1} \\ x_{i+1} &\mapsto (x_{i+1})^{-1} x_i \, x_{i+1} \\ x_j &\mapsto x_j, \ \ j \neq i, i+1. \end{aligned}$$

This action defines a faithful representation $\varrho \colon B_k \longrightarrow \mathrm{Aut}(F_k)$, and we use it to identify B_k with its image in $\mathrm{Aut}(F_k)$ under ϱ. As this is a right action, we will use x_i^σ to denote the image of x_i under $\sigma \in B_k$.

Example 1.1. We explain how to read the action of a braid, which is explained in section 2.4 of [5] for the left action, and we present the details for the right action.
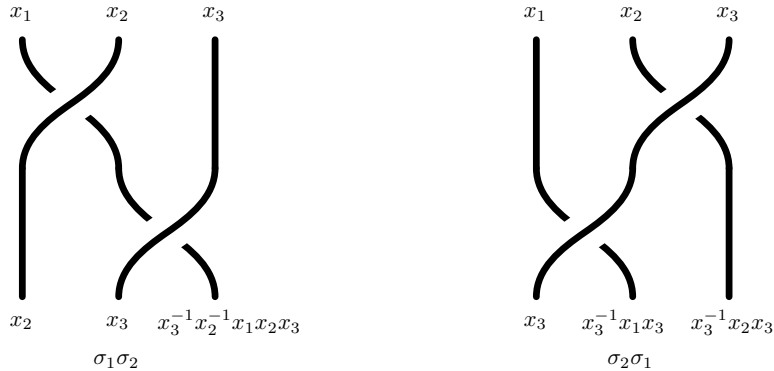
FIGURE 2. Reading the action of the braids $\sigma_1\sigma_2$ and $\sigma_2\sigma_1$

The basic idea is to view the free group F_k as the fundamental group of a 2-disk with k punctures and keep track of basepoints as you move the disk vertically, letting the punctures move along the braid. Specifically, label the top strands $x_1, \ldots, x_k$ from left to right. Then push the labels down, inserting a Wirtinger relation at each

crossing. At the bottom of the braid the strands will be labeled by words $w_1, \ldots, w_k$ in $x_1, \ldots, x_k$, and the right action of σ is given by the automorphism sending x_i to $x_i^\sigma := w_i$.

Using Figure 2, we determine the actions of $\sigma_1\sigma_2$ and $\sigma_2\sigma_1$ on $F_3 = \langle x_1, x_2, x_3 \rangle$ to be given by

$$\begin{cases} x_1^{\sigma_1\sigma_2} = x_2, \\ x_2^{\sigma_1\sigma_2} = x_3, \\ x_3^{\sigma_1\sigma_2} = x_3^{-1}x_2^{-1}x_1x_2x_3, \end{cases} \quad \text{and} \quad \begin{cases} x_1^{\sigma_2\sigma_1} = x_3, \\ x_2^{\sigma_2\sigma_1} = x_3^{-1}x_1x_3, \\ x_3^{\sigma_2\sigma_1} = x_3^{-1}x_2x_3. \end{cases}$$

We point out two facts about the action of B_k on F_k, both of which are easily verified for each generator. Firstly, for any $\sigma \in B_k$, σ acts by conjugation and permutation on the generating set $x_1, \ldots, x_k$ for F_k. Indeed,

$$x_i^\sigma = w\, x_{(i)^{\bar\sigma}}\, w^{-1}, \tag{1}$$

where $w \in F_k$ is some word depending on σ and i. Secondly, every braid $\sigma \in B_k$ preserves the product $x_1 \cdots x_k$,

$$(x_1 \cdots x_k)^\sigma = x_1 \cdots x_k. \tag{2}$$

1.2. **The group of a link.** Every link L in S^3 can be realized as the closure $L = \hat\sigma$ of a braid σ. We regard L as an oriented link, where the strands of the braid σ are oriented in the downward direction. The link group $G_L = \pi_1(S^3 \smallsetminus L)$ admits a standard presentation

$$G_L = \pi_1(S^3 \smallsetminus \hat\sigma) = \langle\, x_1, \ldots, x_k \mid x_i = x_i^\sigma,\ i = 1, \ldots, k \,\rangle. \tag{3}$$

The number of components of the link $L = \hat\sigma$ is the number of disjoint cycles in the permutation $\bar\sigma$. We will be interested in n-component links, that is, the closures of braids σ with

$$\bar\sigma = (i_1, \ldots, i_{k_1})(i_{k_1+1}, \ldots, i_{k_2}) \cdots (i_{k_{n-1}+1}, \ldots, i_{k_n}), \tag{4}$$

where $1 \le k_1 < k_2 < \cdots < k_n = k$. We define multi-indices $I_1, I_2, \ldots, I_n$ by setting $I_j = \{i_{k_{j-1}+1}, \ldots, i_{k_j}\}$ for $j = 1, \ldots, n$, and we denote $\bar\sigma = (I_1) \cdots (I_n)$. If $L = \ell_1 \cup \cdots \cup \ell_n$ is the closure of a braid σ, we will assume that the cycles in the permutation $\bar\sigma = (I_1) \cdots (I_n)$ are written correspondingly, so that the component ℓ_j of L corresponds to the braid closure of the strands in I_j.

1.3. **The special unitary group.** Consider the Lie group $SU(N)$ of unitary $N \times N$ matrices with determinant one. Recall that $SU(N)$ has real dimension $N^2 - 1$ and has center isomorphic to $\mathbb{Z}_N = \{\omega^d \mid d \in \mathbb{Z}\}$, where $\omega = e^{2\pi i/N}$. Notice that we are viewing $\mathbb{Z}_N$ as the subgroup of $U(1)$ consisting of N-th roots of unity, and for this reason we view it as a multiplicative group and identify it with the center of $SU(N)$ via the map defined by sending $\omega^d \mapsto \omega^d I$.

Since every matrix in $SU(N)$ is diagonalizable, conjugacy classes in $SU(N)$ are completely determined by their eigenvalues when considered with multiplicities. Given $A \in SU(N)$ we denote its conjugacy class by C_A. There is a unique conjugacy class C_A which is preserved under multiplication by $\omega = e^{2\pi i/N}$, and this is the conjugacy class of the diagonal matrix A whose eigenvalues are the N distinct N-th roots of $(-1)^{N-1}$. Setting $\xi = e^{2\pi i/2N}$, then A is given by the diagonal matrix $\mathrm{diag}(1, \omega, \ldots, \omega^{N-1})$ when N is odd and by $\mathrm{diag}(\xi, \omega\xi, \ldots, \omega^{N-1}\xi)$ when N is even. In either case, since the eigenvalues of A are all distinct, we see that the stabilizer of A is the standard maximal torus T^{N-1} in $SU(N)$ and that $C_A \cong SU(N)/T^{N-1}$ is the variety of full flags in $\mathbb{C}^N$ and has real dimension $N^2 - N$.

1.4. **$SU(N)$ representations.** For a discrete group G, let $\mathrm{R}(G) = \mathrm{Hom}\,(G, SU(N))$ denote the variety of $SU(N)$ representations of G. For convenience, we set $\mathrm{R}_k = \mathrm{R}(F_k) = SU(N) \times \cdots \times SU(N)$ to be the variety of $SU(N)$ representations of the free group F_k. The faithful representation $\varrho \colon B_k \to \mathrm{Aut}(F_k)$ induces a representation

$$\widetilde{\varrho} \colon B_k \longrightarrow \mathrm{Diff}(\mathrm{R}_k) \tag{5}$$

given by $\widetilde{\varrho}(\sigma)(\alpha) = \alpha \circ \sigma$. We will often abuse notation and simply denote $\widetilde{\varrho}(\sigma)$ by σ.

Remark 1.2. Theorem 2.1 of [15] implies that $\widetilde{\varrho}$ is defined on $\mathrm{Aut}(F_k)$ and is faithful.

Example 1.3. Consider σ_1 and $\sigma_2 \in B_3$. For $X = (X_1, X_2, X_3) \in \mathrm{R}_3$, we have

$$\sigma_1(X) = (X_2, X_2^{-1}X_1X_2, X_3) \qquad \text{and} \qquad \sigma_2(X) = (X_1, X_3, X_3^{-1}X_2X_3).$$

Using this, one can easily compute that

$$\sigma_1^2(X) = (X_2^{-1}X_1X_2, X_2^{-1}X_1^{-1}X_2X_1X_2, X_3)$$

and further that

$$\begin{aligned} \sigma_1\sigma_2(X) &= (X_2, X_3, X_2^{-1}X_3^{-1}X_1X_3X_2) \qquad \text{and} \\ \sigma_2\sigma_1(X) &= (X_3, X_3^{-1}X_1X_3, X_3^{-1}X_2X_3). \end{aligned}$$

Using the standard presentation (3) of the link group, we notice that $\mathrm{R}(G_L)$ can be identified with $\mathrm{Fix}(\sigma) \subset \mathrm{R}_k$,

$$\mathrm{R}(G_L) = \{(X_1, \ldots, X_k) \in \mathrm{R}_k \mid X_i = \sigma(X)_i\}.$$

A k-tuple $(X_1, \ldots X_k) \in \mathrm{R}_k$ is called *reducible* if it can be simultaneously conjugated by an element of $SU(N)$ such that each X_i has the form

$$X_i = \begin{pmatrix} A_i & 0 \\ 0 & B_i \end{pmatrix}, \tag{6}$$

where A_i is a block of size N_1 and B_i is a block of size N_2, and $N_1 + N_2 = N$. A k-tuple $(X_1, \ldots X_k) \in \mathrm{R}_k$ is *irreducible* if it is not reducible.

