DYNAMIC ALPHA-INVARIANTS OF DEL PEZZO SURFACES

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Abstract. For every smooth del Pezzo surface S, smooth curve $C \in |-K_S|$ and $\beta \in (0,1]$, we compute the α-invariant of Tian $\alpha(S,(1-\beta)C)$ and prove the existence of Kähler–Einstein metrics on S with edge singularities along C of angle $2\pi\beta$ for β in certain interval. In particular we give lower bounds for the invariant $R(S,C)$, introduced by Donaldson as the supremum of all $\beta \in (0,1]$ for which such a metric exists.

1. Introduction

Let X be a normal variety of dimension $n \geqslant 1$, and let Δ be an effective $\mathbb{R}$-divisor on X. Suppose that (X,Δ) has at most Kawamata log terminal singularities, and $-(K_X+\Delta)$ is ample. Then (X,Δ) is a log Fano variety. Its α-invariant can be defined as

$$\alpha(X,\Delta)=\sup\left\{\lambda\in\mathbb{R}\ \middle|\ \begin{array}{l}\text{the log pair } \big(X,\Delta+\lambda B\big) \text{ is log canonical}\\ \text{for any effective } \mathbb{R}\text{-divisor } B\sim_{\mathbb{R}}-(K_X+\Delta)\end{array}\right\}\in\mathbb{R}_{>0}.$$

Remark 1.1. For every effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor B on X, the number

$$\mathrm{lct}(X,\Delta;B)=\sup\Big\{\lambda\in\mathbb{R}\ \Big|\ \text{the log pair } \big(X,\Delta+\lambda B\big) \text{ is log canonical}\Big\}$$

is called the *log canonical threshold* of B with respect to (X,Δ). Note that

$$\alpha(X,\Delta)=\inf\Big\{\mathrm{lct}\big(X,\Delta;B\big)\ \Big|\ B \text{ is an effective } \mathbb{R}\text{-divisor such that } B\sim_{\mathbb{R}}-(K_X+\Delta)\Big\}.$$

If $\Delta=0$, we denote $\alpha(X,\Delta)$ by $\alpha(X)$. Tian introduced α-invariants of smooth Fano varieties in [19]. His definition coincides with ours by [5, Theorem A.3]. In [19], Tian also proved

Theorem 1.2 ([19, Theorem 2.1])**.** Let X be a smooth Fano variety of dimension n. If $\alpha(X)>\frac{n}{n+1}$, then X admits a Kähler–Einstein metric.

This theorem gives the initial motivation for the study of $\alpha(X,\Delta)$ in the case when $\Delta=0$. In fact, $\alpha(X,\Delta)$ is also important if $\Delta\neq 0$. When X is smooth and $\mathrm{Supp}(\Delta)$ is a smooth irreducible divisor, Theorem 1.2 has been generalized by Jeffres, Mazzeo and Rubinstein as follows

Theorem 1.3 ([11, Theorem 2, Lemma 6.13])**.** Let X be a smooth projective variety of dimension n, and let D be a smooth irreducible hypersurface in X. Let $\beta\in(0,1]$ and suppose that the divisor $-(K_X+(1-\beta)D)$ is ample. If $\alpha(X,(1-\beta)D)>\frac{n}{n+1}$, then X admits a Kähler–Einstein metric with edge singularities of angle $2\pi\beta$ along D.

Song computed α-invariants of smooth toric Fano varieties in [18, Theorem 1.1]. The same approach can be used to obtain an explicit combinatorial formula for $\alpha(X,\Delta)$ in the case when X is toric and $\mathrm{Supp}(\Delta)$ consists of torus-invariant divisors (cf. [5, Lemma 5.1]).

Example 1.4 ([6, Remark 6.7])**.** Let L_1, L_2 and L_3 be distinct lines on $\mathbb{P}^2$ such that $\bigcap_i L_i=\emptyset$, and let $(\beta_1,\beta_2,\beta_3)$ be any point in $(0,1]^3$. Then

$$\alpha\Big(\mathbb{P}^2,\sum_{i=1}^{3}(1-\beta_i)L_i\Big)=\frac{\max(\beta_1,\beta_2,\beta_3)}{\beta_1+\beta_2+\beta_3}.$$

Throughout this paper, we assume that all considered varieties are projective and defined over $\mathbb{C}$.

For smooth del Pezzo surfaces, α-invariants have been explicitly computed in [2, Theorem 1.7] (see [7] and [16] for analytic approach, see [14] for a characteristic free approach). The proof of this theorem implies

Theorem 1.5. Let S be a smooth del Pezzo surface. Then

$$\alpha(S)=\inf\left\{\mathrm{lct}\big(S,0;B\big)\ \Bigg|\ B\in|-K_S|\text{ and }B=\sum B_i,\text{ where }B_i\cong\mathbb{P}^1\text{ and }-K_S\cdot B_i\leqslant 3\ \forall i\right\}.$$

To apply Theorem 1.3, the divisor $-(K_X+(1-\beta)D)$ must be ample. A natural choice for the pair (X,D) considered by Donaldson in his approach to the Yau-Tian-Donaldson conjecture is to let X be a smooth Fano variety and let D be a smooth anticanonical divisor (see [9]).

Remark 1.6. Let X be a smooth Fano variety of dimension n, and let D be a smooth divisor in $|-K_X|$. By [17, Theorem 1.2], such divisor D always exists when $n\leqslant 3$, which is no longer true in general if $n\geqslant 4$ (see [10, Example 2.12]). One has $\alpha(X,(1-\beta)D)=1>\frac{n}{n+1}$ for all positive $\beta\ll 1$ (see Theorem 1.10). In particular, X admits a Kähler–Einstein metric with edge singularities of angle $2\pi\beta$ along D for all positive $\beta\ll 1$ by Theorem 1.3.

A Kähler–Einstein metric with singularities along D of angle 2π is a Kähler–Einstein metric in the usual sense. So, it is natural to consider the following invariant introduced by Donaldson:

Definition 1.7 ([9])**.** Let X be a smooth Fano variety, and let D be a smooth divisor in $|-K_X|$. Then $R(X,D)$ is the supremum of all $\beta\in(0,1]$ such that X admits a Kähler–Einstein metric with edge singularities along D of angle $2\pi\beta$.

It follows from [11] that the smooth Fano variety X admits a Kähler–Einstein metric with edge singularities of angle $2\pi\beta$ along D for every positive $\beta<R(X,D)$.

Corollary 1.8. Let X be a smooth Fano variety, and let D be a smooth divisor in $|-K_X|$. Suppose that X admits a Kähler-Einstein metric. Then $R(X,D)=1$.

By Tian's theorem (see [20]), a smooth del Pezzo surface S admits a Kähler–Einstein metric if and only if $S\not\cong\mathbb{F}_1$ and $K_S^2\neq 7$. Thus, we have

Corollary 1.9 ([20])**.** Let S be a smooth del Pezzo surface such that $S\not\cong\mathbb{F}_1$ and $K_S^2\neq 7$, and let C be a smooth curve in $|-K_S|$. Then $R(S,C)=1$.

Unless $R(X,D)=1$, we do not know a single example for which the invariant $R(X,D)$ is known precisely (cf. [12, Theorem 1.7]). A lower bound for $R(X,D)$ can be found using

Theorem 1.10 ([1], [15, Corollary 5.5], [6, Proposition 6.10], [6, Remark 6.11])**.** Let X be a smooth Fano variety of dimension n, and let D be a smooth divisor in $|-K_X|$. Let

$$M=\begin{cases}9\text{ if }n=2,\\ 64\text{ if }n=3,\\ 3^n(2^n-1)^n(n+1)^{n(n+2)(2^n-1)}(2n(n+1)(n+2)!)^{n-1}\text{ if }n\geqslant 4.\end{cases}$$

Then $1\geqslant\alpha(X,(1-\beta)D)\geqslant\min\{1,\frac{1}{M\beta}\}$ for every $\beta\in(0,1]$.

Corollary 1.11. In the assumptions and notation of Theorem 1.10, one has $R(X,D)\geqslant\frac{n+1}{nM}$.

The purpose of this paper is to merge Theorem 1.5 with Theorem 1.10 by proving

Theorem 1.12. Let S be a smooth del Pezzo surface, let C be a smooth curve in $|-K_S|$, and let β be a real number in $(0,1]$. Then

$$\alpha\big(S,(1-\beta)C\big)=\inf\left\{\mathrm{lct}\big(S,(1-\beta)C;\beta B\big)\ \middle|\ \begin{array}{l}B\in|-K_S|\text{ such that }B=C\text{ or }B=\sum B_i,\\ \text{where }B_i\cong\mathbb{P}^1\text{ and }-K_S\cdot B_i\leqslant 3\ \forall i\end{array}\right\}.$$

We will prove Theorem 1.12 in Section 4. In Section 2, we will give very explicit formulas for the invariant $\alpha(S,(1-\beta)C)$. Instead of presenting them here, let us consider their applications.

Corollary 1.13. Let S be a smooth del Pezzo surface, and let C be a smooth curve in $|-K_S|$. Then $\alpha(S,(1-\beta)C)$ is a decreasing continuous piecewise smooth function for $\beta\in(0,1]$.

Corollary 1.14. Let S_1 and S_2 be smooth del Pezzo surfaces, let C_1 and C_2 be smooth curves in $|-K_{S_1}|$ and $|-K_{S_2}|$, respectively. Suppose that there is a birational morphism $f\colon S_2\to S_1$ such that $f(C_2)=C_1$. Then $\alpha(S_1,(1-\beta)C_1)\leqslant\alpha(S_2,(1-\beta)C_2)$ for every $\beta\in(0,1]$ except the following cases:

(1) $S_1\cong\mathbb{P}^2$, $S_2\cong\mathbb{F}_1$, and f is the blow up of an inflection point of the cubic curve $C_1\subset\mathbb{P}^2$,
(2) $S_1\cong\mathbb{P}^1\times\mathbb{P}^1$, $K_{S_2}^2=7$, and f is the blow up of a point in C_1.

If S is a smooth del Pezzo surface such that either $S\cong\mathbb{F}_1$ or $K_S^2=7$, and C is a smooth curve in $|-K_S|$, then $R(S,C)\geqslant\frac{1}{6}$ by Corollary 1.11. We improve this bound:

Corollary 1.15. Suppose that $S\cong\mathbb{F}_1$. Let C be a smooth curve in $|-K_S|$. Then $R(S,C)\geqslant\frac{3}{10}$. Furthermore, if C is chosen to be *general* in $|-K_S|$, then $R(S,C)\geqslant\frac{3}{7}$.

Corollary 1.16. Let S be a smooth del Pezzo surface such that $K_S^2=7$, and let C be a smooth curve in $|-K_S|$. Then $R(S,C)\geqslant\frac{3}{7}$. Furthermore, if C does not pass through the intersection point of two intersecting (-1)-curves in S, then $R(S,C)\geqslant\frac{1}{2}$.

In [21, Theorem 1], Székelyhidi proved that $R(S,C)\leqslant\frac{4}{5}$ when $S=\mathbb{F}_1$, and $R(S,C)\leqslant\frac{7}{9}$ when $K_S^2=7$ and C passes through the intersection point of two intersecting (-1)-curves in S.

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2. Explicit formulas

Let S be a smooth del Pezzo surface. If $K_S^2\geqslant 3$, then $-K_S$ is very ample. In this case, we will identify S with its anticanonical image, and we will call a curve $Z\subset S$ such that $Z\cdot(-K_S)=1,2,3$ a line, conic, cubic, respectively. Let C be a smooth curve in $|-K_S|$, and let β be a positive real number in $(0,1]$. Let

$$\breve{\alpha}\big(S,(1-\beta)C\big)=\inf\left\{\mathrm{lct}\big(S,(1-\beta)C;\beta B\big)\ \middle|\ \begin{array}{l}B\in|-K_S|\text{ such that }B=C\text{ or }B=\sum B_i,\\ \text{where }B_i\cong\mathbb{P}^1\text{ and }-K_S\cdot B_i\leqslant 3\ \forall i\end{array}\right\}.$$

Then $\alpha(S,(1-\beta)C)\leqslant\breve{\alpha}(S,(1-\beta)C)$. Theorem 1.12 states that $\alpha(S,(1-\beta)C)=\breve{\alpha}(S,(1-\beta)C)$. In this section, we will define a number $\hat{\alpha}(S,(1-\beta)C)$ such that $\hat{\alpha}(S,(1-\beta)C)\geqslant\breve{\alpha}(S,(1-\beta)C)$. In Section 4, we will prove that $\alpha(S,(1-\beta)C)\geqslant\hat{\alpha}(S,(1-\beta)C)$. The latter inequality implies Theorem 1.12, since $\hat{\alpha}(S,(1-\beta)C)\geqslant\breve{\alpha}(S,(1-\beta)C)\geqslant\alpha(S,(1-\beta)C)$.

