ON A GENERALIZATION OF BEITER CONJECTURE

BARTŁOMIEJ BZDĘGA

Abstract. We prove that for every $\varepsilon > 0$ and a nonnegative integer ω there exist primes $p_1, p_2, \ldots, p_\omega$ such that for $n = p_1 p_2 \ldots p_\omega$ the height of the cyclotomic polynomial Φ_n is at least $(1-\varepsilon)c_\omega M_n$, where $M_n = \prod_{i=1}^{\omega-2} p_i^{2^{\omega-1-i}-1}$ and c_ω is a constant depending only on ω; furthermore $\lim_{\omega\to\infty} c_\omega^{2^{-\omega}} \approx 0.71$. In our construction we can have $p_i > h(p_1 p_2 \ldots p_{i-1})$ for all $i = 1, 2, \ldots, \omega$ and any function $h : \mathbb{R}_+ \to \mathbb{R}_+$.

1. Introduction

Let Φ_n be the nth cyclotomic polynomial, i.e. the unique monic polynomial irreducible over integers, which roots are all primitive nth roots of unity. We assume that $n = p_1 p_2 \ldots p_\omega$ and $2 < p_1 < p_2 < \ldots < p_\omega$ are primes, since $\Phi_{2n}(x) = \Phi_n(-x)$ for odd n and $\Phi_{np}(x) = \Phi_n(x^p)$ for a prime p dividing n. We call the number $\omega = \omega(n)$ the order of Φ_n.

Let A_n denotes the maximal absolute value of a coefficient of Φ_n. We say shortly that A_n is the height of Φ_n. In case of $\omega \in \{0, 1, 2\}$ determining of A_n is easy and we have $A_1 = A_{p_1} = A_{p_1 p_2} = 1$. For $\omega = 3$ it is known that $A_{p_1 p_2 p_3} \le \frac{3}{4} p_1$ [1]. The Corrected Beiter Conjecture states that $A_{p_1 p_2 p_3} \le \frac{2}{3} p_1$ (see [4] and references given there for details). The constant $\frac{2}{3}$ is best possible if the conjecture is true.

For cyclotomic polynomials of any order we put

$$M_n = \prod_{i=1}^{\omega-2} p_i^{2^{\omega-1-i}-1},$$

where the empty product, which happens if $\omega \le 2$, equals 1. P.T. Bateman, C. Pomerance and R.C. Vaughan proved in [2] that $A_n \le M_n$. In [3] the author proved that $A_n \le C_\omega M_n$, where $C_\omega^{2^{-\omega}}$ converges to approximately 0.95 with $\omega \to \infty$. However, so far we have known no good general class of Φ_n for which A_n is close to $C_\omega M_n$.

It has not been even known if M_n gives the optimal order for the upper bound on A_n. For example we have $A_{p_1 \ldots p_5} \le C_5 p_1^7 p_2^3 p_3$, but we did not know whether $A_{p_1 \ldots p_5} \le C_5' p_1^8 p_2^2 p_3$ for some other constant C_5'. All known constructions of Φ_n with large height required that most prime factors of n are of almost the same size.

1991 *Mathematics Subject Classification.* 11B83, 11C08.
Key words and phrases. cyclotomic polynomial, coefficients, height of a polynomial.

One of the main purposes of this paper is to show that M_n is optimal, i.e. in the upper bound on A_n it cannot be replaced by any smaller product of the form $p_1^{\alpha_1}p_2^{\alpha_2}\dots p_\omega^{\alpha_\omega}$ in a sense which we describe below.

For a fixed ω we define the following strict lexicographical order on $\mathbb{R}^\omega$:

$$\begin{aligned}&(\alpha_1,\alpha_2,\dots,\alpha_\omega)\prec(\beta_1,\beta_2,\dots,\beta_\omega)\\ \iff&\alpha_\omega=\beta_\omega,\alpha_{\omega-1}=\beta_{\omega-1},\dots,\alpha_{k+1}=\beta_{k+1}\text{ and }\alpha_k<\beta_k\text{ for some }k\le\omega.\end{aligned}$$

For $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_\omega)$ and $n=p_1p_2\dots p_\omega$ we put $M_n^{(\alpha)}=p_1^{\alpha_1}p_2^{\alpha_2}\dots p_\omega^{\alpha_\omega}$. Note that if $\alpha\prec\beta$ and p_i is large enough compared to $p_1p_2\dots p_{i-1}$ for all $i\le\omega$, then $M_n^{(\alpha)}<M_n^{(\beta)}$.

