GEOMETRICITY FOR DERIVED CATEGORIES
OF ALGEBRAIC STACKS

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To Joseph Bernstein on the occasion of his 70th birthday

ABSTRACT. We prove that the dg category of perfect complexes on a smooth, proper Deligne–Mumford stack over a field of characteristic zero is geometric in the sense of Orlov, and in particular smooth and proper. On the level of triangulated categories, this means that the derived category of perfect complexes embeds as an admissible subcategory into the bounded derived category of coherent sheaves on a smooth, projective variety. The same holds for a smooth, projective, tame Artin stack over an arbitrary field.

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1. INTRODUCTION

The derived category of a variety or, more generally, of an algebraic stack, is traditionally studied in the context of triangulated categories. Although triangulated categories are certainly powerful, they do have some shortcomings. Most notably, the category of triangulated categories seems to have no tensor product, no concept of duals, and categories of triangulated functors have no obvious triangulated structure. A remedy to these problems is to work instead with differential graded categories, also called dg categories. We follow this approach and replace the derived category $D_{pf}(X)$ of perfect complexes on a variety or an algebraic stack X by a certain dg category $D_{pf}^{dg}(X)$ which enhances $D_{pf}(X)$ in the sense that its homotopy category is equivalent to $D_{pf}(X)$.

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The study of dg categories is central in noncommutative geometry, and dg categories are sometimes thought of as categories of sheaves on a hypothetical noncommutative space. Although a variety in general cannot be recovered from its associated dg category, several of its important homological invariants can. These include the algebraic K-theory spectrum as well as various variants of cyclic homology. See [Kel06] and [Tab11] for surveys on dg categories and their invariants.

In noncommutative algebraic geometry, saturated dg categories play a similar role as smooth and proper varieties in usual commutative algebraic geometry. For example, the dg category $\mathrm{D}^{\mathrm{dg}}_{\mathrm{pf}}(X)$ associated to a variety X is saturated if and only if X is smooth and proper. The saturated dg categories have an intrinsic characterization as the homotopy dualizable objects in the category of all dg categories with respect to a certain localization [CT12, §5].

It is natural to ask how far a dg category, thought of as a noncommutative space, is from being commutative. Motivated by the dominant role of smooth, projective varieties, Orlov recently introduced the notion of a geometric dg category [Orl14]. By definition, every dg category of the form $\mathrm{D}^{\mathrm{dg}}_{\mathrm{pf}}(X)$, for X a smooth, projective variety, is geometric, and so are all its "admissible" subcategories (see Definition 5.16 for a precise definition). Every geometric dg category is saturated. Orlov asks whether in fact all saturated dg categories are geometric. To our knowledge, this question is wide open. Our main theorems say that we stay in the realm of geometric dg categories if we consider certain algebraic stacks.

Theorem 6.6. *Let X be a smooth, proper Deligne–Mumford stack over a field of characteristic zero. Then the dg category $\mathrm{D}^{\mathrm{dg}}_{\mathrm{pf}}(X)$ is geometric, and in particular saturated.*

This theorem can be seen as a noncommutative counterpart to [Cho12, Corollary 4.7], which states that the mixed motive of a finite type, smooth Deligne–Mumford stack over a field of characteristic zero is effective geometric.

In the preprint [HLP15], the authors consider a similar problem for certain stacks with positive dimensional stabilizers and even for categories of matrix factorizations on such stacks. They prove a geometricity result in a generalized sense which involves infinite sums of dg categories of smooth Deligne–Mumford stacks [HLP15, Theorem 2.7]. As mentioned in Remark 2.9 of *loc. cit.* our result strengthens their Theorem 2.7 by replacing Deligne–Mumford stacks with varieties.

We also give a version of our main theorem which is valid for stacks over arbitrary fields. Since we do not have resolution of singularities over a field of positive characteristic, we restrict the discussion to *projective* algebraic stacks (see Definition 2.5). Indeed, even for a smooth, proper scheme X over a field of positive characteristic it is not clear whether $\mathrm{D}^{\mathrm{dg}}_{\mathrm{pf}}(X)$ is geometric if X is not projective. Moreover, instead of Deligne–Mumford stacks we consider tame algebraic stacks [AOV08]. Over a field of characteristic zero the class of separated Deligne–Mumford stacks coincides with the class of separated tame algebraic stacks, but in positive characteristic tame stacks are usually better behaved. For example, under mild finiteness assumptions the tame algebraic stacks are scheme-like from a noncommutative perspective in the sense that their derived categories are generated by a single compact object and the compact objects coincide with the perfect complexes [BVdB03, Theorem 3.1.1], [HR14, Theorem A, Remark 4.6].

Theorem 6.4. *Let X be a tame, smooth, projective algebraic stack over an arbitrary field. Then the dg category $\mathrm{D}^{\mathrm{dg}}_{\mathrm{pf}}(X)$ is geometric, and in particular saturated.*

We specify the dg enhancement $\mathrm{D}^{\mathrm{dg}}_{\mathrm{pf}}(X)$ of $\mathrm{D}_{\mathrm{pf}}(X)$ we work with in Example 5.5. A priori, there are other possible choices (cf. Remark 5.3), but it turns out that they are all equivalent. This follows from a recent result by Canonaco and Stellari [CS15] which implies that the derived categories $\mathrm{D}_{\mathrm{pf}}(X)$ for the stacks considered in the theorems above have unique dg enhancements (see Remark 5.9).

Theorem 6.4 has the following equivalent reformulation in terms of varieties with group actions.

Theorem 6.4b. *Let U be a smooth, quasi-projective variety over a field k, and let G be a linear algebraic group over k acting properly on U. Assume that the action admits a geometric quotient $U \to U/G$ (in the sense of [MFK94, Definition 0.6]) such that U/G is projective over k. Also assume that all stabilizers of the action are linearly reductive. Then the dg category enhancing the bounded derived category $\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}^G(U))$ of G-equivariant coherent sheaves on U is geometric, and in particular saturated.*

Proof. See Example 2.6 and Remark 2.12 together with the fact that the category of coherent sheaves on the stack $[U/G]$ is equivalent to the category of G-equivariant coherent sheaves on U. $\square$

Note that the requirement that the action be proper implies that the stabilizers are finite. In particular, the condition that the stabilizers be linearly reductive is superfluous if our base field k has characteristic zero. Also note that if G is finite and U is projective over k, then the action is automatically proper and the existence of a projective geometric quotient U/G is guaranteed.

Outline. The proof of the main results primarily builds on two results – the destackification theorem by Bergh and Rydh [Ber14], [BR15] and the gluing theorem for geometric dg categories by Orlov [Orl14].

The destackification theorem allows us to compare a smooth, tame algebraic stack to a smooth algebraic space via a sequence of birational modifications called *stacky blowups*. We review this theorem as Theorem 6.1 in Section 6. In this section, we also give the main geometric arguments of the proofs of the main theorems.

Stacky blowups play a similar role in the study of the birational geometry of tame stacks as do usual blowups for schemes. They come in two incarnations: usual blowups and so-called *root stacks*. Root stacks are purely stacky operations which have no counterpart in the world of schemes. We review the notion of a root stack in Section 3.

A stacky blowup modifies the derived category of a stack in a predictable way. More specifically, it induces a semiorthogonal decomposition on the derived category on the blowup. For usual blowups this is due to Orlov ([Orl92], [Huy06, Proposition 11.18]) and for root stacks this is due to Ishii–Ueda [IU11]. In Section 4, we reprove the theorem by Ishii–Ueda as Theorem 4.7, but in a more general setting. We also give a combinatorial description of the semiorthogonal decomposition on an iterated root stack as Theorem 4.9.

In Section 5, we provide the dg categorical ingredients for the proofs of our main theorems. We first introduce the dg enhancements $\mathrm{D}^{\mathrm{dg}}_{\mathrm{pf}}(X)$ and lift certain derived

functors to these enhancements. Then we discuss geometric dg categories and state Orlov's gluing theorem as Theorem 5.22.

Some general facts concerning algebraic stacks, derived categories, and semiorthogonal decompositions are assembled in section 2. Appendix A contains some results on bounded derived categories of coherent sheaves on noetherian algebraic stacks.

2. Preliminaries

In this section, we fix our notation and our conventions for algebraic stacks as well as their derived categories of sheaves. We also review the notions of tame and projective stacks. Finally, we review the notion of semiorthogonal decompositions for triangulated categories.

Conventions for algebraic stacks. We will use the definition of algebraic space and algebraic stack from the stacks project [SP16, Tag 025Y, Tag 026O]. In particular, we will always state all separatedness assumptions explicitly. The main results of this article concern tame algebraic stacks which are separated and of finite type over a field k. If k is a field of characteristic zero, the class of these stacks consists precisely of the separated Deligne–Mumford stacks of finite type over k. Thus the reader unwilling to work in full generality could safely assume that all stacks are of the aforementioned kind.

Although algebraic stacks form a 2-category we will follow the common practice to suppress their 2-categorical nature to simplify the exposition if no misunderstanding is likely. In particular, we will often say morphism instead of 1-morphism, isomorphism instead of equivalence, commutative instead of 2-commutative and cartesian instead of 2-cartesian.

Tame algebraic stacks. An algebraic stack X has *finite inertia* if the canonical morphism $I_X \to X$ from its inertia stack is finite. If X has finite inertia, then there is a canonical morphism $\pi\colon X \to X_{\mathrm{cs}}$ to the *coarse (moduli) space*, which is an algebraic space [KM97, Ryd13]. If X is locally of finite type over some base algebraic space S, then the morphism π is proper.

Definition 2.1. An algebraic stack X is called *tame* provided that the stabilizer at each of its geometric points is finite and linearly reductive.

If X has finite inertia, tameness implies that the pushforward $\pi_*\colon \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(X_{\mathrm{cs}})$ is exact. The converse implication holds if X has finite inertia and is quasi-separated [Hal14, Corollary A.3].

Remark 2.2. Tame algebraic stacks are defined in [AOV08] in a slightly less general context. We use the more general definition given in [Hal14].

Example 2.3. Let X be a separated algebraic stack of finite type over a field k. Then X has finite inertia provided that the stabilizer at each geometric point is

finite. If k has characteristic zero, then X is tame if and only if it is a Deligne–Mumford stack. If k has characteristic $p > 0$, the stack X is tame if and only if the stabilizer group at each geometric point is of the form $\Delta \rtimes H$, where Δ is a finite diagonalizable p-group and H is a constant finite group of order prime to p.

Projective algebraic stacks. A definition of *projective* Deligne–Mumford stack is suggested in [Kre09]. Since we also work with some tame stacks which are not Deligne–Mumford stacks, we will need to extend this definition slightly. First we clarify what we mean by a global quotient stack.

An algebraic stack X is a *global quotient stack* if there exists a GL_n-torsor $T \to X$, for some non-negative integer n, such that T is an algebraic space.

Example 2.4. Let k be a field and $G \subset \mathrm{GL}_n$ a linear algebraic group over k. Assume that G acts on an algebraic space U over k. Let T be the quotient of $\mathrm{GL}_n \times U$ by G, with G acting on the right on the factor GL_n via the inclusion $G \subset \mathrm{GL}_n$ and on the left on U. Then T is an algebraic space and the obvious projection $T \to [U/G]$ is a GL_n-torsor. In particular, the stack quotient $[U/G]$ is a global quotient stack.

If a global quotient stack X is separated over some base algebraic space S then the relative diagonal $\Delta_{X/S} \colon X \to X \times_S X$ is affine and proper and hence finite. In particular, such a stack has finite inertia and therefore admits a coarse space $X \to X_{\mathrm{cs}}$.

Definition 2.5. Let X be an algebraic stack over a field k. We say that X is *quasi-projective over* k if it is a global quotient stack which is separated and of finite type over k, and its coarse space X_{cs} is a quasi-projective scheme over k. If in addition X or, equivalently, X_{cs} is proper over k, we say that X is *projective over* k.

Example 2.6. GIT-quotients by proper actions (in the sense of [MFK94]) give rise to quasi-projective algebraic stacks. Let U be a quasi-projective variety over a field k and let G be a reductive linear algebraic group acting properly on U. Assume that U admits a G-linearized line bundle $\mathcal{L}$ such that U is everywhere stable with respect to $\mathcal{L}$ in the sense of [MFK94, Definition 1.7]. Then the GIT-quotient $U \to U/G$ is geometric and U/G is quasi-projective. Since geometric quotients by proper actions are universal among algebraic spaces [Kol97, Corollary 2.15], the canonical morphism $[U/G] \to U/G$ identifies U/G with the coarse space of the stack quotient $[U/G]$. Since the action of G on U is proper, the stack quotient $[U/G]$ is separated. Hence $[U/G]$ is a quasi-projective stack in the sense of Definition 2.5.

Conversely, every quasi-projective algebraic stack can be obtained in this way. Indeed, assume that X is quasi-projective. Since X is a separated global quotient stack, it is of the form $[U/\mathrm{GL}_n]$ where GL_n acts properly on an algebraic space U. Denote the geometric quotient by $q \colon U \to U/\mathrm{GL}_n \cong [U/\mathrm{GL}_n]_{\mathrm{cs}}$. Let $\mathcal{M}$ be an ample line bundle on U/GL_n. This pulls back to a GL_n-linearized line bundle $\mathcal{L} = q^*\mathcal{M}$ on U. Since q is affine (cf. [Kre09, Remark 4.3] or [Kol97, Theorem 3.12]), the bundle $\mathcal{L}$ is ample and U is quasi-projective. Moreover, the space U is everywhere stable with respect to $\mathcal{L}$, since sections of $\mathcal{M}$ pull back to invariant sections of $\mathcal{L}$.

Example 2.7. Let U be a quasi-projective variety over a field k, and let G be a finite group scheme over k acting on U. Then the quotient U/G is quasi-projective by [SGA1, Exposé V, Proposition 1.8] combined with graded prime avoidance (cf.

[SP16, Tag 09NV]). Since U/G coincides with the coarse space of $[U/G]$, it follows that $[U/G]$ is a quasi-projective stack in the sense of Definition 2.5. Moreover, since the natural morphism $[U/G] \to U/G$ is proper, the stack $[U/G]$ is projective if and only if U is projective.

As expected, we have the following permanence property for projective algebraic stacks with respect to morphisms which are projective in the sense of [EGAII, Definition 5.5.2].