1.5. **The wreath product $\mathbb{Z}_N \wr B_k$.** The wreath product $\mathbb{Z}_N \wr B_k$ is the semidirect product of B_k with $(\mathbb{Z}_N)^k$, where B_k acts on $(\mathbb{Z}_N)^k$ by permutation. In other words, $\mathbb{Z}_N \wr B_k$ consists of pairs $(\varepsilon, \sigma) \in (\mathbb{Z}_N)^k \times B_k$, and the group structure is given by

$$(\varepsilon, \sigma) \cdot (\varepsilon', \sigma') \;=\; (\varepsilon \bar{\sigma}(\varepsilon'), \, \sigma\sigma').$$

Here, $\bar{\sigma}$ acts on $\varepsilon' = (\varepsilon'_1, \ldots, \varepsilon'_k)$ by permutation, i.e. $\bar{\sigma}(\varepsilon') = (\varepsilon'_{(1)^{\bar{\sigma}}}, \ldots, \varepsilon'_{(k)^{\bar{\sigma}}})$. In particular, it follows that $\sigma(\varepsilon X) = \bar{\sigma}(\varepsilon)\sigma(X)$.

We extend the representation (5) to the representation

$$\widetilde{\varrho} \colon \mathbb{Z}_N \wr B_k \longrightarrow \mathrm{Diff}(\mathrm{R}_k) \tag{7}$$

defined by sending the pair (ε, σ) to the diffeomorphism $\varepsilon\sigma \colon \mathrm{R}_k \to \mathrm{R}_k$, where

$$\varepsilon\sigma(X) = (\varepsilon_1 \sigma(X)_1, \ldots, \varepsilon_k \sigma(X)_k).$$

Thus ε twists the coordinates of $\sigma(X)$ by elements of the center $\mathbb{Z}_N$.

Example 1.4. For $X = (X_1, X_2, X_3) \in \mathrm{R}_3$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in (\mathbb{Z}_N)^3$, we have

$$
\begin{aligned}
(\varepsilon\sigma_1)(X_1, X_2, X_3) &= (\varepsilon_1\, X_2, \varepsilon_2\, X_2^{-1}X_1X_2, \varepsilon_3\, X_3) && \text{and} \\
\sigma_1(\varepsilon X) = \bar{\sigma}_1(\varepsilon)\sigma_1(X) &= (\varepsilon_2\, X_2, \varepsilon_1\, X_2^{-1}X_1X_2, \varepsilon_1 X_1, \varepsilon_3 X_3).
\end{aligned}
$$

2. Projective representations of the link group

Our goal in this paper is to define invariants of L, and we will do so by performing a signed count of certain irreducible projective $SU(N)$ representations.

2.1. **Projective representations.** Suppose $\sigma \in B_k$ is a braid whose closure $\widehat{\sigma}$ is a link L in S^3. For any k-tuple $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k) \in (\mathbb{Z}_N)^k$, an element $X = (X_1, \ldots, X_k) \in SU(N)^k$ in $\mathrm{Fix}(\varepsilon\sigma)$ determines a $PU(N)$ representation of the link group G_L, i.e. a homomorphism $\widetilde{\alpha} \colon G_L \to PU(N)$. To see this, note that for any $X \in \mathrm{Fix}(\varepsilon\sigma)$, since $\varepsilon_i \sigma(X)_i = X_i$ holds in $SU(N)$ and $\varepsilon_i \in \mathbb{Z}_N$ is central, the equation $\sigma(\widetilde{X})_i = \widetilde{X}_i$ holds for the k-tuple $\widetilde{X} \in PU(N)^k$, which shows that $\widetilde{X}$ determines a representation $\widetilde{\alpha} \colon G_L \to PU(N)$.

Given a discrete group G, we define a *projective representation* of G to be a *function* (not a homomorphism!) $\alpha \colon G \to SU(N)$ such that $\alpha(gh)\alpha(h)^{-1}\alpha(g)^{-1} \in \mathbb{Z}_N$ for all $g, h \in G$. For any projective representation $\alpha \colon G \to SU(N)$, its composition with the surjection $\mathrm{Ad} \colon SU(N) \to PU(N)$ gives rise to a representation $\widetilde{\alpha} = \mathrm{Ad}\,\alpha \colon G \to PU(N)$, and thus every projective representation $\alpha \colon G \to SU(N)$ is the lift of an honest representation $\widetilde{\alpha} \colon G \to PU(N)$. Alternatively, any representation $\widetilde{\alpha} \colon G \to PU(N)$ can be lifted to a projective representation $\alpha \colon G \to SU(N)$, though the lift is generally not unique.

Given a projective representation $\alpha \colon G \to SU(N)$, we can associate a map $c \colon G \times G \to \mathbb{Z}_N$ defined by $c(g, h) = \alpha(gh)\alpha(h)^{-1}\alpha(g)^{-1}$. Notice that the map c satisfies

the condition that $c(gh,k)c(g,h) = c(g,hk)c(h,k)$ for all $g,h,k \in G$, and hence c is a 2-cocyle of G.

For a fixed 2-cocycle $c\colon G \times G \to \mathbb{Z}_N$, let $\mathrm{PR}_c(G)$ denote the set of projective representations $\alpha\colon G \to SU(N)$ whose associated 2-cocycle is c. If G is finitely generated with generating set $\{g_1, \ldots, g_k\}$, then any projective representation $\alpha \in \mathrm{PR}_c(G)$ is completely determined by the 2-cocycle c and the elements $\alpha(g_1), \ldots, \alpha(g_k) \in SU(N)$, and in this way one can realize $\mathrm{PR}_c(G)$ as a subset of $SU(N)^k$. It is a compact real algebraic variety.

2.2. Allowable labels and compatible k-tuples.

Given a link L in S^3 with n components, we can write $L = \ell_1 \cup \cdots \cup \ell_n$. An n-tuple $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ of integers is called *allowable* if the following three conditions are satisfied:

(i) $0 \le a_i < N$ for $i = 1 \ldots, n$,
(ii) $d = \gcd(a_1, \ldots, a_n)$ is relatively prime to N,
(iii) $a_1 + \cdots + a_n$ is a multiple of N.

An allowable n-tuple $(a_1, \ldots, a_n)$ is called an n-tuple of *labels* for L, and a_j is the label corresponding to the j-th component ℓ_j of L.

Suppose now that L is the closure of a braid $\sigma \in B_k$, and write the permutation $\bar{\sigma}$ as a product $(I_1) \cdots (I_n)$ of disjoint cycles in such a way that I_j corresponds to the j-th component ℓ_j of L.

Recall that $\omega = e^{2\pi i/N}$. A k-tuple $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k) \in (\mathbb{Z}_N)^k$ for σ is said to be compatible with the choice of labels $(a_1, \ldots, a_n)$ of L if it satisfies the conditions

$$\prod_{i \in I_j} \varepsilon_i = \omega^{a_j} \tag{8}$$

for $j = 1, \ldots, n$. This effectively labels each strand of the braid σ so that, upon closure of the braid, the j-th component ℓ_j of L is assigned the number a_j for its label. Note that with this choice $\varepsilon\sigma$ also preserves condition (2) since, by Equation (8) and condition (iii), we have

$$\begin{aligned}
(\varepsilon\sigma)(X)_1 \cdots (\varepsilon\sigma)(X)_k &= (\varepsilon_1 \cdots \varepsilon_k)X_1 \cdots X_k \\
&= (\omega^{a_1} \cdots \omega^{a_n})X_1 \cdots X_k = X_1 \cdots X_k.
\end{aligned} \tag{9}$$

2.3. An obstruction to lifting.

For $X \in \mathrm{Fix}(\varepsilon\sigma)$ we will show that the associated representation $\widetilde{\alpha}\colon G_L \to PU(N)$ does not lift to an $SU(N)$ representation. Essential for this conclusion is that the k-tuple ε is compatible with a, the choice of labels for L. In particular, we use Equation (8) and assumption (ii) to give a nonzero obstruction to lifting $\widetilde{\alpha}$ to an $SU(N)$ representation.

Proposition 2.1. *The representation $\widetilde{\alpha}\colon G_L \to PU(N)$ does not lift to an $SU(N)$ representation.*

Proof. Lift $\widetilde{\alpha}$ arbitrarily to a map $\alpha \colon G_L \to SU(N)$. Since α is a lift of $\widetilde{\alpha}$, we see that $\alpha(x_i) = \eta_i X_i$ for some $\eta_i \in \mathbb{Z}_N$. Let $\eta = (\eta_1, \ldots, \eta_k) \in (\mathbb{Z}_N)^k$ be the corresponding k-tuple.