Therefore, we say that $M_n^{(\alpha)}$ is the optimal bound on A_n for a fixed ω if there exists a constant b_ω such that $A_n\le b_\omega M_n^{(\alpha)}$ for all n with $\omega(n)=\omega$ and α is smallest possible in sense of the order $\prec$.

It requires an explanation what it means that p_i is large enough compared to $p_1p_2\dots p_{i-1}$ for all $i\le\omega$. Let $h:\mathbb{R}_+\to\mathbb{R}_+$ be any function, preferably growing fast. We say that a sequence of primes $p_1,p_2,\dots,p_\omega$ is *h-growing* if $p_i\ge h(p_1p_2\dots p_{i-1})$ for $i=1,2,\dots,\omega$ (empty product equals 1). With a small abuse of notation we will also write that the number $n=p_1p_2\dots p_\omega$ is h-growing.

The following theorem is the main result of this paper.

Theorem 1. *For every $\omega\ge3$, $\varepsilon>0$ and $h:\mathbb{R}_+\to\mathbb{R}_+$ there exists an h-growing $n=p_1p_2\dots p_\omega$ such that $A_n>(1-\varepsilon)c_\omega M_n$, where*

$$M_n=\prod_{i=1}^{\omega-2}p_i^{2^{\omega-1-i}-1}\quad\textit{and}\quad c_\omega=\frac{1}{\omega}\cdot\left(\frac{2}{\pi}\right)^{3\cdot2^{\omega-3}}\cdot\left(\prod_{k=3}^{\omega-1}k^{2^{\omega-1-k}}\right)^{-1}.$$

By this theorem and the already mentioned result from [3], M_n is the optimal bound on A_n. Furthermore

$$\lim_{\omega\to\infty}c_\omega^{2^{-\omega}}=\left(\frac{2}{\pi}\right)^{3/8}\cdot\prod_{k=3}^{\infty}k^{-2^{-k-1}}\approx0.71.$$

Let us define the ωth Beiter constant in the following natural way:

$$B_\omega=\limsup_{\omega(n)=\omega}(A_n/M_n).$$

For example we know that $B_0=B_1=B_2=1$ and $\frac{2}{3}\le B_3\le\frac{3}{4}$. If Corrected Beiter Conjecture is true, then $B_3=\frac{2}{3}$.

For all ω we have

$$c+o(1)<B_\omega^{2^{-\omega}}<C+o(1),\quad\omega\to\infty$$

with $c\approx0.71$ and $C\approx0.95$. It would be interesting to know the asymptotics of B_ω. For example, we expect that the following natural conjecture is true.

Conjecture 2. *There exists a limit $\lim_{\omega\to\infty}B_\omega^{2^{-\omega}}$.*

2. Preliminaries and binary case

Let us define the value

$$L_n = \max_{|z|=1} |\Phi_n(z)|.$$

It was already considered by several authors [2, 5, 6] while estimating A_n. If S_n denotes the sum of absolute values of the coefficients of Φ_n, then for $n > 1$

$$A_n \geq \frac{S_n}{\deg \Phi_n + 1} \geq \frac{L_n}{n}.$$

We express $|\Phi_n(z)|$ as a real function of $x = \arg(z)$ for $|z| = 1$. For all $n \geq 1$ let

$$F_n(x) = \prod_{d|n} \left(\sin \frac{d}{2} x \right)^{\mu(n/d)},$$

where we put $\frac{\sin ax}{\sin bx} = \frac{a}{b}$ for $\sin bx = \sin ax = 0$. Note that F_n is periodic with the period 2π. By the following lemma $F_n(x)$ is well defined for all $x \in \mathbb{R}$.