2.1. Projective plane.

Suppose that $S\cong\mathbb{P}^2$. Then C is a smooth cubic curve on S. Let

$$\hat{\alpha}\big(S,(1-\beta)C\big)=\min\Big\{1,\frac{1+3\beta}{9\beta},\frac{1}{3\beta}\Big\}=\begin{cases}1\text{ for }0<\beta\leqslant\dfrac{1}{6},\\[2mm] \dfrac{1+3\beta}{9\beta}\text{ for }\dfrac{1}{6}\leqslant\beta\leqslant\dfrac{2}{3},\\[2mm] \dfrac{1}{3\beta}\text{ for }\dfrac{2}{3}\leqslant\beta\leqslant 1.\end{cases}$$

Let P be an inflection point of the curve C, and let T be the line in $\mathbb{P}^2$ that is tangent to C at the point P. Then $\hat{\alpha}(S,(1-\beta)C)\geqslant\check{\alpha}(S,(1-\beta)C)$, since

$$\hat{\alpha}\big(S,(1-\beta)C\big)=\min\Big\{\mathrm{lct}\big(S,(1-\beta)C;\beta C\big),\mathrm{lct}\big(S,(1-\beta)C;3\beta T\big)\Big\}.$$

2.2. **Smooth quadric surface.** Suppose that $S\cong\mathbb{P}^1\times\mathbb{P}^1$. Let

$$\hat{\alpha}\big(S,(1-\beta)C\big)=\min\Big\{1,\frac{1+2\beta}{6\beta}\Big\}=\begin{cases}1 \text{ for } 0<\beta\leqslant\dfrac{1}{4},\\ \dfrac{1+2\beta}{6\beta} \text{ for } \dfrac{1}{4}\leqslant\beta\leqslant 1.\end{cases}$$

Let T be a divisor of bi-degree $(1,1)$ on S that is a union of two fibers of each projection from S to $\mathbb{P}^1$. Suppose in addition that one component of T is tangent to C at some point, and another component of T passes through this point. Then $\hat{\alpha}(S,(1-\beta)C)\geqslant\check{\alpha}(S,(1-\beta)C)$, since

$$\hat{\alpha}\big(S,(1-\beta)C\big)=\min\Big\{\mathrm{lct}\big(S,(1-\beta)C;\beta C\big),\mathrm{lct}\big(S,(1-\beta)C;2\beta T\big)\Big\}.$$

2.3. **First Hirzebruch surface.** Suppose that $S\cong\mathbb{F}_1$. Let Z be the unique (-1)-curve in S, and let F be the fiber of the natural projection $S\to\mathbb{P}^1$ that passes through the point $C\cap Z$. Then $C\sim 2Z+3F$. If F is tangent to C at the point $C\cap Z$, let

$$\hat{\alpha}\big(S,(1-\beta)C\big)=\min\Big\{1,\frac{1+2\beta}{8\beta},\frac{1}{3\beta}\Big\}=\begin{cases}1 \text{ for } 0<\beta\leqslant\dfrac{1}{6},\\ \dfrac{1+2\beta}{8\beta} \text{ for } \dfrac{1}{6}\leqslant\beta\leqslant\dfrac{5}{6},\\ \dfrac{1}{3\beta} \text{ for } \dfrac{5}{6}\leqslant\beta\leqslant 1.\end{cases}$$

If F is not tangent to C at the point $C\cap Z$, let

$$\hat{\alpha}\big(S,(1-\beta)C\big)=\min\Big\{1,\frac{1+\beta}{5\beta},\frac{1}{3\beta}\Big\}=\begin{cases}1 \text{ for } 0<\beta\leqslant\dfrac{1}{4},\\ \dfrac{1+\beta}{5\beta} \text{ for } \dfrac{1}{4}\leqslant\beta\leqslant\dfrac{2}{3},\\ \dfrac{1}{3\beta} \text{ for } \dfrac{2}{3}\leqslant\beta\leqslant 1.\end{cases}$$

In both cases, we have $\hat{\alpha}(S,(1-\beta)C)\geqslant\check{\alpha}(S,(1-\beta)C)$, because

$$\hat{\alpha}\big(S,(1-\beta)C\big)=\min\Big\{\mathrm{lct}\big(S,(1-\beta)C;\beta C\big),\mathrm{lct}\big(S,(1-\beta)C;\beta(2Z+3F)\big)\Big\}.$$

2.4. **Blow up of $\mathbb{P}^2$ at two points.** Suppose that $K_S^2=7$. Then there exists a birational morphism $\pi\colon S\to\mathbb{P}^2$ that is the blow up of two distinct points in $\mathbb{P}^2$. Denote by E_1 and E_2 two π-exceptional curves, and denote by L the proper transform of the line in $\mathbb{P}^2$ that passes through $\pi(E_1)$ and $\pi(E_2)$. Then E_1, E_2, and L are all (-1)-curves in S.

The pencil $|E_2+L|$ contains a unique curve that passes though $C\cap E_1$. Similarly, $|E_1+L|$ contains a unique curve that passes though $C\cap E_2$. Denote these curves by L_1 and L_2, respectively. Then L_1 is irreducible and smooth unless $L_1=E_2+L$ (in this case $E_1\cap L\in C$). Similarly, the curve L_2 is irreducible and smooth unless $L_2=E_1+L$ and $L\cap E_2\in C$.

If C does not contain the points $E_1\cap L$ nor $E_2\cap L$, then there exists a unique smooth irreducible curve $R\in|E_1+E_2+L|$ such that R passes though $C\cap L$ and is tangent to C at the point $C\cap L$. If either $E_1\cap L\in C$ or $E_2\cap L\in C$, we let $R=E_1+E_2+L$. In the former case, either R and C have simple tangency at the point $C\cap L$ or the curve R is tangent to C at the point $C\cap L$ with multiplicity 3 (in this case, we must have $R\cap C=C\cap L$, because $R\cdot C=3$).

If either $E_1 \cap L \in C$ or $E_2 \cap L \in C$ (but not both, since $C \cdot L = 1$), then we let

$$\hat{\alpha}\big(S,(1-\beta)C\big) = \min\Big\{1, \frac{1+\beta}{5\beta}, \frac{1}{3\beta}\Big\} = \begin{cases} 1 \text{ for } 0 < \beta \leqslant \frac{1}{4}, \\ \frac{1+\beta}{5\beta} \text{ for } \frac{1}{4} \leqslant \beta \leqslant \frac{2}{3}, \\ \frac{1}{3\beta} \text{ for } \frac{2}{3} \leqslant \beta \leqslant 1. \end{cases}$$

If the curve C does not contain the points $E_1 \cap L$ nor $E_2 \cap L$, and either L_1 is tangent to C at the point $C \cap E_1$ or L_2 is tangent to C at the point $C \cap E_2$, then we let

$$\hat{\alpha}\big(S,(1-\beta)C\big) = \min\Big\{1, \frac{1+2\beta}{6\beta}, \frac{1}{3\beta}\Big\} = \begin{cases} 1 \text{ for } 0 < \beta \leqslant \frac{1}{4}, \\ \frac{1+2\beta}{6\beta} \text{ for } \frac{1}{4} \leqslant \beta \leqslant \frac{1}{2}, \\ \frac{1}{3\beta} \text{ for } \frac{1}{2} \leqslant \beta \leqslant 1. \end{cases}$$

If the curve C does not contain the points $E_1 \cap L$ nor $E_2 \cap L$ (this implies that the curve R is smooth), neither L_1 is tangent to C at the point $C \cap E_1$ nor L_2 is tangent to C at the point $C \cap E_2$, and the curve R is tangent to C at the point $C \cap L$ with multiplicity 3, then we let

$$\hat{\alpha}\big(S,(1-\beta)C\big) = \min\Big\{1, \frac{1+3\beta}{7\beta}, \frac{1}{3\beta}\Big\} = \begin{cases} 1 \text{ for } 0 < \beta \leqslant \frac{1}{4}, \\ \frac{1+3\beta}{7\beta} \text{ for } \frac{1}{4} \leqslant \beta \leqslant \frac{4}{9}, \\ \frac{1}{3\beta} \text{ for } \frac{4}{9} \leqslant \beta \leqslant 1. \end{cases}$$

Finally, if the curve C does not contain the points $E_1 \cap L$ nor $E_2 \cap L$ (and hence the curve R is smooth), neither L_1 is tangent to C at the point $C \cap E_1$ nor L_2 is tangent to C at the point $C \cap E_2$, and R is tangent to C at the point $C \cap L$ with multiplicity 2, then we let

$$\hat{\alpha}\big(S,(1-\beta)C\big) = \min\Big\{1, \frac{1}{3\beta}\Big\} = \begin{cases} 1 \text{ for } 0 < \beta \leqslant \frac{1}{3}, \\ \frac{1}{3\beta} \text{ for } \frac{1}{3} \leqslant \beta \leqslant 1. \end{cases}$$

We have $\hat{\alpha}(S,(1-\beta)C) \geqslant \check{\alpha}(S,(1-\beta)C)$. Indeed, if either $E_1 \cap L \in C$ or $E_2 \cap L \in C$, then

$$\hat{\alpha}\big(S,(1-\beta)C\big) = \min\Big\{\mathrm{lct}\big(S,(1-\beta)C;\beta C\big), \mathrm{lct}\big(S,(1-\beta)C;\beta(3L+2E_1+2E_2)\big)\Big\},$$

which implies that $\hat{\alpha}(S,(1-\beta)C) \geqslant \check{\alpha}(S,(1-\beta)C)$. If neither $E_1 \cap L \in C$ nor $E_2 \cap L \in C$, then

$$\min\Big\{\mathrm{lct}\big(S,(1-\beta)C;\beta C\big), \mathrm{lct}\big(S,(1-\beta)C;\beta(3L+2E_1+2E_2)\big)\Big\} = \min\Big\{1, \frac{1}{3\beta}\Big\}.$$

If the curve C does not contain the points $E_1 \cap L$ nor $E_2 \cap L$, and L_1 is tangent to C at the point $C \cap E_1$, then

$$\hat{\alpha}\big(S,(1-\beta)C\big) = \min\Big\{1, \frac{1}{3\beta}, \mathrm{lct}\big(S,(1-\beta)C;\beta(2L_1+2E_1+L)\big)\Big\},$$

and similarly if L_2 is tangent to C at the point $C \cap E_2$. If the curve C does not contain the points $E_1 \cap L$ nor $E_2 \cap L$ (this implies that the curve R is smooth), neither L_1 is tangent to C

at the point $C \cap E_1$ nor L_2 is tangent to C at the point $C \cap E_2$, and the curve R is tangent to C at the point $C \cap L$ with multiplicity 3, then

$$\min\Big\{\mathrm{lct}\big(S,(1-\beta)C;\beta C\big),\mathrm{lct}\big(S,(1-\beta)C;\beta(3L+2E_1+2E_2)\big),\mathrm{lct}\big(S,(1-\beta)C;\beta(L+2R)\big)\Big\}$$

equals $\hat{\alpha}(S,(1-\beta)C)$. We conclude that $\hat{\alpha}(S,(1-\beta)C) \geqslant \check{\alpha}(S,(1-\beta)C)$ in every case.

2.5. **Blow up of $\mathbb{P}^2$ at three points.** Suppose that $K_S^2 = 6$. Then there exists a birational morphism $\pi\colon S \to \mathbb{P}^2$ that is the blow up of three non-colinear points. Denote the π-exceptional curves by E_1, E_2, E_3, denote the proper transform on S of the line in $\mathbb{P}^2$ that passes through $\pi(E_1)$ and $\pi(E_2)$ by L_{12}, denote the proper transform on S of the line in $\mathbb{P}^2$ that passes through $\pi(E_1)$ and $\pi(E_3)$ by L_{13}, and denote the proper transform on S of the line in $\mathbb{P}^2$ that passes through $\pi(E_2)$ and $\pi(E_3)$ by L_{23}. Then E_1, E_2, E_3, L_{12}, L_{13} and L_{23} are all lines in S.

If the curve C contains an intersection point of two intersecting lines in S, then we let

$$\hat{\alpha}\big(S,(1-\beta)C\big) = \min\Big\{1,\frac{1+\beta}{4\beta}\Big\} = \begin{cases} 1 \text{ for } 0<\beta\leqslant\dfrac{1}{3}, \\ \dfrac{1+\beta}{4\beta} \text{ for } \dfrac{1}{3}\leqslant\beta\leqslant 1. \end{cases}$$

If the curve C does not contain the intersection points of any two intersecting lines, and there are a line Z_1 and an irreducible conic Z_2 in S such that Z_2 is tangent to C at the point $C \cap Z_1$, then we let

$$\hat{\alpha}\big(S,(1-\beta)C\big) = \min\Big\{1,\frac{1+2\beta}{5\beta},\frac{1}{2\beta}\Big\} = \begin{cases} 1 \text{ for } 0<\beta\leqslant\dfrac{1}{3}, \\ \dfrac{1+2\beta}{5\beta} \text{ for } \dfrac{1}{3}\leqslant\beta\leqslant\dfrac{3}{4}, \\ \dfrac{1}{2\beta} \text{ for } \dfrac{3}{4}\leqslant\beta\leqslant 1. \end{cases}$$

If C does not contain the intersection point of any two intersecting lines, and for every line Z_1 in S, there exists no irreducible conic Z_2 in S such that Z_2 is tangent to C at $C \cap Z_1$, then we let

$$\hat{\alpha}\big(S,(1-\beta)C\big) = \min\Big\{1,\frac{1}{2\beta}\Big\} = \begin{cases} 1 \text{ for } 0<\beta\leqslant\dfrac{1}{2}, \\ \dfrac{1}{2\beta} \text{ for } \dfrac{1}{2}\leqslant\beta\leqslant 1. \end{cases}$$

One has $\hat{\alpha}(S,(1-\beta)C) \geqslant \check{\alpha}(S,(1-\beta)C)$. Indeed, we have $2E_1+2L_{12}+L_{13}+E_2 \sim -K_S$. Thus, if $E_1 \cap L_{12} \notin C$, $E_1 \cap L_{13} \notin C$ and $E_2 \cap L_{12} \notin C$, then

$$\min\Big\{\mathrm{lct}\big(S,(1-\beta)C;\beta C\big),\mathrm{lct}\big(S,(1-\beta)C;\beta(2E_1+2L_{12}+L_{13}+E_2)\big)\Big\} = \begin{cases} 1 \text{ for } 0<\beta\leqslant\dfrac{1}{2}, \\ \dfrac{1}{2\beta} \text{ for } \dfrac{1}{2}\leqslant\beta\leqslant 1. \end{cases}$$

Otherwise, this minimum is $\hat{\alpha}(S,(1-\beta)C)$. This shows that $\hat{\alpha}(S,(1-\beta)C) \geqslant \check{\alpha}(S,(1-\beta)C)$ except for the case when C does not contain the intersection point of any two intersecting lines, but there are a line Z_1 and a conic Z_2 in S such that Z_2 is tangent to C at the point $C \cap Z_1$. In the latter case, we may assume that $Z_1 = E_1$ and $Z_2 \in |L_{12}+E_2|$, which implies that

$$\hat{\alpha}\big(S,(1-\beta)C\big) = \min\Big\{\mathrm{lct}\big(S,(1-\beta)C;\beta C\big),\mathrm{lct}\big(S,(1-\beta)C;\beta(2Z_2+E_1+L_{23})\big)\Big\},$$

since $2Z_2+E_1+L_{23} \sim -K_S$. Thus, in all cases we have $\hat{\alpha}(S,(1-\beta)C) \geqslant \check{\alpha}(S,(1-\beta)C)$.