Lemma 2.8. *Let $f\colon X \to Y$ be a projective morphism of algebraic stacks with Y being quasi-compact and quasi-separated. Assume that both X and Y have finite inertia. Then the induced morphism $f_{\mathrm{cs}}\colon X_{\mathrm{cs}} \to Y_{\mathrm{cs}}$ between the coarse spaces is projective. In particular, if Y is (quasi-)projective over a field k in the sense of Definition 2.5, then the same holds for X.*

Proof. Note that since Y is assumed to be quasi-compact and quasi-separated, projectivity of f is equivalent to f being proper and admitting an f-ample invertible sheaf (see [Ryd15a, Proposition 8.6]). Hence the statement of the lemma follows from [Ryd15b, Proposition 2]. Less general versions of the lemma can be found in [KV04, Proof of Theorem 1] and [Ols12, Proposition 6.1].   □

Derived categories and derived functors. There are several equivalent ways to define quasi-coherent modules on algebraic stacks. We will follow [LMB00] and view quasi-coherent modules as sheaves on the lisse–étale site. The results on derived categories depend on the techniques of cohomological descent as described in [Ols07] and [LO08]. A concise summary of these results is given in [HR14, Section 1]. We give a brief overview here. Let $(X, \mathcal{O})$ be a ringed topos. We use the notation $\mathrm{Mod}(X, \mathcal{O})$, for the abelian category of $\mathcal{O}$-modules in X and $\mathrm{D}(X, \mathcal{O})$ for its derived category.

Let X be an algebraic stack. We denote the topos of sheaves on the lisse–étale site by $X_{\mathrm{lis\text{-}\acute{e}t}}$. If X is a Deligne–Mumford stack, we denote the topos of sheaves on the small étale site by $X_{\acute{e}t}$. In these situations, we use the short hand notation

$$\mathrm{Mod}(X_\tau) := \mathrm{Mod}(X_\tau, \mathcal{O}_X), \qquad \mathrm{D}(X_\tau) := \mathrm{D}(X_\tau, \mathcal{O}_X),$$

where τ is either lis-ét or ét. By default, we will use the lisse-étale site when considering sheaves on algebraic stacks and simply write $\mathrm{Mod}(X)$ instead of $\mathrm{Mod}(X_{\mathrm{lis\text{-}\acute{e}t}})$ and $\mathrm{D}(X)$ instead of $\mathrm{D}(X_{\mathrm{lis\text{-}\acute{e}t}})$.

Recall that an $\mathcal{O}_X$-module is *quasi-coherent* if it is locally presentable [SP16, Tag 03DL]. We let $\mathrm{Qcoh}(X)$ denote the full subcategory of $\mathrm{Mod}(X)$ of quasi-coherent modules and $\mathrm{D}_{\mathrm{qc}}(X)$ the full subcategory of $\mathrm{D}(X)$ consisting of complexes with quasi-coherent cohomology. Since $\mathrm{Qcoh}(X)$ is a weak Serre subcategory of $\mathrm{Mod}(X)$, the category $\mathrm{D}_{\mathrm{qc}}(X)$ is a thick triangulated subcategory of $\mathrm{D}(X)$.

Also recall that a complex in $\mathrm{Mod}(X)$ is called *perfect* if it is locally quasi-isomorphic to a bounded complex of direct summands of finite free modules [SP16, Tag 08G4]. We denote by $\mathrm{D}_{\mathrm{pf}}(X)$ the subcategory of $\mathrm{D}(X)$ consisting of perfect complexes. The category $\mathrm{D}_{\mathrm{pf}}(X)$ is a thick triangulated subcategory of $\mathrm{D}_{\mathrm{qc}}(X)$.

Remark 2.9. The reader willing to restrict the discussion to Deligne–Mumford stacks could instead use $X_{\acute{e}t}$ as the default topos when considering sheaves on such a stack X. In this situation, the correspondingly defined categories $\mathrm{D}_{\mathrm{qc}}(X_{\acute{e}t})$ and $\mathrm{D}_{\mathrm{pf}}(X_{\acute{e}t})$ are equivalent to $\mathrm{D}_{\mathrm{qc}}(X)$ and $\mathrm{D}_{\mathrm{pf}}(X)$ respectively.

Explicitly, the equivalences are constructed as follows. Let X be a Deligne–Mumford stack. The inclusion of its small étale site into its lisse-étale site induces a morphism

$$(2.1) \qquad \varepsilon = (\varepsilon^*, \varepsilon_*) \colon (X_{\mathrm{lis\text{-}ét}}, \mathcal{O}_X) \to (X_{\mathrm{ét}}, \mathcal{O}_X)$$

of ringed topoi, where ε_* is the restriction functor. Both functors ε^* and ε_* are exact, and the equivalences are obtained by restriction of the induced adjoint pair $\varepsilon^* \colon \mathrm{D}(X_{\mathrm{ét}}) \to \mathrm{D}(X_{\mathrm{lis\text{-}ét}})$ and $\varepsilon_* \colon \mathrm{D}(X_{\mathrm{lis\text{-}ét}}) \to \mathrm{D}(X_{\mathrm{ét}})$.

Let $f \colon X \to Y$ be a morphism of algebraic stacks. Assume, for simplicity, that f is *concentrated*. This means that f is quasi-compact, quasi-separated and has a boundedness condition on its cohomological dimension Y [HR14, Definition 2.4]. For our needs, it suffices to know that a quasi-compact and quasi-separated morphism of algebraic stacks is concentrated provided that its fibers are tame (cf. [HR15, Theorem 2.1]). We get induced adjoint pairs of functors

$$f^* \colon \mathrm{Qcoh}(Y) \to \mathrm{Qcoh}(X), \qquad f_* \colon \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(Y)$$

and

$$\mathsf{L}f^* \colon \mathrm{D}_{\mathrm{qc}}(Y) \to \mathrm{D}_{\mathrm{qc}}(X), \qquad \mathsf{R}f_* \colon \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc}}(Y).$$

Here f^*, f_* and $\mathsf{R}f_*$ are simply the restrictions of the corresponding functors on $\mathrm{Mod}(X)$, $\mathrm{Mod}(Y)$ and $\mathrm{D}(X)$ ([HR14, Theorem 2.6.(2)]).

Remark 2.10. It requires some work to see that the functor $\mathsf{L}f^* \colon \mathrm{D}_{\mathrm{qc}}(Y) \to \mathrm{D}_{\mathrm{qc}}(X)$ actually exists. This is due to the fact that the naturally defined adjoint pair (f^{-1}, f_*) does not induce a morphism $(X_{\mathrm{lis\text{-}ét}}, \mathcal{O}_X) \to (Y_{\mathrm{lis\text{-}ét}}, \mathcal{O}_Y)$ of ringed topoi, owing to the fact that f^{-1} in general is not exact. Hence, we do not get a functor $\mathsf{L}f^* \colon \mathrm{D}(Y) \to \mathrm{D}(X)$ from the general theory.

Remark 2.11. If $f \colon X \to Y$ is a morphism of Deligne–Mumford stacks, then the pair (f^{-1}, f_*) does induce a morphism $(X_{\mathrm{ét}}, \mathcal{O}_X) \to (Y_{\mathrm{ét}}, \mathcal{O}_Y)$ of ringed topoi. Hence we do get a functor $\mathsf{L}f^* \colon \mathrm{D}(Y_{\mathrm{ét}}) \to \mathrm{D}(X_{\mathrm{ét}})$. Furthermore, its restriction to $\mathrm{D}_{\mathrm{qc}}(Y_{\mathrm{ét}})$ is compatible with $\mathsf{L}f^* \colon \mathrm{D}_{\mathrm{qc}}(Y) \to \mathrm{D}_{\mathrm{qc}}(X)$ via the equivalences described in Remark 2.9.

Remark 2.12. Let X be an algebraic stack. Then the category $\mathrm{Qcoh}(X)$ is a Grothendieck abelian category. In particular, the category of complexes of quasi-coherent modules has enough h-injectives and the derived category $\mathrm{D}(\mathrm{Qcoh}(X))$ has small hom-sets.

There is an obvious triangulated functor

$$(2.2) \qquad \mathrm{D}(\mathrm{Qcoh}(X)) \to \mathrm{D}_{\mathrm{qc}}(X)$$

induced by the inclusion $\mathrm{Qcoh}(X) \subset \mathrm{Mod}(X)$. Assume that X is quasi-compact, separated and has finite stabilizers. In particular, this implies that X has finite, and hence affine, diagonal. Then the functor (2.2) is an equivalence of categories. This follows from [HNR14, Theorem 1.2] and [HR14, Theorem A].

Assume that X, in addition, is regular. In particular, this includes the stacks considered in the main theorems of this article. Then the obvious functor induces an equivalence $\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X)) \cong \mathrm{D}_{\mathrm{pf}}(X)$ by Remark A.3. In particular, a complex of $\mathcal{O}_X$-modules is perfect if and only if it is isomorphic in $\mathrm{D}_{\mathrm{qc}}(X)$ to a bounded complex of coherent modules.

Remark 2.13. Let $X \to Y$ be a concentrated morphism of quasi-compact stacks which are separated and have finite stabilizers. Then the derived pushforward $\mathsf{R}f_*\colon \mathrm{D}(\mathrm{Qcoh}(X)) \to \mathrm{D}(\mathrm{Qcoh}(Y))$ corresponds to $\mathsf{R}f_*\colon \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc}}(Y)$ under the equivalences mentioned in Remark 2.12, see [HNR14, Corollary 2.2].

Semiorthogonal decompositions. We recall the definition of admissible subcategories and semiorthogonal decompositions of triangulated categories (cf. [BK89], [LS12, Appendix A]).

Definition 2.14. Let $\mathcal{T}$ be a triangulated category. A *right* (resp. *left*) *admissible subcategory of* $\mathcal{T}$ is a strict full triangulated subcategory $\mathcal{T}'$ of $\mathcal{T}$ such that the inclusion functor $\mathcal{T}' \to \mathcal{T}$ admits a right (resp. left) adjoint. An *admissible* subcategory is a subcategory that is both left and right admissible.

Definition 2.15. A sequence $(\mathcal{T}_1, \ldots, \mathcal{T}_r)$ of subcategories of $\mathcal{T}$ is called *semiorthogonal* provided that $\mathrm{Hom}_{\mathcal{T}}(T_i, T_j) = 0$ for all objects $T_i \in \mathcal{T}_i$ and $T_j \in \mathcal{T}_j$ whenever $i > j$. If, in addition, all $\mathcal{T}_i$ are strict full triangulated subcategories and the category $\mathcal{T}$ coincides with its smallest strict full triangulated subcategory containing all the $\mathcal{T}_i$, then we say that the sequence forms a *semiorthogonal decomposition* of $\mathcal{T}$ and write

$$\mathcal{T} = \langle \mathcal{T}_1, \ldots, \mathcal{T}_r \rangle.$$

We say that a sequence $\Phi_1, \ldots, \Phi_r$ of triangulated functors with codomain $\mathcal{T}$ forms a *semiorthogonal decomposition* of $\mathcal{T}$ and write $\mathcal{T} = \langle \Phi_1, \ldots, \Phi_r \rangle$ if all Φ_i are full and faithful and the essential images of the functors $\Phi_1, \ldots, \Phi_r$ form a semiorthogonal decomposition of $\mathcal{T}$.

3. Root constructions

The root construction can be seen as a way of adjoining roots of one or several divisors on a scheme or an algebraic stack. It has been described in several sources, e.g. [Cad07, §2], [BC10, §2.1], [FMN10, §1.3] and [AGV08, Appendix B]. We recall its definition along with some of its basic properties. Since most of these basic properties are already described in the sources mentioned above or trivial generalizations, we will omit most of the proofs.

It is straightforward to define the root stack in terms of its generalized points (cf. [Cad07, Remark 2.2.2]), but the most economical definition seems to use the *universal root construction*. Let r be a positive integer and consider the commutative diagram

$$(3.1) \qquad \begin{array}{ccc} \mathrm{B}\mathbb{G}_m & \xrightarrow{\iota} & [\mathbb{A}^1/\mathbb{G}_m] \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathrm{B}\mathbb{G}_m & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m]. \end{array}$$

Here $[\mathbb{A}^1/\mathbb{G}_m]$ denotes the stack quotient of $\mathbb{A}^1 = \mathrm{Spec}\,\mathbb{Z}[x]$ by $\mathbb{G}_m$ acting by multiplication. The maps ρ and π are induced by the maps $\mathbb{G}_m \to \mathbb{G}_m$ and $\mathbb{A}^1 \to \mathbb{A}^1$ taking the coordinate x to its r-th power x^r. Note that the diagram above is not cartesian if $r > 1$.

Recall that the stack $[\mathbb{A}^1/\mathbb{G}_m]$ parametrizes pairs consisting of a line bundle together with a global section.

Definition 3.1. Let X be an algebraic stack and E an effective Cartier divisor [SP16, Tag 01WR] on X. Consider the morphism $f_E \colon X \to [\mathbb{A}^1/\mathbb{G}_m]$ corresponding to the line bundle $\mathcal{O}_X(E)$ together with the canonical global section $\mathcal{O}_X \to \mathcal{O}_X(E)$. Given a positive integer r, we construct the *root diagram*

(3.2)
$$
\begin{array}{ccc}
r^{-1}E & \xrightarrow{\ \iota\ } & X_{r^{-1}E} \\
{\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \pi} \\
E & \xrightarrow{\ \kappa\ } & X
\end{array}
$$

as the base change of the universal root diagram (3.1) along the morphism f_E. The stack $X_{r^{-1}E}$ is called the *r-th root stack* of X with respect to E. The notation for the divisor $r^{-1}E$ is motivated by the fact that $r \cdot (r^{-1}E) = \pi^*E$. We refer to this construction as the *root construction* with respect to the datum (X, E, r).

The next example gives a local description of a root stack.

Example 3.2. Assume that $X = \operatorname{Spec} R$ is affine and the effective Cartier divisor $E \hookrightarrow X$ corresponds to a ring homomorphism $R \to R/(f)$ where $f \in R$ is a regular element. Then the r-th root construction yields

(3.3) $r^{-1}E = \mathrm{B}\mu_r \times E, \qquad X_{r^{-1}E} = [\operatorname{Spec} R'/\mu_r], \qquad R' = R[t]/(t^r - f)$

where μ_r denotes the group scheme of r-th roots of unity. The μ_r-action on $\operatorname{Spec} R'$ corresponds to the $\mathbb{Z}/r\mathbb{Z}$-grading on R' with R in degree zero and t homogeneous of degree 1. The closed immersion $r^{-1}E \hookrightarrow X_{r^{-1}E}$ corresponds to the ideal generated by t.