Lemma 3. *For $n > 1$ we have $|\Phi_n(e^{ix})| = |F_n(x)|$.*

Proof. By elementary computations $|1 - z| = 2 \left|\sin \frac{1}{2} x\right|$. Then we use the well known Moebius formula $\Phi_n(z) = \prod_{d|n} (1 - z^d)^{\mu(n/d)}$. Note that $\Phi_n(e^{ix})$ is a bounded continous function of x, so if the product $F_n(x_0)$ is not defined for some x_0 (which happens only for finitely many values of $0 \leq x_0 < 2\pi$), then we can replace it by its limit with $x \to x_0$. □

By Lemma 3 we have

$$L_n = \max_{|z|=1} |\Phi_n(z)| = \max_{0 \leq x < 2\pi} |F_n(x)|$$

as long as $n > 1$. Furthermore $|F_1(x)| = \frac{1}{2} |\Phi_1(e^{ix})|$.

It is easy to determine $L_1 = 1$ and $L_{p_1} = p_1$. Let us consider the case $\omega = 2$.

Theorem 4. *Let $p_1 < p_2$ be primes and let a be the unique integer such that $p_1 \mid p_2 + 2a$ and $|a| < p_1/2$. Then $L_{p_1 p_2} \geq \frac{4(p_1 - 2) p_2}{\pi^2 |2a+1|}$.*

Proof. Put $x = \left(1 + \frac{1}{p_1} + \frac{2a+1}{p_1 p_2}\right) \pi$. Then

$$\left|\sin \frac{p_1 p_2 x}{2}\right| = \left|\sin \frac{p_1 p_2 + p_2 + 2a + 1}{2} \pi\right| = 1,$$

$$\left|\sin \frac{x}{2}\right| = \left|\cos \left(\frac{1}{2p_1} + \frac{2a+1}{2p_1 p_2}\right) \pi\right| \geq 1 - \frac{1}{p_1} - \frac{|2a+1|}{p_1 p_2} \geq 1 - \frac{2}{p_1},$$

where we used the inequality $\cos t \geq 1 - \frac{2}{\pi} \cdot |t|$ for $|t| \leq \pi/2$. Furthermore

$$\left|\sin \frac{p_1 x}{2}\right| = \left|\sin\left(\frac{p_1+1}{2} + \frac{2a+1}{2p_2}\right)\pi\right| = \left|\sin \frac{2a+1}{2p_2}\pi\right| \leq \frac{|2a+1|\pi}{2p_2},$$

$$\left|\sin \frac{p_2 x}{2}\right| = \left|\sin\left(\frac{p_2}{2} + \frac{p_2+2a}{2p_1} + \frac{1}{2p_1}\right)\pi\right| = \left|\sin \frac{\pi}{2p_1}\right| \leq \frac{\pi}{2p_1},$$

where we used the inequality $|\sin t| \leq |t|$ for $t \in \mathbb{R}$. By the above inequalities we obtain

$$L_{p_1p_2} \geq F_{p_1p_2}(x) = \left|\frac{\sin(x/2)\sin(p_1p_2x/2)}{\sin(p_1x/2)\sin(p_2x/2)}\right| \geq \frac{4(p_1-2)p_2}{\pi^2|2a+1|},$$

as desired. □

3. Derivative of F_n

It is not difficult to prove that F_n is a differentiable function. Let $f_n(x)$ be the derivative of $F_n(x)$. The function f_n plays a crucial role in our construction of n with large L_n, especially its minimal absolute values in points x_0 for which $F_n(x_0) = 0$. Let

$$D_n = \min_{x_0 \colon F_n(x_0)=0} |f_n(x_0)|.$$

The aim of this section is to prove the following theorem.

Theorem 5. *For all positive integers ω and all $\varepsilon > 0$ there exists a function $h_{\omega,\varepsilon} : \mathbb{R}_+ \to \mathbb{R}_+$ depending only on ω and ε, such that*

$$\frac{n}{2} \cdot (L_{p_1}L_{p_1p_2}\ldots L_{p_1p_2\ldots p_{\omega-1}})^{-1} \leq D_n < (1+\varepsilon)\frac{n}{2} \cdot (L_{p_1}L_{p_1p_2}\ldots L_{p_1p_2\ldots p_{\omega-1}})^{-1}$$

for all $h_{\omega,\varepsilon}$-growing $n = p_1p_2\ldots p_\omega$.

