PARITY SHEAVES AND TILTING MODULES

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ABSTRACT. We show that tilting modules and parity sheaves on the affine Grassmannian are related through the geometric Satake correspondence, when the characteristic is bigger than an explicit bound.

1. INTRODUCTION

1.1. **Tilting modules for reductive groups.** Let $\mathbf{G}$ be a split reductive group over a field k of characteristic p with a chosen maximal torus and Borel subgroup $\mathbf{T} \subset \mathbf{B} \subset \mathbf{G}$. Let Λ^+ denote the set of dominant weights. To each dominant weight $\lambda \in \Lambda^+$, is associated an induced representation $\nabla_\lambda := \mathrm{ind}_{\mathbf{B}}^{\mathbf{G}} k_\lambda = H^0(\mathbf{G}/\mathbf{B}, \mathcal{O}(\lambda))$ and its dual Δ_λ, the Weyl module.

The rational representations of $\mathbf{G}$ form a highest weight category in which the Weyl modules are the standard objects and the induced modules are the costandard objects. A rational representation is said to be tilting, if it admits two filtrations — one with successive quotients isomorphic to Weyl modules and the other with successive quotients isomorphic to induced modules.

A theorem of Ringel [Rin91, Proposition 2] about general highest weight categories specializes in this setting to the following result [Don93, Theorem 1.1],

Theorem 1.1. *For each $\lambda \in \Lambda^+$, (up to non-canonical isomorphism) there exists a unique indecomposable tilting module $T(\lambda)$ which has a unique highest weight λ. Moreover, λ has multiplicity one as a weight of $T(\lambda)$. Every indecomposable tilting module is isomorphic to $T(\lambda)$ for some $\lambda \in \Lambda^+$.*

An interesting feature of the class of tilting modules is that it is closed under both tensor product and restriction to a Levi subgroup:

Theorem 1.2. *If T and T' are tilting modules for $\mathbf{G}$, then so is the tensor product $T \otimes T'$.*

Theorem 1.3. *Let $\mathbf{L}$ be a Levi subgroup of $\mathbf{G}$. If T is a tilting module for $\mathbf{G}$, then the restriction $\mathrm{Res}_{\mathbf{L}}^{\mathbf{G}} T$ to $\mathbf{L}$ is a tilting module for $\mathbf{L}$.*

The first of these theorems was originally proven by Wang [Wan82] in type A and in large characteristic for other groups. Donkin [Don85] later proved both theorems in almost full generality (he excluded the case when $p = 2$ and G has a component of type E_7 or E_8). The first complete and uniform proof of both theorems is due

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to Mathieu [Mat90] and uses Frobenius splitting techniques. Other approaches to the first theorem appear in [Lit92, Pol93, Par94, Kan98].[1]

1.2. **Parity sheaves for the affine Grassmannian.** Let $\check{G} \supset \check{T}$ be the connected complex algebraic group and maximal torus with root datum dual to that of $\mathbf{G}$.

Let $\mathcal{K} = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$. The affine Grassmannian $\mathcal{G}r$ for $\check{G}$ is an ind-scheme whose complex points form the set $\check{G}(\mathcal{K})/\check{G}(\mathcal{O})$. We consider its complex points as an ind-$\check{G}(\mathcal{O})$-variety. The $\check{G}(\mathcal{O})$-orbits are labeled by the set Λ^+ of dominant weights of $\mathbf{G}$ and we denote the orbit corresponding to a weight λ by $\mathcal{G}r^\lambda$.

The geometric Satake theorem [MV07] shows that the representation theory of $\mathbf{G}$ is encoded in a category of sheaves of k-vector spaces on $\mathcal{G}r$. More precisely, the category of rational representations of $\mathbf{G}$ is equivalent to the category $P(\mathcal{G}r)$: the category of $\check{G}(\mathcal{O})$-equivariant perverse sheaves on $\mathcal{G}r$ with coefficients in k.

The category $P(\mathcal{G}r)$ is the heart of a t-structure on $D(\mathcal{G}r)$, the bounded $\check{G}(\mathcal{O})$-equivariant constructible derived category of sheaves of k-vector spaces on $\mathcal{G}r$. There is a natural convolution product $\star : D(\mathcal{G}r) \times D(\mathcal{G}r) \to D(\mathcal{G}r)$ which is t-exact and produces a tensor structure on $P(\mathcal{G}r)$, corresponding under the equivalence to the tensor product of rational representations of $\mathbf{G}$.

Similarly, for any Levi subgroup $\mathbf{L} \subset \mathbf{G}$, the restriction functor from $\mathbf{G}$ to $\mathbf{L}$ corresponds to a geometrically-defined t-exact functor $R^{\check{G}}_{\check{L}} : D(\mathcal{G}r) \to D(\mathcal{G}r_{\check{L}})$, where $\mathcal{G}r_{\check{L}}$ is the affine Grassmannian for $\check{L} \subset \check{G}$, the Levi subgroup containing $\check{T}$ whose roots are dual to those of $\mathbf{L}$ (see Section 2.3 for more details).

Recall the notion of a parity complex [JMW, Section 2.2].[2] The affine Grassmannian is a Kac-Moody flag variety and hence the results from [JMW, Section 4.1] (see in particular Example 4.2 of loc. cit.) can be used to study $D(\mathcal{G}r)$. Parity complexes on the affine Grassmannian behave very much like the tilting modules for $\mathbf{G}$. In particular, we have the following theorems which mirror the ones for tilting modules.

The starting point is a result [JMW, Theorem 4.6] that is very similar to Theorem 1.1:

Theorem 1.4. *Assume that the characteristic of k is not a torsion prime for $\check{G}$.*[3]

For each $\lambda \in \Lambda^+$, (up to non-canonical isomorphism) there exists a unique indecomposable parity complex $\mathcal{E}(\lambda)$ such that $\mathrm{supp}(\mathcal{E}(\lambda)) = \overline{\mathcal{G}r^\lambda}$ and $\mathcal{E}(\lambda)|_{\mathcal{G}r^\lambda} = \underline{k}_{\mathcal{G}r^\lambda}[\dim \mathcal{G}r^\lambda]$. Every indecomposable parity complex is isomorphic to $\mathcal{E}(\lambda)$ for some $\lambda \in \Lambda^+$.

The indecomposable parity complexes $\mathcal{E}(\lambda)$ are known as parity sheaves.

As a special case of [JMW, Theorem 4.8], we obtain an analogue of Theorem 1.2:

Theorem 1.5. *If $\mathcal{F} \in D(\mathcal{G}r)$ and $\mathcal{G} \in D(\mathcal{G}r)$ are parity complexes, then so is the convolution product $\mathcal{F} \star \mathcal{G} \in D(\mathcal{G}r)$.*

The first part of this paper establishes an analogue of Theorem 1.3. In Section 2, we prove the following result,

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1. In the literature, these theorems appear with the words 'tilting modules' replaced by 'modules admitting a good filtration' or 'modules admitting a Weyl filtration'. In the appendix to this paper, we explain the fact, well-known to experts, that these are equivalent formulations.

2. Unless stated otherwise, in this paper parity complexes are defined with respect to the constant pariversity $\natural$.

3. This restriction can be removed by working in the non-equivariant setting.

Theorem 1.6. *Let $\check{L}$ be the Langlands dual of $\mathbf{L}$ a Levi subgroup of $\mathbf{G}$. If $\mathcal{F} \in D(\mathcal{G}r)$ is a parity complex, then $R_{\check{L}}^{\check{G}}(\mathcal{F})$ is a parity complex on the affine Grassmannian for $\check{L}$.*

The idea of the proof is to replace the purity argument of [Bra03, Theorem 2] by a parity argument.