2.6. **Blow up of $\mathbb{P}^2$ at four points.** Suppose that $K_S^2 = 5$. Then there exists a birational morphism $\pi\colon S \to \mathbb{P}^2$ that contracts four smooth rational curves to four points such that no three of them are colinear. Denote these curves by E_1, E_2, E_3, E_4. For and integers i and j such that $1 \leqslant i < j \leqslant 4$, denote by L_{ij} the proper transform on S via π of the line in $\mathbb{P}^2$ that passes through $\pi(E_i)$ and $\pi(E_j)$. These gives us six lines L_{12}, L_{13}, L_{14}, L_{23}, L_{24} and L_{34}. Moreover, E_1, E_1, E_2, E_3, E_4, L_{12}, L_{13}, L_{14}, L_{23}, L_{24} and L_{34} are all lines in S. Let

$$\hat{\alpha}\big(S,(1-\beta)C\big) = \min\Big\{1, \frac{1}{2\beta}\Big\} = \begin{cases} 1 \text{ for } 0 < \beta \leqslant \dfrac{1}{2}, \\ \dfrac{1}{2\beta} \text{ for } \dfrac{1}{2} \leqslant \beta \leqslant 1. \end{cases}$$

Then $\hat{\alpha}(S,(1-\beta)C) \geqslant \check{\alpha}(S,(1-\beta)C)$, since $2E_1 + L_{12} + L_{13} + L_{14} \sim -K_S$ and

$$\hat{\alpha}\big(S,(1-\beta)C\big) = \min\Big\{\mathrm{lct}\big(S,(1-\beta)C;\beta C\big), \mathrm{lct}\big(S,(1-\beta)C;\beta(2E_1+L_{12}+L_{13}+L_{14})\big)\Big\}.$$

2.7. **Complete intersections of two quadrics.** Suppose that $K_S^2 = 4$. Then there exists a birational morphism $\pi\colon S \to \mathbb{P}^2$ that is the blow up of five points such that no three of them are colinear. Denote by E_1, E_2, E_3, E_4 and E_5 the π-exceptional curves. For any integers i and j such that $1 \leqslant i < j \leqslant 5$, denote by L_{ij} the proper transform via π on S of the line in $\mathbb{P}^2$ that passes through $\pi(E_i)$ and $\pi(E_j)$. Denote by E the proper transform on S of the unique smooth conic in $\mathbb{P}^2$ that passes through $\pi(E_1)$, $\pi(E_2)$, $\pi(E_3)$, $\pi(E_4)$ and $\pi(E_5)$. Then E_1, E_2, E_3, E_4, E_5, L_{12}, L_{13}, L_{14}, L_{15}, L_{23}, L_{24}, L_{25}, L_{34}, L_{35}, L_{45} and E are all the lines in S.

If the curve C contains the intersection point of any two intersecting lines, then we let

$$\hat{\alpha}\big(S,(1-\beta)C\big) = \min\Big\{1, \frac{1+\beta}{3\beta}\Big\} = \begin{cases} 1 \text{ for } 0 < \beta \leqslant \dfrac{1}{2}, \\ \dfrac{1+\beta}{3\beta} \text{ for } \dfrac{1}{2} \leqslant \beta \leqslant 1. \end{cases}$$

If the curve C does not contain the intersection point of any two intersecting lines, but there are two conics C_1 and C_2 in S such that $C_1 + C_2 \sim -K_S$, and C_1 and C_2 both tangent C at one point, then we let

$$\hat{\alpha}\big(S,(1-\beta)C\big) = \min\Big\{1, \frac{1+2\beta}{4\beta}, \frac{2}{3\beta}\Big\} = \begin{cases} 1 \text{ for } 0 < \beta \leqslant \dfrac{1}{2}, \\ \dfrac{1+2\beta}{4\beta} \text{ for } \dfrac{1}{2} \leqslant \beta \leqslant \dfrac{5}{6}, \\ \dfrac{2}{3\beta} \text{ for } \dfrac{5}{6} \leqslant \beta \leqslant 1. \end{cases}$$

Finally, if the curve C does not contain the intersection point of any two intersecting lines, and for every two conics C_1 and C_2 in S such that $C_1 + C_2 \sim -K_S$, the conics C_1 and C_2 do not tangent C at one point, then we let

$$\hat{\alpha}\big(S,(1-\beta)C\big) = \min\Big\{1, \frac{2}{3\beta}\Big\} = \begin{cases} 1 \text{ for } 0 < \beta \leqslant \dfrac{2}{3}, \\ \dfrac{2}{3\beta} \text{ for } \dfrac{2}{3} \leqslant \beta \leqslant 1. \end{cases}$$

We claim that $\hat{\alpha}(S,(1-\beta)C) \geqslant \check{\alpha}(S,(1-\beta)C)$. Indeed, the lines L_{12} and L_{34} intersect at a single point. Let Z be the proper transform on S of the line in $\mathbb{P}^2$ that passes through $\pi(E_5)$

and $\pi(L_{12}\cap L_{34})$. Then $L_{12}+L_{34}+Z\sim -K_S$. Moreover, if $L_{12}\cap L_{34}\in C$, then

$$\min\Big\{\mathrm{lct}\big(S,(1-\beta)C;\beta C\big),\mathrm{lct}\big(S,(1-\beta)C;\beta(L_{12}+L_{34}+Z)\big)\Big\}=\begin{cases}1 \text{ for } 0<\beta\leqslant\dfrac{1}{2},\\ \dfrac{1+\beta}{3\beta} \text{ for } \dfrac{1}{2}\leqslant\beta\leqslant 1.\end{cases}$$

However, if $L_{12}\cap L_{34}\notin C$, then this minimum equals $\min\{1,\frac{2}{3\beta}\}$. Since we can repeat these computations for any pair of intersecting lines in S, we see that $\hat{\alpha}(S,(1-\beta)C)\geqslant\check{\alpha}(S,(1-\beta)C)$ except possibly the case when C does not contain the intersection point of any two intersecting lines, but there are two conics C_1 and C_2 in S such that $C_1+C_2\sim -K_S$, and C_1 and C_2 both tangent C at one point. In the latter case, $\hat{\alpha}(S,(1-\beta)C)$ is equal to

$$\min\Big\{\mathrm{lct}\big(S,(1-\beta)C;\beta C\big),\mathrm{lct}\big(S,(1-\beta)C;\beta(L_{12}+L_{34}+Z)\big),\mathrm{lct}\big(S,(1-\beta)C;\beta(C_1+C_2)\big)\Big\},$$

since $C_1+C_2\sim -K_S$. This shows that $\hat{\alpha}(S,(1-\beta)C)\geqslant\check{\alpha}(S,(1-\beta)C)$ in all three cases.

2.8. **Cubic surfaces.** Suppose that $K_S^2=3$. Then S is a smooth cubic surface in $\mathbb{P}^3$. Recall that an Eckardt point in S is a point of intersection of three lines contained in S. General cubic surface contains no Eckardt points. If S contains an Eckardt point that is contained in C, then we let

$$\hat{\alpha}\big(S,(1-\beta)C\big)=\min\Big\{1,\frac{1+\beta}{3\beta}\Big\}=\begin{cases}1 \text{ for } 0<\beta\leqslant\dfrac{1}{2},\\ \dfrac{1+\beta}{3\beta} \text{ for } \dfrac{1}{2}\leqslant\beta\leqslant 1.\end{cases}$$

If S contains an Eckardt point and C contains no Eckardt points, then we let

$$\hat{\alpha}\big(S,(1-\beta)C\big)=\min\Big\{1,\frac{2}{3\beta}\Big\}=\begin{cases}1 \text{ for } 0<\beta\leqslant\dfrac{2}{3},\\ \dfrac{2}{3\beta} \text{ for } \dfrac{2}{3}\leqslant\beta\leqslant 1.\end{cases}$$

If S contains no Eckardt points, but S contains a line L and a conic M such that L is tangent to M and $L\cap M\in C$, then we let

$$\hat{\alpha}\big(S,(1-\beta)C\big)=\min\Big\{1,\frac{2+\beta}{4\beta}\Big\}=\begin{cases}1 \text{ for } 0<\beta\leqslant\dfrac{2}{3},\\ \dfrac{2+\beta}{4\beta} \text{ for } \dfrac{2}{3}\leqslant\beta\leqslant 1.\end{cases}$$

If S contains no Eckardt points, for every line L and every conic M on S such that L is tangent to M, we have $L\cap M\notin C$, but there is a cuspidal curve $T\in|-K_S|$ such that $T\cap C=\mathrm{Sing}(T)$, then we let

$$\hat{\alpha}\big(S,(1-\beta)C\big)=\min\Big\{1,\frac{2+3\beta}{6\beta},\frac{3}{4\beta}\Big\}=\begin{cases}1 \text{ for } 0<\beta\leqslant\dfrac{2}{3},\\ \dfrac{2+3\beta}{6\beta} \text{ for } \dfrac{2}{3}\leqslant\beta\leqslant\dfrac{5}{6},\\ \dfrac{3}{4\beta} \text{ for } \dfrac{5}{6}\leqslant\beta\leqslant 1.\end{cases}$$

Finally, if S contains no Eckardt points, for every line L and every conic M on S such that L is tangent to M we have $L\cap M\notin C$, and every irreducible cuspidal curve $T\in|-K_S|$ intersects

C by at least two point, then we let

$$\hat{\alpha}\big(S,(1-\beta)C\big) = \min\Big\{1, \frac{3}{4\beta}\Big\} = \begin{cases} 1 \text{ for } 0 < \beta \leqslant \dfrac{3}{4}, \\ \dfrac{3}{4\beta} \text{ for } \dfrac{3}{4} \leqslant \beta \leqslant 1. \end{cases}$$

One can easily check that $\hat{\alpha}(S,(1-\beta)C) \geqslant \check{\alpha}(S,(1-\beta)C)$ (see [13, Theorem 4.9.1]).

2.9. **Double covers of $\mathbb{P}^2$.** Suppose that $K_S^2 = 2$. If $|-K_S|$ contains a tacnodal curve whose singular point is contained in C, then we let

$$\hat{\alpha}\big(S,(1-\beta)C\big) = \min\Big\{1, \frac{2+\beta}{4\beta}\Big\} = \begin{cases} 1 \text{ for } 0 < \beta \leqslant \dfrac{2}{3}, \\ \dfrac{2+\beta}{4\beta} \text{ for } \dfrac{2}{3} \leqslant \beta \leqslant 1. \end{cases}$$

If $|-K_S|$ contains a tacknodal curve, but C does not contain singular points of all tacknodal curves in $|-K_S|$, then we let

$$\hat{\alpha}\big(S,(1-\beta)C\big) = \min\Big\{1, \frac{3}{4\beta}\Big\} = \begin{cases} 1 \text{ for } 0 < \beta \leqslant \dfrac{3}{4}, \\ \dfrac{3}{4\beta} \text{ for } \dfrac{3}{4} \leqslant \beta \leqslant 1. \end{cases}$$

If $|-K_S|$ contains no curves with tacnodal singularities, but C contains the cuspidal singular point of a cuspidal rational curve in $|-K_S|$, then we let

$$\hat{\alpha}\big(S,(1-\beta)C\big) = \min\Big\{1, \frac{3+2\beta}{6\beta}\Big\} = \begin{cases} 1 \text{ for } 0 < \beta \leqslant \dfrac{3}{4}, \\ \dfrac{3+2\beta}{6\beta} \text{ for } \dfrac{3}{4} \leqslant \beta \leqslant 1. \end{cases}$$

Finally, if $|-K_S|$ contains no curves with tacnodal singularities, and C does not contain cuspidal singular points of all cuspidal rational curves in $|-K_S|$, then we let

$$\hat{\alpha}\big(S,(1-\beta)C\big) = \min\Big\{1, \frac{5}{6\beta}\Big\} = \begin{cases} 1 \text{ for } 0 < \beta \leqslant \dfrac{5}{6}, \\ \dfrac{5}{6\beta} \text{ for } \dfrac{5}{6} \leqslant \beta \leqslant 1. \end{cases}$$

One can easily check that $\hat{\alpha}(S,(1-\beta)C) \geqslant \check{\alpha}(S,(1-\beta)C)$ (see [13, Theorem 4.10.1]).

2.10. **Double covers of quadric cones.** Suppose that $K_S^2 = 1$. If $|-K_S|$ contains no cuspidal curves, then we let $\hat{\alpha}(S,(1-\beta)C) = 1$ for every $\beta \in (0,1]$. Otherwise, we let

$$\hat{\alpha}\big(S,(1-\beta)C\big) = \min\Big\{1, \frac{5}{6\beta}\Big\} = \begin{cases} 1 \text{ for } 0 < \beta \leqslant \dfrac{5}{6}, \\ \dfrac{5}{6\beta} \text{ for } \dfrac{5}{6} \leqslant \beta \leqslant 1. \end{cases}$$

In the former case, we have $\hat{\alpha}(S,(1-\beta)C) = \mathrm{lct}(S,(1-\beta)C;\beta C)$. In the latter case, we have

$$\hat{\alpha}\big(S,(1-\beta)C\big) = \min\Big\{\mathrm{lct}\big(S,(1-\beta)C;\beta C\big), \mathrm{lct}\big(S,(1-\beta)C;\beta Z\big)\Big\},$$

where Z is a cuspidal curve in $|-K_S|$. Thus, $\hat{\alpha}(S,(1-\beta)C) \geqslant \check{\alpha}(S,(1-\beta)C)$ in both cases.

3. Local inequalities

Let S be a smooth surface, let D be an effective $\mathbb{R}$-divisor on S, and let P be a point in S.

Lemma 3.1. Suppose that (S, D) is not log canonical at P. Then $\mathrm{mult}_P(D) > 1$.

Proof. This is a well-known fact. See [8, Exercise 6.18], for instance. □

Lemma 3.2. Suppose that (S, D) is not log canonical at P. Let B be an effective $\mathbb{R}$-divisor on S such that (S, B) is log canonical and $B \sim_{\mathbb{R}} D$. Then there exists an effective $\mathbb{R}$-divisor D' on S such that $D' \sim_{\mathbb{R}} D$, the log pair (S, D') is not log canonical at P, and $\mathrm{Supp}(D')$ does not contain at least one irreducible component of $\mathrm{Supp}(B)$.