In order to prove this theorem we will need some lemmas.

Lemma 6. *If $F_n(x_0) = 0$, then*

$$|f_n(x_0)| = \frac{n}{2} \prod_{d|n,\ d\neq n} \left|\sin \frac{d}{2} x_0\right|^{\mu(n/d)}.$$

Proof. Since $x_0 = \frac{2t_0\pi}{n}$ with some integer t_0 coprime to n, we have

$$\begin{aligned} f_n(x_0) &= \lim_{\epsilon\to\infty} \frac{1}{\epsilon} \prod_{d|n} \left(\sin \frac{d}{2}(x_0+\epsilon)\right)^{\mu(n/d)} \\ &= \lim_{\epsilon\to\infty} \frac{\sin(t_0\pi + n\epsilon/2)}{\epsilon} \prod_{d|n,\ d\neq n} \left(\sin \frac{d}{2}(x_0+\epsilon)\right)^{\mu(n/d)} \\ &= \pm\frac{n}{2} \prod_{d|n,\ d\neq n} \left(\sin \frac{d}{2} x_0\right)^{\mu(n/d)}, \end{aligned}$$

as desired. □

Lemma 7. *Let p be a prime not dividing n. If $F_{np}(x_1) = 0$, then $f_{np}(x_1) = \frac{p|f_n(x_1 p)|}{|F_n(x_1)|}$.*

Proof. By Lemma 6

$$\begin{aligned}
|f_{np}(x_1)| &= \frac{np}{2} \prod_{d|np,\, d\neq np} \left|\sin \frac{d}{2} x_1\right|^{\mu(np/d)} \\
&= \frac{np}{2} \cdot \left(\prod_{d|n} \left|\sin \frac{d}{2} x_1\right|^{\mu(n/d)}\right)^{-1} \cdot \left(\prod_{d|n,\, d\neq n} \left|\sin \frac{dp}{2} x_1\right|^{\mu(n/d)}\right) \\
&= \frac{np}{2} \cdot |F_n(x_1)|^{-1} \cdot \frac{2}{n} \cdot |f_n(px_1)| = \frac{p|f_n(px_1)|}{|F_n(x_1)|},
\end{aligned}$$

which completes the proof. □

Lemma 8. *We have $D_{np} \geq p \cdot \frac{D_n}{L_n}$. Moreover, for all $\varepsilon > 0$ there exists a function $h_\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ depending only on ε, such that $D_{np} < (1+\varepsilon) \cdot p \cdot \frac{D_n}{L_n}$ for all $p > h_\varepsilon(n)$.*

Proof. Let x_0 and x_1 be such that $F_n(x_0) = F_{np}(x_1) = 0$, $|f_n(x_0)| = D_n$ and $|f_{np}(x_1)| = D_{np}$. Since $x_1 = \frac{2t_1\pi}{np}$ for some t_1 coprime to np, we have $px_1 = \frac{2t_1\pi}{n}$. Therefore $F_n(px_1) = 0$ and hence $|f_p(px_1)| \geq D_n$. By applying this inequality and Lemma 7 we obtain

$$D_{np} = |f_{np}(x_1)| = \frac{p|f_n(px_1)|}{|F_n(x_1)|} \geq p \cdot \frac{D_n}{L_n}.$$

For obtaining the opposite inequality, let $x_0 = \frac{2t_0\pi}{n}$ and $x_1' = \frac{x_0+2t\pi}{p} = \frac{2(t_0+tn)\pi}{np}$ with any $t \not\equiv -\frac{t_0}{n} \pmod{p}$. Then $F_{np}(x_1') = 0$ and $f_n(px_1') = D_n$. Again by Lemma 7

$$D_{np} \leq |f_{np}(x_1')| = \frac{p|f_n(px_1')|}{|F_n(x_1')|} = p \cdot \frac{D_n}{\left|F_n\left(\frac{x_0+2t\pi}{p}\right)\right|}.$$

By choosing an appropriate t we can have $\left|F_n\left(\frac{x_0+2t\pi}{p}\right)\right|$ as close to L_n as we wish when $p \to \infty$. □

Now we are ready to prove the main theorem of this section.