1.3. **Tilting equals parity.** In Section 3, which can be read independently of Section 2, we prove our main result, which explains the similarities between the theorems stated above. Our result shows that, for most characteristics, the Theorems 1.2 and 1.3 about tilting modules are equivalent to the Theorems 1.5 and 1.6 about parity sheaves.

Recall [MV07, Prop. 13.1] that, for $\lambda \in \Lambda^+$ a dominant weight, the Weyl module Δ_λ (resp. ∇_λ) goes under the geometric Satake equivalence to the standard sheaf ${}^p\mathcal{J}_!(\lambda) := {}^p j_{\lambda!}\underline{k}_\lambda[d_\lambda]$ (resp. costandard sheaf ${}^p\mathcal{J}_*(\lambda) := {}^p j_{\lambda*}\underline{k}_\lambda[d_\lambda]$) where $j_\lambda : \mathcal{G}r^\lambda \to \mathcal{G}r$ denotes the inclusion, $\underline{k}_\lambda$ the constant sheaf on $\mathcal{G}r^\lambda$ and d_λ the dimension of $\mathcal{G}r^\lambda$.

We say $\mathcal{F} \in P(\mathcal{G}r)$ is a tilting sheaf if it corresponds to a tilting module for $\mathbf{G}$. This is equivalent to admitting two filtrations — one with standard successive quotients and the other with costandard successive quotients. [4] We denote by $\mathcal{T}(\lambda)$ the tilting sheaf corresponding to the indecomposable tilting module $T(\lambda)$.

Our main theorem is the following geometric characterization of the tilting sheaves on the affine Grassmannian. We will need to assume that the characteristic p is bigger than some bound depending only on the root system Φ of $\mathbf{G}$.

Definition 1.7. If the root system Φ is irreducible, let $b(\Phi)$ be given by the following table:

Type of Φ	A_n	B_n, D_n	C_n	G_2, F_4, E_6	E_7	E_8
$b(\Phi)$	1	2	n	3	19	31

TABLE 1. Table of bounds

In the general case, let $\Phi = \sqcup_{i=1}^s \Phi_i$ be the decomposition into irreducible components. Then we set $b(\Phi) := \max_{1 \le i \le s} b(\Phi_i)$.

Theorem 1.8. *If $p > b(\Phi)$, then the group $\mathbf{G}$ satisfies*

$$(*) \qquad\qquad \forall \lambda \in \Lambda^+, \quad \mathcal{E}(\lambda) = \mathcal{T}(\lambda).$$

In particular, for p as in the Theorem, every $\mathcal{E}(\lambda)$ is perverse. Note that we know examples where $\mathcal{E}(\lambda)$ fails to be perverse, for bad primes (see Lemma 3.7). However, Lemma 3.7(4) suggests that the following may always be true:

Conjecture 1.9. *In arbitrary characteristic, for every dominant weight $\lambda \in \Lambda^+$, the perverse sheaf ${}^p\mathcal{H}^0\mathcal{E}(\lambda)$ is tilting.*

On the other hand, note that by Proposition 3.3, the property $(*)$ is actually equivalent to all $\mathcal{E}(\lambda)$ being perverse.

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4. Warning: this definition of tilting sheaf is more general than that of [BBM04], which does not apply to this setting.

1.4. **Applications.** One motivation for this work was a conjecture of Mirković and Vilonen [MV07, Conjecture 13.3]. They conjectured that the stalks of standard sheaves with $\mathbb{Z}$-coefficients on the affine Grassmannian are torsion free. This conjecture is equivalent to the standard sheaves being $*$-parity for all fields (i.e. their non-zero stalks should be concentrated in one parity). Actually the minimal nilpotent orbit singularities provide counterexamples to this conjecture, in all types but in type A_n: see [Jut08], where the conjecture is modified to exclude bad primes. We get the following reformulation:

Conjecture 1.10. *If p is a good prime for $\mathbf{G}$, then the standard sheaves with coefficients in a field of characteristic p are $*$-parity.*

Note that if Conjecture 1.10 is true, then it implies that $\mathbf{G}$ satisfies $(*)$ whenever p is a good prime.

Conversely, since an earlier draft of the current paper was circulated, Achar-Rider [AR13] proved that if $\mathbf{G}$ satisfies $(*)$, then the Mirković-Vilonen conjecture is correct. In particular, using our Theorem 1.8, they settle the conjecture for all but a handful of cases.

Our results also may be used to obtain new proofs of Theorems 1.2 and 1.3 in most characteristics. To see this note that for all groups for which the hypothesis of Theorem 1.8 applies, our Theorems 1.5 and 1.6 imply Theorems 1.2 and 1.3. The careful reader will observe that in proving Theorem 1.8 we do not use any results that rely on Theorem 1.2 or 1.3.

Lastly, in Section 4, we observe that our main result implies the existence (for most characteristics) of q-characters for tilting modules, meaning a natural q-analogue of the characters of tilting modules.

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2. Hyperbolic Localization

The goal of this section is to prove Theorem 1.6. In 2.1, we recall the notion of hyperbolic localization and Braden's Theorem. In 2.2, we introduce some simplifying assumptions and study the hyperbolic localization of parity sheaves in this setting. In 2.3, we apply the result of 2.2 to the affine Grassmannian and prove Theorem 1.6.

2.1. **Braden's Theorem.** Let T be a complex torus and X a complex T-variety.
We make the following assumption, which is automatic if X is normal by Sumihiro's theorem [Sum74, KKLV89]:

(C) X has a covering by T-stable affine open subvarieties.

Let $\chi : \mathbb{G}_m \to T$ be a cocharacter of T. Our goal is to understand the hyperbolic localization of $\pi_* \underline{k}_Y$ with respect to the $\mathbb{G}_m$-action defined by χ. We begin by recalling Braden's definition.

Let $Z \subset X$ denote the variety of χ-fixed points and $Z_1, \ldots, Z_m$ its connected components. Consider the attracting and repelling varieties for each component Z_i, $1 \leq i \leq m$:

$$Z_i^+ = \{x \in X | \lim_{s \to 0} \chi(s) \cdot x \in Z_i\},$$

$$Z_i^- = \{x \in X | \lim_{s \to \infty} \chi(s) \cdot x \in Z_i\}.$$

Let Z^+ (respectively Z^-) be the disjoint, disconnected union of the Z_i^+ (respectively Z_i^-) and the maps $f^\pm : Z \to Z^\pm$ and $g^\pm : Z^\pm \to X$ be the component-wise inclusions. Define projection maps $p^\pm : Z^\pm \to Z$ by $p^+(x) = \lim_{t \to 0} \chi(t) \cdot x$ and $p^-(x) = \lim_{t \to \infty} \chi(t) \cdot x$. These are algebraic maps by [Hes81, Proposition 4.2].[5]

For this section, we let $D(X)$ denote the constructible derived category of sheaves of k-vector spaces on X.

Recall that the hyperbolic localization functors $(-)^{!*}, (-)^{*!} : D(X) \to D(Z)$ for the character χ are defined by

$$\mathcal{F}^{!*} := (f^+)^!(g^+)^*\mathcal{F},$$

$$\mathcal{F}^{*!} := (f^-)^*(g^-)^!\mathcal{F}.$$

We will use the following results of Braden:

Theorem 2.1 ([Bra03], Theorem 1). *For any $\mathcal{F} \in D(X)$, there is a natural morphism $\iota_\mathcal{F} : (\mathcal{F})^{*!} \to (\mathcal{F})^{!*}$. If $\mathcal{F}$ is weakly equivariant (e.g., comes from an object in the equivariant derived category), then*

(i) *there are natural isomorphisms $\mathcal{F}^{!*} \cong (p^+)_!(g^+)^*\mathcal{F}$ and $\mathcal{F}^{*!} \cong (p^-)_*(g^-)^!\mathcal{F}$ and*

(ii) *the morphism $\iota_\mathcal{F} : \mathcal{F}^{*!} \to \mathcal{F}^{!*}$ is an isomorphism.*

Using this result, Braden proves that for $k = \mathbb{Q}$, hyperbolic localization of the intersection cohomology complex $\mathbf{IC}(X; \mathbb{Q})$ is a direct sum of shifted intersection cohomology complexes. Our goal here is to prove a similar result for certain parity complexes.