Proposition 3.3. *The root construction described in Definition 3.1 has the following basic properties:*

(a) *The morphism ι in diagram (3.2) is a closed immersion realizing $r^{-1}E$ as an effective Cartier divisor on $X_{r^{-1}E}$.*

(b) *The morphism π is a universal homeomorphism which is proper, faithfully flat and birational with exceptional locus contained in $r^{-1}E$.*

(c) *The morphism ρ turns $r^{-1}E$ into a μ_r-gerbe over E with trivial Brauer class. In particular, the morphism ρ is smooth.*

(d) *If X is an algebraic space, then π identifies X with the coarse space of $X_{r^{-1}E}$. More generally, if X is an algebraic stack then the morphism π is a relative coarse space. In particular, if X is an algebraic stack having a coarse space $X \to X_{\mathrm{cs}}$, then the composition $X_{r^{-1}E} \to X \to X_{\mathrm{cs}}$ is a coarse space for $X_{r^{-1}E}$.*

(e) *The pushforward $\pi_* \colon \operatorname{Qcoh}(X_{r^{-1}E}) \to \operatorname{Qcoh}(X)$ is exact, and $X_{r^{-1}E}$ is tame provided that the same holds for X.*

(f) *If X is a Deligne–Mumford stack and r is invertible in $\mathcal{O}_X$, then $X_{r^{-1}E}$ is a Deligne–Mumford stack.*

The notion of projectivity is well-behaved under taking roots of effective Cartier divisors.

Lemma 3.4. *Let X be a quasi-projective algebraic stack over a field k, and let E be an effective Cartier divisor on X. Then the root stack $X_{r^{-1}E}$ is quasi-projective over k for any positive integer r. In particular, if X is projective over k, then so is $X_{r^{-1}E}$.*

Proof. Assume that X is quasi-projective over k. Since $X_{r^{-1}E} \to X$ is proper (Proposition 3.3.(b)), the root stack $X_{r^{-1}E}$ is separated and of finite type over k. Since X and $X_{r^{-1}E}$ have isomorphic coarse spaces (Proposition 3.3.(d)), it is enough to verify that $X_{r^{-1}E}$ is a global quotient stack. But this follows from the general fact, proved below, that the fibre product of two global quotient stacks over any algebraic stack is a global quotient stack. This we apply to the morphism $f_E \colon X \to [\mathbb{A}^1/\mathbb{G}_m]$ corresponding to E (cf. Definition 3.1) and the morphisms $\pi \colon [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ from the universal root diagram (3.1).

Now we prove the general fact about fiber products of global quotient stacks. Let $Y \to S$ and $Z \to S$ be morphisms of algebraic stacks and assume that Y and Z are global quotients. Then there exist a GL_n-torsor $U \to Y$ and a GL_m-torsor $V \to Z$ such that U and V are algebraic spaces. Then $U \times_S V$ is an algebraic space and the canonical morphism $U \times_S V \to Y \times_S Z$ is a $\mathrm{GL}_n \times \mathrm{GL}_m$-torsor. Extending this torsor along the obvious embedding $\mathrm{GL}_n \times \mathrm{GL}_m \to \mathrm{GL}_{n+m}$ yields a GL_{n+m}-torsor over $Y \times_S Z$ which is an algebraic space. This shows that $Y \times_S Z$ is a global quotient. $\qquad\square$

There is also the more general concept of a root stack in a simple normal crossing (snc) divisor. In the next section, we will obtain a semiorthogonal decomposition for such root stacks. Since this decomposition also exists in the case that the ambient algebaic stack is not smooth, will work with a (non-standard) generalized notion of snc divisor. This generality will not be needed in the applications we have in mind, but it comes at no extra cost and reveals the true relative nature of the root construction and the induced semiorthogonal decomposition.

Definition 3.5. Let X be an algebraic stack. A *generalized snc divisor* on X is a finite family $E = (E_i)_{i \in I}$ of effective Cartier divisors on X such that for each subset $J \subseteq I$ and each element $i \in J$ the inclusion

$$\cap_{j \in J} E_j \to \cap_{j \in J \setminus \{i\}} E_j$$

is an effective Cartier divisor. We call the divisors E_i the *components* of E. If X is smooth over a field k, we call a generalized snc divisor an *snc divisor* if all intersections $\cap_{i \in J} E_j$ are smooth over k.

Note that the definition asserts that the non-empty components of a generalized snc divisor are distinct.

Remark 3.6. Note that we do not require the components to be irreducible, reduced or non-empty. This will somewhat simplify the exposition in the proof of Theorem 4.9. In the smooth case, our definition of an snc divisor coincides with the standard one, possibly with the subtle difference that we make the splitting of E into components part of the structure.

When dealing with the combinatorics of iterated root stacks, it is convenient to use multi-index notation. Given a finite set I a *multi-index* (with respect to I) is an element $a = (a_i)$ of $\mathbb{Z}^I$. Multiplication of multi-indexes is defined coordinatewise. We will also consider the partial ordering on $\mathbb{Z}^I$ defined by $a \leq b$ if and only if $a_i \leq b_i$ for all $i \in I$. We write $a < b$ if and only if $a_i < b_i$ for all $i \in I$. If a is a multi-index and E is a (generalized) snc divisor indexed by I, then aE denotes the Cartier divisor given by $\sum_{i \in I} a_i E_i$.

Definition 3.7 (Iterated root construction)**.** Let X be an algebraic stack and $E = (E_i)_{i \in I}$ a generalized snc divisor on X. Fix a multi-index $r > 0$ in $\mathbb{Z}^I$. The *r-th root stack*, denoted by $X_{r^{-1}E}$, of X with respect to E and r is defined as the fiber product of the root stacks $X_{r_i^{-1}E_i}$ over X for $i \in I$. The *transform* of E is defined as the family $r^{-1}E = (\widetilde{E}_i)_{i \in I}$, where $\widetilde{E}_i$ denotes the pull-back of $r_i^{-1}E_i$ along the projection $X_{r^{-1}E} \to X_{r_i^{-1}E_i}$.

Remark 3.8. If X is smooth and E is an snc divisor, then the root stack $X_{r^{-1}E}$ only depends on the divisor rE. The corresponding statement for generalized snc divisors is not true. For instance, consider the coordinate axes $V(x)$ and $V(y)$ in the affine plane $\mathbb{A}^2 = \operatorname{Spec}\mathbb{Z}[x, y]$. Then the $(2, 2)$-th root stack of $\mathbb{A}^2$ in $(V(x), V(y))$ is smooth whereas the 2-nd root stack of $\mathbb{A}^2$ in $(V(xy))$ is not (cf. [BC10, §2.1]).

The next proposition is well-known and stated, in a slightly different form, in [BC10, §2.1]. We include a proof since none is given in *loc. cit.*

Proposition 3.9. *Let X be an algebraic stack and E a generalized snc divisor indexed by I. Given a multi-index $r > 0$ in $\mathbb{Z}^I$, the r-th root construction of X in E has the following properties:*

> *(a) Given $s > 0$ in $\mathbb{Z}^I$, the s-th root stack $(X_{r^{-1}E})_{s^{-1}(r^{-1}E)}$ of $X_{r^{-1}E}$ in the transform $r^{-1}E$ is canonically isomorphic to the (rs)-th root stack $X_{(sr)^{-1}E}$ in E, and this isomorphism identifies $s^{-1}(r^{-1})E$ with $(sr)^{-1}E$.*
> *(b) The transform $r^{-1}E$ is a generalized snc divisor on $X_{r^{-1}E}$.*

If furthermore X is smooth over a field k and E is an snc divisor, then we have the following:

> *(c) The stack $X_{r^{-1}E}$ is smooth and $r^{-1}E$ is an snc divisor.*

Proof. Assume that $r = (r_1, \ldots, r_n)$. By identifying $[\mathbb{A}^n/\mathbb{G}_m^n]$ with $[\mathbb{A}^1/\mathbb{G}_m]^n$, we get a canonical morphism

$$\pi_r = \pi_{r_1} \times \cdots \times \pi_{r_n} \colon [\mathbb{A}^n/\mathbb{G}_m^n] \to [\mathbb{A}^n/\mathbb{G}_m^n]$$

where each π_{r_i} corresponds to the morphism π in the universal r_i-th root diagram 3.1. The generalized snc divisor $E = (E_1, \ldots, E_n)$ gives rise to a morphism

$$f_E \colon X \xrightarrow{\Delta} X^n \to [\mathbb{A}^n/\mathbb{G}_m^n],$$

where the second morphism is the product $f_{E_1} \times \cdots \times f_{E_n}$ with each f_{E_i} as in Definition 3.1. It is now easy to see that the structure map $X_{r^{-1}E} \to X$ of the root stack is canonically isomorphic to the pull-back of π_r along f_E (cf. [FMN10, §1.3.b]). Using this description, statement (a) follows from the diagram

$$
\begin{array}{ccc}
(X_{r^{-1}E})_{s^{-1}(r^{-1}E))} & \longrightarrow & [\mathbb{A}^n/\mathbb{G}_m^n] \\
\downarrow & & \downarrow{\scriptstyle \pi_s} \\
X_{r^{-1}E} & \xrightarrow{\ f_{r^{-1}E}\ } & [\mathbb{A}^n/\mathbb{G}_m^n] \\
\downarrow & & \downarrow{\scriptstyle \pi_r} \\
X & \xrightarrow{\ f_E\ } & [\mathbb{A}^n/\mathbb{G}_m^n]
\end{array}
$$

with cartesian squares and the fact that $\pi_r \circ \pi_s = \pi_{rs}$.

Next we prove statement (b). We may work locally on X and assume that $X = \operatorname{Spec} R$ and $E = (V(f_1), \ldots, V(f_n))$ where $f_1, \ldots, f_n$ is a regular sequence in R. Then $X_{r^{-1}E}$ is given by $[\operatorname{Spec} R'/A]$ with $R' = R[t_1, \ldots, t_n]/(t_1^{r_1} - f_1, \cdots t_n^{r_n} - f_n)$ and $A = \mu_{r_1} \times \cdots \times \mu_{r_n}$ (cf. Example 3.2). The divisor $r^{-1}E$ is given by $(V(t_1), \ldots, V(t_n))$. It is now easy to verify that $t_1, \ldots, t_n$ is a regular sequence in R', which proves the statement.

Now assume in addition that the ring $R/(f_1, \ldots, f_n)$ is regular. Then the same holds for the rings $R'/(t_1, \ldots, t_r)$ for $0 \leq r \leq n$ since $R'/(t_1, \ldots, t_n) \cong R/(f_1, \ldots, f_n)$ and $t_1, \ldots, t_n$ is a regular sequence. In particular, this implies (c) since smoothness over k is equivalent to regularity after base change to an algebraically closed field. $\square$

Due to Proposition 3.9.(a) root constructions in generalized snc divisors are sometimes referred to as *iterated root constructions*.

4. Semiorthogonal decompositions for root stacks

In many aspects, root stacks behave like blowups. For example, they give rise to semiorthogonal decompositions. This was observed by Ishii–Ueda in [IU11, Theorem 1.6]. In this section, we reprove this theorem in a more general setting as Theorem 4.7. We also give an explicit, combinatorial description of the semiorthogonal decomposition of the derived category of an iterated root stack.

Lemma 4.1 (cf. [HR14, Theorem 4.14.(1)]). *Let $\iota \colon E \to X$ be an effective Cartier divisor on an algebraic stack X. Then the functor $\iota_* = \mathrm{R}\iota_* \colon \mathrm{D}_{\mathrm{qc}}(E) \to \mathrm{D}_{\mathrm{qc}}(X)$ admits a right adjoint $\iota^\times$, and both ι_* and $\iota^\times$ preserve perfect complexes.*

Before we prove the lemma, we introduce some auxiliary notation. Let Z be an algebraic stack and $\mathcal{R}$ a quasi-coherent sheaf of commutative $\mathcal{O}_Z$-algebras. We call a sheaf of $\mathcal{R}$-modules quasi-coherent if it is quasi-coherent as an $\mathcal{O}_Z$-module. Let $\mathrm{D}(Z, \mathcal{R})$ denote the derived category of sheaves of $\mathcal{R}$-modules in the topos $Z_{\mathrm{lis\text{-}ét}}$ and let $\mathrm{D}_{\mathrm{qc}}(Z, \mathcal{R})$ be the full subcategory of objects with quasi-coherent cohomology. More generally, the definitions of $\mathrm{D}(Z, \mathcal{R})$ and $\mathrm{D}_{\mathrm{qc}}(Z, \mathcal{R})$ generalize in the obvious way to the case that $\mathcal{R}$ is a quasi-coherent sheaf of commutative dg $\mathcal{O}_Z$-algebras.

Proof. Consider the sheaf of commutative dg $\mathcal{O}_X$-algebras $\mathcal{R} = (\mathcal{O}_X(-E) \to \mathcal{O}_X)$ where $\mathcal{O}_X$ sits in degree zero. It comes with a quasi-isomorphism $\mathcal{R} \to \iota_* \mathcal{O}_E$ of sheaves of dg $\mathcal{O}_X$-algebras. The pushforward

$$\iota_* \colon \mathrm{D}_{\mathrm{qc}}(E) = \mathrm{D}_{\mathrm{qc}}(E, \mathcal{O}_E) \to \mathrm{D}_{\mathrm{qc}}(X) = \mathrm{D}_{\mathrm{qc}}(X, \mathcal{O}_X)$$

factors as

$$(4.1) \qquad \mathrm{D}_{\mathrm{qc}}(E, \mathcal{O}_E) \xrightarrow{\sim} \mathrm{D}_{\mathrm{qc}}(X, \iota_* \mathcal{O}_E) \xrightarrow{\sim} \mathrm{D}_{\mathrm{qc}}(X, \mathcal{R}) \xrightarrow{\alpha_*} \mathrm{D}_{\mathrm{qc}}(X, \mathcal{O}_X).$$

Indeed, the first functor is an equivalence since ι is affine [HR14, Corollary 2.7]. The second equivalence is induced by restriction along the quasi-isomorphism $\mathcal{R} \to \iota_* \mathcal{O}_E$ [Ric10, Proposition 1.5.6]. The third functor α_* is induced by restriction along the structure morphism $\mathcal{O}_X \to \mathcal{R}$. This reduces the problem of finding a right adjoint to ι_* to finding a right adjoint to α_*.

On the level of complexes, the functor $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{R}, -)$ is easily seen to be right adjoint to restriction along $\mathcal{O}_X \to \mathcal{R}$. Since $\mathcal{R}$ is strictly perfect as a complex of $\mathcal{O}_X$-modules, the functor $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{R}, -)$ takes acyclic complexes to acyclic complexes

and descends to a right adjoint $\alpha^\times \colon \mathrm{D}_{\mathrm{qc}}(\mathcal{O}_X, X) \to \mathrm{D}_{\mathrm{qc}}(\mathcal{R}, X)$ of α_*. This proves the existence of $i^\times$.