Proof. Let μ be the greatest real number such that $D' := (1 + \mu)D - \mu B$ is effective. Since $D \neq B$, the number μ does exist. Then $D' \sim_{\mathbb{R}} D$, the log pair (S, D') is not log canonical at P, and $\mathrm{Supp}(D')$ does not contain at least one irreducible component of $\mathrm{Supp}(B)$. □

Let $\pi_1 \colon S_1 \to S$ be a blow up of the point P, let F_1 be the π-exceptional curve, and let D^1 be the proper transform of D via π_1. Then $K_{S_1} + D^1 + (\mathrm{mult}_P(D) - 1)F_1 \sim_{\mathbb{R}} \pi_1^*(K_S + D)$.

Lemma 3.3. Suppose that (S, D) is not log canonical at P. Then $\mathrm{mult}_P(D) > 1$ and there exists a point $P_1 \in F_1$ such that $(S_1, D^1 + (\mathrm{mult}_P(D) - 1)F_1)$ is not log canonical at P_1. Moreover, one has $\mathrm{mult}_P(D) + \mathrm{mult}_{P_1}(D^1) > 2$. If, in addition, $\mathrm{mult}_P(D) \leqslant 2$, then such point P_1 is unique.

Proof. This is a well-known fact. See, for example, [4, Remark 2.5]. □

Let C be an irreducible curve on S that contains P. Suppose that C is smooth at P. Write $D = aC + \Omega$, where $a \in \mathbb{R}_{\geqslant 0}$, and Ω is an effective $\mathbb{R}$-divisor on S with $C \not\subset \mathrm{Supp}(\Omega)$.

Theorem 3.4. If $(S, aC + \Omega)$ is not log canonical at P and $a \leqslant 1$, then $\mathrm{mult}_P(\Omega \cdot C) > 1$.

Proof. See, for example, [8, Exercise 6.31], [14, Lemma 2.5] or [3, Theorem 7]. □

Denote the proper transform of the curve C on the surface S_1 by C^1, and denote the proper transform of the $\mathbb{R}$-divisor Ω on the surface S_1 by Ω^1.

Lemma 3.5. Suppose that $a \leqslant 1$, the log pair $(S, aC + \Omega)$ is not log canonical at the point P, and $\mathrm{mult}_P(\Omega) \leqslant 1$. Then $(S_1, aC^1 + \Omega^1 + (a + \mathrm{mult}_P(\Omega) - 1)F_1)$ is not log canonical at $C^1 \cap F_1$, it is log canonical at every point in $E_1 \setminus (C^1 \cap F_1)$, and $\mathrm{mult}_P(\Omega \cdot C) > 2 - a$.

Proof. Since $a \leqslant 1$ and $\mathrm{mult}_P(\Omega) \leqslant 1$, we have $\mathrm{mult}_P(D) \leqslant 2$. By Lemma 3.3, there exists a unique point $P_1 \in F_1$ such that the log pair $(S_1, aC^1 + \Omega^1 + (a + \mathrm{mult}_P(\Omega) - 1)F_1)$ is not log canonical at P_1. If $P_1 \notin C^1$, then $\mathrm{mult}_P(\Omega) = F_1 \cdot \Omega^1 \geqslant \mathrm{mult}_{P_1}(\Omega^1 \cdot F_1) > 1$ by Theorem 3.4, which is impossible, since $\mathrm{mult}_P(\Omega) \leqslant 1$. Thus, $P_1 \in C^1$. Then, by Theorem 3.4 again:

$$\mathrm{mult}_P\Big(\Omega \cdot C\Big) \geqslant \mathrm{mult}_P(\Omega) + \mathrm{mult}_{P_1}\Big(\Omega^1 \cdot C^1\Big) > 2 - a. \qquad \square$$

Let us consider an *infinite* sequence of blow ups

$$\cdots \xrightarrow{\pi_{n+1}} S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_3} S_2 \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S$$

such that each π_n is the blow up of the point in the proper transform of the curve C on the surface S_{n-1} that dominates P. Denote the π_n-exceptional curve by F_n, and denote the proper transform of C on S_n by C^n. For every $n \geqslant 1$, write $P_n = C^n \cap F_n$, denote the proper transform of the divisor Ω on S_n by Ω^n, let $m_n = \mathrm{mult}_{P_n}(\Omega^n)$ and let $m_0 = \mathrm{mult}_P(\Omega)$. For every positive integers $k \leqslant n$, denote the proper transform of the curve F_k on S_n by F_k^n. Finally, we let

$$D^{S_n} = aC^n + \Omega^n + \sum_{k=1}^{n} \Big(ka - k + \sum_{i=0}^{k-1} m_i\Big) F_k^n$$

for every $n \geqslant 1$. Then $K_{S_n} + D^{S_n} \sim_{\mathbb{R}} (\pi_1 \circ \pi_2 \circ \cdots \circ \pi_n)^*(K_S + D)$ for every $n \geqslant 1$.

Theorem 3.6. Suppose that $(S, aC+\Omega)$ is not log canonical at P and $a \leqslant 1$. Then $m_0+a>1$ and $\mathrm{mult}_P(\Omega\cdot C)>1$. Moreover, the following additional assertions hold:

(i) if $m_0 \leqslant 1$, then the log pair (S_1, D^{S_1}) is not log canonical at P_1,

(ii) if (S_n, D^{S_n}) is not log canonical at some point in F_n, then D^{S_n} is an effective divisor,

(iii) if (S_n, D^{S_n}) is not log canonical at some point in F_n and $\sum_{i=0}^{n-1} m_i \leqslant n+1-na$, then such point in F_n is unique,

(iv) if (S_n, D^{S_n}) is not log canonical at P_n, then $(n+1)a+\sum_{i=0}^{n} m_i > n+2$, the log pair $(S_{n+1}, D^{S_{n+1}})$ is not log canonical at some point in F_{n+1}, and $\mathrm{mult}_P(\Omega\cdot C)>n+1-na$,

(v) if $n\geqslant 2$, $m_{n-1}\leqslant 1$ and $\sum_{i=0}^{n-1} m_i \leqslant n+1-na$, then (S_n, D^{S_n}) is log canonical at every point of F_n different from P_n and $F_n\cap F_{n-1}^n$,

(vi) if $n\geqslant 2$ and $\sum_{i=0}^{n-1} m_i \leqslant n-(n-1)a$, then (S_n, D^{S_n}) is log canonical at $F_n\cap F_{n-1}^n$,

(vii) if $n\geqslant 2$, $\sum_{i=0}^{n-2} m_i\leqslant n-(n-1)a$, and $\sum_{i=0}^{n-3} m_i+2m_{n-2}\leqslant n+1-na$, then (S_n, D^{S_n}) is log canonical at $F_n\cap F_{n-1}^n$.

Proof. By Lemma 3.1, we have $m_0+a>1$. By Theorem 3.4, we have $\mathrm{mult}_P(\Omega\cdot C)>1-a$. Assertion (i) follows from Lemma 3.5. If (S_n, D^{S_n}) is not log canonical at some point in F_n, then $(S_{n-1}, D^{S_{n-1}})$ is not log canonical at P^{n-1}. Thus, assertion (ii) follows from Lemma 3.1. Inequality $\sum_{i=0}^{n-1} m_i\leqslant n+1-na$ is equivalent to $\mathrm{mult}_{P_{n-1}}(D^{S_{n-1}})\leqslant 2$. Thus, assertion (iii) follows from Lemma 3.3. If (S_n, D^{S_n}) is not log canonical at P_n, then $(n+1)a+\sum_{i=0}^{n} m_i>n+2$ by Lemma 3.1, the pair $(S_{n+1}, D^{S_{n+1}})$ is not log canonical at some point in F_{n+1} by Lemma 3.3, and

$$\mathrm{mult}_P\Big(\Omega\cdot C\Big)-\sum_{i=0}^{n-1} m_i=\mathrm{mult}_{P_n}\Big(\Omega^n\cdot C^n\Big)>1-\Big(na-n+\sum_{i=0}^{n-1} m_i\Big),$$

by Theorem 3.4. This proves assertion (iv).

Suppose that $n\geqslant 2$. Let $O=F_n\cap F_{n-1}^n$. If $\sum_{i=0}^{n-1} m_i\leqslant n+1-na$ and (S_n, D^{S_n}) is not log canonical at some point in $F_n\setminus(P_n\cup O)$, then $m_{n-1}=F_n\cdot\Omega^n>1$ by Theorem 3.4, which implies assertion (v). If (S_n, D^{S_n}) is not log canonical at O and $\sum_{i=0}^{n-1} m_i\leqslant n+1-na$, then

$$m_{n-1}=F_n\cdot\Omega^n\geqslant \mathrm{mult}_O\Big(F_n\cdot\Omega^n\Big)>1-\Big((n-1)a-n+1+\sum_{i=0}^{n-2} m_i\Big)$$

by Theorem 3.4. If (S_n, D^{S_n}) is not log canonical at O and $\sum_{i=0}^{n-2} m_i\leqslant n-(n-1)a$, then

$$m_{n-2}-m_{n-1}=F_{n-1}^n\cdot\Omega^n\geqslant \mathrm{mult}_O\Big(F_{n-1}^n\cdot\Omega^n\Big)>1-\Big(na-n+\sum_{i=0}^{n-1} m_i\Big)$$

by Theorem 3.4. This proves assertions (vi) and (vii). □

Corollary 3.7. Suppose that $(S, aC+\Omega)$ is not log canonical at P, $C\not\subset \mathrm{Supp}(\Omega)$, $a\leqslant 1$ and $m_0\leqslant \min\{1, 1+\frac{1}{n}-na\}$ for some integer $n\geqslant 1$. Then $\mathrm{mult}_P(\Omega\cdot C)>n+1-na$.

Corollary 3.8. Suppose that $(S, aC+\Omega)$ is not log canonical at P, $a\leqslant 1$ and $m_0\leqslant 1$. Suppose that $2m_0\leqslant 3-2a$ or $m_0+m_1\leqslant 2-a$. Suppose that $m_0+2m_1\leqslant 4-3a$ or $m_0+m_1+m_2\leqslant 3-2a$. Then $\mathrm{mult}_P(\Omega\cdot C)>4-3a$. If $m_0+m_1+2m_2\leqslant 5-4a$ or $m_0+m_1+m_2+m_3\leqslant 4-3a$, then $\mathrm{mult}_P(\Omega\cdot C)>5-4a$.

Let us conclude this section by recalling

Theorem 3.9 ([3, Theorem 13])**.** Let C_1 and C_2 be two irreducible curves on S that are both smooth at P and intersect transversally at P. Let $D=a_1C_1+a_2C_2+\Delta$, where a_1 and a_2 are non-negative real numbers, and Δ is an effective $\mathbb{R}$-divisor on S whose support does not contain the curves C_1 and C_2. If (S,D) is not log canonical at P and $\mathrm{mult}_P(\Delta)\leqslant 1$, then $\mathrm{mult}_P(\Delta\cdot C_1)>2(1-a_2)$ or $\mathrm{mult}_P(\Delta\cdot C_2)>2(1-a_1)$.

4. The proof

Let us use the notation of Section 2. The goal of this section is to prove

Theorem 4.1. One has $\alpha(S,(1-\beta)C)=\hat{\alpha}(S,(1-\beta)C)$ for every $\beta\in(0,1]$.

This theorem implies Theorem 1.12, since $\hat{\alpha}(S,(1-\beta)C)\geqslant\check{\alpha}(S,(1-\beta)C)$ (see Section 2) and $\check{\alpha}(S,(1-\beta)C)\geqslant\alpha(S,(1-\beta)C)$ (by definition) for every $\beta\in(0,1]$.

Let D be *any* effective $\mathbb{R}$-divisor such that $D\sim_{\mathbb{R}}-K_S$, and let P be *any* point in S. Since $\alpha(S,(1-\beta)C)\leqslant\hat{\alpha}(S,(1-\beta)C)$, to prove Theorem 4.1, it is enough to show that the log pair

$$\Big(S,(1-\beta)C+\hat{\alpha}(S,(1-\beta)C)\beta D\Big) \tag{4.2}$$

is log canonical at P for every $\beta\in(0,1]$. We will do this in several steps.

Lemma 4.3. Suppose that (4.2) is not log canonical at P. Then $P\in C$, we have

$$\mathrm{mult}_P(D)>\frac{1}{\hat{\alpha}(S,(1-\beta)C)}\geqslant 1,$$

and (4.2) is log canonical outside of the point P. Moreover, if there exists a (-1)-curve $Z\subset S$ such that $P\in Z$, then $Z\subset\mathrm{Supp}(D)$. Furthermore, there exists an effective $\mathbb{R}$-divisor $D'\sim_{\mathbb{R}}D$ such that $C\not\subset\mathrm{Supp}(D')$ and $(S,(1-\beta)C+\hat{\alpha}(S,(1-\beta)C)\beta D')$ is not log canonical at P.

Proof. If $P\notin C$, then $(S,\hat{\alpha}(S,(1-\beta)C)\beta D)$ is not log canonical at P, which is impossible, since $\alpha(S)\leqslant\beta\hat{\alpha}(S,(1-\beta)C)$ by [2, Theorem 1.7]. We have $\hat{\alpha}(S,(1-\beta)C)\mathrm{mult}_P(D)>1$ by Lemma 3.1. In particular, if there exists a (-1)-curve $Z\subset S$ such that $P\in Z$, then Z must be contained in $\mathrm{Supp}(D)$, because otherwise we would have $1=Z\cdot D\geqslant\mathrm{mult}_P(D)>1$.

We see that (4.2) is log canonical outside of the curve C. Moreover, the coefficient of the curve C in the divisor $(1-\beta)C+\hat{\alpha}(S,(1-\beta)C)\beta D$ does not exceed 1, since $D\sim_{\mathbb{R}}C$. Hence, the log pair (4.2) is log canonical outside of finitely many points. Now the connectedness principle (see, for example, [8, Theorem 6.32]) implies that (4.2) is log canonical outside of P.

Since $(S,(1-\beta)C+\hat{\alpha}(S,(1-\beta)C)\beta C)$ is log canonical, it follows from Lemma 3.2 that there is an effective $\mathbb{R}$-divisor $D'\sim_{\mathbb{R}}D$ such that $C\not\subset\mathrm{Supp}(D')$ and $(S,(1-\beta)C+\hat{\alpha}(S,(1-\beta)C)\beta D')$ is not log canonical at P. ∎

Thus, to prove that (4.2) is log canonical at P, we may assume that $P\in C\not\subset\mathrm{Supp}(D)$.