2.2. **Parity of stalks at T-fixed points.** We keep the notation of §2.1.

Let Y be a smooth projective T-variety and $\pi : Y \to X$ a T-equivariant proper morphism. Furthermore, we will assume that:

(1) The sets of T-fixed points, X^T and Y^T, are finite.

(2) For any T-fixed point $x \in X^T$, there exists a cocharacter $\mathbb{G}_m \to T$ such that $\mathbb{G}_m$ acts attractively on a neighborhood of x.

(3) There exists a T-module V, for which Y admits a closed T-equivariant embedding into the projective space $\mathbb{P}(V)$.

Proposition 2.2. *The cohomology of the stalk of $(\pi_*\underline{k}_Y)^{!*}$ at any $z \in X^T$ is concentrated in even degrees.*

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5. In this article we only consider attractive sets, which corresponds to the speed $m = 1$ case in the setting of [Hes81]. One can reduce to the affine case using [Hes81, Lemma 4.4] because of our standing assumption that X admits a covering by T-stable open affine subvarieties.

Proof. The push-forward $\pi_* \underline{k}_Y$ is weakly equivariant and thus by (i) of Theorem 2.1, $(\pi_* \underline{k}_Y)^{!*} \cong (p^+)_!(g^+)^* \pi_* \underline{k}_Y$.

Let C^+ be defined by the Cartesian square

$$
\begin{array}{ccc}
C^+ & \longrightarrow & Y \\
\downarrow & & \downarrow \\
Z^+ & \longrightarrow & X.
\end{array}
$$

In other words, it is the disjoint union of the preimages $\pi^{-1}(Z_i^+)$, $1 \le i \le m$. We abuse notation and also write π for the induced map $\pi : C^+ \to Z^+$. The map π is proper and so by base change we have

$$(\pi_* \underline{k}_Y)^{!*} = (p^+)_! \pi_! \underline{k}_{C^+}.$$

The stalk of $(\pi_* \underline{k}_Y)^{!*}$ at z thus has cohomology $H_c^*((p^+ \circ \pi)^{-1}(z))$. Using the following lemma, we will construct a Białynicki-Birula decomposition on Y which restricts to one on $(p^+ \circ \pi)^{-1}(z)$.

Lemma 2.3. *For any $z \in X^T$, consider the direct sum decomposition of the Zariski tangent space $T_z X \cong V^+ \oplus V^0 \oplus V^-$, such that χ acts on V^+, V^0, and V^- with positive, zero, and negative weights respectively. There exists a cocharacter $\zeta : \mathbb{G}_m \to T$ whose action on $T_z X$ is attractive on V^+ and repulsive on $V^0 \oplus V^-$, and such that $Y^\zeta = Y^T$.*

Proof. Let $\mathbf{Y}(T)$ denote the cocharacters of T. The set of cocharacters which act on $T_z X$ with negative weights is the intersection of $\mathbf{Y}(T)$ with an open cone in $\mathbf{Y}(T) \otimes \mathbb{R}$. By assumption 2.2(2), this intersection is nonempty. In other words, there exists a $\rho \in \mathbf{Y}(T)$ that acts on $T_z X$ with negative weights.

At each fixed point $y \in Y^T$, the action of T on $T_y Y$ splits as a direct sum of characters. By assumption 2.2(1), the set $\{\beta_i\}$ of all characters of T obtained in this way is finite. As Y is smooth, for any cocharacter $\sigma \in \mathbf{Y}(T)$, $Y^\sigma = Y^T$ if and only if σ is not contained in one of the (finitely many) hyperplanes $\langle \sigma, \beta_i \rangle = 0$. Thus we can choose ρ above such that it acts on $T_z X$ with negative weights and $Y^\rho = Y^T$.

Let $\{\alpha_k^+\}$ denote the set of characters of T occurring in V^+. Thus $\langle \chi, \alpha_k^+ \rangle$ is positive and $\langle \rho, \alpha_k^+ \rangle$ is negative. For $m \in \mathbb{Z}$ sufficiently large, $\langle m\chi + \rho, \alpha_k^+ \rangle = m \langle \chi, \alpha_k^+ \rangle + \langle \rho, \alpha_k^+ \rangle$ is positive for all k. Thus for m large enough the cocharacter $m\chi + \rho$ acts attractively on V^+. On the other hand, for any $m > 0$, $m\chi + \rho$ has strictly negative weights on $V^0 \oplus V^-$.

Lastly, as ρ has been chosen such that $\langle \rho, \beta_i \rangle \neq 0$ for all i, $\langle m\chi + \rho, \beta_i \rangle$ will also be non-zero for m sufficiently large.

For such an m, we may define ζ to be $m\chi + \rho$. $\qquad \square$

Consider the attracting Białynicki-Birula decomposition of Y with respect to ζ. By the assumption (C) in Section 2.1, a T-stable neighborhood of z embeds T-equivariantly into the Zariski tangent space $T_z X$. By the construction of ζ, $(p^+)^{-1}(z)$ is thus the attracting set of z for the action of ζ. If $y \in Y$, then $\lim_{t \to 0} \zeta(t)y \in \pi^{-1}(z)$ if and only if $y \in (p^+ \circ \pi)^{-1}(z)$. It follows that the space $(p^+ \circ \pi)^{-1}(z)$ is a union of cells in the decomposition.

By assumption 2.2(3), the cell decomposition is filtrable [BB76], hence the fundamental classes of the cells give a basis for $H_c^*((p^+ \circ \pi)^{-1}(z))$, which is therefore concentrated in even degrees.

This concludes the proof of Proposition 2.2. $\hfill\square$

Proposition 2.4. *The cohomology of the costalk of $(\pi_* \underline{k}_Y)^{!*}$ at any $z \in X^T$ is concentrated in even degrees.*

Proof. It suffices to show that the stalk of the Verdier dual $\mathbb{D}((\pi_* \underline{k}_Y)^{!*})$ is concentrated in even degrees. For the duration of this proof, let us write $(-)^{!*,\chi}$ for the hyperbolic localization with respect to χ to emphasize the dependence on χ. Then we have:

$$\mathbb{D}((\pi_* \underline{k}_Y)^{!*,\chi}) \cong \mathbb{D}((\pi_* \underline{k}_Y)^{*!,\chi}) = (\mathbb{D}(\pi_* \underline{k}_Y))^{!*,-\chi} = (\pi_* \underline{k}_Y)^{!*,-\chi}[2 \dim Y],$$

where the first isomorphism is given by Theorem 2.1(ii), the second by the definition of hyperbolic localization and the third because π is proper and Y is smooth. The cohomology of the stalk at $z \in X^T$ of the right hand side is concentrated in even degrees by Proposition 2.2. $\hfill\square$

2.3. **Hyperbolic localization on the affine Grassmannian.** We now specialize to the case: X is the affine Grassmannian $\mathcal{G}r$ and T is the maximal torus $\check{T} \subset \check{G}$.