Proof of Theorem 5. Let $\varepsilon > 0$ be fixed and let $\varepsilon' = \sqrt[\omega]{1+\varepsilon} - 1$. Let $h_{\varepsilon'}$ be a function given by Lemma 8, which implies that if $n = p_1 p_2 \dots p_\omega$ is $h_{\varepsilon'}$-growing, then

$$p_i \cdot \frac{D_{p_1 p_2 \dots p_{i-1}}}{L_{p_1 p_2 \dots p_{i-1}}} \leq D_{p_1 p_2 \dots p_i} < (1+\varepsilon') p_i \cdot \frac{D_{p_1 p_2 \dots p_{i-1}}}{L_{p_1 p_2 \dots p_{i-1}}}$$

for $i = 1, 2 \dots, \omega$ (empty product equals 1). By these inequalities

$$\frac{nD_1}{L_1 L_{p_1} L_{p_1 p_2} \dots L_{p_1 p_2 \dots p_{\omega-1}}} \leq D_n < (1+\varepsilon')^\omega \frac{nD_1}{L_1 L_{p_1} L_{p_1 p_2} \dots L_{p_1 p_2 \dots p_{\omega-1}}}.$$

Note that $(1+\varepsilon')^\omega = 1+\varepsilon$, $L_1 = 1$ and $D_1 = \frac{1}{2}$. So the theorem holds with the function $h_{\omega,\varepsilon} = h_{\varepsilon'} = h_{\sqrt[\omega]{1+\varepsilon}-1}$, which clearly depends only on ω and ε. □

4. Proof of main result

In the following lemma we give a lower bound on L_{np} which depends on the residue class of p modulo n.

Lemma 9. *Let $\varepsilon > 0$ and $n = p_1p_2\dots p_\omega$ be fixed. Put $x_M \in [0, 2\pi)$ such that $F_n(x_M) = L_n$ and $x_0 = \frac{2t_0\pi}{n}$ for which $F_n(x_0) = 0$ and $|f_n(x_0)| = D_n$. Let $b = \min_{k\in\mathbb{Z}} \left|\frac{nx_M}{2\pi} - pt_0 + nk\right|$. Then*

$$L_{np} > (1-\varepsilon)L_n \cdot \frac{np}{2b\pi D_n}$$

for every p large enough. Furthermore, if $p_1 > \omega$ and r is an integer coprime to n such that $\left|\frac{nx_M}{2\pi} - r\right|$ is smallest possible, then

$$L_{np} > (1-\varepsilon)L_n \cdot \frac{1}{\pi(\omega+1)} \cdot \frac{np}{D_n}$$

for every sufficiently large $p \equiv \frac{r}{t_0} \pmod{n}$.

Proof. We have $F_n(x) = \frac{F_n(px)}{F_n(x)}$, so

$$L_{np} = \max_{0\le x<2\pi}\left|\frac{F_n(px)}{F_n(x)}\right| \ge \max_{k\in\mathbb{Z}} \frac{|F_n(x_M+2k\pi)|}{\left|F_n\left(\frac{x_M+2k\pi}{p}\right)\right|} = \frac{L_n}{\min_{k\in\mathbb{Z}}\left|F_n\left(\frac{x_M+2k\pi}{p}\right)\right|}.$$

Let k_0 be a integer for which $\left|\frac{x_M+2k_0\pi}{p} - x_0\right|$ is smallest possible. Then

$$\begin{aligned}
\min_{k\in\mathbb{Z}}\left|F_n\left(\frac{x_M+2k\pi}{p}\right)\right| &\le \left|F_n\left(\frac{x_M+2k_0\pi}{p}\right)\right| \\
&\sim |f_n(x_0)| \cdot \left|\frac{x_M+2k_0\pi}{p} - x_0\right| \quad (\text{with } p\to\infty) \\
&= D_n \cdot \frac{2\pi}{np}\cdot\left|\frac{nx_M}{2\pi} - t_0p + k_0n\right| \\
&= D_n \cdot \frac{2b\pi}{np}.
\end{aligned}$$