Lemma 4.4. If $S\cong\mathbb{P}^2$, then (4.2) is log canonical at P.

Proof. Suppose (4.2) is not log canonical at P. Let L be a general line in S that contains P. Then $\mathrm{mult}_P(D)\leqslant D\cdot L=3$. But $3\hat{\alpha}(S,(1-\beta)C)\beta\leqslant\frac{1}{3}+\beta$ (see §2.1). Thus, if $\beta\leqslant\frac{2}{3}$, then

$$\hat{\alpha}(S,(1-\beta)C)\beta\mathrm{mult}_P(D)\leqslant 3\hat{\alpha}(S,(1-\beta)C)\beta\leqslant\frac{1}{3}+\beta\leqslant 1.$$

Similarly, if $\frac{2}{3}\leqslant\beta\leqslant 1$, then $\hat{\alpha}(S,(1-\beta)C)\beta\mathrm{mult}_P(D)\leqslant\frac{1}{3}\mathrm{mult}_P(D)\leqslant 1$. Applying Corollary 3.7 with $n=3$ to (4.2), we get

$$9\beta\hat{\alpha}(S,(1-\beta)C)=\hat{\alpha}(S,(1-\beta)C)\beta(C\cdot D)\geqslant\hat{\alpha}(S,(1-\beta)C)\beta\mathrm{mult}_P\Big(C\cdot D\Big)>1+3\beta,$$

which contradicts the definition of $\hat{\alpha}(S,(1-\beta)C)$ in §2.1. ∎

Lemma 4.5. Suppose that $S\cong\mathbb{P}^1\times\mathbb{P}^1$. Then (4.2) is log canonical at P.

Proof. Suppose that (4.2) is not log canonical at P. Let L_1 and L_2 be the fibers of two different projections $S\to\mathbb{P}^1$ that both pass through P. Since $(S,(1-\beta)C+\hat{\alpha}(S,(1-\beta)C)\beta(2L_1+2L_2))$ is log canonical and $2L_1+2L_2 \sim_{\mathbb{R}} D$, we may assume that either $L_1 \not\subset \mathrm{Supp}(D)$ or $L_2 \not\subset \mathrm{Supp}(D)$ by Lemma 3.2. This implies that $\mathrm{mult}_P(D) \leqslant 2$, since $D\cdot L_1 = D\cdot L_2 = 2$. Then

$$\hat{\alpha}(S,(1-\beta)C)\beta\mathrm{mult}_P(D) \leqslant 2\hat{\alpha}(S,(1-\beta)C)\beta \leqslant \min\Big\{1, \frac{1}{4}+\beta\Big\},$$

(see §2.2). Applying Corollary 3.7 with $n=4$, we get

$$8\hat{\alpha}(S,(1-\beta)C)\beta = \hat{\alpha}(S,(1-\beta)C)\beta(C\cdot D) \geqslant \hat{\alpha}(S,(1-\beta)C)\beta\mathrm{mult}_P\Big(C\cdot D\Big) > 1+4\beta,$$

which contradicts the definition of $\hat{\alpha}(S,(1-\beta)C)$ in §2.2. □

Lemma 4.6. Suppose that $K_S^2 \leqslant 3$. Then (4.2) is log canonical at P.

Proof. Suppose that (4.2) is not log canonical at P. By [4, Theorem 1.12], there is $T \in |-K_S|$ such that (S,T) is not log canonical at P, and *all* irreducible components of the curve T are contained in the support of the divisor D. Moreover, such T is unique.

Since (S,T) is not log canonical at P, we have very limited number of choices for $T\in|-K_S|$. Going through all of them, we see that $(S,(1-\beta)C+\hat{\alpha}(S,(1-\beta)C)\beta T)$ is log canonical at P (for details, see the proofs of [13, Theorems 4.9.1, 4.10.1, 4.11.1]).

By Lemma 3.2, there is an effective $\mathbb{R}$-divisor D' on the surface S such that $D' \sim_{\mathbb{R}} D$, the log pair $(S,(1-\beta)C+\hat{\alpha}(S,(1-\beta)C)\beta D')$ is not log canonical at P, and $\mathrm{Supp}(D')$ does not contain at least one irreducible component of T. The latter contradicts [4, Theorem 1.12]. □

Corollary 4.7. Theorem 4.1 holds in the following cases: $S\cong\mathbb{P}^2$, $S\cong\mathbb{P}^1\times\mathbb{P}^1$ and $K_S^2\leqslant 3$.

Lemma 4.8. Suppose that $4 \leqslant K_S^2 \leqslant 7$, and P is the intersection point of two intersecting (-1)-curves in S. Then (4.2) is log canonical at P.

Proof. Suppose that (4.2) is not log canonical at P. Denote by Z_1 and Z_2 two (-1)-curves in S that contains P. We write $D = aZ_1+bZ_2+\Omega$, where a and b are non-negative real numbers, and Ω is an effective $\mathbb{R}$-divisor that whose support does not contain Z_1 and Z_2. By Lemma 4.3, one has $a>0$ and $b>0$. Let $x=\mathrm{mult}_P(\Omega)$. Then $1-b+a=\Omega\cdot Z_1\geqslant x$, which gives $b-a+x\leqslant 1$. Similarly, we obtain $a-b+x\leqslant 1$. Then $a\leqslant 1+b$, $b\leqslant 1+a$ and $x\leqslant 1$. Thus, we have

$$\mathrm{mult}_P\Big((1-\beta)C+\hat{\alpha}(S,(1-\beta)C)\beta\Omega\Big) = 1-\beta+\hat{\alpha}(S,(1-\beta)C)\beta x \leqslant 1-\beta+\hat{\alpha}(S,(1-\beta)C)\beta \leqslant 1,$$

because $\hat{\alpha}(S,(1-\beta)C)\leqslant 1$. Applying Theorem 3.9 to (4.2), we see that

$$2\Big(1-\hat{\alpha}(S,(1-\beta)C\Big)\beta a) < Z_1\cdot\Big(\hat{\alpha}(S,(1-\beta)C)\beta\Omega+(1-\beta)C\Big) = \hat{\alpha}(S,(1-\beta)C)\beta(1-a+b)+1-\beta,$$

or

$$2\Big(1-\hat{\alpha}(S,(1-\beta)C)\beta b\Big) < Z_2\cdot\Big(\hat{\alpha}(S,(1-\beta)C)\beta\Omega+(1-\beta)C\Big) = \hat{\alpha}(S,(1-\beta)C)\beta\Big(1-b+a\Big)+1-\beta.$$

In both cases, we obtain $\hat{\alpha}(S,(1-\beta)C)\beta(1+a+b) > 1+\beta$.

Suppose that $K_S^2=7$. Let us use the notation of §2.4. We may assume that $Z_1=E_1$ and $Z_2=L$. Since $3L+2E_1+2E_2\sim -K_S$ and $(S,(1-\beta)C+\hat{\alpha}(S,(1-\beta)C)\beta(3L+2E_1+2E_2)$ is log canonical, we may also assume that $E_2\not\subset\mathrm{Supp}(\Omega)$ by Lemma 3.2. Then $1-b=E_2\cdot\Omega\geqslant 0$, which gives $b\leqslant 1$. Since $a\leqslant 1+b$, we get $a+b\leqslant 3$. Thus, we have

$$4\beta\hat{\alpha}(S,(1-\beta)C) \geqslant \hat{\alpha}(S,(1-\beta)C)\beta\Big(1+a+b\Big) > 1+\beta,$$

which contradicts the definition of $\hat{\alpha}(S,(1-\beta)C)$.

Suppose that $K_S^2=6$. Let us use the notation of §2.5. Without loss of generality, we may assume that $Z_1=E_1$ and $Z_2=L_{12}$. Since $(S,(1-\beta)C+\hat{\alpha}(S,(1-\beta)C)\beta(2L_{12}+2E_1+L_{13}+E_2))$ is log canonical and $2L_{12}+2E_1+L_{13}+E_2\sim -K_S$, we may assume that $\mathrm{Supp}(\Omega)$ does not contain L_{13} or E_2 by Lemma 3.2. If $L_{13}\not\subset\mathrm{Supp}(\Omega)$, then $1-a=\Omega\cdot L_{13}\geqslant 0$, which implies that $a \leqslant 1$. Similarly, if $E_2 \not\subset \mathrm{Supp}(\Omega)$, then $b \leqslant 1$. Since $a \leqslant 1+b$ and $b \leqslant 1+a$, we see that $a+b \leqslant 3$. Thus, we have

$$4\beta\hat{\alpha}(S,(1-\beta)C) \geqslant \hat{\alpha}(S,(1-\beta)C)\beta\Big(1+a+b\Big) > 1+\beta,$$

which contradicts the definition of $\hat{\alpha}(S,(1-\beta)C)$.

Suppose that $K_S^2 = 5$. Let us use the notation of §2.6. Without loss of generality, we may assume that $Z_1 = E_1$ and $Z_2 = L_{12}$. Since $(S,(1-\beta)C+\hat{\alpha}(S,(1-\beta)C)\beta(2E_1+L_{12}+L_{13}+L_{14}))$ is log canonical and $2E_1+L_{12}+L_{13}+L_{14} \sim -K_S$, we may assume that $\mathrm{Supp}(\Omega)$ does not contain L_{13} or L_{14} by Lemma 3.2. Since $(S,(1-\beta)C+\hat{\alpha}(S,(1-\beta)C)\beta(E_1+2L_{12}+E_2+L_{34}))$ is log canonical and $E_1+2L_{12}+E_2+L_{34} \sim -K_S$, we may assume that $\mathrm{Supp}(\Omega)$ does not contain E_2 or L_{34} by Lemma 3.2. If $L_{13} \not\subset \mathrm{Supp}(\Omega)$, then $1-a = \Omega\cdot L_{13} \geqslant 0$, which gives $a \leqslant 1$. Similarly, if $L_{14} \not\subset \mathrm{Supp}(\Omega)$, then $a \leqslant 1$. If $E_2 \not\subset \mathrm{Supp}(\Omega)$, then $1-b = \Omega\cdot E_2 \geqslant 0$, which gives $b \leqslant 1$. Similarly, if $L_{34} \not\subset \mathrm{Supp}(\Omega)$, then $b \leqslant 1$. Thus, we have $a \leqslant 1$ and $b \leqslant 1$. Then

$$3\beta\hat{\alpha}(S,(1-\beta)C) \geqslant \hat{\alpha}(S,(1-\beta)C)\beta\Big(1+a+b\Big) > 1+\beta,$$

which contradicts the definition of $\hat{\alpha}(S,(1-\beta)C)$.

We have $K_S^2 = 4$. Let us use the notation of §2.7. Without loss of generality, we may assume that $Z_1 = L_{12}$ and $Z_2 = L_{34}$. Let Z be the proper transform on S of the line in $\mathbb{P}^2$ that passes through $\pi(E_5)$ and $\pi(L_{12}\cap L_{34})$. Since $(S,(1-\beta)C+\hat{\alpha}(S,(1-\beta)C)\beta(L_{12}+L_{34}+Z))$ is log canonical and $L_{12}+L_{34}+Z \sim -K_S$, we may assume that $Z \not\subset \mathrm{Supp}(\Omega)$ by Lemma 3.2. Then $2-a-b = \Omega\cdot Z \geqslant 0$, which implies that $3\beta\hat{\alpha}(S,(1-\beta)C) \geqslant \hat{\alpha}(S,(1-\beta)C)\beta(1+a+b) > 1+\beta$. The latter contradicts the definition of $\hat{\alpha}(S,(1-\beta)C)$. □

Lemma 4.9. Suppose $S \cong \mathbb{F}_1$, and P is contained in a unique (-1)-curve in S. Then (4.2) is log canonical at P.

Proof. Let us use the notation of §2.3. Then $P = Z\cap C$, since $P \in C$. Suppose that (4.2) is not log canonical at P. By Lemma 4.3, we have $Z \subset \mathrm{Supp}(D)$. By Lemma 3.2, we may assume that $F \not\subset \mathrm{Supp}(D)$, since $(S,(1-\beta)C+\hat{\alpha}(S,(1-\beta)C)\beta(2Z+3F))$ is log canonical and $2Z+3F \sim -K_S$. Then $\mathrm{mult}_P(D) \leqslant F\cdot D = 2$. On the other hand, we have $2\hat{\alpha}(S,(1-\beta)C)\beta \leqslant \frac{1}{4}+\beta$ and $2\hat{\alpha}(S,(1-\beta)C)\beta \leqslant 1$. Applying Corollary 3.7 with $n=4$ to (4.2), we get

$$8\hat{\alpha}(S,(1-\beta)C)\beta = \hat{\alpha}(S,(1-\beta)C)\beta(C\cdot D) \geqslant \hat{\alpha}(S,(1-\beta)C)\beta\mathrm{mult}_P\Big(C\cdot D\Big) > 1+4\beta,$$

which contradicts the definition of $\hat{\alpha}(S,(1-\beta)C)$. □

Lemma 4.10. Suppose that $4 \leqslant K_S^2 \leqslant 7$, and P is contained in a (-1)-curve in S. Then (4.2) is log canonical at P.

Proof. See Section 5. □

The following result implies Corollary 1.14 *modulo* Theorem 4.1.

Theorem 4.11. Let S_1 and S_2 be smooth del Pezzo surfaces, let C_1 and C_2 be smooth curves in $|-K_{S_1}|$ and $|-K_{S_2}|$, respectively. Suppose that there exists a birational morphism $f\colon S_2 \to S_1$ such that $f(C_2) = C_1$. Then $\hat{\alpha}(S_1,(1-\beta)C_1) \leqslant \hat{\alpha}(S_2,(1-\beta)C_2)$ for every $\beta \in (0,1]$ except the following cases:

(1) $S_1 \cong \mathbb{P}^2$, $S_2 \cong \mathbb{F}_1$, and f is the blow up of an inflection points of the cubic curve $C_1 \subset \mathbb{P}^2$,
(2) $S_1 \cong \mathbb{P}^1\times\mathbb{P}^1$, $K_{S_2}^2 = 7$, and f is the blow up of a point in C_1.