Recall that the connected components of $\mathcal{G}r$ are parametrized by the group $Z(\mathbf{G})^\vee$ of characters of the center $Z(\mathbf{G}) \subset \mathbf{G}$. For any $\zeta \in Z(\mathbf{G})^\vee$, let $\mathcal{G}r_\zeta$ denote the corresponding connected component. For any $\mathcal{F} \in D(\mathcal{G}r)$, we write $\mathcal{F} = \oplus_{\zeta \in Z(\mathbf{G})^\vee} \mathcal{F}_\zeta$ where $\mathcal{F}_\zeta$ is supported on $\mathcal{G}r_\zeta$.

Fix a Levi subgroup $\mathbf{L} \subset \mathbf{G}$ containing the maximal torus $\mathbf{T}$. Correspondingly, there is a Levi subgroup $\check{L}$ of $\check{G}$ containing $\check{T}$ whose roots are dual to those of $\mathbf{L}$. Let χ be the cocharacter of $\check{T}$ defined by $2\rho_\mathbf{G} - 2\rho_\mathbf{L}$, where $\rho_\mathbf{G}$ (resp. $\rho_\mathbf{L}$) denotes the half-sum of the positive roots of $\mathbf{G}$ (resp. $\mathbf{L}$). The set of χ-fixed points of $\mathcal{G}r$ is $\mathcal{G}r_{\check{L}}$. We denote by $R_{\check{L}}^{\check{G}} : D(\mathcal{G}r) \to D(\mathcal{G}r_{\check{L}})$ the shifted hyperbolic localization functor:

$$R_{\check{L}}^{\check{G}}(\mathcal{F}) = \bigoplus_{\zeta \in Z(\mathbf{L})^\vee} (\mathcal{F}^{*!})_\zeta [\langle \zeta, 2\rho_{\check{L}} - 2\rho_{\check{G}} \rangle].$$

As shown in [BD, 5.3.27-31], $R_{\check{L}}^{\check{G}}$ is t-exact and corresponds under geometric Satake to the restriction functor $\mathrm{Rep}(\mathbf{G}) \to \mathrm{Rep}(\mathbf{L})$. (This generalizes the Mirković-Vilonen weight functors which are the case when $\check{L} = \check{T}$.)

We can now use Proposition 2.2 to prove Theorem 1.6.

Proof of Theorem 1.6. Recall from [JMW, Theorem 4.6] and its proof, that every indecomposable parity complex is a direct summand of the push forward of the constant sheaf from a generalized Bott-Samelson resolution $f : \mathrm{BS} \to \overline{\mathcal{G}r^\lambda}$. Thus it suffices to show that $R_{\check{L}}^{\check{G}}(f_* \underline{k}_{\mathrm{BS}})$ is a parity complex.

Let $\check{T} \times \mathbb{C}^*$ act on BS and $\overline{\mathcal{G}r^\lambda}$, where $\mathbb{C}^*$ acts by 'loop-rotation'. We will now show that $\check{T} \times \mathbb{C}^*$ and $f : \mathrm{BS} \to \overline{\mathcal{G}r^\lambda}$ satisfy the assumptions of 2.1 and 2.2 on T, X, and Y. Then by Proposition 2.2 (resp. 2.4), $((f_* \underline{k}_{\mathrm{BS}})^{*!})_\zeta$ has $*$-even stalks (resp. costalks) at the $\check{T} \times \mathbb{C}^*$-fixed points (equivalently the $\check{T}$-fixed points) of $\mathcal{G}r_{\check{L}}$. But $(f_* \underline{k}_{\mathrm{BS}})^{*!}$ is also $\check{L}(\mathcal{O})$-equivariant and every $\check{L}(\mathcal{O})$-orbit contains a $\check{T}$-fixed point, thus $(f_* \underline{k}_{\mathrm{BS}})^{*!}$ is an even complex and $R_{\check{L}}^{\check{G}}(f_* \underline{k}_{\mathrm{BS}})$ is parity.

Thus it remains to check the assumptions (C) of Section 2.1 and (1–3) of Section 2.2. We will outline why they are valid for any Schubert variety in any partial flag variety of a Kac-Moody group (of which the varieties $\overline{\mathcal{G}r^\lambda}$ are a special case).

Assume that $\mathcal{G}$ is a Kac-Moody group with maximal torus $\mathcal{T}$, Weyl group W and simple reflections S. In [Kum02, Chapter VII] it is shown that any Schubert variety in a partial flag variety for $\mathcal{G}$ embeds into the projectivization of a finite dimensional $\mathcal{T}$-representation. Also, the $\mathcal{T}$-fixed points on any Schubert variety are parametrized by an ideal in the Bruhat order on W/W_I, where W denotes the Weyl group and $W_I \subset W$ is a standard parabolic subgroup. In particular, the $\mathcal{T}$-fixed points on any Schubert variety are finite. Recall (see e.g. [GL05, §7, Def.-Prop. 1]) that generalized Bott-Samelson resolutions may be embedded as closed subvarieties of products of Schubert varieties for $\mathcal{G}$. It follows that 2.1 (C) and 2.2(1) and (3) hold for Schubert varieties and their Bott-Samelson resolutions.

Finally, the $\mathcal{T}$-fixed point corresponding to $w \in W/W_I$ in any Schubert variety is attractive. Indeed, all weights in the tangent space belong to the set $-w(R^+)$, where $R^+ \subset \mathbf{X}(\mathcal{T})$ denotes the positive real roots. Hence if $\chi \in \mathbf{Y}(\mathcal{T})$ is a cocharacter which is negative on all simple roots (which exists because the simple roots are linearly independent in $\mathbf{X}(\mathcal{T})$) then $w \cdot \chi$ acts attractively at the fixed point corresponding to w. Hence 2.2(2) is satisfied for any Kac-Moody Schubert variety. $\square$

3. Tilting modules

The aim of this section, which may be read independently of the previous one, is to prove our main result, Theorem 1.8.

3.1. Tilting objects in highest weight categories.
Let $\mathcal{C}$ be a highest weight category with poset Λ and standard (resp. costandard) objects Δ_λ (resp. ∇_λ) for each $\lambda \in \Lambda$.

Let $\mathcal{F}(\Delta) \subset \mathcal{C}$ (resp. $\mathcal{F}(\nabla) \subset \mathcal{C}$) denote the full subcategory of (co)standard filtered objects, meaning $X \in \mathcal{C}$ and X has a filtration whose successive quotients are (co)standard objects. Thus $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) \subset \mathcal{C}$ is the full subcategory of tilting objects.

The following theorem is due to Ringel [Rin91, Theorem 4 and 4*] and gives a useful criterion for determining if an object is tilting.

Theorem 3.1. *An object $X \in \mathcal{F}(\Delta)$ if and only if $\mathrm{Ext}^1(X, \nabla_\lambda) = 0$ for all $\lambda \in \Lambda$. Dually, $X \in \mathcal{F}(\nabla)$ if and only if $\mathrm{Ext}^1(\Delta_\lambda, X) = 0$ for all $\lambda \in \Lambda$.*

We also mention the following result of Donkin [Don98, Proposition A4.4] that will be used in the appendix:

Proposition 3.2. *An object X of $\mathcal{C}$ is in $\mathcal{F}(\nabla)$ (resp. $\mathcal{F}(\Delta)$) if and only if it admits a finite left (resp. right) resolution by tilting modules.*

3.2. Parity and Tilting.
In this section we apply the tilting criterion, Theorem 3.1, to parity sheaves. We first briefly recall the setting of [JMW].