Therefore

$$L_{np} > (1+o(1))\frac{L_n}{D_n\cdot\frac{2b\pi}{np}} \sim L_n \cdot \frac{np}{2b\pi D_n}$$

with $p\to\infty$, which completes the proof of the first statement.

For $p \equiv \frac{r}{t_0} \pmod{n}$ we have

$$b = \min_{k\in\mathbb{Z}}\left|\frac{nx_M}{2\pi} - pt_0 + nk\right| = \left|\frac{nx_M}{2\pi} - r\right| \le \frac{\omega+1}{2}$$

since, in view of $p_1 > \omega$, at most ω consecutive integers are not coprime to p. □

Simple calculations show that Theorem 4 gives a better lower bound for $L_{p_1p_2}$ than Lemma 9. Therefore we use Theorem 4 in the proof of the main result. By the fact that $A_n \geq L_n/n$ for $n > 1$, Theorem 1 is an immediate consequence of the following theorem.

Theorem 10. *For every $\omega \geq 3$, $\varepsilon > 0$ and $h : \mathbb{R}_+ \to \mathbb{R}_+$ there exists an h-growing $n = p_1p_2 \dots p_\omega$ such that $L_n > (1-\varepsilon)c_\omega n M_n$, where c_ω and M_n are defined in Theorem 1.*

Proof. We prove this by a strong induction on $\omega = \omega(n)$. The induction starts with $\omega = 2$.

Our inductive assumption is that for all $\varepsilon' > 0$ and a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ there exists an h-growing $n = p_1p_2 \dots p_\omega$ such that $L_{p_1p_2} > (1-\varepsilon')\frac{4}{\pi^2}p_1p_2$ and $L_{p_1p_2\dots p_i} > (1-\varepsilon')c_ip_1p_2\dots p_iM_{p_1p_2\dots p_i}$ for $3 \leq i \leq \omega$. By Theorem 4 it is true for $\omega = 2$ with $p_1 \mid q_1 - 2$ (note that the second part of the inductive assumption is empty when $\omega = 2$).

Now we show the inductive step. Let $\omega \geq 2$. Without loss of generality we may assume that $h(1) \geq \omega$. By Lemma 9 and Dirichlet's theorem on primes in arithmetic progressions, there exists $p_{\omega+1} > h(p_1p_2\dots p_\omega)$ for which

$$L_{p_1p_2\dots p_{\omega+1}} > (1-\varepsilon')L_n \cdot \frac{np_{\omega+1}}{\pi(\omega+1)D_n}.$$

By Theorem 5 there exists a function $h_1 : \mathbb{R}_+ \to \mathbb{R}_+$ depending only on ω and ε', such that for all h_1-growing n

$$D_n > (1-\varepsilon')^{-1}\frac{n}{2} \cdot \frac{1}{L_{p_1}L_{p_1p_2}\dots L_{p_1p_2\dots p_\omega}}.$$

Again without loss of generality we can assume that $h(x) > h_1(x)$ for all $x \in \mathbb{R}_+$. In this situation all h-growing numbers are also h_1-growing, so the above inequality holds for every h-growing n.