Next we prove that ι_* preserves perfect complexes. Let $\mathcal{F} \in \mathrm{D}_{\mathrm{pf}}(E)$. The question whether $\iota_* \mathcal{F}$ is perfect is local on X. Since vector bundles on E trivialize locally on X, we may assume that $\mathcal{F}$ is a bounded complex of finite free modules. But now the fact that $\iota_* \mathcal{O}_E$ is perfect implies that $\iota_* \mathcal{F}$ is perfect.

Finally, let us prove that $\iota^\times$ preserves perfect complexes. Since the question is local on X, it is enough to verify that $\iota^\times(\mathcal{O}_X)$ is perfect. Observe first that $\alpha^\times(\mathcal{O}_X) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{R}, \mathcal{O}_X) \cong \mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{O}_X(E)[-1]$ in $\mathrm{D}_{\mathrm{qc}}(X, \mathcal{R})$. Under the equivalences of (4.1), this object corresponds to the object $\iota_* \mathcal{O}_E \otimes_{\mathcal{O}_X} \mathcal{O}_X(E)[-1] \cong \iota_*(\mathcal{O}_E \otimes_{\mathcal{O}_E} \iota^*(\mathcal{O}_X(E)))[-1] \cong \iota_*(\iota^*(\mathcal{O}_X(E))[-1]$ of $\mathrm{D}_{\mathrm{qc}}(X, \iota_* \mathcal{O}_E)$ and to the object $\iota^*(\mathcal{O}_X(E))[-1]$ of $\mathrm{D}_{\mathrm{qc}}(E)$. This latter object is obviously perfect and isomorphic to $\iota^\times(\mathcal{O}_X)$. $\qquad\square$

Lemma 4.2. *In the setting of Lemma 4.1, given any object $\mathcal{F}$ of $\mathrm{D}_{\mathrm{qc}}(E)$, the adjunction counit $\mathsf{L}\iota^* \iota_* \mathcal{F} \to \mathcal{F}$ fits into a triangle*

$$\mathcal{F} \otimes_{\mathcal{O}_E} \iota^* \mathcal{O}_X(-E)[1] \to \mathsf{L}\iota^* \iota_* \mathcal{F} \to \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_E} \iota^* \mathcal{O}_X(-E)[2]$$

Proof. Recall the factorization (4.1) of ι_* from the proof of Lemma 4.1. The functor α_* occuring there is restriction of scalars along $\mathcal{O}_X \to \mathcal{R}$. Extension of scalars $(- \otimes_{\mathcal{O}_X} \mathcal{R})$ preserves acyclic complexes and therefore defines a left adjoint $(- \otimes_{\mathcal{O}_X} \mathcal{R}) \colon \mathrm{D}_{\mathrm{qc}}(X, \mathcal{O}_X) \to \mathrm{D}_{\mathrm{qc}}(X, \mathcal{R})$ to α_*. Modulo the two equivalences in (4.1) this functor is isomorphic to $\mathsf{L}\iota^*$, and the adjunction counit $\mathsf{L}\iota^* \iota_* \to \mathrm{id}$ corresponds to the adjunction counit $(- \otimes_{\mathcal{O}_X} \mathcal{R}) \to \mathrm{id}$.

Let $\mathcal{M}$ be a dg $\mathcal{R}$-module. The adjunction counit $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{R} \to \mathcal{M}$ is given by multiplication. We denote its kernel by $\mathcal{K}$ and obtain a short exact sequence

$$\mathcal{K} \hookrightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{R} \twoheadrightarrow \mathcal{M}$$

of dg $\mathcal{R}$-modules. As complexes of dg $\mathcal{O}_X$-modules, we have an obvious isomorphism

$$\mathcal{K} \cong \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-E)[1].$$

Assume that $\mathcal{M}$ is obtained from a complex of dg $\iota_* \mathcal{O}_E$-modules by restriction along $\mathcal{R} \to \iota_* \mathcal{O}_E$. Then this isomorphism is even an isomorphism of dg $\mathcal{R}$-modules. Since $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-E)$ and $\mathcal{M} \otimes_{\mathcal{R}} (\mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-E))$ are isomorphic as dg $\mathcal{R}$-modules, we obtain a triangle

$$\mathcal{M} \otimes_{\mathcal{R}} (\mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-E))[1] \to \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{R} \to \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{R}} (\mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-E))[2]$$

in $\mathrm{D}(X, \mathcal{R})$. Since restriction of scalars $\mathrm{D}_{\mathrm{qc}}(X, \iota_* \mathcal{O}_E) \to \mathrm{D}_{\mathrm{qc}}(X, \mathcal{R})$ is an equivalence, we obtain such a triangle in $\mathrm{D}_{\mathrm{qc}}(X, \mathcal{R})$ for any object $\mathcal{M}$ of $\mathrm{D}_{\mathrm{qc}}(X, \mathcal{R})$. The claim follows. $\qquad\square$

Lemma 4.3. *Let $f \colon X \to Y$ be a concentrated morphism of algebraic stacks such that $\mathsf{R}f_* \colon \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc}}(Y)$ preserves perfect complexes. Then $\mathsf{L}f^* \colon \mathrm{D}_{\mathrm{pf}}(Y) \to \mathrm{D}_{\mathrm{pf}}(X)$ has a left adjoint $f_\times$ given by*

$$f_\times \colon \mathrm{D}_{\mathrm{pf}}(X) \to \mathrm{D}_{\mathrm{pf}}(Y), \quad \mathcal{F} \mapsto (\mathsf{R}f_*(\mathcal{F}^\vee))^\vee$$

where $(-)^\vee$ denotes the dual $\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X)$ on $\mathrm{D}_{\mathrm{pf}}(X)$, and similarly for $\mathrm{D}_{\mathrm{pf}}(Y)$.

Proof. This statement is a formal consequence of the dual $(-)^\vee$ being an involutive anti-equivalence which respects derived pullbacks. Explicitly, for $\mathcal{F} \in D_{\mathrm{pf}}(X)$ and $\mathcal{G} \in D_{\mathrm{pf}}(Y)$ we have

$$\begin{aligned}
\operatorname{Hom}_{D(X)}(\mathcal{F}, \mathsf{L}f^*\mathcal{G}) &\cong \operatorname{Hom}_{D(X)}((\mathsf{L}f^*\mathcal{G})^\vee, \mathcal{F}^\vee) \\
&\cong \operatorname{Hom}_{D(X)}(\mathsf{L}f^*(\mathcal{G}^\vee), \mathcal{F}^\vee) \\
&\cong \operatorname{Hom}_{D(Y)}(\mathcal{G}^\vee, \mathsf{R}f_*(\mathcal{F}^\vee)) \\
&\cong \operatorname{Hom}_{D(Y)}((\mathsf{R}f_*(\mathcal{F}^\vee))^\vee, (\mathcal{G}^\vee)^\vee) \\
&\cong \operatorname{Hom}_{D(Y)}(f_\times \mathcal{F}, \mathcal{G}).
\end{aligned}$$

$\square$

Lemma 4.4. *Let $f\colon X \to Y$ be a concentrated morphism of algebraic stacks. Then $\mathsf{L}f^*\colon D_{\mathrm{qc}}(Y) \to D_{\mathrm{qc}}(X)$ is full and faithful if and only if the natural morphism $\mathcal{O}_Y \to \mathsf{R}f_*\mathcal{O}_X$ is an isomorphism.*

Proof. A left adjoint functor is full and faithful if and only if the adjunction unit is an isomorphism. In particular, one implication is trivial. For the other implication, assume that $\mathcal{O}_Y \to \mathsf{R}f_*\mathcal{O}_X$ is an isomorphism. Then the projection formula [HR14, Corollary 4.12] gives

$$\mathcal{G} \xrightarrow{\sim} \mathsf{R}f_*\mathcal{O}_X \otimes \mathcal{G} \xrightarrow{\sim} \mathsf{R}f_*(\mathcal{O}_X \otimes \mathsf{L}f^*\mathcal{G}) \xrightarrow{\sim} \mathsf{R}f_*\mathsf{L}f^*\mathcal{G}, \qquad \mathcal{G} \in D_{\mathrm{qc}}(Y).$$

This shows that the adjunction unit is an isomorphism. $\square$

Lemma 4.5. *Let X be a tame algebraic stack with finite inertia, and let $\pi\colon X \to X_{\mathrm{cs}}$ denote the canonical morphism to its coarse space. Then the natural morphism $\mathcal{O}_{X_{\mathrm{cs}}} \to \mathsf{R}\pi_*\mathcal{O}_X$ is an isomorphism. Moreover, if π is flat and of finite presentation, then $\mathsf{R}\pi_*\colon D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(X_{\mathrm{cs}})$ preserves perfect complexes.*

Proof. To check that a morphism in the derived category $D_{\mathrm{qc}}(X_{\mathrm{cs}})$ is an isomorphism, we may pass to an fppf covering by affine schemes. Since the derived pushforward and the formation of the coarse space commute with flat base change, we reduce to the situation where X_{cs} is affine. Since $\pi\colon X \to X_{\mathrm{cs}}$ is the canonical morphism to the coarse space, it is separated and quasi-compact. Furthermore X has finite stabilizers. By Remark 2.12 and 2.13, we can therefore identify $D_{\mathrm{qc}}(X)$ with $D(\mathrm{Qcoh}(X))$ and similarly for X_{cs}. The canonical morphism $\mathcal{O}_{X_{\mathrm{cs}}} \to \pi_*\mathcal{O}_X$ is an isomorphism, again because π is the structure morphism to the coarse space. Hence the first statement follows from the exactness of $\pi_*\colon \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(X_{\mathrm{cs}})$ which is a consequence of the tameness hypothesis.

For the other statement, we may again work locally on X_{cs}, since perfectness of a complex is a local property. Hence we may again assume that X_{cs} is affine. We can also assume that we have a finite locally free covering $\alpha\colon U \to X$ by an affine scheme U (cf. [Ryd13, Theorem 6.10 and Proposition 6.11]).

In this situation, we claim that the object $\alpha_*\mathcal{O}_U$ is a compact projective generator for $\mathrm{Qcoh}(X)$. In particular, the category $\mathrm{Qcoh}(X)$ is equivalent to the category of modules for a not necessarily commutative ring.

Now we prove the statement claimed above. First note that since α is affine, the functor α_* has a right adjoint $\alpha^\times$ with the property that $\alpha_*\alpha^\times = \mathcal{H}om_{\mathcal{O}_X}(\alpha_*\mathcal{O}_U, -)$. Since $\alpha_*\mathcal{O}_U$ is finite locally free, it follows that the functor $\alpha_*\alpha^\times$ is exact, faithful, and commutes with filtered colimits. Since α is affine, the functor α_* reflects these

properties, which implies that also $\alpha^\times$ is exact, faithful and commutes with filtered colimits. Finally, since U is affine, it follows that also the functor

$$\mathrm{Hom}_{\mathcal{O}_X}(\alpha_*\mathcal{O}_U, -) \cong \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \alpha^\times(-))$$

has these properties, so $\alpha_*\mathcal{O}_U$ is indeed a compact, projective generator for $\mathrm{Qcoh}(X)$.

It follows that the compact objects of the derived category $\mathrm{D}_{\mathrm{qc}}(X)$, which coincides with $\mathrm{D}(\mathrm{Qcoh}(X))$ by Remark 2.12, are precisely those isomorphic to bounded complexes of compact projective objects. By tameness, the perfect objects of $\mathrm{D}_{\mathrm{qc}}(X)$ coincide with the compact objects [HR14, Remark 4.6]. Hence it suffices to show that $\pi_*\colon \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(X_{\mathrm{cs}})$ preserves compact projective objects.

To prove this, we assume that $\mathcal{P}$ is a compact, projective object in $\mathrm{Qcoh}(X)$. Since $\alpha_*(\mathcal{O}_U)$ is a compact, projective generator of $\mathrm{Qcoh}(X)$, there exists a split surjection $\alpha_*(\mathcal{O}_U)^{\oplus n} \to \mathcal{P}$ for some positive integer n. Hence also the pushforward $\pi_*\alpha_*(\mathcal{O}_U)^{\oplus n} \to \pi_*\mathcal{P}$ is a split surjection. But $\pi_*\alpha_*(\mathcal{O}_U)$ is finite locally free, and hence compact and projective in $\mathrm{Qcoh}(X_{\mathrm{cs}})$, by our assumption that $\pi\colon X \to X_{\mathrm{cs}}$ is flat and of finite presentation. It follows that $\pi_*\mathcal{P}$ is compact and projective, which concludes the proof. $\qquad\square$

Example 4.6. We give some examples of concentrated morphisms $f\colon X \to Y$ of algebraic stacks such that the functor $\mathrm{R}f_*\colon \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc}}(Y)$ preserves perfect complexes and the natural morphism $\mathcal{O}_Y \to \mathrm{R}f_*\mathcal{O}_X$ is an isomorphism (cf. Lemmas 4.3 and 4.4). Note that these properties are fppf local on Y. Examples where f is representable are

(a) blow-ups of smooth algebraic stacks over a field in a smooth locus;
(b) more generally, proper birational morphisms between smooth algebraic stacks over a field;
(c) projective bundles.

Examples where f is not necessarily representable are

(d) the morphism ρ in the root diagram (3.2);
(e) the morphism π in the root diagram (3.2).

The last two items follow by applying Lemma 4.5 after an appropriate base change, and using part (b), (c), (d), and (e) of Proposition 3.3.

Theorem 4.7. *Let X be an algebraic stack and $E \subset X$ an effective Cartier divisor. Fix a positive integer r and let $\pi\colon \widetilde{X} = X_{r^{-1}E} \to X$ be the r-th root construction of X in E with ι and ρ as in the root diagram (3.2). Then the functors*

$$(4.2) \qquad\qquad \pi^*\colon \mathrm{D}_{\mathrm{pf}}(X) \to \mathrm{D}_{\mathrm{pf}}(\widetilde{X}),$$

$$(4.3) \qquad\quad \Phi_a := \mathcal{O}_{\widetilde{X}}\left(ar^{-1}E\right) \otimes \iota_*\rho^*(-)\colon \mathrm{D}_{\mathrm{pf}}(E) \to \mathrm{D}_{\mathrm{pf}}(\widetilde{X})$$

for $a \in \{1, \ldots, r-1\}$, are full and faithful and admit left and right adjoints. Furthermore, the category $\mathrm{D}(\widetilde{X})$ has the semiorthogonal decomposition

$$(4.4) \qquad\qquad \mathrm{D}(\widetilde{X}) = \left\langle \Phi_{r-1}, \ldots, \Phi_1, \pi^* \right\rangle$$

into admissible subcategories.