Let H denote a connected linear complex algebraic group. Let X be a complex algebraic variety (resp. H-variety) together with an algebraic stratification $X = \sqcup_{\lambda \in \Lambda} X_\lambda$ into smooth locally closed (resp. H-stable) subsets. We let $D(X)$, or $D(X; k)$, denote the bounded (equivariant) constructible derived category of k-sheaves on X.

Let $P(X)$ or $P(X; k)$ denote the abelian subcategory of $D(X)$ obtained as the heart of the perverse t-structure for the middle perversity. The objects in this category are (equivariant) perverse sheaves and we denote the simple objects, which are parametrized by strata X_λ and irreducible (equivariant) local systems $\mathcal{L}$, by $\mathbf{IC}(\lambda, \mathcal{L})$, or simply $\mathbf{IC}(\lambda)$ when $\mathcal{L} = \underline{k}_{X_\lambda}$ is the constant sheaf.

Recall that for any choice of a function $\dagger : \Lambda \to \mathbb{Z}/2$, which we refer to as a pariversity, there are notions of ($*$- or !-) even, odd and parity complexes. Unless stated otherwise, the pariversity is assumed to be the trivial function $\natural : \Lambda \to \mathbb{Z}/2$, $\natural(\lambda) = 0$. Recall that the dimension (or diamond) pariversity $\Diamond : \Lambda \to \mathbb{Z}/2$ is the function for which $\Diamond(\lambda)$ is given by the parity of the dimension of the stratum X_λ. Note that if all of the strata are even dimensional, it is equal to the $\natural$-pariversity.

We assume that for each stratum X_λ and each (H-equivariant) local system $\mathcal{L}$ on X_λ, the (equivariant) cohomology $H^i(X_\lambda, \mathcal{L}) = 0$ vanishes for all i odd.

The following criterion provides a technique for showing that parity sheaves are tilting.

Proposition 3.3. *Let X be as above. Assume moreover that the strata of X are simply connected and that $P(X)$ is a highest weight category with highest weight poset equal to the closure ordering on the set of strata and whose standard (resp. costandard) objects are given by the perverse extension sheaves, ${}^p\mathcal{J}_!(\lambda) := {}^p j_{\lambda!}\underline{k}_\lambda[d_\lambda]$ (resp. ${}^p\mathcal{J}_*(\lambda) := {}^p j_{\lambda*}\underline{k}_\lambda[d_\lambda]$).*

If a complex $\mathcal{E}$ on X is perverse and parity with respect to the dimension pariversity $\Diamond$, then it is tilting.

Proof. By Proposition 3.1, it suffices to demonstrate for any $\lambda \in \Lambda$ the vanishing

$$\mathrm{Ext}^1(\mathcal{E}, {}^p\mathcal{J}_*(\lambda)) = 0 = \mathrm{Ext}^1({}^p\mathcal{J}_!(\lambda), \mathcal{E}).$$

We prove the first equality. The second follows by duality.

Consider the distinguished triangle

$$ {}^p\mathcal{J}_*(\lambda) \to j_{\lambda*}\underline{k}_\lambda[d_\lambda] \to A \to $$

where $A = {}^p\tau_{>0} j_{\lambda*}\underline{k}_\lambda[d_\lambda] \in {}^pD^{>0}$. By applying $\mathrm{Ext}^i(\mathcal{E}, -)$ to this distinguished triangle in the constructible or equivariant derived category, one obtains a long exact sequence

$$\ldots \to \mathrm{Hom}(\mathcal{E}, A) \to \mathrm{Ext}^1(\mathcal{E}, {}^p\mathcal{J}_*(\lambda)) \to \mathrm{Ext}^1(\mathcal{E}, j_{\lambda*}\underline{k}_\lambda[d_\lambda]) \to \ldots$$

The term $\mathrm{Hom}(\mathcal{E}, A) = 0$ because $\mathcal{E}$ is perverse and $A \in {}^pD^{>0}$. By adjunction $\mathrm{Ext}^1(\mathcal{E}, j_{\lambda*}\underline{k}_\lambda[d_\lambda]) = \mathrm{Ext}^1(j_\lambda^*\mathcal{E}, \underline{k}_\lambda[d_\lambda])$. By the parity assumptions and the fact that X_λ is a single stratum, $j_\lambda^*\mathcal{E}$ is $\Diamond$-even. On the other hand $\underline{k}_\lambda[d_\lambda]$ is also $\Diamond$-even, which implies that the Ext^1 between them vanishes. We have shown that the left and right terms in the sequence above vanish and therefore the middle term does too. $\square$

Remark 3.4. The assumption that the strata be simply connected is made purely for the sake of exposition. The obvious analogue with that assumption removed is true and proven by the same method.

3.3. **A key observation.** We now restrict our attention to the affine Grassmannian for $\check{G}$. Recall the notation from Section 1.2. As mentioned there, the affine Grassmannian satisfies the conditions needed to define parity sheaves. On the

other hand, $P(\mathcal{G}r)$ is also a highest weight category by the geometric Satake theorem. Thus, we can use the previous proposition in this setting. Together with Theorem 1.5 it gives a weak version of Theorem 1.2.

Lemma 3.5. *Suppose T_1 and T_2 are tilting modules for $\mathbf{G}$ such that the corresponding tilting sheaves, $\mathcal{T}_1$ and $\mathcal{T}_2$, on $\mathcal{G}r$ are parity. Then*

(1) *The convolution product $\mathcal{T}_1 \star \mathcal{T}_2$ is perverse and parity, and*

(2) *The tensor product $T_1 \otimes T_2$ is tilting.*

Proof. (1) The convolution product $\mathcal{T}_1 \star \mathcal{T}_2$ is both parity by Theorem 1.5 and perverse as $\star$ is t-exact.

(2) The tensor product $T_1 \otimes T_2$ corresponds under geometric Satake to the convolution product $\mathcal{T}_1 \star \mathcal{T}_2$. By Proposition 3.3 and (1), it is therefore tilting. $\square$

3.4. **Reduction to simple simply-connected groups.** Our goal is to prove Theorem 1.8. We begin with the following reduction.

Lemma 3.6.   (1) $\mathbf{G}$ *satisfies $(\ast)$ if and only if its derived group $\mathcal{D}(\mathbf{G})$ does.*

(2) *Let $Z \subset \mathbf{G}$ be a finite central subgroup and $\mathbf{H} = \mathbf{G}/Z$ be the quotient. If $\mathbf{G}$ satisfies $(\ast)$, then so does $\mathbf{H}$.*

(3) *If $\mathbf{G} = \mathbf{G}_1 \times \ldots \times \mathbf{G}_k$ is a product of connected reductive groups $\mathbf{G}_i$ and each $\mathbf{G}_i$ satisfies $(\ast)$, then $\mathbf{G}$ satisfies $(\ast)$.*

Proof. (1) The short exact sequence $1 \to \mathcal{D}(\mathbf{G}) \to \mathbf{G} \to \mathbf{G}/\mathcal{D}(\mathbf{G}) \to 1$ is Langlands dual to the short exact sequence $1 \to Z(\check{G})^0 \to \check{G} \to \check{G}/Z(\check{G})^0 \to 1$. The latter gives rise to maps

$$\mathcal{G}r_{Z(\check{G})^0} \to \mathcal{G}r \to \mathcal{G}r_{\check{G}/Z(\check{G})^0},$$

which express $\mathcal{G}r$ as a trivial cover of $\mathcal{G}r_{\check{G}/Z(\check{G})^0}$ with fiber $\mathcal{G}r_{Z(\check{G})^0}$. Thus every orbit closure in $\mathcal{G}r$ is isomorphic to an orbit closure in $\mathcal{G}r_{\check{G}/Z(\check{G})^0}$ and vice versa.