We assume that α is a representation. This implies that $\eta X \in \mathrm{Fix}(\sigma)$. Since X is also a fixed point of $\varepsilon\sigma$ we have that

$$\eta_i X_i = \sigma(\eta_i X_i) = \eta_{(i)^{\bar{\sigma}}}\sigma(X)_i = (\varepsilon_i)^{-1}\eta_{(i)^{\bar{\sigma}}}X_i.$$

By condition (ii), some $a_j \neq 0$, and we assume (wlog) that $a_1 \neq 0$. Consider the component ℓ_1 associated with the multi-index $I_1 = (i_1, \ldots, i_{k_1})$, then Equation (8) implies that

$$\eta_{i_1} = (\varepsilon_{i_1})^{-1}\eta_{i_2} = (\varepsilon_{i_1})^{-1}(\varepsilon_{i_2})^{-1}\eta_{i_3} = \cdots = (\varepsilon_{i_1})^{-1}\cdots(\varepsilon_{i_k})^{-1}\eta_{i_1} = \omega^{-a_1}\eta_{i_1},$$

which is a contradiction since $\omega^{-a_1} \neq 1$. $\qquad\square$

2.4. **Irreducibility for elements in Fix($\varepsilon\sigma$).** We will now show that for any allowable n-tuple $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ of labels and compatible k-tuple $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k) \in (\mathbb{Z}_N)^k$, every $X \in \mathrm{Fix}(\varepsilon\sigma)$ is irreducible. The key to the proof is condition (ii) on the labels.

Proposition 2.2. *If $X \in \mathrm{Fix}(\varepsilon\sigma)$, then X is irreducible.*

Proof. Suppose to the contrary that $X \in \mathrm{Fix}(\varepsilon\sigma)$ is reducible, which means that up to conjugation, we can assume

$$X_i = \begin{pmatrix} A_i & 0 \\ 0 & B_i \end{pmatrix},$$

where A_i has size N_1 and B_i has size N_2.

The first step is to consider the component ℓ_1 of L. It is obtained by closing the strands of σ associated with the cycle $I_1 = (i_1, \ldots, i_{k_1})$ of $\bar{\sigma}$. By (1), there are words $W_1, \ldots, W_{k_1}$ in $X_1, \ldots, X_k$ such that

$$\begin{aligned}
X_{i_1} &= \varepsilon_{i_1} W_1 X_{i_2} W_1^{-1} \\
&= (\varepsilon_{i_1}\varepsilon_{i_2})W_1 W_2\, X_{i_3}\, W_2^{-1}W_1^{-1} = \cdots \\
&= (\varepsilon_{i_1}\cdots\varepsilon_{i_{k_1}})W_1\cdots W_{k_1}\, X_{i_1}\, W_{k_1}^{-1}\cdots W_1^{-1} \\
&= \omega^{a_1}W X_{i_1} W^{-1},
\end{aligned}$$

where the last step follows by setting $W = W_1\cdots W_{k_1}$ and applying Equation (8).

Since W is a word in the X_i, and each X_i is block diagonal, it follows that W is also block diagonal so we can write

$$W = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}.$$

Applying this to the equation above, we see that the following relationship must hold for the blocks, so

$$A_{i_1} = \omega^{a_1} P A_{i_1} P^{-1}, \tag{10}$$

and taking the determinant of both sides of (10), we see that

$$\det(A_{i_1}) = \omega^{a_1 N_1} \det(A_{i_1}).$$

Because $\det(A_{i_1}) \neq 0$, this implies $\omega^{a_1 N_1} = 1$.

Now repeat the argument for the other components of the link L. For the component ℓ_j, which is the one obtained by closing the strands of σ associated with the cycle I_j, Equation (8) implies that $\omega^{a_j N_1} = 1$, and we see this holds for each $j = 1, \ldots, n$. However, since $\omega = e^{2\pi i / N}$ is a primitive N-th root of unity, this can only happen if N divides $a_j N_1$ for each $j = 1, \ldots, n$. This contradicts condition (ii) on the labels, and we conclude that each $X \in \mathrm{Fix}(\varepsilon\sigma)$ is in fact irreducible. $\qquad\square$

Remark 2.3. We would like to thank the referee for the following observation. Suppose $L = \ell_1 \cup \cdots \cup \ell_n$ is a link and let $\Lambda_i \cong \langle \mu_i \rangle \times \langle \lambda_i \rangle \cong \mathbb{Z} \times \mathbb{Z}$ denote the i-th peripheral subgroup of G_L, where μ_i, λ_i denote the meridian and longitude of ℓ_i. Given a representation $\widetilde{\alpha} \colon G_L \to PU(N)$, let $\omega(\widetilde{\alpha}) \in H^2(G_L, \mathbb{Z}_N)$ denote the obstruction cocycle, which is related to the commutator pairing of the restriction $\widetilde{\alpha}|_{\Lambda_i}$ as follows. If $\alpha \colon G_L \to SU(N)$ is a set-theoretic lift of $\widetilde{\alpha}$, then the *commutator pairing* of Λ_i is the map $c_i \colon \Lambda_i \times \Lambda_i \to \mathbb{Z}_N$ given by $c_i(x, y) = [\alpha(x), \alpha(y)]$. Since Λ_i is free abelian of rank two,

$$\theta \colon H^2(\Lambda_i, \mathbb{Z}_N) \xrightarrow{\ \cong\ } \mathrm{Hom}\,(\Lambda_i \wedge \Lambda_i, \mathbb{Z}_N) \cong \mathbb{Z}_N,$$

and one can show that $\theta(\omega(\widetilde{\alpha}|_{\partial_i})) = c_i$.

In the previous proof, the element $W = W_1 \ldots W_{k_1}$ is the image of the longitude λ_1 of ℓ_1, thus our computation that $[X_{i_1}, W] = \omega^{a_1}$ determined the commutator pairing $c_1 = \theta(\omega(\widetilde{\alpha}_{\partial_1}))$ by showing that $c_1(\mu_1, \lambda_1) = \omega^{a_1}$. The labels $a_1, \ldots, a_n$ thus determine the commutator pairings associated to the peripheral subgroups $\Lambda_1, \ldots, \Lambda_n$.

3. THE LINK INVARIANTS

Throughout this section, we assume that σ is a braid with closure $\widehat{\sigma} = L$ a link with n components $L = \ell_1 \cup \cdots \cup \ell_n$, and that $a = (a_1, \ldots, a_n)$ is an n-tuple of allowable labels, with a_j the label for the component ℓ_j.

In this section we define $h_{N,a}(\varepsilon\sigma)$ for compatible k-tuples ε, and we show that it gives rise to an invariant of n-component links in S^3.

We define $h_{N,a}(\varepsilon\sigma)$ as an algebraic count of certain projective $SU(N)$ representations in $\mathrm{Fix}(\varepsilon\sigma)$, namely those that satisfy the monodromy condition $X_i \in C_A$. In other words, we require each X_i to be in the conjugacy class of matrices with characteristic polynomial $p_A(t) = t^N + (-1)^N$.

We will first show that $h_{N,a}(\varepsilon\sigma)$ is independent of choice of ε, and then we prove that $h_{N,a}(\varepsilon\sigma)$ gives rise to a well-defined invariant of the underlying link L by showing that it is invariant under the Markov moves.

3.1. The definition of $h_{N,a}(\varepsilon\sigma)$.

Recall that A is the diagonal matrix consisting of the N-th roots of $(-1)^{N-1}$, i.e.

$$A = \begin{cases} \text{diag}(1, \omega, \ldots, \omega^{N-1}) & \text{if } N \text{ is odd} \\ \text{diag}(\xi, \omega\xi, \ldots, \omega^{N-1}\xi) & \text{if } N \text{ is even.} \end{cases}$$

We impose the following monodromy condition and restrict to k-tuples lying in the subset $Q_k \subset \mathrm{R}_k$ given by

$$Q_k = \{(X_1, \ldots, X_k) \in \mathrm{R}_k \mid X_i \in C_A\}.$$

Since $Q_k = (C_A)^k$ is a just a k-fold product of C_A, we see that Q_k is a manifold of dimension $k(N^2 - N)$.

Let $\Delta_k = \{(X, X)\} \subset Q_k \times Q_k$ be the diagonal and $\Gamma_{\varepsilon\sigma} = \{(X, \varepsilon\sigma(X))\} \subset Q_k \times Q_k$ be the graph of $\varepsilon\sigma$. Notice that we can identify points in the intersection $\Delta_k \cap \Gamma_{\varepsilon\sigma}$ with elements in $Q_k \cap \text{Fix}(\varepsilon\sigma)$.

For certain choices of labels, it will follow that $\text{Fix}(\varepsilon\sigma) \subset Q_k$, i.e. that these monodromy conditions are automatically satisfied. This will occur whenever the labels have the property that each a_i is relatively prime to N. For a simple example, suppose N is prime, n is a positive multiple of N, and d is any positive integer less than N. Then one can easily verify that $a = (d, d, \ldots, d)$ is an allowable n-tuple of labels, and the next result implies that $\text{Fix}(\varepsilon\sigma) \subset Q_k$ for any k-tuple $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ compatible with these labels.