Recall that π and ρ are flat and that ι is the embedding of the Cartier divisor $r^{-1}E$ (Proposition 3.3, part (a), (b), (c)) and that $\mathcal{O}_{\widetilde{X}}(ar^{-1}E)$ is a line bundle. Therefore we omitted the usual decorations for derived functors in (4.2) and (4.3). Also note that Φ_a is well-defined by Lemma 4.1.

Proof of Theorem 4.7. Both functors $\rho^*\colon \mathrm{D}_{\mathrm{pf}}(E) \to \mathrm{D}_{\mathrm{pf}}(r^{-1}E)$ and $\pi^*\colon \mathrm{D}_{\mathrm{pf}}(X) \to \mathrm{D}_{\mathrm{pf}}(\widetilde{X})$ are full and faithful and admit left and right adjoints by part (d) and (e) of Example 4.6 and Lemmas 4.3 and 4.4. Since tensoring with a line bundle is an autoequivalence and since $\iota_*\colon \mathrm{D}_{\mathrm{pf}}(r^{-1}E) \to \mathrm{D}_{\mathrm{pf}}(\widetilde{X})$ admits left and right adjoints, by Lemma 4.1, we deduce that the functors Φ_a admit left and right adjoints.

The stack $r^{-1}E$ is a μ_r-gerbe over E. Therefore the category $\mathrm{Mod}(r^{-1}E)$ splits as a direct sum $\bigoplus_{\chi=0}^{r-1} \mathrm{Mod}(r^{-1}E)_\chi$ according to the characters of the inertial action. This induces a corresponding decomposition $\bigoplus_{\chi=0}^{r-1} \mathrm{D}_{\mathrm{pf}}(r^{-1}E)_\chi$ of the triangulated category $\mathrm{D}_{\mathrm{pf}}(r^{-1}E)$. The essential image of ρ^* is $\mathrm{D}_{\mathrm{pf}}(r^{-1}E)_0$.

Consider
$$\mathrm{Hom}_{\mathrm{D}(\widetilde{X})}(\iota_*\mathcal{F}, \iota_*\mathcal{G}) \cong \mathrm{Hom}_{\mathrm{D}(r^{-1}E)}(\mathsf{L}\iota^*\iota_*\mathcal{F}, \mathcal{G})$$
for $\mathcal{F}, \mathcal{G} \in \mathrm{D}_{\mathrm{pf}}(r^{-1}E)$. Since ι is the inclusion of an effective Cartier divisor, Lemma 4.2 provides a triangle
$$\mathcal{F} \otimes \mathcal{N}[1] \to \mathsf{L}\iota^*\iota_*\mathcal{F} \to \mathcal{F} \to \mathcal{F} \otimes \mathcal{N}[2]$$
where $\mathcal{N} = \iota^*\mathcal{O}_{\widetilde{X}}(-r^{-1}E)$ is the conormal line bundle of the closed immersion ι.

Now assume that $\mathcal{F} \in \mathrm{D}_{\mathrm{pf}}(r^{-1}E)_\chi$ and $\mathcal{G} \in \mathrm{D}_{\mathrm{pf}}(r^{-1}E)_\psi$. Since $\mathcal{N} \in \mathrm{D}_{\mathrm{pf}}(r^{-1}E)_1$, the above triangle enables us to compute

$$(4.5) \quad \mathrm{Hom}_{\mathrm{D}(\widetilde{X})}(\iota_*\mathcal{F}, \iota_*\mathcal{G}) \cong \begin{cases} \mathrm{Hom}_{\mathrm{D}(r^{-1}E)}(\mathcal{F}, \mathcal{G}) & \text{if } \chi = \psi, \\ \mathrm{Hom}_{\mathrm{D}(r^{-1}E)}(\mathcal{F}[1] \otimes \mathcal{N}, \mathcal{G}) & \text{if } \chi + 1 = \psi \text{ in } \mathbb{Z}/r, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we see that the restriction of the functor ι_* to the category $\mathrm{D}_{\mathrm{pf}}(r^{-1}E)_\chi$ is full and faithful for each χ. As a consequence, all functors Φ_a are full and faithful.

Moreover, given $\mathcal{H} \in \mathrm{D}_{\mathrm{pf}}(X)$, we have
$$\mathrm{Hom}_{\mathrm{D}(\widetilde{X})}(\pi^*\mathcal{H}, \iota_*\mathcal{G}) \cong \mathrm{Hom}_{\mathrm{D}(r^{-1}E)}(\mathsf{L}\iota^*\pi^*\mathcal{H}, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{D}(r^{-1}E)}(\rho^*\mathsf{L}\kappa^*\mathcal{H}, \mathcal{G}),$$
which vanishes if $\psi \neq 0$ since the essential image of ρ^* is $\mathrm{D}_{\mathrm{pf}}(r^{-1}E)_0$.

This, together with the third equality in (4.5) shows that
$$(4.6) \qquad \iota_*\mathrm{D}_{\mathrm{pf}}(r^{-1}E)_1, \ldots, \iota_*\mathrm{D}_{\mathrm{pf}}(r^{-1}E)_{r-1}, \pi^*\mathrm{D}_{\mathrm{pf}}(X)$$
is a semiorthogonal sequence. The projection formula [HR14, Corollary 4.12] shows that $\Phi_a \cong \iota_*(\mathcal{N}^{\otimes(-a)} \otimes \rho^*(-))$. Hence the essential image of Φ_a lies in $\iota_*\mathrm{D}_{\mathrm{pf}}(r^{-1}E)_{-a}$ and the essential images of the functors in (4.4) form a semiorthogonal sequence.

Let $\mathcal{T}$ denote the smallest strict full triangulated subcategory of $\mathrm{D}_{\mathrm{pf}}(r^{-1}E)$ which contains all these essential images. Then
$$\mathcal{T} = \big\langle \Phi_{r-1}, \ldots, \Phi_1, \pi^* \big\rangle$$
is a semiorthogonal decomposition into admissible subcategories. It remains to prove that $\mathcal{T} = \mathrm{D}_{\mathrm{pf}}(X_{r^{-1}E})$. This can be done fppf locally on X by conservative descent [BS16]. Hence we may work with the local description given in Example 3.2. Using the notation from the example, the category $\mathrm{Qcoh}(X_{r^{-1}E})$ is equivalent to the category of $\mathbb{Z}/r\mathbb{Z}$-graded R'-modules. We use the symbol $\langle - \rangle$ to denote shifts with respect to the $\mathbb{Z}/r\mathbb{Z}$-grading. More precisely, given a graded R'-module $M = \bigoplus M^n$, we write $M\langle i \rangle$ for the graded R'-module with components $(M\langle i \rangle)^n = M^{i+n}$.

Note that $P = R'\langle 0 \rangle \oplus \cdots \oplus R'\langle r-1 \rangle$ is a compact projective generator of $\mathrm{Qcoh}(X_{r^{-1}E})$. This implies that P is a classical generator of $\mathrm{D}_{\mathrm{pf}}(X_{r^{-1}E})$. Since each of the semiorthogonal summands of $\mathcal{T}$ is idempotent complete, the same holds

for $\mathcal{T}$. Therefore, it is enough to prove that $R'\langle i \rangle$ is contained in $\mathcal{T}$ for each i. But $\mathcal{T}$ contains $\pi^* \mathcal{O}_X = R'\langle 0 \rangle$, and $\Phi_i \mathcal{O}_E = R'/(t)\langle i \rangle = R/(f)\langle i \rangle$ for $i \in \{1, \dots r-1\}$, so this follows from the triangles

$$R'\langle i-1 \rangle \xrightarrow{t} R'\langle i \rangle \to R/(f)\langle i \rangle \to R'\langle i-1 \rangle[1]$$

and induction on i starting with $i = 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 4.8. In [IU11], Ishii and Ueda state Theorem 4.7 for bounded derived categories of coherent sheaves. They assume that X and E are quasi-compact, separated Deligne–Mumford stacks which are smooth over $\mathbb{C}$ (although not all of these conditions are explicitly mentioned). Under these hypotheses the triangulated categories $D^{\mathrm{b}}(\mathrm{Coh}(X))$, $D^{\mathrm{b}}(\mathrm{Coh}(E))$, and $D^{\mathrm{b}}(\mathrm{Coh}(X_{r^{-1}E}))$ are equivalent to the categories $\mathrm{D}_{\mathrm{pf}}(X)$, $\mathrm{D}_{\mathrm{pf}}(E)$, and $\mathrm{D}_{\mathrm{pf}}(X_{r^{-1}E})$ respectively (cf. Remark A.3).

Next, we generalize Theorem 4.7 to iterated root stacks.

Theorem 4.9. *Let X be an algebraic stack and E a generalized snc divisor on X with components indexed by I. Fix a multi-index $r > 0$ in $\mathbb{Z}^I$ and let $X_{r^{-1}E}$ be the r-th root stack as in Definition 3.7. For any multi-index a satisfying $r > a \geq 0$ denote its support by $I_a \subseteq I$ and consider the diagram*

$$(4.7) \qquad \begin{array}{ccc} r^{-1}E(I_a) & \xrightarrow{\iota_a} & X_{r^{-1}E} \\ {\scriptstyle \rho_a}\downarrow & & \downarrow{\scriptstyle \pi} \\ E(I_a) & \longrightarrow & X, \end{array}$$

where $E(I_a) := \cap_{i \in I_a} E_i$ and $r^{-1}E(I_a) := \cap_{i \in I_a}(r^{-1}E)_i$. Then all the functors

$$(4.8) \qquad \Phi_a := \mathcal{O}_{X_{r^{-1}E}}(ar^{-1}E) \otimes (\iota_a)_* \rho_a^*(-) \colon \mathrm{D}_{\mathrm{pf}}(E(I_a)) \to \mathrm{D}_{\mathrm{pf}}(X_{r^{-1}E})$$

are full and faithful and admit left and right adjoints. Furthermore, the category $\mathrm{D}_{\mathrm{pf}}(X_{r^{-1}E})$ has the semiorthogonal decomposition

$$(4.9) \qquad\qquad\qquad \langle \Phi_a \mid r > a \geq 0 \rangle$$

into admissible subcategories. Here the multi-indexes a with $r > a \geq 0$ are arranged into any sequence $a^{(1)}, a^{(2)}, \dots, a^{(m)}$ such that $a^{(s)} \geq a^{(t)}$ implies $s \leq t$ for all $s, t \in \{1, \dots, m\}$ where $m = \prod_{i \in I} r_i$.

Remark 4.10. If r has at most one coordinate which is strictly bigger than one, then the root stack $X_{r^{-1}E}$ is isomorphic to a non-iterated root stack, and we recover Theorem 4.7.

Example 4.11. If our generalized snc divisor E has two components $E_1 = D$, $E_2 = F$ and $r_1 = 4$ and $r_2 = 3$, the Hasse diagram of the poset $\{a \in \mathbb{Z}^2 \mid r > a \geq 0\}$

(with arrows pointing to smaller elements) looks as follows.

(4.10)

$$
\begin{array}{c}
(0,2)_F \\
\nearrow \quad \searrow \\
(1,2)_{D\cap F} \qquad (0,1)_F \\
\nearrow \quad \searrow \quad \nearrow \quad \searrow \\
(2,2)_{D\cap F} \quad (1,1)_{D\cap F} \qquad (0,0)_X \\
\nearrow \quad \searrow \quad \nearrow \quad \searrow \quad \nearrow \\
(3,2)_{D\cap F} \quad (2,1)_{D\cap F} \qquad (1,0)_D \\
\searrow \quad \nearrow \quad \searrow \quad \nearrow \\
(3,1)_{D\cap F} \qquad (2,0)_D \\
\searrow \quad \nearrow \\
(3,0)_D
\end{array}
$$

The index at a vertex $a = (a_1, a_2)$ is $E(I_a)$. If we think of such a vertex as representing the essential image of $\mathrm{D}_{\mathrm{pf}}(E(I_a))$ under the fully faithful functor Φ_a, this gives a nice way to visualize the semiorthogonal decompositions (4.9) for all allowed sequences $a^{(1)}, \ldots, a^{(12)}$ at once. If there is a nonzero morphism from an object of the category represented by a vertex a to an object of the category represented by a vertex b, then there is a directed path from a to b in the Hasse diagram. Of course, we could have used the concept of a semiorthogonal decomposition indexed by a poset.

Proof of Theorem 4.9. We use induction on the number of coordinates of r which are strictly bigger than one. In light of Remark 4.10, we may assume that $r_{i_0} > 1$ for some index $i_0 \in I$ which we fix. Then the multi-index r factors as $s \cdot t$, where $t = (t_i)$ satisfies $t_{i_0} = r_{i_0}$ and $t_i = 1$ for all $i \neq i_0$. By Proposition 3.9.(a), the root stack $X_{r^{-1}E} \to X$ decomposes into a sequence

$$
(4.11) \qquad\qquad X_{r^{-1}E} \to X_{s^{-1}E} \to X
$$

of root stacks.

Any multi-index $r > a \geq 0$ can be uniquely written as a sum $a = a' + a''$ with $t > a' \geq 0$ and $s > a'' \geq 0$, and this gives a bijective correspondence between the set of multi-indexes a with $r > a \geq 0$ and pairs (a', a'') of multi-indexes satisfying $t > a' \geq 0$ and $s > a'' \geq 0$. The support $I_{a'}$ of such a multi-index a' is contained in $\{i_0\}$.

For any such multi-index $a = a' + a''$ the sequence (4.11) induces the decomposition

$$
(4.12) \qquad
\begin{array}{ccc}
r^{-1}E(I_a) & \longrightarrow r^{-1}E(I_{a'}) & \longrightarrow X_{r^{-1}E} \\
\downarrow & \downarrow & \downarrow \\
s^{-1}E(I_a) & \longrightarrow s^{-1}E(I_{a'}) & \longrightarrow X_{s^{-1}E} \\
\downarrow & \downarrow & \downarrow \\
E(I_a) & \longrightarrow E(I_{a'}) & \longrightarrow X
\end{array}
$$

of the diagram (4.7).