Proof. Since $f(C_2) = C_1$, the morphism f is the blow up of $K_{S_1}^2 - K_{S_2}^2 \geqslant 0$ distinct points on the curve C_2. Suppose that $\hat{\alpha}(S_1,(1-\beta)C_1) > \hat{\alpha}(S_2,(1-\beta)C_2)$. Going through all possible cases considered in Section 2, we end up with the following possibilities:

(1) $S_1 \cong \mathbb{P}^2$, $S_2 \cong \mathbb{F}_1$, and f is the blow up of an inflection points of the cubic curve $C_1 \subset \mathbb{P}^2$,

(2) $S_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$, $K_{S_2}^2 = 7$, and f is the blow up of a point in C_1,
(3) $K_{S_1}^2 = 4$, $K_{S_2}^2 = 3$, the morphism f is the blow up of a point in C_1, the curve C_1 does not contain intersection points of any two lines, for every two conics Z_1 and Z_2 in S_1 such that $Z_1 + Z_2 \sim -K_{S_1}$, the conics Z_1 and Z_2 do not tangent C_1 at one point, and S_2 contains an Eckardt point and this point is contained in C_2,
(4) $K_{S_1}^2 = 3$, $K_{S_2}^2 = 2$, the morphism f is the blow up of a point in C_1, the surface S_1 contains no Eckardt points, for every line L and every conic M on S_1 such that L is tangent to M we have $L \cap M \notin C_1$, and every irreducible cuspidal curve $T \in |-K_{S_1}|$ intersects C_1 by at least two point, the linear system $|-K_{S_2}|$ contains a curve with a tacnodal singularity and this tacnodal singular point is contained in C_2.

The first two cases are indeed possible. Let us show that the last two cases are impossible. Denote by E the f-exceptional curve. Then $f(E) \in C_1$.

Suppose that $K_{S_1}^2 = 4$ and $K_{S_2}^2 = 3$. Then C_2 contains an Eckardt point O. Denote by L_1, L_2, L_3 the lines in S_2 that passes through O. Then either E is one of these three lines, or E intersects exactly one of them. Without loss of generality, we may assume that either $E = L_3$ or $E \cap L_1 = E \cap L_3 = \varnothing$. In the former case, $f(L_1)$ and $f(L_2)$ are two conics in S_1 such that $f(L_1) + f(L_2) \sim -K_{S_2}$, and both $f(L_1)$ and $f(L_2)$ tangent the curve $C_1 = f(C_2)$ at the point $f(P) \in C_1$. Since we know that such conics do not exist by assumption, we conclude that $E \cap L_1 = E \cap L_3 = \varnothing$. Then $f(L_1)$ and $f(L_2)$ are two lines in S_1 that both pass through the point $f(P) \in C_1$. Such lines do not exist either. Thus, this case is impossible.

Now we suppose that $K_{S_1}^2 = 3$ and $K_{S_2}^2 = 2$. Let Z be a curve in $|-K_{S_2}|$ such that Z has tacnodal singularity $Q \in C_2$. Then $Z = L_1 + L_2$, where L_1 and L_2 are two (-1)-curves in S_2 that are tangent each other at the point $Q \in C_2$. Then either E is one of these two curves, or E intersects exactly one of them. Without loss of generality, we may assume that either $E = L_2$ or $E \cap L_1 = \varnothing$. In the former case, $f(L_1)$ is a cuspidal curve in $|-K_{S_1}|$ whose intersection with the curve C_1 consists of the point $f(Q) = \mathrm{Sing}(f(L_1))$. By assumption, such a cuspidal curve does not exist. Thus, $E \cap L_1 = \varnothing$. Then $f(L_1)$ is a line, and $f(L_2)$ is a conic. Moreover, the line $f(L_1)$ tangents to $f(L_2)$ at the point $f(Q) \in C_1$. The latter is impossible by assumption. ☐

To prove Theorem 4.1, we have to prove that (4.2) is log canonical at P, where P is a point in $C \not\subset \mathrm{Supp}(D)$. The latter follows from Corollary 4.7, Lemmas 4.8, 4.9, 5.9, 4.10 and

Lemma 4.12. Suppose that $K_S^2 \geqslant 3$, and neither $S \cong \mathbb{P}^2$ nor $S \cong \mathbb{P}^1 \times \mathbb{P}^1$. Suppose that P is not contained in any (-1)-curve in S. If Theorem 4.1 holds for all smooth del Pezzo surfaces of degree $K_S^2 - 1$, then (4.2) is log canonical at P.

Proof. Suppose that (4.2) is not log canonical at P. Let $f\colon \tilde{S} \to S$ be a blow up of P. Then $\tilde{S}$ is a smooth del Pezzo surface of degree $K_{\tilde{S}}^2 = K_S^2 - 1$, since P is not contained in any (-1)-curve in S. Denote the f-exceptional curve by E, denote the proper transform of C on $\tilde{S}$ by $\tilde{C}$, and denote the proper transform of D on $\tilde{S}$ by $\tilde{D}$. Then $\tilde{C} \in |-K_{\tilde{S}}|$, since $P \in C$. The log pair

$$(4.13)\qquad \Bigg(\tilde{S}, (1-\beta)\tilde{C} + \hat{\alpha}(S,(1-\beta)C)\beta\Big(\tilde{D} + \Big(\mathrm{mult}_P(D) - \frac{1}{\hat{\alpha}(S,(1-\beta)C)}\Big)E\Big)\Bigg)$$

is not log canonical by Lemma 3.3. Let $\tilde{D}' = \tilde{D} + (\mathrm{mult}_P(D) - 1)E$. Then $\tilde{D}' \sim_{\mathbb{R}} -K_{\tilde{S}}$, and $\tilde{D}'$ is effective by Lemma 4.3. Furthermore, the log pair $(\tilde{S}, (1-\beta)\tilde{C} + \hat{\alpha}(S,(1-\beta)C)\beta\tilde{D}')$ is not log canonical, because (4.13) is not log canonical. This shows that $\hat{\alpha}(S,(1-\beta)C) > \alpha(\tilde{S},(1-\beta)\tilde{C})$. But it follows from Theorem 4.11 that $\hat{\alpha}(\tilde{S},(1-\beta)\tilde{C}) \geqslant \hat{\alpha}(S,(1-\beta)C)$. Thus, we see that $\hat{\alpha}(\tilde{S},(1-\beta)\tilde{C}) > \alpha(\tilde{S},(1-\beta)\tilde{C})$. Hence, Theorem 4.1 does not hold for $\tilde{S}$. ☐

This completes the proof of Theorem 4.1 *modulo* Lemma 4.10.

5. The proof of Lemma 4.10

In this section, we will prove Lemma 4.10. Let us use its notation and assumptions. Then $4 \leqslant K_S^2 \leqslant 7$ and P is a point in $C \not\subset \mathrm{Supp}(D)$ that is contained in a (-1)-curve in S. Let us denote this (-1)-curve by $\mathcal{L}$. We must prove that (4.2) is log canonical at P. By Lemma 4.8, we may assume that $\mathcal{L}$ is the only (-1)-curve in S that contains P. We write $D = a\mathcal{L} + \Omega$, where a is a non-negative real number, and Ω is an effective $\mathbb{R}$-divisor such that $\mathcal{L} \not\subset \mathrm{Supp}(\Omega)$. By Lemma 4.3, we have $a > 0$. Let $x = \mathrm{mult}_P(\Omega)$. Then $1 + a = \mathcal{L} \cdot \Omega \geqslant x$.

Corollary 5.1. One has $x \leqslant 1 + a$.

Let $\lambda = \hat{\alpha}(S, (1-\beta)C)$. Consider a sequence of 4 blow ups

$$S_4 \xrightarrow{\pi_4} S_3 \xrightarrow{\pi_3} S_2 \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S$$

such that π_1 is the blow up of the point P, π_2 is the blow up of the intersection point of the π_1-exceptional curve and the proper transform of the curve C on S_1, π_3 is the blow up of the intersection point of the π_2-exceptional curve and the proper transform of the curve C on S_2, and π_4 is the blow up of the intersection point of the π_3-exceptional curve and the proper transform of the curve C on S_3. Denote by F_1, F_2, F_3 and F_4 the exceptional curves of the blow ups π_1, π_2, π_3 and π_4, respectively. Denote by C^1, C^2, C^3 and C^4 the proper transforms of the curve C on the surfaces S_1, S_2, S_3 and S_4, respectively. Let $P_1 = C^1 \cap F_1$, $P_2 = C^2 \cap F_2$, $P_3 = C^3 \cap F_3$ and $P_4 = C^4 \cap F_4$. Denote the proper transform of the divisor Ω on the surfaces S_1, S_2, S_3 and S_4 by Ω^1, Ω^2, Ω^3 and Ω^4, respectively. Let $x_1 = \mathrm{mult}_{P_1}(\Omega)$, $x_2 = \mathrm{mult}_{P_2}(\Omega)$ and $x_3 = \mathrm{mult}_{P_3}(\Omega)$.

Lemma 5.2. Suppose that (4.2) is not log canonical at P. Then at least one of the following four conditions is not satisfied:

(i) $\lambda\beta(a + x) \leqslant 1$,
(ii) $2\lambda\beta(a + x) - 2\beta \leqslant 1$ or $\lambda\beta(a + x + x_1) - \beta \leqslant 1$,
(iii) $\lambda\beta(a + x + 2x_1) - 3\beta \leqslant 1$ or $\lambda\beta(a + x + x_1 + x_2) - 2\beta \leqslant 1$,
(iv) $\lambda\beta(a + x + x_1 + 2x_2) - 4\beta \leqslant 1$ or $\lambda\beta(a + x + x_1 + x_2 + x_3) - 3\beta \leqslant 1$.

If $\lambda\beta K_S^2 \leqslant 1 + 3\beta$, then at least one of the conditions (i), (ii) or (iii) is not satisfied.

Proof. If conditions (i), (ii), (iii) and (iv) are satisfied, then Corollary 3.8 gives

$$K_S^2 = D \cdot C \geqslant \mathrm{mult}_P\Big(D \cdot C\Big) > \frac{1 + 4\beta}{\lambda\beta},$$

which is impossible, since $\lambda\beta K_S^2 \leqslant 1+4\beta$ by the definition of $\lambda = \hat{\alpha}(S,(1-\beta)C)$ for $4 \leqslant K_S^2 \leqslant 7$. Similarly, if conditions (i), (ii), (iii) are satisfied, then $\lambda\beta K_S^2 > 1 + 3\beta$ by Corollary 3.8. □

Lemma 5.3. Suppose that $K_S^2 = 7$. Then (4.2) is log canonical at P.

Proof. Suppose that (4.2) is not log canonical at P. Let us use the notation of §2.4. Without loss of generality, we may assume that either $\mathcal{L} = E_1$ or $\mathcal{L} = L$ (but not both).

Suppose that $\mathcal{L} = L$. Since $P \notin E_1 \cup E_2$, the curve R is smooth and irreducible. Since $(S, (1-\beta)C, \lambda\beta(L+2R))$ is log canonical and $L+2R \sim -K_S$, we may assume that $R \not\subset \mathrm{Supp}(\Omega)$. Denote the proper transform of the curve R on S_1 by R^1, and denote its proper transform on S_2 by R^2. Then $3 - a - x - x_1 = R^2 \cdot \Omega^2 \geqslant 0$, which gives $a + x + x_1 \leqslant 3$. Since $x - a \leqslant 1$ by Corollary 5.1, then $x_1 \leqslant \frac{4}{3}$ and all conditions of Lemma 5.2 are satisfied, giving a contradiction.

We have $\mathcal{L} = E_1$. Then L_1 is irreducible, since $P \notin L$. Since $(S, (1-\beta)C, \lambda\beta(2L_1 + 2E_1 + L))$ is log canonical and $2L_1 + 2E_1 + L \sim -K_S$, we may assume that L_1 or L is not contained in $\mathrm{Supp}(\Omega)$ by Lemma 3.2. We write $\Omega = bL_1 + \Delta$, where b is a non-negative real number, and Δ is an effective $\mathbb{R}$-divisor on S such that $L_1 \not\subset \mathrm{Supp}(\Delta)$ and $E_1 \not\subset \mathrm{Supp}(\Delta)$. Then $1 - b + a = E_1 \cdot \Delta \geqslant y$, which gives $b + y \leqslant 1 + a$. If $b > 0$, then $a \leqslant 1$. Indeed, if $L \not\subset \mathrm{Supp}(\Delta)$, then $1 - a = L \cdot \Delta \geqslant 0$.

Denote the proper transform of the divisor Δ on S_1 by Δ^1 , denote the proper transform of the divisor Δ on S_2 by Δ^2, and denote the proper transform of the divisor Δ on S_3 by Δ^3 . Let $y = \mathrm{mult}_P(\Delta)$, $y_1 = \mathrm{mult}_{P_1}(\Delta^1)$, $y_2 = \mathrm{mult}_{P_2}(\Delta^2)$ and $y_3 = \mathrm{mult}_{P_3}(\Delta^3)$. Then $x = b + y$. Since $L_1 \cdot C = 2$, either $\mathrm{mult}_P(L_1 \cdot C) = 1$ or $\mathrm{mult}_P(L_1 \cdot C) = 2$. Thus, we have, $x_2 = y_2$ and $x_3 = y_3$.

Suppose that $\mathrm{mult}_P(L_1 \cdot C) = 1$. Then $x_1 = y_1$ and $2 - a = L_1 \cdot \Delta \geqslant y$. We have $b + y \leqslant 1 + a$ by Corollary 5.1. If $b > 0$, then $a \leqslant 1$. Therefore, we have $\lambda\beta(a + x) \leqslant 1$, $\lambda\beta(a + x + x_1) - \beta \leqslant 1$, $\lambda\beta(a + x + 2x_1) - 3\beta \leqslant 1$ and $\lambda\beta(a + x + x_1 + 2x_2) - 4\beta \leqslant 1$, which contradicts Lemma 5.2.