Proposition 3.1. *Suppose $(a_1, \ldots, a_n)$ is an allowable n-tuple of labels such that each a_j is relatively prime to N, and suppose $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k) \in (\mathbb{Z}_N)^k$ is compatible with a. Then $\text{Fix}(\varepsilon\sigma) \subset Q_k$, i.e. if $X \in \text{Fix}(\varepsilon\sigma)$, then each X_j is conjugate to A.*

Proof. The condition on a_j ensures that ω^{a_j} generates $\mathbb{Z}_N$ for each $j = 1, \ldots, n$, where $\omega = e^{2\pi i/N}$. Write the induced permutation $\bar{\sigma} = (I_1) \cdots (I_n)$ as a product of disjoint cycles, where I_j corresponds to the j-th component of $L = \hat{\sigma}$. Then for any $i \in I_j$, we can apply the same argument as used to prove Proposition 2.2 to see that

$$X_i = \omega^{a_j} W X_i W^{-1}.$$

Thus, the set of eigenvalues of X_i is invariant under multiplication by ω^{a_j}, and this shows the eigenvalues of X are given by the set

$$\{\xi, \omega^{a_j}\xi, \ldots, \omega^{(N-1)a_j}\xi\} = \{\xi, \omega\xi, \ldots, \omega^{N-1}\xi\}$$

for some ξ satisfying $\xi^N = (-1)^{N-1}$. When N is odd, one can take $\xi = 1$, and when N is even, one can take $\xi = e^{2\pi i/2N}$, and this shows that X_i is conjugate to A.

Alternative argument: consider the characteristic polynomial of both sides of the above equation we see that $p_{X_i}(t) = p_{\omega^{a_j} X_i}(t) = p_{X_i}(\omega^{-a_j} t)$. Since ω^{-a_j} has order N, $p_{X_i}(t)$ must be a polynomial in t^N, and indeed the only possibility is that $p_{X_i}(t) = t^N + (-1)^N$. $\qquad\square$

We define
$$H_k = \{(X,Y) \in Q_k \times Q_k \mid X_1 \cdots X_k = Y_1 \cdots Y_k\},$$
and we note that H_k is not a manifold because of the presence of reducibles. Recall that $(X,Y) \in Q_k \times Q_k$ is called *reducible* if all X_i and Y_i can be simultaneously conjugated into block diagonal form as in (6). We note that the subset $S_k \subset Q_k \times Q_k$ of reducibles is closed, and that $(Q_k \times Q_k)^* = (Q_k \times Q_k) \smallsetminus S_k$ is an open manifold of dimension $2k(N^2 - N)$.

Proposition 3.2. *The subset $H_k^* = H_k \smallsetminus S_k$ of irreducible representations is an open manifold of dimension $2k(N^2 - N) - (N^2 - 1)$.*

Proof. Clearly $H_k^* = f^{-1}(I)$, where $f \colon (Q_k \times Q_k)^* \to SU(N)$ is the map defined by $f(X,Y) = X_1 \cdots X_k Y_k^{-1} \cdots Y_1^{-1}$. We will show that I is a regular value of f, i.e. that $df_{(X,Y)}$ is surjective for all $(X,Y) \in f^{-1}(I)$. It is enough to prove this statement for the map $f \colon Q_\ell^* \to SU(N)$ given by $f(X_1, \ldots, X_\ell) = X_1 \cdots X_\ell$.

Clearly the matrix A, since it is diagonal, lies on the standard maximal torus $T^{N-1} \subset SU(N)$ with Lie algebra

$$\mathfrak{t} = \left\{ \begin{pmatrix} ia_1 & & 0 \\ & \ddots & \\ 0 & & ia_N \end{pmatrix} \;\middle|\; a_1 + \cdots + a_N = 0 \right\}.$$

Since A has the standard maximal torus as its stablizer group, we can identify the tangent space $T_A(C_A)$ with the orthogonal complement $\mathfrak{t}^\perp$ in $su(N)$, which is the subspace

$$\mathfrak{t}^\perp = \left\{ \begin{pmatrix} 0 & z_{12} & \cdots & z_{1N} \\ -\bar{z}_{12} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & z_{N-1,N} \\ -\bar{z}_{1N} & \cdots & -\bar{z}_{N-1,N} & 0 \end{pmatrix} \;\middle|\; z_{ij} \in \mathbb{C} \right\}.$$

There is a similar decomposition of $su(N)$ at each X_i in the ℓ-tuple $(X_1, \ldots, X_\ell)$. Because each X_i has N distinct eigenvalues, it lies on a unique maximal torus $T_i \cong T^{N-1}$ in $SU(N)$. We let $\mathfrak{t}_i \subset su(N)$ denote the corresponding Lie subalgebra, which is the Lie algebra of the stabilizer group of X_i. Using the decomposition $su(N) = \mathfrak{t}_i \oplus \mathfrak{t}_i^\perp$, we can identify the tangent space $T_{X_i}(C_A)$ with $\mathfrak{t}_i^\perp X_i$, the right translation of the subspace $\mathfrak{t}_i^\perp \subset su(N)$ by X_i. It is helpful to note that, in terms of the specific subspaces identified above, we have $\mathfrak{t}_i = \mathrm{Ad}_{P_i} \mathfrak{t}$ and $\mathfrak{t}_i^\perp = \mathrm{Ad}_{P_i} \mathfrak{t}^\perp$ for any matrix $P_i \in SU(N)$ such that $X_i = P_i A P_i^{-1}$.

Using the fact that $\mathfrak{t}_i$ is the Lie algebra of the stabilizer subgroup of X_i, one can see that irreducibility of the ℓ-tuple $(X_1, \ldots, X_\ell)$ is equivalent to the condition that $\mathfrak{t}_1 \cap \cdots \cap \mathfrak{t}_\ell = \{0\}$.

For $u_i \in \mathfrak{t}_i^\perp$, we set $x_i = u_i X_i \in \mathfrak{t}_i^\perp X_i = T_{X_i}(C_A)$. Differentiating and using the fact that $X_1 \cdots X_\ell = I$, we obtain

$$\frac{d}{dt}(X_1 + tx_1)(X_2 + tx_2) \cdots (X_\ell + tx_\ell)|_{t=0}$$
$$= x_1 X_2 \cdots X_\ell + X_1 x_2 X_2 \cdots X_\ell + \cdots + X_1 \cdots X_{\ell-1} x_\ell$$
$$= u_1 + X_2 u_2 X_2^{-1} + \cdots + (X_2 \cdots X_\ell) u_\ell (X_\ell^{-1} \cdots X_2^{-1}).$$

In order to show that the map df_X is onto, we claim that, given any $v \in su(N)$, we can find $u_i \in \mathfrak{t}_i^\perp$ for $i = 1, \ldots, \ell$ such that

$$v = u_1 + X_2 u_2 X_2^{-1} + \cdots + (X_2 \cdots X_\ell) u_\ell (X_\ell^{-1} \cdots X_2^{-1}). \tag{11}$$

Notice that we can solve (11) for any

$$v \in \mathfrak{t}_1^\perp \cap \left(X_1 \mathfrak{t}_2^\perp X_1^{-1}\right) \cap \cdots \cap (X_1 \cdots X_{\ell-1}) \mathfrak{t}_\ell^\perp \left(X_{\ell-1}^{-1} \cdots X_1^{-1}\right).$$

Notice further that since $\mathfrak{t}_i$ is the Lie algebra of the maximal torus containing X_i, we have $\mathfrak{t}_i \cap \mathfrak{t}_{i+1} = \mathfrak{t}_i \cap \left(X_i \mathfrak{t}_{i+1} X_i^{-1}\right)$. More generally, for any subspace $V \subset su(N)$, we have $\mathfrak{t}_i \cap V = \mathfrak{t}_i \cap \left(X_i V X_i^{-1}\right)$. Repeated application gives that

$$
\begin{aligned}
\mathfrak{t}_1 \cap \cdots \cap \mathfrak{t}_\ell &= \mathfrak{t}_1 \cap \cdots \cap \left(X_{\ell-1} \mathfrak{t}_\ell X_{\ell-1}^{-1}\right) \\
&= \mathfrak{t}_1 \cap \cdots \cap \left(X_{\ell-2} \mathfrak{t}_{\ell-1} X_{\ell-2}^{-1}\right) \cap \left(X_{\ell-2} X_{\ell-1} \mathfrak{t}_\ell X_{\ell-1}^{-1} X_{\ell-2}^{-1}\right) \\
&\vdots \\
&= \mathfrak{t}_1 \cap \left(X_1 \mathfrak{t}_2 X_1^{-1}\right) \cap \cdots \cap (X_1 \cdots X_{\ell-1}) \mathfrak{t}_\ell \left(X_{\ell-1}^{-1} \cdots X_1^{-1}\right).
\end{aligned}
$$

The condition of irreducibility implies that $\mathfrak{t}_1 \cap \cdots \cap \mathfrak{t}_\ell = \{0\}$, and it then follows from the above that (11) can be solved for any $v \in su(N)$. This concludes the argument that df_X is a surjection whenever the ℓ-tuple $X = (X_1, \ldots, X_\ell)$ is irreducible. $\qquad \square$

Since both Δ_k and $\Gamma_{\varepsilon\sigma}$ preserve the product $X_1 \cdots X_k$, see (9), we can restrict from $Q_k \times Q_k$ to H_k.

Now we are in a position to define the invariant $h_{N,a}(\varepsilon\sigma)$. Set $\Gamma_{\varepsilon\sigma}^* = \Gamma_{\varepsilon\sigma} \cap H_k^*$ and $\Delta_k^* = \Delta_k \cap H_k^*$. Since S_k is closed, it follows that both Γ^* and Δ_k^* are open submanifolds of H_k^* of dimension $k(N^2 - N)$.

Both Δ_k and $\Gamma_{\varepsilon\sigma}$ are compact, and so is their intersection $\Delta_k \cap \Gamma_{\varepsilon\sigma}$. Consequently, as Proposition 2.2 implies that every point in this intersection is irreducible, we have the following result.