In the rest of this proof, we call a diagram of the form (4.7) a transform diagram. The upper right square in (4.12) depends on the support of a' (but not on a and

a'') and is a transform diagram for the t-th root of the divisor $s^{-1}E$ in $X_{s^{-1}E}$. If $I_{a'} = \{i_0\}$ it is, in fact, by the special form of t, a root diagram for the r_{i_0}-th root of the effective Cartier divisor $(s^{-1}E)_{i_0}$ on $X_{s^{-1}E}$. Denote the functors corresponding to (4.8) for this root construction by $\Phi'_{a'} \colon \mathrm{D}_{\mathrm{pf}}(s^{-1}E(I_{a'})) \to \mathrm{D}_{\mathrm{pf}}(X_{r^{-1}E})$. Then Theorem 4.7 yields the semiorthogonal decomposition

$$(4.13) \qquad \mathrm{D}_{\mathrm{pf}}(X_{r^{-1}E}) = \langle \Phi'_{a'} \mid t > a' \geq 0 \rangle.$$

Fix $t > a' \geq 0$ for a moment. Note that the lower right square of (4.12) is cartesian. This is trivial if $I_{a'} = \emptyset$ and otherwise follows from the fact that $s_{i_0} = 1$. As a consequence, $s^{-1}E(I_{a'}) \to E(I_{a'})$ is an s'-th root of the generalized snc divisor $E' = (E_i \cap E(I_{a'}))_{i \in I - \{i_0\}}$ where $s' = (s_i)_{i \in I - \{i_0\}}$. The corresponding transform diagram for $s > a'' \geq 0$ is the lower left square of diagram (4.12) where $a = a' + a''$. Let $\Phi''_{a''} \colon \mathrm{D}_{\mathrm{pf}}(E(I_a)) \to \mathrm{D}_{\mathrm{pf}}(s^{-1}E(I_{a'}))$ denote the functors corresponding to (4.8) for this iterated root construction; here a'' is identified with its restriction to $I - \{i_0\}$. By the induction hypothesis, we obtain the semiorthogonal decomposition

$$(4.14) \qquad \mathrm{D}_{\mathrm{pf}}(s^{-1}E(I_{a'})) = \langle \Phi''_{a''} \mid s > a'' \geq 0 \rangle.$$

Combining the decompositions (4.13) and (4.14) yields the semiorthogonal decomposition

$$(4.15) \qquad \mathrm{D}_{\mathrm{pf}}(X_{r^{-1}E}) = \langle \Phi'_{a'} \circ \Phi''_{a''} \mid t > a' \geq 0, \ s > a'' \geq 0 \rangle.$$

Next, we establish an isomorphism $\Phi_a \cong \Phi'_{a'} \circ \Phi''_{a''}$ for $a = a' + a''$ as above. Let

$$\mathcal{L}' = \mathcal{O}_{X_{r^{-1}E}}(a'r^{-1}E), \qquad \mathcal{L}'' = \mathcal{O}_{X_{s^{-1}E}}(a''s^{-1}E).$$

Furthermore, we denote the horizontal arrows in (4.12) by ι_{ij} and the vertical arrows by ρ_{ij}, where i denotes the row and j the column of domain of the morphism as viewed in the diagram. With this notation, we have identities

$$\Phi'_{a'} = \mathcal{L}' \otimes (\iota_{12})_* \rho_{12}^*(-), \qquad \Phi''_{a''} = \iota_{22}^* \mathcal{L}'' \otimes (\iota_{21})_* \rho_{21}^*(-).$$

Consider the composition $\Phi'_{a'} \circ \Phi'_{a''}$ of these two functors. By the projection formula for ι_{12} ([HR14, Corollary 4.12]) and the fact that pullbacks and tensor products commute, we see that this composition is isomorphic to

$$(4.16) \qquad \mathcal{L}' \otimes \rho_{13}^* \mathcal{L}'' \otimes (\iota_{12})_* \rho_{12}^* (\iota_{21})_* \rho_{21}^*(-).$$

Since the upper left square in (4.12) is cartesian, flat base change ([HR14, Theorem 2.6.(4)]) along the flat morphism ρ_{12} (Proposition 3.3.(c)) shows that our composition (4.16) is isomorphic to

$$(4.17) \qquad \mathcal{L}' \otimes \rho_{13}^* \mathcal{L}'' \otimes (\iota_{12} \circ \iota_{11})_* (\rho_{11} \circ \rho_{21})^*.$$

Now

$$\mathcal{L}' \otimes \rho_{13}^* \mathcal{L}'' = \mathcal{O}_{X_{r^{-1}E}} \left((a' + ta'')r^{-1}E \right).$$

But $a' + ta'' = a' + a'' = a$ since $t_i = 1$ for i in the support of a'', so (4.17) is indeed isomorphic to Φ_a. This shows $\Phi_a \cong \Phi'_{a'} \circ \Phi''_{a''}$.

Hence Φ_a is full and faithful and admits left and right adjoints, and the semiorthogonal decomposition (4.15) simplifies to

$$(4.18) \qquad \mathrm{D}_{\mathrm{pf}}(X_{r^{-1}E}) = \langle \Phi_a \mid t > a' \geq 0, \ s > a'' \geq 0, \ a = a' + a'' \rangle.$$

Since i_0 was arbitrary with $r_{i_0} > 1$ the above shows: if a and b are two multi-indexes with $r > a \geq 0$ and $r > b \geq 0$ such that a nonzero morphism from an

object of the essential image of Φ_a to an object of the essential image of Φ_b exists, then $a \geq b$. This proves the theorem. $\square$

5. Differential graded enhancements and geometricity

Many triangulated categories are homotopy categories of certain differential graded (dg) categories. This observation leads to the notion of a dg enhancement of a triangulated category. We introduce obvious dg enhancements of the derived categories considered in this article and explain how to lift certain derived functors to dg functors between these enhancements. We then recall Orlov's notion of a geometric dg category and state his main glueing result.

We assume that the reader has some familiarity with differential graded categories, see for example [Kel06, Toë11]. In this section, we will work over a fixed field k and assume that all our triangulated categories and all our dg categories are k-linear.

DG enhancements. We introduce the dg enhancements we will use in the rest of this article.

The homotopy category of a dg category $\mathcal{A}$ is denoted by $[\mathcal{A}]$. Recall that if $\mathcal{A}$ is a *pretriangulated* dg category, then the homotopy category $[\mathcal{A}]$ has a canonical structure of a triangulated category.

Definition 5.1. A *dg enhancement* of a triangulated category $\mathcal{T}$ is a pair $(\mathcal{E}, \varepsilon)$ consisting of a pretriangulated dg category $\mathcal{E}$ together with an equivalence $\varepsilon \colon [\mathcal{E}] \xrightarrow{\sim} \mathcal{T}$ of triangulated categories.

Example 5.2. Let $(X, \mathcal{O})$ be a ringed topos over k. In the dg category of complexes of $\mathcal{O}$-modules, consider the full dg subcategory $\mathrm{D}^{\mathrm{dg}}(X, \mathcal{O})$ consisting of h-injective complexes of injective $\mathcal{O}$-modules. This pretriangulated dg category together with the obvious equivalence

$$(5.1) \qquad\qquad [\mathrm{D}^{\mathrm{dg}}(X, \mathcal{O})] \xrightarrow{\sim} \mathrm{D}(X, \mathcal{O})$$

forms a dg enhancement of $\mathrm{D}(X)$. We chose to work with these dg enhancements in this article.

Remark 5.3. Another dg enhancement of $\mathrm{D}(X, \mathcal{O})$ is provided by the Drinfeld dg quotient of the dg category of complexes of $\mathcal{O}$-modules by its full dg subcategory of acyclic complexes.

Remark 5.4. If $(\mathcal{E}, \varepsilon)$ is a dg enhancement of $\mathcal{T}$ then any strict full triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ has an induced dg enhancement: just take the full dg subcategory of $\mathcal{E}$ of objects that go to objects of $\mathcal{S}$ under ε, and restrict ε appropriately.

Example 5.5. Let X be an algebraic stack over k and consider the ringed topos $(X_{\text{lis-ét}}, \mathcal{O}_X)$. The derived category $\mathrm{D}(X) = \mathrm{D}(X_{\text{lis-ét}}, \mathcal{O}_X)$ has the dg enhancement $\mathrm{D}^{\mathrm{dg}}(X) := \mathrm{D}^{\mathrm{dg}}(X_{\text{lis-ét}}, \mathcal{O}_X)$. By Remark 5.4, the strict triangulated subcategories $\mathrm{D}_{\mathrm{qc}}(X)$ and $\mathrm{D}_{\mathrm{pf}}(X)$ have induced dg enhancements which we denote by $\mathrm{D}_{\mathrm{qc}}^{\mathrm{dg}}(X)$ and $\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(X)$, respectively.

Example 5.6. If X is a Deligne–Mumford stack over k we could instead consider the ringed topos $(X_{\text{ét}}, \mathcal{O}_X)$ and define the dg enhancements $\mathrm{D}^{\mathrm{dg}}(X_{\text{ét}})$, $\mathrm{D}_{\mathrm{qc}}^{\mathrm{dg}}(X_{\text{ét}})$ and $\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(X_{\text{ét}})$ for the triangulated categories $\mathrm{D}(X_{\text{ét}})$, $\mathrm{D}_{\mathrm{qc}}(X_{\text{ét}})$ and $\mathrm{D}_{\mathrm{pf}}(X_{\text{ét}})$ in a similar way as in the previous example.

Remark 5.7. Let X be a Deligne–Mumford stack. As described in Remark 2.9, the morphism (2.1) of ringed topoi induces a triangulated equivalence $\varepsilon_* \colon D_{qc}(X_{lis\text{-}\acute{e}t}) \xrightarrow{\sim} D_{qc}(X_{\acute{e}t})$. Since ε^* is exact, the functor $\varepsilon_* \colon \mathrm{Mod}(X_{lis\text{-}\acute{e}t}, \mathcal{O}) \to \mathrm{Mod}(X_{\acute{e}t}, \mathcal{O})$ preserves injectives and h-injective complexes. Therefore we obtain quasi-equivalences $D_{qc}^{dg}(X_{lis\text{-}\acute{e}t}) \to D_{qc}^{dg}(X_{\acute{e}t})$ and $D_{pf}^{dg}(X_{lis\text{-}\acute{e}t}) \to D_{pf}^{dg}(X_{\acute{e}t})$ lifting the equivalences $D_{qc}(X_{lis\text{-}\acute{e}t}) \xrightarrow{\sim} D_{qc}(X_{\acute{e}t})$ and $D_{pf}(X_{lis\text{-}\acute{e}t}) \xrightarrow{\sim} D_{pf}(X_{\acute{e}t})$ to dg enhancements.

Uniqueness of dg enhancements. We would like to point out that the derived categories we are mainly interested in have unique dg enhancements in the sense of the following definition.

Definition 5.8 (cf. [LO10, CS15]). We say that a triangulated category $\mathcal{T}$ *has a unique dg enhancement* if it has a dg enhancement and given any two dg enhancements $(\mathcal{E}, \varepsilon)$ and $(\mathcal{E}', \varepsilon')$ of $\mathcal{T}$, the dg categories $\mathcal{E}$ and $\mathcal{E}'$ are quasi-equivalent. That is, they are connected by a zig-zag of quasi-equivalences.

Remark 5.9. By [CS15, Proposition 6.10], the derived category $D_{pf}(X)$ for any separated, tame algebraic stack X which is smooth and of finite type over k has a unique dg enhancement. In particular, this includes the stacks considered in the main theorems of this article. Indeed, Proposition 6.10 from *loc. cit.* applies since every coherent $\mathcal{O}_X$-module on X is perfect by the assumption that X is regular (cf. Proposition A.2). Furthermore, the category $\mathrm{Qcoh}(X)$ is generated by a set of coherent $\mathcal{O}_X$-modules as a Grothendieck category since every quasi-coherent $\mathcal{O}_X$-module is the filtered colimit of its coherent submodules [LMB00, Proposition 15.4].

Lifts of some derived functors to dg enhancements. We need to lift some derived functors to the level of dg enhancements. Since our main results concern algebraic stacks over a field we chose to use the methods of [Sch15]. We briefly recall the results we need.

If $(X, \mathcal{O})$ is a ringed topos over the field k, we have replacement dg functors $\mathbf{i}$ and $\mathbf{e}$ on the dg category of complexes of $\mathcal{O}$-modules. The functor $\mathbf{i}$ replaces a complex with a quasi-isomorphic h-injective complex of injective $\mathcal{O}$-modules, and $\mathbf{e}$ replaces a complex with an h-flat complex of flat $\mathcal{O}$-modules ([Sch15, Theorem 4.17]).

Let $f \colon (X, \mathcal{O}) \to (Y, \mathcal{O}')$ be a morphism of ringed topoi over k. Then the dg functors

$$\underline{f}^* := \mathbf{i} f^* \mathbf{e} \colon D^{dg}(Y) \to D^{dg}(X),$$
$$\underline{f}_* := \mathbf{i} f_* \colon D^{dg}(X) \to D^{dg}(Y)$$

make the diagrams

(5.2)

$$\begin{array}{ccc}
[D^{dg}(Y)] & \xrightarrow{[\underline{f}^*]} & [D^{dg}(X)] \\
\downarrow{\sim} & & \downarrow{\sim} \\
D(Y) & \xrightarrow{Lf^*} & D(X),
\end{array} \qquad
\begin{array}{ccc}
[D^{dg}(X)] & \xrightarrow{[\underline{f}_*]} & [D^{dg}(Y)] \\
\downarrow{\sim} & & \downarrow{\sim} \\
D(X) & \xrightarrow{Rf_*} & D(Y)
\end{array}$$

commutative up to isomorphisms of triangulated functors, by [Sch15, Proposition 6.5]. The vertical arrows in these diagrams are given by the functor (5.1). Here we abbreviate $D(X) = D(X, \mathcal{O})$ and $D^{dg}(X) = D^{dg}(X, \mathcal{O})$ to ease the notation, and similarly for Y.

Similarly, if $E \in \mathrm{D}(X)$ is any object, the dg functor

$$(E \underline{\otimes} -) := \mathbf{i}(E \otimes \mathbf{e}(-)) \colon \mathrm{D}^{\mathrm{dg}}(X) \to \mathrm{D}^{\mathrm{dg}}(X)$$

makes the diagram

$$(5.3) \qquad \begin{array}{ccc} [\mathrm{D}^{\mathrm{dg}}(X)] & \xrightarrow{\ [E \underline{\otimes} -]\ } & [\mathrm{D}^{\mathrm{dg}}(X)] \\ {\scriptstyle \sim}\downarrow & & \downarrow{\scriptstyle \sim} \\ \mathrm{D}(X) & \xrightarrow{\ E \otimes^{\mathsf{L}} -\ } & \mathrm{D}(X) \end{array}$$

commutative up to an isomorphism of triangulated functors; this follows immediately from [Sch15, Section 6.3].

These three diagrams say that the dg functors $\underline{f}^*$, $\underline{f}_*$ and $(E \underline{\otimes} -)$ lift the triangulated functors $\mathsf{L}f^*$, $\mathsf{R}f_*$ and $(E \otimes^{\mathsf{L}} -)$ to dg enhancements.