Thus we see that $\mathrm{mult}_P(L_1 \cdot C) = 2$. Then $x_1 = y_1 + b$ and $2 - a = L_1 \cdot \Delta \geqslant y + y_1$, which gives $a + y + y_1 \leqslant 2$. Since L_1 is tangent to C at the point P, we have

$$\lambda = \hat{\alpha}\big(S, (1 - \beta)C\big) \leqslant \min\Big\{1, \frac{1 + 2\beta}{7\beta}, \frac{1}{3\beta}\Big\}.$$

Moreover, we have $b + y \leqslant 1 + a$ by Corollary 5.1. Furthermore, if $b > 0$, then $a \leqslant 1$. This gives $\lambda\beta(a+x) \leqslant 1$, $2\lambda\beta(a+x)-2\beta \leqslant 1$, $\lambda\beta(a+x+x_1+x_2)-2\beta \leqslant 1$ and $\lambda\beta(a+x+x_1+2x_2)-4\beta \leqslant 1$, which is impossible by Lemma 5.2. □

Lemma 5.4. Suppose that $K_S^2 = 6$. Then (4.2) is log canonical at P.

Proof. Suppose that (4.2) is not log canonical at P. Let us use the notation of §2.5. Without loss of generality, we may assume that $\mathcal{L} = E_1$. Denote the proper transform of the curve E_1 on the surface S_1 by E_1^1. Let L be the proper transform on S of the line in $\mathbb{P}^2$ that is tangent to $\pi(C)$ at the point $\pi(P)$. Then $-K_S \cdot L = 2$, since $P \notin L_{12} \cup L_{13} \cup L_{23}$. Denote the proper transform of the curve L on S_1 by L^1, denote the proper transform of the curve L on S_2 by L^2, and denote the proper transform of the curve L on S_3 by L^3.

We claim that $L \subset \mathrm{Supp}(\Omega)$. Indeed, suppose that $L \not\subset \mathrm{Supp}(\Omega)$. Then $a + x \leqslant 2$, since $2 - a = \Omega \cdot L \geqslant x$. But $x \leqslant 1 + a$ by Corollary 5.1. Therefore, we have $x_1 \leqslant x \leqslant \frac{3}{2}$. These inequalities give $\lambda\beta(a + x) \leqslant 1$, $2\lambda\beta(a + x) - \beta \leqslant 1$ and $\lambda\beta(a + x + 2x_1) - 3\beta \leqslant 1$. Therefore, $\lambda\beta(a + x + x_1 + 2x_2) - 4\beta > 1$ and $6\lambda\beta > 1 + 3\beta$ by Lemma 5.2. The former inequality implies that $a + x + x_1 + 2x_2 > 6$. The latter inequality implies that L is not tangent to C at the point P (see §2.5).

Let Z be the proper transform on S of the conic in $\mathbb{P}^2$ that passes through the points $\pi(E_1)$, $\pi(E_2)$, $\pi(E_3)$, and is tangent to $\pi(C)$ at the point $\pi(P)$. Then Z is irreducible, $E_1 + L + Z \sim -K_S$ and $-K_S \cdot Z = 3$, since L is not tangent to C at P. Then $\mathrm{mult}_P(Z \cdot C) \leqslant 3$, since $-K_S \cdot Z = 3$.

We write $\Omega = cZ + \Upsilon$, where c is a non-negative real number, and Υ is an effective $\mathbb{R}$-divisor on S whose support does not contain Z. Denote the proper transform of the divisor Υ on S_1 by Υ^1, denote the proper transform of the divisor Υ on S_2 by Υ^2, and denote the proper transform of the divisor Υ on S_3 by Υ^3. Let $z = \mathrm{mult}_P(\Upsilon)$, $z_1 = \mathrm{mult}_{P_1}(\Upsilon^1)$, $z_2 = \mathrm{mult}_{P_2}(\Upsilon^2)$, $z_3 = \mathrm{mult}_{P_3}(\Upsilon^3)$. Then $x = c + z$, $x_1 = c + z_1$, $x_3 = z_3$. If $\mathrm{mult}_P(Z \cdot C) = 2$, then $x_2 = z_2$ and $3 - a - c - z = Z^1 \cdot \Upsilon^1 \geqslant \mathrm{mult}_{P_1}(Z^1 \cdot \Upsilon^1) \geqslant z_1$, which implies that

$$6 < a + x + x_1 + 2x_2 = a + z + z_1 + 2z_2 + 2c \leqslant 3 + 2z_2 + c \leqslant 3 + 2z_2 + 2c \leqslant 3 + 2x \leqslant 6,$$

since $z + c \leqslant \frac{3}{2}$ and $a + c + z \leqslant 2$. Thus, we see that $\mathrm{mult}_P(Z \cdot C) = 3$. Then $x_2 = c + z_2$ and $3 - a - c - z - z_1 = Z^2 \cdot \Upsilon^2 \geqslant \mathrm{mult}_{P_2}(Z^2 \cdot \Upsilon^2) \geqslant z_2$, which gives $a + c + z + z_1 + z_2 \leqslant 3$. Then

$$6 < a + x + x_1 + 2x_2 = a + z + z_1 + 2z_2 + 3c < 3 + z_2 + 2c \leqslant 3 + 2z_2 + 2c \leqslant 3 + 2x \leqslant 6,$$

which is absurd. This shows that $L \subset \mathrm{Supp}(\Omega)$.

We write $\Omega = bL + \Delta$, where b is a positive real number, and Δ is an effective $\mathbb{R}$-divisor on S such that $L \not\subset \mathrm{Supp}(\Delta)$. Let $y = \mathrm{mult}_P(\Delta)$. Then $2 - a = \Delta \cdot L \geqslant y$. Denote the proper transform of the divisor Δ on S_1 by Δ^1, denote the proper transform of the divisor Δ on S_2 by Δ^2, and denote the proper transform of the divisor Δ on S_3 by Δ^3. Let $y_1 = \mathrm{mult}_{P_1}(\Delta^1)$, $y_2 = \mathrm{mult}_{P_2}(\Delta^2)$ and $y_3 = \mathrm{mult}_{P_3}(\Delta^3)$. Then $x = b + y$, $x_2 = y_2$ and $x_3 = y_3$, which implies that $b+y\leqslant 1+a$ by Corollary 5.1. Then

$$\Big(S_1,(1-\beta)C^1+\lambda\beta aE_1^1+\lambda\beta bL^1+\lambda\beta\Delta^1+\Big(\lambda\beta(a+b+y)-\beta\Big)F_1\Big)\tag{5.5}$$

is not log canonical at some point $Q_1\in F_1$ by Lemma 3.3.

We claim that L is tangent to C at the point P. Indeed, suppose that L is not tangent to C at P. Then $x_1=y_1$. Let Z be the proper transform on S of the conic in $\mathbb{P}^2$ that passes through $\pi(E_1)$, $\pi(E_2)$, $\pi(E_3)$, and is tangent to $\pi(C)$ at $\pi(P)$. Then Z is irreducible and $-K_S\cdot Z=3$. Moreover, we have $E_1+L+Z\sim -K_S$, and the log pair $(S,(1-\beta)C+\lambda\beta(E_1+L+Z))$ is log canonical. Thus, we may assume that $Z\not\subset\mathrm{Supp}(D)$ by Lemmas 3.2. Then $3-a-b-y=Z^1\cdot\Delta^1\geqslant\mathrm{mult}_{P_1}(Z^1\cdot\Delta^1)\geqslant y_1$. Since we also have $b+y\leqslant 1+a$, $a+y\leqslant 2$, $x=y+b$, $x_1=y_1$ and $x_2=y_2$, we see that

$$\begin{gathered}\lambda\beta y_1\leqslant 1,\qquad \lambda\beta(a+b+y)-\beta\leqslant\lambda\beta(a+b+y+y_1)-\beta\leqslant 1,\\ \lambda\beta(a+b+y+2y_1)-3\beta\leqslant 1,\qquad \lambda\beta(a+b+y_1+2y_2)-4\beta\leqslant 1.\end{gathered}\tag{5.6}$$

In particular, (5.5) is log canonical at every point of F_1 that is different from Q_1 by Lemma 3.3. If $Q_1\neq L^1\cap F_1$ and $Q_1\neq P_1$, then $\lambda\beta(a+y)=F_1\cdot(\lambda\beta(aE_1+\Delta^1))>1$, by Theorem 3.4. But $\lambda\beta(a+y)\leqslant 1$, since $a+y\leqslant 2$. This shows that $Q_1=L^1\cap F_1$ or $Q_1=P_1$. Since $b-a+y\leqslant 1$ and $a+b+y+y_1\leqslant 3$, we have $b+y\leqslant 2$. So, if $Q_1=L^1\cap F_1$, then

$$1<\lambda\beta F_1\cdot\Big(bL^1+\Delta^1\Big)=\lambda\beta(b+y)\leqslant 2\lambda\beta\leqslant 1,$$

by Theorem 3.4. If $Q_1=P_1$, then $6=D\cdot C>\frac{1+4\beta}{\lambda\beta}$ by (5.6) and Theorem 3.6. The latter contradicts $6\lambda\beta\leqslant 1+4\beta$.

We see that L is tangent to C at the point P. Then $x_1=y_1+b$ and

$$\lambda\leqslant\min\Big\{1,\frac{1+2\beta}{5\beta},\frac{1}{2\beta}\Big\},$$

which gives $6\lambda\beta\leqslant 1+3\beta$. Moreover, we have $a+y+y_1\leqslant 2$, because $2-a-y-y_1=L^2\cdot\Delta^2\geqslant 0$. Furthermore, since $2L+L_{23}+E_1\sim -K_S$ and $(S,(1-\beta)C+\lambda\beta(2L+L_{23}+E_1)$ is log canonical, we may assume that $L_{23}\not\subset\mathrm{Supp}(\Delta)$ by Lemma 3.2. This gives us $b\leqslant 1$, because $1-b=\Delta\cdot L_{23}\geqslant 0$. Since $L+L_{12}+L_{13}+2E_1\sim -K_S$ and $(S,(1-\beta)C+\lambda\beta(L+L_{12}+L_{13}+2E_1))$ is log canonical, we may assume that $L_{12}\not\subset\mathrm{Supp}(\Delta)$ or $L_{13}\not\subset\mathrm{Supp}(\Delta)$ by Lemma 3.2. If $L_{12}\not\subset\mathrm{Supp}(\Delta)$, then $1-a=\Delta\cdot L_{12}\geqslant 0$, which gives $a\leqslant 1$. Similarly, we get $a\leqslant 1$ if $L_{13}\not\subset\mathrm{Supp}(\Delta)$. Thus, we have

$$a\leqslant 1,\qquad b\leqslant 1,\qquad b-a+y\leqslant 1,\qquad a+y+y_1\leqslant 2,\tag{5.7}$$

which implies that $\lambda\beta(a+b+y)-\beta\leqslant 1$. In particular, (5.5) is log canonical at every point of F_1 that is different from Q_1 by Lemma 3.3. If $Q_1\neq P_1$ and $Q_1\neq E_1^1\cap F_1$, then $\lambda\beta y=\lambda\beta\Delta^1\cdot F_1>1$ by Theorem 3.4. The latter is impossible, since $\lambda\beta y\leqslant 2\lambda\beta\leqslant 1$ by (5.7). If $Q_1=E_1^1\cap F_1$, then

$$1<E_1^1\cdot\Big(\lambda\beta\Delta^1+\big(\lambda\beta(a+b+y)-\beta\big)F_1\Big)=\lambda\beta(1+2a)-\beta$$

by Theorem 3.4. The latter is impossible, since $\lambda\beta(1+2a)-\beta\leqslant 3\lambda\beta-\beta\leqslant 1$ by (5.7). Thus, we see that $Q_1=P_1$.

By (5.7), one has $a+2b+y+y_1\leqslant 4$. This implies that $\lambda\beta(a+2b+y+y_1)-2\beta\leqslant 1$. Then

$$\Big(S_2,(1-\beta)C^2+\lambda\beta bL^2+\lambda\beta\Delta^2+\big(\lambda\beta(a+b+y)-\beta\Big)F_1^2+\big(\lambda\beta(a+2b+y+y_1)-2\beta\Big)F_2\Big)$$

is not log canonical at a unique point $Q_2\in F_2$ by Lemma 3.3. If $Q_2\notin L^2\cup F_1^2\cup C^2$, then $\lambda\beta y_2=\lambda\beta\Delta^2\cdot F_2>1$ by Theorem 3.4, which is impossible, since $\lambda\beta y_2\leqslant 1$ by (5.7). Similarly, if $Q_2=F_2\cap L^2$, then $\lambda\beta(b+y_2)=\lambda\beta(bL^2+\Delta^2)\cdot F_2>1$ by Theorem 3.4, which is impossible, because $b+y_2\leqslant b+y\leqslant 2$ by (5.7). If $Q_2=F_2\cap F_1^2$, then

$$\lambda\beta(y+y_1+a+b)-\beta=\Big(\lambda\beta\Delta^2+\big(\lambda\beta(a+b+y)-\beta\big)F_1^2\Big)\cdot F_2>1$$

by Theorem 3.4, which is impossible, since $y+y_1+a+b\leqslant 3$ by (5.7). Then $Q_2=P_2$.

We have $\lambda\beta(a+2b+y+y_1+y_2)-3\beta\leqslant 1$, since $a+2b+y+y_1+y_2\leqslant 5$ by (5.7). Then

$$\Big(S_3,(1-\beta)C^3+\lambda\beta\Delta^3+\big(\lambda\beta(a+2b+y+y_1)-2\beta\Big)F_2^3+\big(\lambda\beta(a+2b+y+y_1+y_2)-3\beta\Big)F_3\Big).$$

is not log canonical at a *unique* point $Q_3\in F_3$ by Lemma 3.3. If $Q_3\notin F_2^3\cup C^3$, then $\lambda\beta y_3=\lambda\beta\Delta^3\cdot F_3>1$ by Theorem 3.4, which is impossible, because $\lambda\beta y_3\leqslant 1$ by (5.7). If $Q_3=F_3\cap F_2^3$, then Theorem 3.4 gives

$$1<F_2^3\cdot\Big(\lambda\beta\Delta^3+(\lambda\beta(a+2b+y+y_1+y_2)-3\beta)F_3\Big)=\lambda\beta(a+2b+y+2y_1)-3\beta\leqslant 5\lambda\beta-3\beta,$$

which is impossible, since $a+2b+y+2y_1\leqslant 5$ by (5.7). Thus, we see that $Q_3=P_3$. By Theorem 3.6 (iv), we have $6=D\cdot C\geqslant \mathrm{mult}_P(D\cdot C)>\frac{1+3\beta}{\lambda\beta}$. The latter is impossible, since we already proved earlier that $6\lambda\beta\leqslant 1+3\beta$. □

Lemma 5.8. Suppose that $K_S^2=5$. Then (4.2) is log canonical at P.