Corollary 3.3. *The intersection $\Delta_k^* \cap \Gamma_{\varepsilon\sigma}^* \subset H_k^*$ is compact.*

The group $PU(N)$ acts freely by conjugation on each of H_k^*, Δ_k^*, and $\Gamma_{\varepsilon\sigma}^*$, and the quotients by this action are the manifolds we denote as

$$\widehat{H}_k = H_k^*/PU(N), \quad \widehat{\Delta}_k = \Delta_k^*/PU(N), \quad \text{and} \quad \widehat{\Gamma}_{\varepsilon\sigma} = \Gamma_{\varepsilon\sigma}^*/PU(N).$$

Here, the dimension of $\widehat{H}_k$ equals $2k(N^2 - N) - 2(N^2 - 1)$, and both $\widehat{\Delta}_k$ and $\widehat{\Gamma}_{\varepsilon\sigma}$ are half-dimensional submanifolds of $\widehat{H}_k$. Since the intersection $\widehat{\Delta}_k \cap \widehat{\Gamma}_{\varepsilon\sigma}$ is compact, we can isotope $\widehat{\Gamma}_{\varepsilon\sigma}$ into a submanifold $\widetilde{\Gamma}_{\varepsilon\sigma}$ using an isotopy with compact support so that the intersection $\widehat{\Delta}_k \cap \widetilde{\Gamma}_{\varepsilon\sigma}$ is transverse and consists of finitely many points. Define

$$h_{N,a}(\varepsilon\sigma) = \#_{\widehat{H}_k} \left(\widehat{\Delta}_k \cap \widetilde{\Gamma}_{\varepsilon\sigma} \right)$$

as the oriented intersection number. We will describe the orientations in the following subsection. The intersection number $h_{N,a}(\varepsilon\sigma)$ is independent of the choice of isotopy of $\widehat{\Gamma}_{\varepsilon\sigma}$, and we denote

$$h_{N,a}(\varepsilon\sigma) = \langle \widehat{\Delta}_k, \widehat{\Gamma}_{\varepsilon\sigma} \rangle_{\widehat{H}_k}.$$

3.2. **Orientations.** The following argument is similar to the one found in section 3.4 of [7], and we include it here for completeness.

First, observe that the conjugacy class $C_A \subset SU(N)$ is orientable, which follows for instance by identifying it with a flag variety. So choose an orientation for C_A and give $Q_k = (C_A)^k$ and $Q_k \times Q_k$ the induced product orientations. The diagonal Δ_k and the graph $\Gamma_{\varepsilon\sigma}$ are naturally diffeomorphic to Q_k via projection and so an orientation for Q_k determines orientations for both Δ_k and $\Gamma_{\varepsilon\sigma}$.

Using the standard orientation of $SU(N)$, we obtain an orientation on $H_k^* = f^{-1}(I)$ using the base-fiber rule. Since the adjoint action of $PU(N)$ on C_A is orientation preserving, the quotients $\widehat{H}_k, \widehat{\Delta}_k$, and $\widehat{\Gamma}_{\varepsilon\sigma}$ are all orientable, and we orient them using the base-fiber rule.

Reversing the orientation of C_A reverses the orientation of Q_k only when k is odd, and in this case it reverses the orientations of both $\widehat{\Delta}_k$, and $\widehat{\Gamma}_{\varepsilon\sigma}$ but it does not affect the oriented intersection number $\langle \widehat{\Delta}_k, \widehat{\Gamma}_{\varepsilon\sigma} \rangle_{\widehat{H}_k}$. This shows that the intersection number is actually independent of the choice of orientation on the conjugacy class C_A.

3.3. **Independence of ε.** The next result shows that $h_{N,a}(\varepsilon\sigma)$ is independent of the choice of ε compatible with a.

Proposition 3.4. *Fix a link L with $n > 1$ components and an allowable n-tuple $a = (a_1, \ldots, a_n)$ of labels. Fix also a braid $\sigma \in B_k$ with closure $\widehat{\sigma} = L$. If $\varepsilon, \varepsilon' \in (\mathbb{Z}_N)^k$ are k-tuples compatible with a, i.e. satisfying Equation (8), then $h_{N,a}(\varepsilon\sigma) = h_{N,a}(\varepsilon'\sigma)$.*

Proof. We will define an orientation preserving automorphism $\varphi \colon \widehat{H}_k \to \widehat{H}_k$ such that $\varphi(\widehat{\Delta}_k) = \widehat{\Delta}_k$ and $\varphi(\widehat{\Gamma}_{\varepsilon\sigma}) = \widehat{\Gamma}_{\varepsilon'\sigma}$. Write the permutation

$$\bar{\sigma} = (i_1, \ldots, i_{k_1})(i_{k_1+1}, \ldots, i_{k_2}) \cdots (i_{k_{n-1}+1}, \ldots, i_{k_n}) \tag{12}$$

as a product of disjoint cycles as in Equation (4) and define $\delta = (\delta_1, \ldots, \delta_k) \in (\mathbb{Z}_N)^k$ recursively with initial values

$$\delta_{i_1} = 1 = \delta_{i_{k_1+1}} = \cdots = \delta_{i_{k_{n-1}+1}} \tag{13}$$

and by setting

$$\delta_{(j)^{\bar{\sigma}}} = \delta_j \, \varepsilon_j \, (\varepsilon_j')^{-1}. \tag{14}$$

Writing $\bar{\sigma}$ as a product of disjoint cycles as in (12) and noting that ε and ε' both satisfy Equation (8), repeated application of the recursion (14) shows that the definition of $\delta = (\delta_1, \ldots, \delta_k)$ is compatible with the initial values taken in (13).

Define the diffeomorphism $\tau \colon Q_k \to Q_k$ by

$$\tau(X) = \delta X = (\delta_1 X_1, \ldots, \delta_k X_k).$$

Note that τ may be orientation preserving or reversing. Furthermore, τ preserves irreducibility and commutes with conjugation.

Consider the product map $\tau \times \tau \colon Q_k \times Q_k \to Q_k \times Q_k$. Observe that $\tau \times \tau$ preserves the orientation of $Q_k \times Q_k$ and hence the induced map $\varphi \colon \widehat{H}_k \to \widehat{H}_k$ is orientation preserving.

Since τ may be orientation reversing, φ restricted to $\widehat{\Delta}_k$ or $\widehat{\Gamma}_{\varepsilon\sigma}$ may be orientation reversing. The key observation is that if φ is orientation reversing on one, then it must be orientation reversing on the other. Hence, φ preserves the intersection number $h_{N,a}(\varepsilon\sigma)$,

$$\langle \, \widehat{\Delta}_k, \widehat{\Gamma}_{\varepsilon\sigma} \, \rangle_{\widehat{H}_k} = \langle \, \varphi(\widehat{\Delta}_k), \, \varphi(\widehat{\Gamma}_{\varepsilon\sigma}) \, \rangle_{\widehat{H}_k}.$$

Clearly, $\varphi(\widehat{\Delta}_k) = \widehat{\Delta}_k$, so to finish off the proof we check that $\varphi(\widehat{\Gamma}_{\varepsilon\sigma}) = \widehat{\Gamma}_{\varepsilon'\sigma}$, or that the pair $(\delta X, \delta\varepsilon\sigma(X)) \in \widehat{\Gamma}_{\varepsilon'\sigma}$. By the following calculation

$$(\delta X, \delta\varepsilon\sigma(X)) = (\delta X, \delta\varepsilon\sigma(\delta^{N-1}\delta X)) = (\delta X, \delta\varepsilon\bar{\sigma}(\delta^{N-1}) \, \sigma(\delta X)),$$

this will follow once we verify that $\delta\varepsilon\bar{\sigma}(\delta^{N-1}) = \varepsilon'$. Since $\delta_j^{N-1} = \delta_j^{-1}$, this is equivalent to showing that $\delta_j \varepsilon_j = \bar{\sigma}(\delta_j)\varepsilon_j'$ for $j = 1, \ldots, k$, which follows directly from Equation (14), and this completes the proof of the proposition. $\qquad\square$

3.4. **Independence under Markov moves.** Based on the previous result, we denote $h_{N,a}(\varepsilon\sigma)$ by $h_{N,a}(\sigma)$ assuming that a choice of compatible ε has been made. In this subsection, we show that $h_{N,a}$ defines an invariant of n-component links, and this is achieved by showing that $h_{N,a}(\sigma)$ is invariant under the Markov moves.

Recall that two braids $\sigma, \tau \in B_k$ have isotopic closures $\widehat{\sigma} = \widehat{\tau}$ if and only if σ can be obtained from τ by a finite sequence of Markov moves; see for example [2]. The first Markov move replaces $\sigma \in B_k$ by $\xi^{-1}\sigma\xi \in B_k$ for $\xi \in B_k$, and the second Markov move exchanges $\sigma \in B_k$ with $\sigma\sigma_k^{\pm 1} \in B_{k+1}$.

The following propositions give the $SU(N)$ analogues of the $SU(2)$ results in [7, Propositions 4.2 & 4.3] (cf. the proof of [14, Theorem 1.8]).