Remark 5.10. In the above situation assume that $\mathrm{D}_\diamond(X)$ and $\mathrm{D}_\diamond(Y)$ are strict triangulated subcategories of $\mathrm{D}(X)$ and $\mathrm{D}(Y)$, respectively. By Remark 5.4, these subcategories have induced dg enhancements $\mathrm{D}_\diamond^{\mathrm{dg}}(X)$ and $\mathrm{D}_\diamond^{\mathrm{dg}}(Y)$. If $\mathsf{L}f^* \colon \mathrm{D}(Y) \to \mathrm{D}(X)$ maps $\mathrm{D}_\diamond(Y)$ to $\mathrm{D}_\diamond(X)$, then $\underline{f}^* \colon \mathrm{D}^{\mathrm{dg}}(Y) \to \mathrm{D}^{\mathrm{dg}}(X)$ maps $\mathrm{D}_\diamond^{\mathrm{dg}}(Y)$ to $\mathrm{D}_\diamond^{\mathrm{dg}}(X)$, and the induced dg functor $\underline{f}^* \colon \mathrm{D}_\diamond^{\mathrm{dg}}(Y) \to \mathrm{D}_\diamond^{\mathrm{dg}}(X)$ lifts the induced triangulated functor $\mathsf{L}f^* \colon \mathrm{D}_\diamond(Y) \to \mathrm{D}_\diamond(X)$: diagram (5.2) restricts to

$$(5.4) \qquad \begin{array}{ccc} [\mathrm{D}_\diamond^{\mathrm{dg}}(Y)] & \xrightarrow{\ [\underline{f}^*]\ } & [\mathrm{D}_\diamond^{\mathrm{dg}}(X)] \\ {\scriptstyle \sim}\downarrow & & \downarrow{\scriptstyle \sim} \\ \mathrm{D}_\diamond(Y) & \xrightarrow{\ \mathsf{L}f^*\ } & \mathrm{D}_\diamond(X). \end{array}$$

Similar remarks apply to the functors $\mathsf{R}f_*$ and $(E \otimes^{\mathsf{L}} -)$.

Example 5.11. Let $f \colon X \to Y$ be a concentrated morphism of Deligne–Mumford stacks over the field k. As stated in Remark 2.11, we get an induced morphism of ringed topoi. Hence Remark 5.10 applies and the functors $\mathsf{L}f^*$ and $\mathsf{R}f_*$ lift to dg functors $\underline{f}^* \colon \mathrm{D}_{\mathrm{qc}}^{\mathrm{dg}}(Y_{\mathrm{ét}}) \to \mathrm{D}_{\mathrm{qc}}^{\mathrm{dg}}(X_{\mathrm{ét}})$ and $\underline{f}_* \colon \mathrm{D}_{\mathrm{qc}}^{\mathrm{dg}}(X_{\mathrm{ét}}) \to \mathrm{D}_{\mathrm{qc}}^{\mathrm{dg}}(Y_{\mathrm{ét}})$ between the dg enhancements of Example 5.6. If $E \in \mathrm{D}_{\mathrm{qc}}(X_{\mathrm{ét}})$ then $(E \otimes^{\mathsf{L}} -) \colon \mathrm{D}_{\mathrm{qc}}(X_{\mathrm{ét}}) \to \mathrm{D}_{\mathrm{qc}}(X_{\mathrm{ét}})$ lifts to a dg functor $(E \underline{\otimes} -) \colon \mathrm{D}_{\mathrm{qc}}^{\mathrm{dg}}(X_{\mathrm{ét}}) \to \mathrm{D}_{\mathrm{qc}}^{\mathrm{dg}}(X_{\mathrm{ét}})$.

Example 5.12. Let $f \colon X \to Y$ be a concentrated morphism of arbitrary algebraic stacks over the field k. Then the functors

$$(5.5) \qquad\qquad \mathsf{L}f^* \colon \mathrm{D}_{\mathrm{qc}}(Y) \rightleftarrows \mathrm{D}_{\mathrm{qc}}(X) \colon \mathsf{R}f_*$$

lift to dg functors

$$(5.6) \qquad\qquad \underline{f}^* \colon \mathrm{D}_{\mathrm{qc}}^{\mathrm{dg}}(Y) \rightleftarrows \mathrm{D}_{\mathrm{qc}}^{\mathrm{dg}}(X) \colon \underline{f}_*$$

between the dg enhancements of Example 5.5 as we explain below. Given a complex $E \in \mathrm{D}_{\mathrm{qc}}(X)$, we also get a lift of the triangulated functor $(E \otimes^{\mathsf{L}} -) \colon \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc}}(X)$ to a dg functor $(E \underline{\otimes} -) \colon \mathrm{D}_{\mathrm{qc}}^{\mathrm{dg}}(X) \to \mathrm{D}_{\mathrm{qc}}^{\mathrm{dg}}(X)$. Moreover, if the functors $\mathsf{L}f^*$, $\mathsf{R}f_*$ and $(E \otimes^{\mathsf{L}} -)$ restrict to the triangulated categories $\mathrm{D}_{\mathrm{pf}}(X)$ and $\mathrm{D}_{\mathrm{pf}}(Y)$, then the lifts $\underline{f}^*$, $\underline{f}_*$, $(E \underline{\otimes} -)$ restrict to the dg categories $\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(X)$ and $\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(Y)$.

Due to the fact that f does not induce a morphism between the lisse-étale topoi, this situation is more complicated than the situation in Example 5.11. To circumvent the problem one can use the technique of cohomological descent from [Ols07] and [LO08]. Choose smooth hyper-coverings $\pi_X \colon X_\bullet \to X$ and $\pi_Y \colon Y_\bullet \to Y$ together with a morphism $X_\bullet \to Y_\bullet$ over $f \colon X \to Y$. Passing to the associated strictly simplicial algebraic spaces we obtain a morphism $\widetilde{f} \colon X_\bullet^+ \to Y_\bullet^+$ augmenting f. This gives us a diagram

$$
\begin{array}{ccccc}
(X_{\text{lis-ét}}, \mathcal{O}_X) & \xleftarrow{\ \pi_X\ } & (X_{\bullet,\text{lis-ét}}^+, \mathcal{O}_X) & \xrightarrow{\ \varepsilon_X\ } & (X_{\bullet,\text{ét}}^+, \mathcal{O}_X) \\
 & & & & \Big\downarrow{\widetilde{f}} \\
(Y_{\text{lis-ét}}, \mathcal{O}_Y) & \xleftarrow{\ \pi_Y\ } & (Y_{\bullet,\text{lis-ét}}^+, \mathcal{O}_Y) & \xrightarrow{\ \varepsilon_Y\ } & (Y_{\bullet,\text{ét}}^+, \mathcal{O}_Y)
\end{array}
$$

of ringed topoi, where ε_X and ε_Y are restriction morphism similar to the morphisms (2.1) from Remark 2.9. The dg functor $\underline{f}^* \colon \mathrm{D}^{\mathrm{dg}}(Y) \to \mathrm{D}^{\mathrm{dg}}(X)$ is defined as the composition

$$(\underline{\pi}_X)_* \underline{\varepsilon}_X^* \underline{\widetilde{f}}^* (\underline{\varepsilon}_Y)_* \underline{\pi}_Y^*$$

and similarly for $\underline{f}_*$. Again using Remark 5.10 we see that the restrictions of $\underline{f}^*$ and $\underline{f}_*$ to $\mathrm{D}_{\mathrm{qc}}^{\mathrm{dg}}(X)$ and $\mathrm{D}_{\mathrm{qc}}^{\mathrm{dg}}(Y)$ give the lifts (5.6) of the triangulated functors (5.5) (cf. [LO08, Example 2.2.5], [HR14, Section 1]).

Remark 5.13. As we have seen in Example 5.12, the triangulated functors $\mathsf{L}f^*$, $\mathsf{R}f_*$ and $(E \otimes^{\mathsf{L}} -)$ lift to dg functors $\underline{f}^*$, $\underline{f}_*$ and $(E\underline{\otimes}-)$ when working over a field. Over an arbitrary base ring R, dg R-linear (or even additive) replacement functors similar to **e** and **i** need not exist (see [Sch15, Lemma 4.4]). However, it is presumably possible to define morphisms in the homotopy category of R-linear dg categories (where the quasi-equivalences are inverted) which lift these functors when considered as morphisms in the homotopy category of triangulated categories (where equivalences are inverted).

Geometric dg categories. After recalling some standard notions for dg categories we discuss geometric dg categories. Then we state Orlov's gluing result as Theorem 5.22. We keep the assumption that k is a field.

Definition 5.14 (cf. [TV07, Definition 2.4], [Toë09, Definition 2.3], [LS13, Sections 2.2, 2.5]). Let $\mathcal{A}$ be a k-linear dg category.

(a) $\mathcal{A}$ is *triangulated* if it is pretriangulated and the triangulated category $[\mathcal{A}]$ is idempotent complete.

(b) $\mathcal{A}$ is *locally cohomologically bounded* if $\mathcal{A}(A, B)$ is cohomologically bounded for all $A, B \in \mathcal{A}$.

(c) $\mathcal{A}$ is *locally perfect* if $\mathcal{A}(A, B)$ is a perfect complex of k-vector spaces, for all $A, B \in \mathcal{A}$. That is, all complexes $\mathcal{A}(A, B)$ have bounded and finite dimensional cohomology.

(d) $\mathcal{A}$ *has a compact generator* if its derived category $\mathrm{D}(\mathcal{A})$ of dg $\mathcal{A}$-modules has a compact generator.

(e) $\mathcal{A}$ is *proper* if it is locally perfect and has a compact generator.

(f) $\mathcal{A}$ is *smooth* if $\mathcal{A}$ is compact as an object of the derived category of dg $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$-modules.

(g) $\mathcal{A}$ is *saturated* if it is triangulated, smooth and proper.

We say that two dg categories are quasi-equivalent if they are connected by a zig-zag of quasi-equivalences. The above notions are all well-defined on quasi-equivalence classes of dg categories (cf. [LS13, Lemma 2.12]). In fact, properties (b)–(f) are well-defined on Morita equivalence classes of dg categories (cf. [LS13, Lemma 2.13]).

Remark 5.15. Orlov's definition of a *(derived) non-commutative scheme* ([Orl14, Definition 3.3]) can be reformulated using the terms above. A non-commutative scheme is precisely a locally cohomologically bounded, triangulated dg category with a compact generator.

Indeed, a triangulated dg category $\mathcal{A}$ has a compact generator if and only if it is quasi-equivalent to a dg category of perfect dg A-modules for some dg algebra A ([LS13, Definition 2.2, Lemma 2.3, Corollary 2.4, Proposition 2.16]). In this case, $\mathcal{A}$ is locally cohomologically bounded if and only if A is (locally) cohomologically bounded (the property of being "locally cohomologically bounded" can be added to the list in [LS13, Lemma 2.13].

Definition 5.16 (cf. [Orl14, Definition 4.3]). A dg category $\mathcal{A}$ is *geometric* if there exists a smooth projective scheme X over k and an admissible subcategory $\mathcal{S}$ of $\mathrm{D}_{\mathrm{pf}}(X)$ such that $\mathcal{A}$ and the full dg subcategory of $\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(X)$ consisting of objects of $\mathcal{S}$ are quasi-equivalent.

Remark 5.17. Geometric dg categories are saturated. Indeed, if X is any scheme, then $\mathrm{D}_{\mathrm{pf}}(X)$ is idempotent complete [SP16, Tag 08GA], and so is any admissible subcategory. This shows that a geometric dg category is triangulated. If X is smooth and proper over the field k, then $\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(X)$ is smooth and proper, by [LS14, Theorem 1.2 and 1.4] or [Orl14, Proposition 3.31]. Moreover, smoothness and properness are inherited to dg subcategories of $\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(X)$ enhancing admissible subcategories of $\mathrm{D}_{\mathrm{pf}}(X)$, by [LS13, Proposition 2.20].

This shows, together with Remark 5.15, that our geometric dg categories coincide with Orlov's geometric noncommutative schemes as defined in [Orl14, Definition 4.3].

Example 5.18. Not all geometric dg categories are of the form $\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(X)$ for some smooth projective variety X over k. Let $\Lambda = \mathsf{k}[\bullet \to \bullet]$ be the path algebra of a Dynkin quiver of type A_2. Then the standard enhancement of the bounded derived category $\mathrm{D}^{\mathrm{b}}(\Lambda)$ is geometric by [Orl14, Corollary 5.4].

On the other hand, the third power of the Serre functor on $\mathrm{D}^{\mathrm{b}}(\Lambda)$ is isomorphic to the shift [1] (cf. [HI11, Proposition 3.1]). Hence $\mathrm{D}^{\mathrm{b}}(\Lambda)$ cannot be equivalent to $\mathrm{D}_{\mathrm{pf}}(X)$ for any smooth projective variety X over k.

Lemma 5.19. *Let $\mathcal{B}$ be a dg subcategory of a geometric dg category $\mathcal{A}$ such that $[\mathcal{B}]$ is an admissible subcategory of the triangulated category $[\mathcal{A}]$. Then $\mathcal{B}$ is geometric.*

We will see in Corollary 5.21 below that it is enough to assume that $[\mathcal{B}]$ is right or left admissible in $[\mathcal{A}]$.

Proof. Let X be a smooth projective scheme and $\mathcal{E}$ a dg subcategory of $\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(X)$ such that $[\mathcal{E}]$ is an admissible subcategory of $[\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(X)]$ and there is a zig-zag of quasi-equivalences connecting $\mathcal{A}$ and $\mathcal{E}$. Transfering $\mathcal{B}$ along such a zig-zag yields a zig-zag of quasi-equivalences connecting $\mathcal{B}$ with a dg subcategory $\mathcal{F}$ of $\mathcal{E}$ such that

$[\mathcal{F}]$ is an admissible subcategory of $[\mathcal{E}]$. But then $[\mathcal{F}]$ is also admissible in $[\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(X)]$. Hence $\mathcal{B}$ is geometric. $\qquad\square$

Proposition 5.20. *Let $\mathcal{B}$ be a dg subcategory of a saturated dg category $\mathcal{A}$ such that $[\mathcal{B}]$ is a right (resp. left) admissible subcategory of the triangulated category $[\mathcal{A}]$. Then $[\mathcal{B}]$ is admissible in $[\mathcal{A}]$ and $\mathcal{B}$ is saturated.*

Proof. We have a semiorthogonal decomposition $[\mathcal{A}] = \langle [\mathcal{B}]^{\perp}, [\mathcal{B}] \rangle$ (resp. $[\mathcal{A}] = \langle [\mathcal{B}], {}^{\perp}[\mathcal{B}] \rangle$). So our claim follows from the proof of [LS13, Proposition 2.26]. $\qquad\square$

Corollary 5.21. *Let $\mathcal{B}$ be a dg subcategory of a geometric dg category $\mathcal{A}$ such that $[\mathcal{B}]$ is a right (resp. left) admissible subcategory of the triangulated category $[\mathcal{A}]$. Then $[\mathcal{B}]$ is admissible in $[\mathcal{A}]$ and $\mathcal{B}$ is geometric.*

Proof. Since $\mathcal{A}$ is saturated, by Remark 5.17, this follows from Proposition 5.20 and Lemma 5.19. $\qquad\square$

We reformulate Orlov's result [Orl14, Theorem 4.15] that the gluing of geometric dg categories is again geometric; for completeness we add the implication in the other direction.