(2) Let $\check{H}$ denote the Langlands dual of $\mathbf{H}$. The map $\mathbf{G} \to \mathbf{H}$ is dual to a finite covering map $\check{H} \to \check{G}$. The latter induces an inclusion of connected components $\mathcal{G}r_{\check{H}} \hookrightarrow \mathcal{G}r$. Thus every orbit closure in $\mathcal{G}r_{\check{H}}$ is isomorphic to an orbit closure in $\mathcal{G}r$.

(3) A dominant weight λ of $\mathbf{G}$ is a tuple $(\lambda_1, \ldots, \lambda_k)$ of dominant weights for each $\mathbf{G}_i$. The closure of $\mathcal{G}r^\lambda$ in $\mathcal{G}r$ is the product of the closures of $\mathcal{G}r^{\lambda_i}$ in each $\mathcal{G}r_{\check{G}_i}$. Consider the box product $\mathcal{E} = \mathcal{E}(\lambda_1) \boxtimes \ldots \boxtimes \mathcal{E}(\lambda_k)$ of parity sheaves. It is parity, indecomposable, supported on the closure of $\mathcal{G}r^\lambda$ and $\mathcal{E}|_{\mathcal{G}r^\lambda} = \underline{k}_{\mathcal{G}r^\lambda}[\dim \mathcal{G}r^\lambda]$. Thus $\mathcal{E} = \mathcal{E}(\lambda)$. By assumption, the $\mathcal{E}(\lambda_i)$ are each perverse and tilting. Thus $\mathcal{E} = \mathcal{E}(\lambda)$ is also perverse and hence tilting. $\square$

By part (1) of the Lemma we can replace $\mathbf{G}$ by its derived subgroup $\mathcal{D}(\mathbf{G})$, which is semisimple. Any semisimple group is a quotient of a product of simple simply connected groups by a finite central subgroup. Thus by parts (2) and (3), it suffices to determine for each simple simply-connected group if $(\ast)$ is satisfied.

3.5. **Minuscule weights and the highest short root.** We now assume that $\mathbf{G}$ is simple and simply-connected. We first check the theorem in two special cases.

Lemma 3.7. *Let μ be a minuscule highest weight and α_0 denote the highest short root of $\mathbf{G}$. Then*

(1) $\mathcal{E}(\mu) = \mathcal{T}(\mu) = \mathbf{IC}(\mu)$;

(2) *if p is a good prime for* **G***, then $\mathcal{E}(\alpha_0)$ is perverse and $\mathcal{E}(\alpha_0) = \mathcal{T}(\alpha_0)$.*

(3) *if p is a good prime for* **G** *and moreover $p \nmid n+1$ in type A_n, resp. $p \nmid n$ in type C_n, then $\mathcal{E}(\alpha_0) = \mathcal{T}(\alpha_0) = \mathbf{IC}(\alpha_0)$.*

Although it is not necessary for what follows, we also note:

(4) *In any characteristic, ${}^p\mathcal{H}^0\mathcal{E}(\alpha_0)$ is tilting.*

Proof. (1) The $\check{G}(\mathcal{O})$-orbit in $\mathcal{G}r$ corresponding to the minuscule highest weight μ is closed, thus $\mathbf{IC}(\mu) = {}^p\mathcal{J}_!(\mu) = {}^p\mathcal{J}_*(\mu) = \underline{k}_\mu[d_\mu]$, which implies $\mathcal{E}(\mu) = \mathcal{T}(\mu) = \underline{k}_\mu[d_\mu]$.

(2) Recall [MOV05, 2.3.3] that the orbit closure $\overline{\mathcal{G}r}^{\alpha_0}$ consists of two strata, a point $\mathcal{G}r^0$ and its complement $\mathcal{G}r^{\alpha_0}$, and the singularity is equivalent to that of the orbit closure of the minimal orbit of the corresponding nilpotent cone of $\check{\mathfrak{g}} = \mathrm{Lie}(\check{G})$. Therefore we can apply the results of Section 4.3 of [JMW], for the group $\check{G}$.

When p is a good prime for **G**, it is in particular not one of the primes listed in [JMW, 4.22(5)]. It follows from Proposition 4.22 of [JMW], that there is a perverse parity extension of the constant sheaf on the minimal orbit in $\check{\mathfrak{g}}$. By gluing such a parity complex with the constant sheaf on $\mathcal{G}r^{\alpha_0}$, we obtain an indecomposable perverse parity complex, constructible with respect to the $\check{G}(\mathcal{O})$-orbits. It is thus parity and perverse, so by Proposition 3.3, $\mathcal{E}(\alpha_0) = \mathcal{T}(\alpha_0)$.

(3) As a consequence of [JMW, Proposition 4.23], we have a short exact sequence:

$$0 \longrightarrow i_*(k \otimes_{\mathbb{Z}} H) \longrightarrow {}^p\mathcal{J}_!(\alpha_0) \longrightarrow \mathbf{IC}(\alpha_0) \longrightarrow 0$$

(where H is defined as the fundamental group of the root system consisting of the long roots of $\check{G}$, see [JMW, Proposition 4.23].) So ${}^p\mathcal{J}_!(\alpha_0) \simeq \mathbf{IC}(\alpha_0) \simeq {}^p\mathcal{J}_*(\alpha_0) \simeq \mathcal{T}(\alpha_0)$ as soon as p does not divide H. Assuming that p is good, we only need to add the conditions stated for A_n and C_n.

(4) Let us regard $\mathcal{G}r$ as a flag variety for the affine Kac-Moody group $\mathcal{G}$ associated to $\check{G}$ (see e.g. [JMW, Example 4.2]). Let us parametrize the simple roots of $\mathcal{G}$ by $\{0, \ldots, \ell\}$ so that $\Delta = \{1, \ldots, \ell\}$ corresponds to the simple roots of $\check{G}$. For any subset $I \subset \{0, \ldots, \ell\}$ one has a standard parabolic subgroup $\mathcal{P}_I \subset \mathcal{G}$. Let J denote the subset of Δ of simple roots which are orthogonal to the highest root of $\check{G}$ (i.e. those simple roots corresponding to nodes which are not connected to the exceptional node in the affine Dynkin diagram of $\check{G}$). Now consider the Bott-Samelson space

$$BS := \mathcal{P}_\Delta \times_{\mathcal{P}_J} \mathcal{P}_{J \cup \{\alpha_0\}} \times_{\mathcal{P}_J} \mathcal{P}_\Delta / \mathcal{P}_\Delta.$$

Then BS is a $\mathcal{P}_{J \cup \{\alpha_0\}}/\mathcal{P}_J \cong \mathbb{P}^1$ bundle over $\mathcal{P}_\Delta/\mathcal{P}_J = \check{G}/P$ (where $P \subset \check{G}$ denotes the standard parabolic subgroup corresponding to $J \subset \Delta$). From this one can deduce that $\dim_{\mathbb{C}} BS = \dim \check{G}/P + 1$ and that the $\check{T}$-fixed points on BS are parametrized by $W/W_J \times \{id, s_0\}$, where W denotes the Weyl group of $\check{G}$ and s_0 denotes the affine simple reflection. Now consider the map induced by multiplication:

$$m : BS \to \mathcal{G}/\mathcal{P}_\Delta = \mathcal{G}r.$$

We leave it as an exercise for the reader to show that m is a resolution of $\overline{\mathcal{G}r^{\alpha_0}}$.