Proposition 3.5. *The quantity $h_{N,a}(\sigma)$ is invariant under type 1 Markov moves.*

Proof. Suppose $\sigma \in B_k$ is a braid with

$$\bar{\sigma} = (I_1) \cdots (I_n) = (i_1, \ldots, i_{k_1}) \cdots (i_{k_{n-1}+1}, \ldots, i_{k_n})$$

in multi-index notation. Given a braid $\xi \in B_k$, let $\sigma' = \xi^{-1}\sigma\xi$ and note that $\overline{\sigma'}$ has the same cycle structure as $\bar{\sigma}$, in fact it is given by

$$\overline{\sigma'} = \left(I_1^\xi\right) \cdots \left(I_n^\xi\right) = \left((i_1)^{\bar\xi}, \ldots, (i_{k_1})^{\bar\xi}\right) \cdots \left((i_{k_{n-1}+1})^{\bar\xi}, \ldots, (i_{k_n})^{\bar\xi}\right).$$

We choose $\varepsilon' \in (\mathbb{Z}_N)^k$ compatible with the given labels, which means that ε' satisfies Equation (8) with respect to the braid σ', namely

$$\prod_{i \in I_j^\xi} \varepsilon'_i = \omega^{a_j}$$

holds for $j = 1, \ldots, k$. Notice that if we define the k-tuple ε by setting $\varepsilon_i = \varepsilon'_{(i)\bar\xi}$, then one can show that ε satisfies Equation (8) with respect to $\bar{\sigma} = (I_1) \cdots (I_n)$, hence ε is also compatible with the given labels.

The braid ξ determines a map $\xi \colon Q_k \to Q_k$, and since it acts by permutation and conjugation on each of the factors in $Q_k = C_A \times \cdots \times C_A$, the fact that C_A is even-dimensional implies that this map is orientation preserving. This induces the map $\xi \times \xi$ on $Q_k \times Q_k$ preserving irreducibility, commuting with the adjoint action of $PU(n)$, and preserving Equation (2), thus we obtain a well-defined orientation preserving map $\xi \times \xi \colon \widehat{H}_k \to \widehat{H}_k$.

Clearly, $(\xi \times \xi)(\widehat{\Delta}_k) = \widehat{\Delta}_k$, so the diagonal is preserved, and we consider the effect of $\xi \times \xi$ on the graph $\widehat{\Gamma}_{\varepsilon'\sigma'}$. If $(X, \varepsilon'\sigma'(X)) \in \widehat{\Gamma}_{\varepsilon'\sigma'}$, then

$$(\xi \times \xi)(X, \varepsilon'\sigma'(X)) = (\xi \times \xi)(X, \varepsilon'\xi^{-1}\sigma\xi(X)) = (\xi(X), \xi(\varepsilon')\sigma\xi(X)) \in \widehat{\Gamma}_{\varepsilon\sigma},$$

since $\xi(\varepsilon')_i = \varepsilon'_{(i)\bar\xi} = \varepsilon_i$. Thus $(\xi \times \xi)(\widehat{\Gamma}_{\varepsilon'\sigma'}) = \widehat{\Gamma}_{\varepsilon\sigma}$, and we see that

$$\begin{aligned} h_{N,a}(\sigma') = \langle \widehat{\Delta}_k, \widehat{\Gamma}_{\varepsilon'\sigma'} \rangle_{\widehat{H}_k} &= \langle (\xi \times \xi)(\widehat{\Delta}_k), (\xi \times \xi)(\widehat{\Gamma}_{\varepsilon'\sigma'}) \rangle_{\xi \times \xi(\widehat{H}_k)} \\ &= \langle \widehat{\Delta}_k, \widehat{\Gamma}_{\varepsilon\sigma} \rangle_{\widehat{H}_k} = h_{N,a}(\sigma). \end{aligned}$$

$\square$

The next result is established using the same argument that is used to prove [7, Proposition 4.3] and [14, Theorem 1.8]. We leave the details of the proof to the reader.

Proposition 3.6. *The quantity $h_{N,a}(\sigma)$ is invariant under type 2 Markov moves.*

4. Computations

In this section, we perform computations of $h_{N,a}(L)$ for various links L and we prove a vanishing condition for $h_{N,a}(L)$ for split links.

4.1. **The Hopf link and chain links.** The chain link L is obtained as the closure of the braid $\sigma = \sigma_1^2 \sigma_2^2 \cdots \sigma_{n-1}^2 \in B_n$. In this subsection, we compute $h_{N,a}(L)$ for L the Hopf link and the chain link with $N = n$ components. In particular, if d is chosen relatively prime to N and $a = (d, \ldots, d)$, then we will show that $h_{N,a}(L) = 0$ for the chain link with $n > 2$ components. For $n = 2$, L is just the Hopf link, which we denote by $H \subset S^3$. In [7], it is proved that $h_{2,a}(H) = \pm 1$ for $a = (1,1)$. We generalize this by showing that $h_{N,a}(H) = \pm 1$ if $a = (d, N - d)$, where d satisfies $1 \leq d < N$ and is relatively prime to N.

The next result will be used repeatedly in the computations that follow.

Theorem 4.1. *Suppose $N \geq 2$ and set $\omega = e^{2\pi i/N}$ and $\xi = e^{2\pi i/2N}$. Any pair of matrices $(X, Y) \in SU(N) \times SU(N)$ satisfying $[X, Y] = \omega I$ is, up to conjugation, given by*

$$X = \begin{cases} \mathrm{diag}(1, \omega, \ldots, \omega^{N-1}) & \text{if } N \text{ is odd,} \\ \mathrm{diag}(\xi, \xi\omega, \ldots, \xi\omega^{N-1}) & \text{if } N \text{ is even,} \end{cases}$$

and

$$Y = \begin{pmatrix} 0 & \cdots & & \pm 1 \\ 1 & \ddots & & \vdots \\ & & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}.$$

The pair $(X, Y) \in SU(N) \times SU(N)$ determines an irreducible projective $SU(N)$ representation of the free abelian group $\mathbb{Z} \oplus \mathbb{Z}$ of rank 2.

Proof. First notice that $XYX^{-1}Y^{-1} = \omega I$ if and only if $Y^{-1}XY = \omega X$. Every element of $SU(N)$ is conjugate to a diagonal matrix, and so we can write

$$X = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix},$$

where $\lambda_i \in U(1)$ and $\lambda_1 \cdots \lambda_N = 1$. However, because X is conjugate to ωX, we must have

$$\{\lambda_1, \ldots, \lambda_N\} = \{\omega\lambda_1, \ldots, \omega\lambda_N\}.$$

Reordering the terms, we can arrange that $\lambda_i = \omega^{i-1}\lambda_1$ for $i = 1, \ldots, N$. Since $\det X = 1$, we have $\lambda_1^N = (-1)^{N-1}$, and so without loss of generality we can take

$$\lambda_1 = \begin{cases} 1 & \text{if } N \text{ is odd,} \\ \xi & \text{if } N \text{ is even.} \end{cases}$$

This shows that X is of the required form.

Next, observe that $XYX^{-1}Y^{-1} = \omega I$ if and only if $XY = \omega YX$. Writing $Y = (y_{ij})$ and comparing the (ij) entries on right and left, it follows that

$$\omega^{i-1} y_{ij} = y_{ij}\omega^j.$$

This implies that $y_{ij} = 0$ unless $i \equiv j + 1 \mod N$. Furthermore, since Y has only one nonzero entry in each row and column, each entry must lie in $U(1)$ and we find that

$$Y = \begin{pmatrix} 0 & \cdots & & & \mu_1 \\ \mu_2 & \ddots & & & \vdots \\ & & \ddots & & \\ 0 & & & \mu_N & 0 \end{pmatrix},$$

where $\mu_i \in U(1)$ satisfy $\mu_1 \cdots \mu_N = (-1)^{N-1}$ (since $\det Y = 1$). Because X is diagonal with N distinct eigenvalues, the stabilizer subgroup $\mathrm{Stab}(X)$ is a copy of the standard maximal torus, i.e.

$$\mathrm{Stab}(X) = \{\mathrm{diag}(\theta_1, \ldots, \theta_N) \mid \theta_i \in U(1), \theta_1 \cdots \theta_N = 1\} \cong T^{N-1}.$$

A matrix $P = \mathrm{diag}(\theta_1, \ldots, \theta_N) \in \mathrm{Stab}(X)$ acts on Y by

$$PYP^{-1} = \begin{pmatrix} 0 & \cdots & & & \theta_1\theta_N^{-1}\mu_1 \\ \theta_2\theta_1^{-1}\mu_2 & \ddots & & & \vdots \\ & & \ddots & & \\ 0 & & & \theta_N\theta_{N-1}^{-1}\mu_N & 0 \end{pmatrix}.$$

Setting

$$\theta_1 = \mu_1^{-1}$$
$$\theta_2 = \theta_1\,\mu_2^{-1} = \mu_1^{-1}\mu_2^{-1}$$
$$\theta_3 = \theta_2\,\mu_3^{-1} = \mu_1^{-1}\mu_2^{-1}\mu_3^{-1}$$
$$\vdots$$
$$\theta_N = \theta_{N-1}\,\mu_N^{-1} = \mu_1^{-1}\cdots\mu_N^{-1} = (-1)^{N-1},$$

it follows that

$$PYP^{-1} = \begin{pmatrix} 0 & \cdots & & & (-1)^{N-1} \\ 1 & \ddots & & & \vdots \\ & & \ddots & & \\ 0 & & & 1 & 0 \end{pmatrix}.$$

Since $PXP^{-1} = X$, this shows that, up to conjugation, Y is of the required form. Irreducibility of the pair (X, Y) follows from the fact that $\mathrm{Stab}(X) \cap \mathrm{Stab}(Y) = \mathbb{Z}_N$. $\qquad\square$

Remark 4.2. If $XYX^{-1} = \omega Y$, then $X^d Y X^{-d} = \omega^d Y$ by induction. This shows that if (X, Y) are as in Theorem 4.1, then (X^d, Y) satisfies $[X^d, Y] = \omega^d I$. Using this observation, one can show that solutions (X', Y') to $[X', Y'] = \omega^d I$ are irreducible and unique up to conjugation provided d is relatively prime to N. This fails if d is not relatively prime to N; when $N = 4$ and $d = 2$, one can construct non-conjugate families of pairs $(X, Y) \in SU(4) \times SU(4)$ satisfying $[X, Y] = -I$. All of these pairs are reducible.