For given $\varepsilon > 0$ we choose $\varepsilon' = 1 - \sqrt[\omega+1]{1-\varepsilon}$. By the above inequalities and the inductive assumption

$$\begin{aligned} L_{p_1p_2\dots p_{\omega+1}} &> (1-\varepsilon')^2 \cdot \frac{2p_{\omega+1}}{\pi(\omega+1)} \cdot L_{p_1}L_{p_1p_2}\dots L_{p_1p_2\dots p_\omega} \\ &> (1-\varepsilon')^{\omega+1} \cdot \frac{2p_{\omega+1}}{\pi(\omega+1)} \cdot p_1 \cdot \frac{4}{\pi^2}p_1p_2 \cdot \prod_{i=3}^{\omega}(c_ip_1p_2\dots p_iM_{p_1p_2\dots p_i}) \\ &= (1-\varepsilon)\left(\frac{8}{\pi^3(\omega+1)} \cdot \prod_{i=3}^{\omega} c_i\right)\left(p_{\omega+1}\prod_{i=1}^{\omega}(p_1p_2\dots p_iM_{p_1p_2\dots p_i})\right). \end{aligned}$$

The exponent of p_k in $\prod_{i=1}^{\omega}(p_1p_2\dots p_iM_{p_1p_2\dots p_i})$ for $k \leq \omega$ equals

$$\omega - k + 1 + \sum_{i=k+2}^{\omega}(2^{i-k-1} - 1) = 2^{\omega-k},$$

so

$$p_{\omega+1}\prod_{i=1}^{\omega}(p_1p_2\dots p_iM_{p_1p_2\dots p_i}) = p_1p_2\dots p_{\omega+1}M_{p_1p_2\dots p_{\omega+1}}.$$

It remains to evaluate the constant by using a similar method:

$$\begin{aligned}\frac{8}{\pi^3(\omega+1)}\cdot\prod_{i=3}^{\omega}c_i &= \frac{8}{\pi^3(\omega+1)}\cdot\prod_{i=3}^{\omega}\left(\frac{1}{i}\cdot\left(\frac{2}{\pi}\right)^{3\cdot 2^{i-3}}\cdot\left(\prod_{k=3}^{i-1}k^{2^{i-1-k}}\right)^{-1}\right)\\ &= \frac{1}{\omega+1}\cdot\left(\frac{2}{\pi}\right)^{3\cdot 2^{\omega-2}}\cdot\frac{1}{3\cdot 4\cdot\ldots\cdot\omega}\cdot\left(\prod_{i=3}^{\omega}\prod_{k=3}^{i-1}k^{2^{i-1-k}}\right)^{-1}\\ &= \frac{1}{\omega+1}\cdot\left(\frac{2}{\pi}\right)^{3\cdot 2^{\omega+1-3}}\cdot\left(\prod_{t=3}^{\omega+1-1}t^{2^{\omega+1-1-t}}\right)^{-1} = c_{\omega+1}\end{aligned}$$

for $\omega \geq 2$.

Note that $\varepsilon' < \varepsilon$, so by the inductive assumption also $L_{p_1p_2} > (1-\varepsilon)\frac{4}{\pi^2}p_1p_2$ and $L_{p_1p_2\dots p_i} > (1-\varepsilon)c_ip_1p_2\dots p_iM_{p_1p_2\dots p_i}$ for all $3 \leq i \leq \omega$. It completes the inductive step. □

Acknowledgment

The author is partially supported by NCN grant no. 2012/07/D/ST1/02111.

The author would like to thank Pieter Moree for the invitation to Max Planck Institute in Bonn and for the good time spent there.

References

[1] G. Bachman, *On the coefficients of ternary cyclotomic polynomials*, J. Number Theory **100** (2003), 104–116.

[2] P.T. Bateman, C. Pomerance, R.C. Vaughan, *On the size of the coefficients of cyclotomic polynomials*, Coll. Math. Soc. J. Bolyai **34** (1981), 171–202.

[3] B. Bzdęga, *On the height of cyclotomic polynomials*, Acta Arith. **152** (2012), 349–359.

[4] Y. Gallot, P. Moree, *Ternary cyclotomic polynomials having a large coefficient*, J. Reine Angew. Math. **632** (2009), 105–125.

[5] S. Konyagin, H. Maier, E. Wirsing, *Cyclotomic polynomials with many primes dividing their orders*, Periodica Math. Hungar. **49** (2004), 99–106.

[6] H. Maier, *Anatomy of integers and cyclotomic polynomials*, CRM Proc. and Lecture Notes **46** (2008), 89–95.

Adam Mickiewicz University, Poznań, Poland

E-mail address: exul@amu.edu.pl