Let $\mathcal{P} := f_* \underline{k}_{\mathrm{BS}}[\dim BS]$. By [JMW, Lemma 4.21(1)] we have a short exact sequence:

(3.1) $$0 \to {}^p\mathcal{J}_!(\alpha_0) \to {}^p\mathcal{H}^0\mathcal{P} \to {}^p\mathcal{H}^0 j_{0*}j_0^*\mathcal{P} \to 0$$

This is a filtration by standard sheaves and, as ${}^p\mathcal{H}^0\mathcal{P}$ is self-dual (${}^p\mathcal{H}^0$ is preserved by duality), duality gives a filtration by costandard sheaves. Thus ${}^p\mathcal{H}^0\mathcal{P}$ is tilting.
$\square$

3.6. Fundamental weights.

Proposition 3.8. *Let* **G** *be simple and simply connected with root system* Φ. *If* $p > b(\Phi)$ *(see Table 1), then for each fundamental weight* ϖ_i,

$$\mathcal{E}(\varpi_i) = \mathcal{T}(\varpi_i).$$

Proof. We use the following method: first we express the Weyl modules with fundamental highest weights in characteristic zero as direct summands of tensor products of Weyl modules corresponding to minuscule weights or the highest short root.

By part (1) of Lemma 3.7, we have that $\mathcal{E}(\mu) = \mathcal{T}(\mu) = {}^p\mathcal{J}_!(\mu)$ for any minuscule weight μ, and by part (3), if p is good and Φ not of type A_n or C_n, then $\mathcal{E}(\alpha_0) = \mathcal{T}(\alpha_0) = {}^p\mathcal{J}_!(\alpha_0)$. Thus, in characteristic p, the analogous tensor product of Weyl modules corresponds under to the geometric Satake theorem to a perverse sheaf $\mathcal{E}$ obtained as a convolution product of parity sheaves. The perverse sheaf $\mathcal{E}$ is therefore parity by Theorem 1.5 and tilting by Lemma 3.5. In particular, any summand of $\mathcal{E}$ is perverse, parity and tilting.

On the other hand, if we know that the Weyl modules appearing as direct summands in the tensor products in characteristic 0 remain simple in characteristic p (and hence are indecomposable tilting modules), then by comparing the characters we can conclude that the same decomposition occurs as in characteristic 0.

Thus, we need to know for which primes the Weyl modules remain simple. This has already been done for us and the answers may be found in [Jan03], [Jan91], [Lüb01] or [McN00]. The careful reader should observe that these results are logically independent of Theorems 1.2 and 1.3. In particular, they are obtained via Jantzen's sum formula.

In what follows, we use Bourbaki's notation [Bou68, Planches] for roots, simple roots, fundamental weights, etc. For $\lambda \in \Lambda^+$, we denote by $V(\lambda)$ the Weyl module of highest weight λ over $\mathbb{Q}$.

3.6.1. *Type* A_n. All fundamental weights are minuscule, so there is nothing to prove.

3.6.2. *Type* B_n. The weight ϖ_n is minuscule, and we have

$$V(\varpi_n)^{\otimes 2} \simeq V(2\varpi_n) \oplus V(\varpi_{n-1}) \oplus \ldots \oplus V(\varpi_1) \oplus V(0),$$

all these Weyl modules being simple modulo p as soon as $p > 2$ (see [Jan03, II.8.21] and the references therein, particularly [McN00, Remark 3.4]). So we can generate all fundamental tilting modules when $p > 2$, as claimed.

3.6.3. *Type* C_n. The weight ϖ_1 is minuscule, and for $1 \leq i \leq n$, we have

$$\Lambda^i V(\varpi_1) \simeq V(\varpi_i) \oplus V(\varpi_{i-2}) \oplus \cdots$$

where for convenience we set $\varpi_0 = 0$. All these Weyl modules remaining simple modulo p as soon as $p > n$ (again, see [Jan03, II.8.21] and [McN00, Remark 3.4]). Moreover, the i-th exterior power splits as a summand of the i-th tensor product for $p > i$, so $p > n$ is always sufficient. It follows that we can generate all fundamental tilting modules when $p > n$.

3.6.4. *Type D_n.* The weights ϖ_{n-1} and ϖ_n are minuscule, and we have

$$V(\varpi_n)^{\otimes 2} \quad \simeq \quad V(2\varpi_n) \oplus V(\varpi_{n-2}) \oplus V(\varpi_{n-4}) \oplus \cdots$$

$$V(\varpi_n) \otimes V(\varpi_{n-1}) \quad \simeq \quad V(\varpi_n + \varpi_{n-1}) \oplus V(\varpi_{n-3}) \oplus V(\varpi_{n-5}) \oplus \cdots$$

all these Weyl modules remaining simple modulo p as soon as $p > 2$ (again, see [Jan03, II.8.21] and [McN00, Remark 3.4]). Hence we can generate all fundamental tilting modules when $p > 2$.

3.6.5. *Type E_6.* The minuscule weights are ϖ_1 and ϖ_6, and the highest (short) root is ϖ_2. Moreover, we have

$$\begin{array}{rcl} \Lambda^2 V(\varpi_1) & \simeq & V(\varpi_3) \\ \Lambda^2 V(\varpi_6) & \simeq & V(\varpi_5) \\ \Lambda^2 V(\varpi_2) & \simeq & V(\varpi_4) \oplus V(\varpi_2) \end{array}$$

and all these Weyl modules remain simple modulo p as soon as $p > 3$ [Jan91]. Hence we can generate all fundamental tilting modules when $p > 3$.

3.6.6. *Type E_7.* The weight ϖ_7 is minuscule, and the highest (short) root is ϖ_1. Moreover, we have

$$V(\varpi_1)^{\otimes 2} \quad \simeq \quad V(2\varpi_1) \oplus V(\varpi_1) \oplus V(\varpi_3) \oplus V(\varpi_6) \oplus V(0)$$

$$\begin{array}{rcl} V(\varpi_6) \otimes V(\varpi_7) & \simeq & V(\varpi_6 + \varpi_7) \oplus V(\varpi_1 + \varpi_7) \oplus V(\varpi_2) \oplus \\ & & V(\varpi_5) \oplus V(\varpi_7) \end{array}$$

$$\begin{array}{rcl} V(\varpi_5) \otimes V(\varpi_7) & \simeq & V(\varpi_5 + \varpi_7) \oplus V(\varpi_2 + \varpi_7) \oplus V(\varpi_1 + \varpi_6) \oplus \\ & & V(\varpi_3) \oplus V(\varpi_6) \oplus V(\varpi_4) \end{array}$$

and all these Weyl modules remain simple modulo p as soon as $p > 19$, because then all the weights involved lie in the fundamental alcove.[6] Thus we can generate all fundamental tilting modules when $p > 19$.