We now use this to evaluate $h_{N,a}(H)$ for the Hopf link H.

Proposition 4.3. *Suppose H is the Hopf link and $1 \le d < N$ is relatively prime to N. Then $h_{N,a}(H) = \pm 1$ for $a = (d, N - d)$.*

Proof. We motivate the proof with the following argument. The Hopf link H has link group $G_H = \langle x, y \mid [x, y] = 1 \rangle$, and Theorem 4.1 implies there is a unique irreducible projective representation $\varrho \colon G_H \to SU(N)$ with $[\varrho(x), \varrho(y)] = \omega^d I$. Uniqueness of ϱ up to conjugacy implies that $h_{N,a}(H) = \pm 1$.

More precisely, notice that the Hopf link is the closure of the braid $\sigma_1^2 \in B_2$ and fix the labels $a = (d, N - d)$ for H, where $1 \le d < N$ is relatively prime to N. The braid $\sigma = \sigma_1^2$ acts on pairs $(X_1, X_2) \in \mathrm{R}_2 = SU(N) \times SU(N)$ in the usual way (see Example 1.3), and for $\varepsilon = (\varepsilon_1, \varepsilon_2)$ we have

$$\varepsilon \sigma(X_1, X_2) = (\varepsilon_1 X_2^{-1} X_1 X_2, \varepsilon_2 X_2^{-1} X_1^{-1} X_2 X_1 X_2).$$

For $(\varepsilon_1, \varepsilon_2) = (\omega^d, \omega^{N-d})$, one can easily see that $(X_1, X_2) \in \mathrm{Fix}(\varepsilon \sigma)$ if and only if $[X_1, X_2] = \omega^d$. By Theorem 4.1 and the preceding remarks, this equation has one solution which is irreducible and unique up to conjugation. Lemma 4.4 below shows that the solution is non-degenerate, and this implies that $h_{N,a}(H) = \pm 1$ for the Hopf link. $\square$

The next result establishes the non-degeneracy result required for the above computation of $h_{N,a}(H)$.

Lemma 4.4. *Let H be the Hopf link, G_H its link group, and $1 \le d < N$ relatively prime to N. Suppose $\varrho \colon G_H \to SU(N)$ is the projective representation, unique up to conjugation, of the link group G_H with $a = (d, N - d)$. Then $H^1(G_H; su(N)_{\mathrm{Ad}\, \varrho}) = 0$.*

Proof. Let $Z = S^3 \smallsetminus \tau H$ be the link exterior, and recall that the exterior of every non-split link in S^3 is a $K(\pi, 1)$. Thus $H^i(Z; su(N)_{\mathrm{Ad}\, \varrho}) = H^i(G_H; su(N)_{\mathrm{Ad}\, \varrho})$, where $G_H = \pi_1(Z)$ is the link group.

For the Hopf link, the link group $G_H = \mathbb{Z} \times \mathbb{Z}$ is the free abelian group of rank two. Since ϱ is irreducible, it follows that $H^0(G_H; su(N)_{\mathrm{Ad}\, \varrho}) = 0$, and Poincaré duality implies that $H^2(G_H; su(N)_{\mathrm{Ad}\, \varrho}) = 0$. Using $\chi(Z) = 0$, this shows that $H^1(G_H; su(N)_{\mathrm{Ad}\, \varrho}) = 0$, which completes the proof of the lemma. $\square$

Next, we consider a chain link L and we establish the following vanishing result for $h_{N,a}(L)$.

Proposition 4.5. *Suppose L is a chain link with $n > 2$ components and that $n = N$. Then $h_{N,a}(L) = 0$ for $a = (d, \ldots, d)$, where d is relatively prime to N.*

Proof. We start with the chain link L with $n = 3$ components. It has link group with presentation
$$G_L = \langle x, y, z \mid [x, y] = 1 = [y, z] \rangle.$$
We will parameterize all triples $(X, Y, Z) \in SU(3) \times SU(3) \times SU(3)$ satisfying $[X, Y] = \omega I = [Y, Z]$, and we will use this to show that $h_{3,a}(L) = 0$ for $a = (1, 1, 1)$.

Applying Theorem 4.1, up to conjugacy, there is a unique irreducible pair $(X, Y) \in SU(3) \times SU(3)$ satisfying the equation $[X, Y] = \omega I$. This pair is given by

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \quad \text{and } Y = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In a general group, the commutator satisfies the relations
$$[x, y]^{-1} = [y, x] = y\,[x, y^{-1}]\,y^{-1} = x\,[x^{-1}, y]\,x^{-1}.$$

Setting $Z = X^{-1}$, this shows that $[Y, Z] = \omega I$, and thus the triple (X, Y, Z) gives rise to a projective representation $\varrho \colon G_L \to SU(3)$ satisfying
$$[\varrho(x), \varrho(y)] = \omega I = [\varrho(y), \varrho(z)].$$

If $P \in \mathrm{Stab}(Y)$, then $[Y, PZP^{-1}] = P[Y, Z]P^{-1} = \omega I$, hence the action of $\mathrm{Stab}(Y)$ on triples given by $(X, Y, Z) \mapsto (X, Y, PZP^{-1})$ preserves the relations and is nontrivial on conjugacy classes. It follows that the solution set is 2-dimensional and parameterized by $\mathrm{Stab}(Y)/\mathbb{Z}_3$, which has Euler characteristic zero since $\mathrm{Stab}(Y) \cong T^2$ is a copy of a maximal torus. A calculation similar to the one in the proof of Lemma 4.4 shows that the solution set is a nondegenerate critical submanifold, and a standard argument then shows that its contribution to the invariant is given by $\pm$ its Euler characteristic (cf. the proof of Proposition 8 in [3]). It follows that $h_{3,a}(L) = 0$ for $a = (1, 1, 1)$, and a similar argument shows that $h_{3,a}(L) = 0$ for $a = (2, 2, 2)$.

One can also prove this via a direct approach making use of the fact that L is the closure of the braid $\sigma = \sigma_1^2 \sigma_2^2$ and parameterizing the fixed point set $\mathrm{Fix}(\varepsilon\sigma)$ as was done for the Hopf link. We leave the details to the reader.

Next, consider the chain link L with 4 components. It has link group with presentation
$$G_L = \langle x, y, z, w \mid [x, y] = 1 = [y, z] = [z, w] \rangle.$$
By Theorem 4.1, up to conjugacy, there is a unique irreducible pair $(X, Y) \in SU(4) \times SU(4)$ satisfying the equation $[X, Y] = \omega I$. This pair is given by

$$X = \begin{pmatrix} \xi & 0 & 0 & 0 \\ 0 & \xi^3 & 0 & 0 \\ 0 & 0 & \xi^5 & 0 \\ 0 & 0 & 0 & \xi^7 \end{pmatrix} \quad \text{and } Y = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Taking $Z = X^{-1}$ and $W = Y^{-1}$, one can show that the 4-tuple (X, Y, Z, W) gives rise to a projective representation $\varrho \colon G_L \to SU(3)$ with $[\varrho(x), \varrho(y)] = \omega I = [\varrho(y), \varrho(z)] = [\varrho(z), \varrho(w)]$. The two groups $\mathrm{Stab}(Y)$ and $\mathrm{Stab}(Z)$ act on 4-tuples by $(X, Y, Z, W) \mapsto (X, Y, PZP^{-1}, PWP^{-1})$ for $P \in \mathrm{Stab}(Y)$ and $(X, Y, Z, W) \mapsto (X, Y, Z, QWQ^{-1})$ for $Q \in \mathrm{Stab}(Z)$, and these actions preserve the relations and are nontrivial on conjugacy classes. It follows that the solution set is 6-dimensional and parameterized by $\mathrm{Stab}(Y)/\mathbb{Z}_4 \times \mathrm{Stab}(Z)/\mathbb{Z}_4$, which has Euler characteristic zero. By similar considerations as in the previous case, it follows that $h_{4,a}(L) = 0$ for $a = (1,1,1,1)$, and a similar argument shows that $h_{4,a}(L) = 0$ for $a = (3,3,3,3)$.

As before, one can perform these computations directly by noting that L is the closure of the braid $\sigma = \sigma_1^2 \sigma_2^2 \sigma_3^2$ and parameterizing the fixed point set $\mathrm{Fix}(\varepsilon\sigma)$.