Proof. Suppose that (4.2) is not log canonical at P. Let us use the notation of §2.5. Then $\lambda=\min\{1,\frac{1}{2\beta}\}$. This implies that $5\lambda\beta\leqslant 1+3\beta$. By Lemma 5.2, at least one of the conditions (i), (ii) and (iii) in Lemma 5.2 is not satisfied. In particular, if $a+x\leqslant 2$, then $\lambda\beta(a+x+2x_1)-3\beta>1$.

Without loss of generality, we may assume that $\mathcal{L}=L_{12}$. Let B_3 be the proper transform on S of the line in $\mathbb{P}^2$ that passes through $\pi(P)$ and $\pi(E_3)$, and let B_4 be the proper transform on S of the line in $\mathbb{P}^2$ that passes through $\pi(P)$ and $\pi(E_4)$. Since $L_{12}+B_3+B_4\sim -K_S$ and $(S,(1-\beta)C+\lambda\beta(L_{12}+B_3+B_4))$ is log canonical, we may assume that at least one curve among B_3 and B_4 is not contained in $\mathrm{Supp}(\Omega)$. Intersecting this curve with Ω, we get $a+x\leqslant 2$. Then $\lambda\beta(a+x+2x_1)-3\beta>1$. This implies that $a+x+2x_1>5$.

Denote the proper transform of the curve B_3 on the surface S_1 by B_3^1, and denote the proper transform of the curve B_4 on the surface S_1 by B_4^1. Recall $P_1=C^1\cap F_1$.

Suppose that $P_1\notin B_3^1\cup B_4^1$. Then B_3 and B_4 do not tangent C at P. Let R be the proper transform on S of the line in $\mathbb{P}^2$ that is tangent to $\pi(C)$ at the point $\pi(P)$, let R_1 be the proper transform on S of the conic in $\mathbb{P}^2$ that tangents to $\pi(C)$ at the point $\pi(P)$ and passes through the points $\pi(E_2)$, $\pi(E_3)$ and $\pi(E_4)$, and let R_2 be the proper transform on S of the conic in $\mathbb{P}^2$ that tangents to $\pi(C)$ at the point $\pi(P)$ and passes through the points $\pi(E_1)$, $\pi(E_3)$ and $\pi(E_4)$. Since $P_1\notin B_3^1\cup B_4^1$, the curves R_1 and R_2 are irreducible. Hence $\frac{1}{2}L_{12}+\frac{1}{2}R+\frac{1}{2}R_1+\frac{1}{2}R_2\sim_{\mathbb{R}}-K_S$ and $(S,(1-\beta)C+\lambda\beta(\frac{1}{2}L_{12}+\frac{1}{2}R+\frac{1}{2}R_1+\frac{1}{2}R_2))$ is log canonical. By Lemma 3.2, we may assume that one curve among R, R_1 and R_2 is not contained in $\mathrm{Supp}(D)$. Denote this curve by Z, and denote its proper transform on S_1 by Z^1. Then $P_1\in Z^1$ and $3-a-x=Z^1\cdot\Omega^1\geqslant x_1$, which is impossible, since $a+x\leqslant 2$ and $a+x+2x_1>5$.

We see that $P_1=B_3^1\cap F_1$ or $P_1=B_4^1\cap F_1$. Without loss of generality, we may assume that $P_1=B_3^1\cap F_1$. Then $B_3\subset\mathrm{Supp}(\Omega)$, since otherwise we would have $2-a-x=B_3^1\cdot\Omega^1\geqslant x_1$, which is impossible, since $a+x\leqslant 2$. We write $\Omega=bB_3+\Delta$, where $b\in\mathbb{R}_{>0}$ and Δ is an effective $\mathbb{R}$-divisor on S such that $B_3\not\subset\mathrm{Supp}(\Delta)$. Denote the proper transform of the divisor Δ on S_1 by Δ^1. Let $y=\mathrm{mult}_P(\Delta)$ and $y_1=\mathrm{mult}_{P_1}(\Delta^1)$. Then $x=b+y$ and $x_1=b+y_1$. We have $b-a+y\leqslant 1$ by Corollary 5.1 and $a+b+y=a+x\leqslant 2$, which implies a contradiction $a+x+2x_1\leqslant 2+2y+2b\leqslant 5$. □

Lemma 5.9. Suppose that $K_S^2=4$. Then (4.2) is log canonical at P.

Proof. Suppose that (4.2) is not log canonical at P. Let us use the notation §2.7. Then $\lambda\beta<\frac{2}{3}$. Without loss of generality, we may assume that $P\in E$. Then $P=E\cap C$. By Lemma 4.8, the point P is not contained in any other (-1)-curve. By Lemma 4.3, we have $E\subset\mathrm{Supp}(D)$.

The log pair $(S,(1-\beta)C+\lambda\beta(\frac{3}{2}E+\frac{1}{2}(E_1+E_2+E_3+E_4+E_5)))$ is log canonical and $\frac{3}{2}E+\frac{1}{2}(E_1+E_2+E_3+E_4+E_5)\sim_{\mathbb{R}}-K_S$. By Lemma 3.2, we may assume that $\mathrm{Supp}(\Omega)$ does not contain one curve among E_1, E_2, E_3, E_4, E_5. Intersecting this curve with Ω, we get $a \leqslant 1$. Let L_1, L_2, L_3, L_4, L_5 be the proper transforms on S of the lines in $\mathbb{P}^2$ that pass through $\pi(P)$ and $\pi(E_1)$, $\pi(E_2)$, $\pi(E_3)$, $\pi(E_4)$, $\pi(E_5)$, respectively. Then $\frac{2}{3}E+\frac{1}{3}(L_1+L_2+L_3+L_4+L_5) \sim_{\mathbb{R}} -K_S$, and $(S,(1-\beta)C+\lambda\beta(\frac{2}{3}E+\frac{1}{3}(L_1+L_2+L_3+L_4+L_5)))$ is log canonical. By Lemma 3.2, we may assume that $\mathrm{Supp}(\Omega)$ does not contain one curve among L_1, L_2, L_3, L_4, L_5. Intersecting this curve with Ω, we get $a+x \leqslant 2$. Recall that $a \leqslant 1$ by Corollary 5.1. Thus, we have

$$a \leqslant 1, \qquad x-a \leqslant 1, \qquad a+x \leqslant 2, \tag{5.10}$$

which implies that $x \leqslant \frac{3}{2}$ and $\lambda\beta(a+x)-\beta \leqslant 1$. In particular, we have $\lambda\beta x \leqslant 1$.

Denote the proper transform of the curve E on S_1 by E^1. Then $\lambda\beta(a+x)-\beta \leqslant 1$, since $a+x \leqslant 2$. Thus, the log pair $(S_1,(1-\beta)C^1+\lambda\beta aE^1+\lambda\beta\Omega^1+(\lambda\beta(a+x)-\beta)F_1)$ is not log canonical at the unique point $Q_1 \in F_1$ by Lemma 3.3. Note that $\lambda\beta(a+x)-\beta>0$ by Lemma 3.1. Moreover, either $Q_1=P_1$ or $Q_1=E^1\cap F_1$, since otherwise we would have $\lambda x=\lambda\beta\Omega^1\cdot F_1>1$ by Theorem 3.4. If $Q_1=E^1\cap F_1$, then Theorem 3.9 implies

$$\lambda\beta(1+a-x)=\lambda\beta\Omega^1\cdot E^1>2\Big(1+\beta-\lambda\beta(x+a)\Big)$$

or $\lambda\beta x=\lambda\beta\Omega^1\cdot F_1>2(1-\lambda\beta a)$. The former inequality gives $\lambda\beta(1+3a+x)>2+2\beta$, which is impossible since $1+3a+x\leqslant 5$ by (5.10). The latter inequality gives that $\lambda\beta(x+2a)>2$, which is impossible since $x+2a\leqslant 3$ by (5.10). Thus, we see that $Q_1=P_1$.

Let R be the proper transform on S of a line in $\mathbb{P}^2$ that is tangent to $\pi(C)$ at the point $\pi(P)$. Then either $-K_S\cdot R=3$ or $-K_S\cdot R=2$. Moreover, $-K_S\cdot R=3$ if and only if $\pi(R)$ does not contain any of the points $\pi(E_1)$, $\pi(E_2)$, $\pi(E_3)$, $\pi(E_4)$, $\pi(E_5)$.

Suppose that $-K_S\cdot R=2$. Without loss of generality, we may assume that $R=L_1$. We write $\Omega=bL_1+\Delta$, where b is a non-negative real number, and Δ is an effective $\mathbb{R}$-divisor on S whose support does not contain the curve L_1. Denote the proper transform of the curve L_1 on S_1 by L_1^1, and denote the proper transform of Δ on S_1 by Δ^1. Let $y=\mathrm{mult}_P(\Delta)$ and $y_1=\mathrm{mult}_{P_1}(\Delta^1)$. Then $x=y+b$. Since $(S,(1-\beta)C+\lambda\beta(E+E_1+L_1))$ is log canonical and $E+E_1+L_1\sim -K_S$, we may assume that $b=0$ or $\mathrm{Supp}(\Delta)$ does not contain E_1 by Lemma 3.2. Thus, if $b\neq 0$, then $1-a-b=\Delta\cdot E_1\geqslant 0$. With (5.10), this gives $y+2b\leqslant 2$ and $2+a+y+2b\leqslant\frac{9}{2}$. On the other hand, we have $2-a-y=\Delta^1\cdot L_1^1\geqslant y_1$, which implies that $a+2y_1\leqslant 2$, since $y\geqslant y_1$. Thus, we see that $y_1\leqslant 1$. Then $\mathrm{mult}_{P_1}((1-\beta)C^1+\lambda\beta\Delta^1)=1-\beta+\lambda\beta y_1\leqslant 1$. Applying Theorem 3.9, we see that

$$1-\beta+\lambda\beta(2-a-y)=\Big((1-\beta)C^1+\lambda\beta\Delta^1\Big)\cdot L_1^1>2\Big(1+\beta-\lambda\beta(a+b+y)\Big)$$

or $1-\beta+\lambda\beta y=((1-\beta)C^1+\lambda\beta\Delta^1)\cdot F_1>2(1-\lambda\beta b)$. This gives $\lambda\beta(2+a+y+2b)>1+3\beta$ or $\lambda\beta(y+2b)>1+\beta$. The former inequality is impossible, because $2+a+y+2b\leqslant\frac{9}{2}$. The latter inequality is also impossible, because $y+2b\leqslant 2$.

We have $-K_S\cdot R=3$. Then R is irreducible and $R+E\sim -K_S$. Since $(S,(1-\beta)C+\lambda\beta(R+E))$ is log canonical, we may assume that $R\not\subset\mathrm{Supp}(\Omega)$ by Lemma 3.2. Denote the proper transform of the curve R on the surface S_1 by R^1. Then $3-2a-x=\Omega^1\cdot R^1\geqslant x_1$, which gives $x+x_1+2a\leqslant 3$. Then $\lambda\beta(a+x+x_1)-2\beta\leqslant 1$ by (5.10). Thus, the log pair

$$\Big(S_2,(1-\beta)C^2+\lambda\beta\Omega^2+\big(\lambda\beta(a+x)-\beta\big)F_1^2+\big(\lambda\beta(a+x+x_1)-2\beta\big)F_2\Big)$$

is not log canonical at a unique point $Q_2\in F_2$ by Lemma 3.3. Note that $\lambda\beta(a+x+x_1)-2\beta>0$ by Lemma 3.1. If $Q_2\neq P_2$ and $Q_2\neq F_1^2\cap F_2$, then Theorem 3.4 gives $\lambda\beta x_1=\lambda\beta\Omega^2\cdot F_2>1$, which is impossible, since $\lambda\beta x_1\leqslant\lambda\beta x\leqslant 1$ by (5.10). If $Q_2=F_1^2\cap F_2$, then Theorem 3.4 gives

$$\lambda\beta(a+2x)-2\beta\geqslant\Big(\lambda\beta\Omega^2+(\lambda\beta(a+x+x_1)-2\beta)F_2\Big)\cdot F_1^2>1$$

which is impossible, since $a+2x\leqslant\frac{7}{2}$, by (5.10). Hence, we see that $Q_2=P_2$.

One has $\lambda\beta(a+x+x_1+x_2)-3\beta\leqslant 1$ by (5.10), since $x+x_1+2a\leqslant 3$ and $x_2\leqslant x_1\leqslant x$. Thus, it follows from Lemma 3.3 that

$$\Big(S_3, (1-\beta)C^3+\lambda\beta\Omega^3+\big(\lambda\beta(a+x+x_1)-2\beta\big)F_2^3+\big(\lambda\beta(a+x+x_1+x_2)-3\beta\big)F_3\Big)$$

is not log canonical at a unique point $Q_3\in F_3$. Note that $\lambda\beta(a+x+x_1+x_2)-3\beta>0$ by Lemma 3.1. If $Q_3\neq P_3$ and $Q_3\neq F_2^3\cap F_3$, then Theorem 3.4 gives $\lambda\beta x_2=\lambda\beta\Omega^3\cdot F_3>1$, which is impossible, since $\lambda\beta x_2\leqslant\lambda\beta x\leqslant 1$ by (5.10). If $Q_3=F_2^3\cap F_3$, then Theorem 3.4 gives

$$\lambda\beta(a+x+2x_1)-3\beta=\Big(\lambda\beta\Omega^3+(\lambda\beta(a+x+x_1+x_2)-3\beta)F_3\Big)\cdot F_2^3>1$$

which contradicts (5.10), since $x+x_1+2a\leqslant 3$. Thus, we have $Q_3=P_3$. Then Theorem 3.4 gives

$$\beta\geqslant 4\lambda\beta-3\beta=C^3\cdot\Big(\lambda\beta\Omega^3+(\lambda\beta(a+x+x_1+x_2)-3\beta)F_3\Big)>1,$$

which is impossible, since $\beta\in(0,1]$. □

This completes the proof of Lemma 4.10.

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