Theorem 5.22 (cf. [Orl14, Theorem 4.15]). *Let $\mathcal{A}$ be a pretriangulated dg category with full dg subcategories $\mathcal{B}_1, \ldots, \mathcal{B}_n$ such that $[\mathcal{A}] = \langle [\mathcal{B}_1], \ldots, [\mathcal{B}_n] \rangle$ is a semiorthogonal decomposition. Then $\mathcal{A}$ is geometric if and only if $\mathcal{A}$ is locally perfect and all dg categories $\mathcal{B}_1, \ldots, \mathcal{B}_n$ are geometric.*
Moreover, if these conditions are satisfied then all $[\mathcal{B}_i]$ are admissible in $[\mathcal{A}]$.

Proof. If $\mathcal{A}$ is locally perfect and all dg categories $\mathcal{B}_1, \ldots \mathcal{B}_n$ are geometric then $\mathcal{A}$ is geometric by [Orl14, Theorem 4.15]. (The "proper" dg categories in [Orl14] are usually called locally perfect, cf. [Orl14, Remark 3.15]).

Conversely assume that $\mathcal{A}$ is geometric. Then $\mathcal{A}$ is saturated by Remark 5.17 and in particular locally perfect. Corollary 5.21 and an easy induction (using [LS12, Lemma A.11]) shows that all $\mathcal{B}_i$ are geometric and that all $[\mathcal{B}_i]$ are admissible in $[\mathcal{A}]$. $\qquad\square$

6. Geometricity for dg enhancements of algebraic stacks

In this section, we combine Orlov's result on gluing of geometric dg categories with a geometric argument to obtain the results about geometricity for dg enhancements of algebraic stacks stated in the introduction. The geometric argument depends on the existence of *destackifications* in the sense of [Ber14, BR15]. We start by briefly recalling this notion.

In this section, we will mostly work with tame stacks which are separated and of finite type over a field k. Such a stack will be called an *orbifold* provided that it is smooth over k and contains an open dense substack which is an algebraic space.

Let X be an orbifold over k. Although the stack X is smooth, the same need not hold for its coarse space X_{cs}. However, it is possible to modify the stack via a sequence of birational modifications such that the coarse space of the modified stack becomes smooth. It suffices to use two kinds of modifications: blowups in smooth centers and root stacks in smooth divisors. Collectively, we refer to such modifications as *smooth stacky blowups*.

Theorem 6.1 (Destackification). *Let X be a tame, separated algebraic stack which is smooth and of finite type over a field k. Assume that X contains an open dense substack which is an algebraic space. Then there exists a morphism $f\colon Y \to X$, which is a composition of smooth stacky blowups, such that Y_{cs} is smooth over k and such that $Y \to Y_{\mathrm{cs}}$ is an iterated root construction in an snc divisor E on Y_{cs}.*

Proof. The case where X has abelian stabilizers is treated in [Ber14, Theorem 1.2]. In the discussion before [Ber14, Corollary 1.4] it is shown how the abelian hypothesis can be removed if k has characteristic zero. For k of arbitrary characteristic, the theorem is shown in [BR15]. □

Proposition 6.2. *Let $f\colon X \to Y$ be a concentrated morphism of algebraic stacks over a field k such that the natural morphism $\mathcal{O}_Y \to \mathsf{R}f_*\mathcal{O}_X$ is an isomorphism and $\mathsf{R}f_*$ preserves perfect complexes. If the k-linear dg category $\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(X)$ is geometric then so is $\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(Y)$.*

Proof. Lemma 4.4 shows that $\mathsf{L}f^*\colon \mathrm{D}_{\mathrm{pf}}(Y) \to \mathrm{D}_{\mathrm{pf}}(X)$ is full and faithful. Example 5.12 and Remark 5.10 provide the dg functor $\underline{f}^*\colon \mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(Y) \to \mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(X)$ lifting $\mathsf{L}f^*$ to dg enhancements. It defines a quasi-equivalence from $\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(Y)$ to $\mathcal{E}$ where $(\mathcal{E}, \varepsilon)$ is the induced dg enhancement of the essential image of $\mathsf{L}f^*\colon \mathrm{D}_{\mathrm{pf}}(Y) \to \mathrm{D}_{\mathrm{pf}}(X)$ (cf. [LS13, Lemma 2.5]). By assumption, this essential image is a right admissible subcategory. Therefore, if $\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(X)$ is geometric, so are $\mathcal{E}$ and $\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(Y)$; either use Corollary 5.21, or the easier Lemma 5.19 together with the fact that the essential image of $\mathsf{L}f^*$ is left admissible, by Lemma 4.3. □

Proposition 6.3. *Let X be a smooth projective scheme over a field k. Assume that E is an snc divisor on X and that $r > 0$ is a multi-index (with respect to the indexing set of E). Then the k-linear dg category $\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(X_{r^{-1}E})$ associated to the root stack $X_{r^{-1}E}$ is geometric.*

Proof. The root stack $X_{r^{-1}E}$ is tame (Proposition 3.3.(e)) and proper (Proposition 3.3.(b)) over k. This implies that the cohomology of coherent sheaves is bounded ([HR15, Theorem 2.1]) and coherent ([Fal03, Theorem 1]). Therefore, the dg category $\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(X_{r^{-1}E})$ is locally perfect.

Recall the semiorthogonal decomposition for iterated root constructions from Theorem 4.9 and observe that the functors (4.8) involved in this decomposition lift to dg functors

$$(6.1) \qquad \mathcal{O}_{X_{r^{-1}E}}(ar^{-1}E)\underline{\otimes}(\underline{\iota_a})_*(\underline{\rho_a})^*(-)\colon \mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(E(I_a)) \to \mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(X_{r^{-1}E})$$

between dg enhancements, by Example 5.12, Lemma 4.1 and Remark 5.10. Since all intersections $E(I_a)$ are smooth, projective schemes over k, all dg categories $\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(E(I_a))$ are geometric. The claim now follows from Orlov's glueing Theorem 5.22. □

We are now ready to prove our first result on geometricity for dg enhancements of algebraic stacks.

Theorem 6.4. *Let X be a tame, smooth, projective algebraic stack over an arbitrary field k. Then the k-linear dg category $\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(X)$ is geometric, and in particular saturated.*

Proof. Since X is a global quotient stack, there is a projectivized vector bundle $P \to X$ such that P contains an open dense substack which is an algebraic space (cf. [KV04, Proof of Theorem 1]). Explicitly, we can construct such a bundle as follows. Let $T \to X$ be a GL_n-torsor where T is an algebraic space. Consider the corresponding vector bundle $\mathcal{E}$ of rank n on X. Then we have dense open immersions $T \hookrightarrow V \hookrightarrow P$, where $V = \mathbb{V}(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))$ and $P = \mathbb{P}(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \oplus \mathcal{O}_X)$. The stack P is tame since $P \to X$ is representable. Since also T is representable, it follows that P is an orbifold.

Now we apply Theorem 6.1 and get a proper birational morphism $Y \to P$ which is a composition of smooth stacky blowups such that Y and Y_{cs} are smooth and the canonical map $Y \to Y_{\mathrm{cs}}$ is an iterated root construction in an snc divisor E on Y_{cs}.

Note that Y is a projective algebraic stack. Indeed, the map $Y \to P$ is a composition of root stacks and blowups and $P \to X$ is projective, so this follows from Lemma 3.4 and Lemma 2.8. In particular, Y_{cs} is a smooth projective scheme. Hence Proposition 6.3 shows that $\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(Y)$ is geometric.

Denote the composition $Y \to P \to X$ by π. Since π is a composition of root stacks, blow-ups and the structure morphism of a projective bundle, the canonical morphism $\mathcal{O}_X \to \mathsf{R}\pi_*\mathcal{O}_Y$ is an isomorphism, and $\mathsf{R}\pi_*$ preserves perfect complexes by part (a), (c), (e) of Example 4.6. In particular, the morphism π satisfies the assumptions of Proposition 6.2. Therefore $\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(X)$ is geometric since the same holds for $\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(Y)$. □

If we work over a field k which admits resolution of singularities, we have the following version of Chow's Lemma.

Proposition 6.5 (Chow's Lemma)**.** *Let X be a separated Deligne–Mumford stack which is smooth and of finite type over a field k of characteristic zero. Then there exists a morphism $\pi\colon Y \to X$ which is a composition of (non-stacky) blowups in smooth centers such that Y is a quasi-projective algebraic stack.*

Proof. By [Cho12, Theorem 4.3], which is attributed to Rydh, we can find a sequence of (non-stacky) blowups $Y \to \cdots \to X$ in smooth centers such that Y_{cs} is quasi-projective. By [Kre09, 4.4] any smooth Deligne–Mumford stack of finite type over a field is automatically a global quotient if its coarse space is quasi-projective. In particular, the stack Y is quasi-projective. □

In particular, over a field of characteristic zero, we can replace the projectivity assumption from Theorem 6.4 by a properness assumption.

Theorem 6.6. *Let X be a smooth, proper Deligne–Mumford stack over a field k of characteristc zero. Then the k-linear dg category $\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(X)$ is geometric, and in particular saturated.*

Proof. By Proposition 6.5, there is a composition $\pi\colon Y \to X$ of blow-ups in smooth centers such that Y is a projective algebraic stack. Hence $\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(Y)$ is geometric by Theorem 6.4. By arguing as in the final paragraph of the proof of the same theorem, we see that also $\mathrm{D}_{\mathrm{pf}}^{\mathrm{dg}}(X)$ is geometric. □

Appendix A. Bounded derived category of coherent modules

Our aim is to show that the bounded derived category of coherent modules on a regular, quasi-compact, separated algebraic stack with finite stabilizers is equivalent to the derived category of perfect complexes (see Remark A.3).

The category of coherent $\mathcal{O}_X$-modules on a locally noetherian algebraic stack X is denoted by $\mathrm{Coh}(X)$. We use the usual decorations for full subcategories of derived categories. For example, the symbol $\mathrm{D}_{\mathrm{Coh}}^{-}(\mathrm{Qcoh}(X))$ denotes the full subcategory of the derived category $\mathrm{D}(\mathrm{Qcoh}(X))$ of quasi-coherent modules whose objects have bounded above coherent cohomology modules.

The following proposition generalizes a well-known result for noetherian schemes [SGA6, Exposé II, Proposition 2.2.2], [Huy06, Proposition 3.5] to noetherian algebraic stacks.

Proposition A.1. *Let X be an noetherian algebraic stack. Then the obvious functor defines an equivalence*

$$\mathrm{D}^{-}(\mathrm{Coh}(X)) \xrightarrow{\sim} \mathrm{D}_{\mathrm{Coh}}^{-}(\mathrm{Qcoh}(X)).$$

Proof. It is certainly enough to show that each bounded above complex of quasi-coherent modules with coherent cohomology modules has a quasi-isomorphic subcomplex of coherent modules. This is an easy consequence of the proof of [Huy06, Proposition 3.5] as soon as we know the following fact: given any epimorphism $G \to F$ from a quasi-coherent module G to a coherent module F, there is a coherent submodule G' of G such that the composition $G' \subset G \to F$ is still an epimorphism. This latter statement follows from the fact that every quasi-coherent module is the filtered colimit of its coherent submodules [LMB00, Proposition 15.4] and [SGA6, Exposé II, Lemma 2.1.1.a)]. $\square$

Proposition A.2. *Let X be a regular and quasi-compact algebraic stack. Then we have an equality $\mathrm{D}_{\mathrm{pf}}(X) = \mathrm{D}_{\mathrm{Coh}}^{\mathrm{b}}(X)$.*

Proof. Since X is quasi-compact we have $\mathrm{D}_{\mathrm{pf}}(X) \subset \mathrm{D}_{\mathrm{Coh}}^{\mathrm{b}}(X)$. In order to show equality it is enough to prove that any coherent module is perfect. Let $\mathrm{Spec}\, A \to X$ be any smooth morphism where A is a ring. Then A is regular. It is enough to prove that any finitely generated A-module M has a finite resolution by finitely generated projective A-modules. Let $P \to M$ be a resolution by finitely generated projective A-modules. Let $\mathfrak{p} \in \mathrm{Spec}\, A$. Since $A_{\mathfrak{p}}$ is regular, it has finite global dimension by the Auslander–Buchsbaum–Serre theorem. Therefore, there is a natural number $n = n(\mathfrak{p})$ such that the kernel of the differential $d^{-n} \colon (P^{-n})_{\mathfrak{p}} \to (P^{-n+1})_{\mathfrak{p}}$ is a finitely generated projective $A_{\mathfrak{p}}$-module. Since A is noetherian, there is some open neighborhood $\mathrm{Spec}\, A_f$ of $\mathfrak{p}$ in $\mathrm{Spec}\, A$ such that the kernel of $d^{-n} \colon (P^{-n})_f \to (P^{-n+1})_f$ is a finitely generated projective A_f-module. Then also all kernels $d^{-i} \colon (P^{-i})_f \to (P^{-i+1})_f$, for $i \geq n$, are finitely generated projective A_f-modules. Since $\mathrm{Spec}\, A$ is quasi-compact there is a natural number N such that the kernel of $d^{-N} \colon P^{-N} \to P^{-N+1}$ is a finitely generated projective A-module. $\square$

Remark A.3. If X is a noetherian, separated algebraic stack with finite stabilizers we have equivalences

$$\mathrm{D}^{-}(\mathrm{Coh}(X)) \xrightarrow{\sim} \mathrm{D}_{\mathrm{Coh}}^{-}(\mathrm{Qcoh}(X)) \xrightarrow{\sim} \mathrm{D}_{\mathrm{Coh}}^{-}(X).$$

This follows immediately from Proposition A.1 and the equivalence $D(\mathrm{Qcoh}(X)) \xrightarrow{\sim} D_{\mathrm{qc}}(X)$ from (2.2). If we assume in addition that X is regular then Proposition A.2 together with the above equivalences shows that

$$D^{\mathrm{b}}(\mathrm{Coh}(X)) \xrightarrow{\sim} D^{\mathrm{b}}_{\mathrm{Coh}}(X) = D_{\mathrm{pf}}(X)$$

is an equivalence.

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