This argument generalizes to the n-component chain link in a straightforward manner, as we now explain. The chain link L with n components has link group with presentation

$$G_L = \langle x_1, \ldots, x_n \mid [x_1, x_2] = \cdots = [x_{n-1}, x_n] = 1 \rangle.$$

By Theorem 4.1, up to conjugacy, there is a unique irreducible pair $(X_1, X_2) \in SU(N) \times SU(N)$ satisfying the equation $[X_1, X_2] = \omega I$. A solution is obtained by taking $X_1 = X$ and $X_2 = Y$ for X, Y as in the statement of the theorem, and setting $X_{i+2} = X_i^{-1}$ for $i = 1, \ldots, n-2$, the n-tuple

$$(X_1, \ldots, X_n) \in SU(N) \times \cdots \times SU(N)$$

is easily seen to satisfy the relations

$$[X_1, X_2] = \cdots = [X_{n-1}, X_n] = \omega I.$$

For $i = 3, \ldots, n$, the group $\mathrm{Stab}(X_i)$ acts on n-tuples by

$$(X_1, \ldots, X_n) \mapsto (X_1, \ldots, X_i, PX_{i+1}P^{-1}, \ldots, PX_nP^{-1})$$

for $P \in \mathrm{Stab}(X_i)$. These actions preserve the relations and are nontrivial on conjugacy classes. Since each $\mathrm{Stab}(X_i) \cong T^{N-1}$ is a maximal torus, it follows that the solution set has dimension $(N-1)(N-2)$ and is parameterized by $\mathrm{Stab}(X_3)/\mathbb{Z}_N \times \cdots \times \mathrm{Stab}(X_n)/\mathbb{Z}_N$, which has Euler characteristic zero. It follows that $h_{N,a}(L) = 0$ for $a = (1, \ldots, 1)$, and a similar argument shows that $h_{N,a}(L) = 0$ for $a = (d, \ldots, d)$ for any d relatively prime to N. $\qquad\qquad\square$

4.2. **Split Links.** In this section, we consider links $L \subset S^3$ that are geometrically split and prove a vanishing result for $h_{N,a}(L)$ provided that the labels satisfy the following condition. Assume L is a link with n components, and suppose it is split. Denoting the components of L by $\ell_1 \cup \cdots \cup \ell_n$, this means that $L = L_1 \cup L_2$, where up to reordering $L_1 = \ell_1 \cup \cdots \cup \ell_{n_1}$ and $L_2 = \ell_{n_1+1} \cup \cdots \cup \ell_n$ are sublinks that are separated by a 2-sphere. We shall assume that the labels $(a_1, \ldots, a_n)$ satisfy the additional condition:

$$a_1 + a_2 + \cdots + a_{n_1} \text{ is not a multiple of } N. \tag{15}$$

Using Markov moves we can always find a *split braid* representative $\beta \in B_k$ of L, see Figure 3. This means that $\beta = \beta_1\beta_2$ where $\beta_1 \in \mathrm{Im}(B_{k_1} \overset{i_1}{\hookrightarrow} B_k)$ and $\beta_2 \in \mathrm{Im}(B_{k_2} \overset{i_2}{\hookrightarrow} B_k)$ for $k = k_1 + k_2$ and i_1, i_2 are injective maps obtained by stabilizing on the right and left, respectively. More precisely, i_1 takes a braid in B_{k_1} and adds k_2 trivial strands on the right, and i_2 takes a braid in B_{k_2} and adds k_1 trivial strands on the left. Any link L obtained as the closure $\widehat{\beta}$ of a split braid is obviously a split link, and any split link L can be obtained as the closure of a split braid.
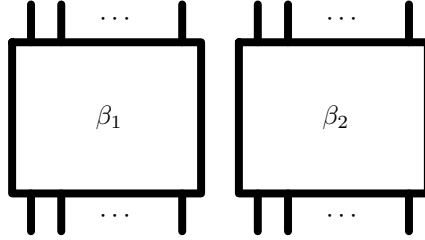

FIGURE 3. A split braid

Proposition 4.6. *Suppose L is a split link and that β is a split braid with $\widehat{\beta} = L$. Suppose further that $a = (a_1, \ldots, a_n)$ is an n-tuple of labels satisfying Equation (15), and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ is a compatible k-tuple. Then the intersection $\Delta_k \cap \Gamma_{\varepsilon\beta} = \varnothing$, and consequently $h_{N,a}(L) = 0$.*

Proof. Let $X \in \Delta_k \cap \Gamma_{\varepsilon\beta}$, then by (8)
$$X_1 \cdots X_{k_1} = \omega^{dn_1}\beta(X)_1 \cdots \beta(X)_{k_1}.$$
Since $\beta = \beta_1\beta_2$ is a split braid with $\beta_1 \in B_{k_1}$, by (2) we have that
$$\beta(X)_1 \cdots \beta(X)_{k_1} = \beta_1(X)_1 \cdots \beta_1(X)_{k_1} = X_1 \cdots X_{k_1},$$
and this implies
$$X_1 \cdots X_{k_1} = \omega^{a_1 + \cdots + a_{n_1}} X_1 \cdots X_{k_1}.$$
But $\omega^{a_1 + \cdots + a_{n_1}} \neq 1$ by the assumption of Equation (15), and this gives the desired contradiction. $\qquad\square$

4.3. **Concluding remarks.** One can give an alternative interpretation of the invariants $h_{N,a}(L)$ in terms of a signed count of conjugacy classes of representations $\varrho \colon G_L \to PU(N)$ of the link group as follows. We begin by recalling the classification of principal $PU(N)$ bundles from [18]. The classifying space $BPU(N)$ is simply connected and has $\pi_2(BPU(N)) = \mathbb{Z}_N$, and an application of the main theorem of [18] implies that principal $PU(N)$ bundles $P \to X$ over a 3-complex X are determined by the characteristic class $w(P) \in H^2(X; \mathbb{Z}_N)$. In case $N = 2$, $PU(2) = SO(3)$ and $w(P)$ coincides with the second Stiefel-Whitney class.

Let $L \subset S^3$ be a link and $M_L = S^3 \smallsetminus \tau(L)$ its exterior. A projective $SU(N)$ representation induces a representation $\varrho\colon G_L \to PU(N)$, and we denote the associated cohomology class by $w(\varrho) \in H^2(G_L; \mathbb{Z}_N)$. The class $w(\varrho)$ vanishes if and only if ϱ lifts to an $SU(N)$ representation. Further, there is a canonical injection $H^2(G_L; \mathbb{Z}_N) \to H^2(M_L; \mathbb{Z}_N)$ which is an isomorphism if and only if M_L is aspherical, i.e. if and only if L is non-split.

By reduction mod N, an allowable n-tuple $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ determines a unique cohomology class $\bar{w}(a_1, \ldots, a_n) \in H^2(\partial M_L; \mathbb{Z}_N) \cong (\mathbb{Z}_N)^n$. The exact sequence in cohomology for the pair $(M_L, \partial M_L)$ gives

$$\to H^2(M_L; \mathbb{Z}_N) \xrightarrow{i^*} H^2(\partial M_L; \mathbb{Z}_N) \to H^3(M_L, \partial M_L; \mathbb{Z}_N) \to 0;$$

and condition (iii) from §2.2 guarantees that $\bar{w}(a_1, \ldots, a_n)$ lies in the image of i^* and hence determines a class $w(a_1, \ldots, a_n) \in H^2(M_L; \mathbb{Z}_N)$. Condition (ii) implies that the class $w(a_1, \ldots, a_n)$ has order N.

From this point of view, the invariant $h_{N,a}(L)$ is closely related to the signed count of conjugacy classes of representations $\varrho\colon G_L \to PU(N)$ such that $w(\varrho) = w(a_1, \ldots, a_n)$. Proposition 4.6 is therefore a direct consequence of the fact that for split links L and allowable n-tuples $(a_1, \ldots, a_n)$ satisfying condition (15), the associated cohomology class $w(a_1, \ldots, a_n)$, which is not unique, does not lie in the image of $H^2(G_L; \mathbb{Z}_N) \to H^2(M_L; \mathbb{Z}_N)$.

As mentioned in the introduction, it would be interesting to investigate the relationship between the $SU(N)$ Casson-Lin invariants studied here and the $SU(N)$ instanton Floer groups constructed by Kronheimer and Mrowka in [12, 13]. It would also be interesting to understand the relationship between the $SU(N)$ Casson-Lin invariants and classical link invariants. For example, the main result of [7] equates the $SU(2)$ Casson-Lin invariant $h_2(L)$ of a two component link $L = \ell_1 \cup \ell_2$ with the linking number $\mathrm{lk}\,(\ell_1, \ell_2)$. The following conjecture, if true, would give a generalization to the higher rank setting.

Conjecture 4.7. *If $L = \ell_1 \cup \ell_2$ is a two component link in S^3, then the $SU(N)$ Casson-Lin invariant satisfies*

$$h_{N,a}(L) = \mathrm{lk}\,(\ell_1, \ell_2)^{N-1}.$$

This conjecture is consistent with all known computations of the $SU(N)$ Casson-Lin invariants, and it's possible that the invariants $h_{N,a}(L)$ are generally invariant under link homotopy. We hope to explore these topics in future work.

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MATHEMATICS & STATISTICS, MCMASTER UNIVERSITY, HAMILTON, ONTARIO
E-mail address: boden@mcmaster.ca

MATHEMATICS & STATISTICS, MCMASTER UNIVERSITY, HAMILTON, ONTARIO
E-mail address: eharper@math.mcmaster.ca