3.6.7. *Type E_8.* There is no minuscule weight. The highest (short) root is ϖ_8. We have

$$V(\varpi_8)^{\otimes 2} \quad \simeq \quad V(2\varpi_8) \oplus V(\varpi_7) \oplus V(\varpi_1) \oplus V(\varpi_8) \oplus V(0)$$

$$\begin{array}{rcl} V(\varpi_7) \otimes V(\varpi_8) & \simeq & V(\varpi_7 + \varpi_8) \oplus V(\varpi_1 + \varpi_8) \oplus V(2\varpi_8) \oplus \\ & & V(\varpi_8) \oplus V(\varpi_7) \oplus V(\varpi_6) \oplus V(\varpi_2) \oplus V(\varpi_1) \end{array}$$

$$\begin{array}{rcl} V(\varpi_6) \otimes V(\varpi_8) & \simeq & V(\varpi_6 + \varpi_8) \oplus V(\varpi_7 + \varpi_8) \oplus V(\varpi_2 + \varpi_8) \oplus \\ & & V(\varpi_1 + \varpi_8) \oplus V(\varpi_1 + \varpi_7) \oplus V(\varpi_7) \oplus \\ & & V(\varpi_6) \oplus V(\varpi_5) \oplus V(\varpi_3) \oplus V(\varpi_2) \end{array}$$

$$\begin{array}{rcl} V(\varpi_5) \otimes V(\varpi_8) & \simeq & V(\varpi_5 + \varpi_8) \oplus V(\varpi_1 + \varpi_6) \oplus V(\varpi_2 + \varpi_8) \oplus \\ & & V(\varpi_6 + \varpi_8) \oplus V(\varpi_1 + \varpi_2) \oplus V(\varpi_3 + \varpi_8) \oplus \\ & & V(\varpi_2 + \varpi_7) \oplus V(\varpi_7) \oplus V(\varpi_6) \oplus \\ & & V(\varpi_5) \oplus V(\varpi_4) \oplus V(\varpi_3) \end{array}$$

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6. We remark that $V(\varpi_1 + \varpi_7)$ is reducible modulo 19, according to [Lüb01].

and all these Weyl modules remain simple modulo p as soon as $p > 31$, because then all the weights involved lie in the fundamental alcove.[7] Thus we can generate all fundamental tilting modules when $p > 31$.

3.6.8. *Type F_4.* The short dominant root is ϖ_4, and we have

$$\begin{array}{rcl}
\Lambda^2 V(\varpi_4) & \simeq & V(\varpi_1) \oplus V(\varpi_3) \\
\Lambda^3 V(\varpi_4) & \simeq & V(\varpi_2) \oplus V(\varpi_1 + \varpi_4) \oplus V(\varpi_3)
\end{array}$$

and all these Weyl modules remain simple modulo p as soon as $p > 3$ [Jan91, Lüb01].

So, for $p > 3$, we can get $T(\varpi_1)$ and $T(\varpi_3)$ as direct summands of $T(\varpi_4)^{\otimes 2}$, and $T(\varpi_2)$ as a direct summand of $T(\varpi_4)^{\otimes 3}$.

3.6.9. *Type G_2.* The short dominant root is ϖ_1. We have

$$\Lambda^2 V(\varpi_1) \simeq V(\varpi_1) \oplus V(\varpi_2),$$

and these Weyl modules remain simple modulo p for $p > 3$. Thus we can generate all fundamental tilting modules when $p > 3$.

$\square$

3.7. **Arbitrary weights.** We can now complete the proof of our main theorem.

Proof of Theorem 1.8. Recall that any dominant weight $\lambda \in \Lambda^+$ can be expressed as

$$\lambda = \sum_i a_i \varpi_i$$

for non-negative integers a_i. By Proposition 3.8, for every fundamental weight ϖ_i, $\mathcal{T}(\varpi_i) = \mathcal{E}(\varpi_i)$. Thus the tensor product $\bigotimes T(\varpi_i)^{\otimes a_i}$ corresponds to a perverse sheaf that is also parity (as it is a convolution of parity sheaves) and tilting (by Lemma 3.5). The tensor product is therefore tilting and as it is of highest weight λ, contains $T(\lambda)$ as a direct summand. We conclude that $\mathcal{T}(\lambda)$ is a summand of the corresponding (parity) convolution product and thus $\mathcal{T}(\lambda) = \mathcal{E}(\lambda)$. $\square$

3.8. **A remark on the bound in type C_n.** We do not know in all cases the exact bound on p for which the property $(*)$ holds. Recall that if Conjecture 1.10 is true, then $(*)$ holds whenever p is a good prime.

Stephen Donkin has pointed out to us that in type C_n there exist p with $2 < p \leq n$ such that not all indecomposable tilting modules can be obtained as direct summands of tensor products of the minuscule tilting module $T(\varpi_1)$ and $T(\alpha_0)$. Thus, while it is possible that $(*)$ may hold in type C_n for all $p > 2$, a different method of proof will be required. Let us explain Donkin's argument.

So consider $\mathbf{G} = \mathbf{Sp}_{2n}$, and assume $p > 2$. The only minuscule fundamental weight is ϖ_1, and $E := L(\varpi_1) = T(\varpi_1)$ is the natural $\mathbf{G}$-module, of dimension $2n$. Now the highest short root is ϖ_2, and $T(\varpi_2)$ is a direct summand of $\Lambda^2 E$ which itself is a direct summand of $E^{\otimes 2}$ since $p > 2$. Hence the question is whether every indecomposable tilting module is a direct summand of a tensor power of E.

Now assume p divides n. Thus p divides $\dim E$. By [BC86, Proposition 2.2] (which is stated for representations of finite groups but is valid with the same proof here), any indecomposable summand of a tensor power $E^{\otimes a}$ (with $a \geq 1$) also has dimension divisible by p. Hence an affirmative answer would imply that all indecomposable tilting modules except $T(0) = k$ have dimension divisible by p.

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7. We remark that $V(2\varpi_8)$ is reducible modulo 31, according to [Lüb01].

From this it would follow by induction that the dimension of each standard module ∇_λ, where λ is not in the principal block, would be divisible by p. Indeed, there is a short exact sequence

$$0 \longrightarrow R \longrightarrow T(\lambda) \longrightarrow \nabla_\lambda \longrightarrow 0$$

where R is filtered by ∇_μ's with $\mu < \lambda$. Hence by induction, p divides $\dim R$, and by hypothesis also $\dim T(\lambda)$, hence also $\dim \nabla_\lambda$. The case where $R = 0$ is the base of the induction: then $\nabla_\lambda = T(\lambda)$, and $\lambda \neq 0$, since λ is not in the principal block.

Now by Weyl's dimension formula, for any $m \in \mathbb{N}$ we have

$$\dim \nabla_{(m-1)\rho_{\mathbf{G}}} = m^{n^2}$$

(in general type, one gets $m^{|\Phi^+|}$). Taking m prime to p, this would force $(m-1)\rho_{\mathbf{G}}$ to be in the principal block. In particular we could take $m = 2$ (since $p > 2$), and so $\rho_{\mathbf{G}}$ itself should be in the principal block. However this is not true when n is congruent to 1 or 2 modulo 4, since in the basis of roots the coefficient of ρ along α_n is $\frac{1}{4}m(m+1)$.

Since we have reached a contradiction, we cannot sharpen the bound of the Proposition in type C_n.

4. q-Characters for tilting modules

When p satisfies the conditions of the Theorem 1.8 we are able to deduce that the character of a tilting module has a natural graded refinement. More precisely,

Corollary 4.1. *Suppose that $\mathcal{T}(\lambda) = \mathcal{E}(\lambda)$. Then the total cohomology of the stalk of the tilting sheaf $\mathcal{T}(\lambda)$ at a point in $\mathcal{G}r^\nu$ has the same dimension as the weight space $T(\lambda)^\nu$. Thus the dimension of the weight space has a natural graded refinement.*

Proof. Recall from 2.3 that the weight space functor F_ν corresponds under geometric Satake to the summand $(\mathcal{F}^{*!})_\nu[\langle \nu, 2\rho \rangle]$ of $R_{\check{T}}^{\check{G}}$ supported at the $\check{T}$-fixed point $t^\nu \in \mathcal{G}r^\nu$.

As explained in [Bra03, Prop. 3], the local Euler characteristic of a sheaf $\mathcal{F}$ at a torus fixed point x is equal to the local Euler characteristic of any hyperbolic localization of $\mathcal{F}$. Therefore, the stalk of a perverse sheaf in the Satake category at the point t^ν has an Euler characteristic of absolute value equal to the dimension of the ν-th weight space of the corresponding representation of $\check{G}$. On the other hand, the cohomology of the stalk of the parity sheaf $\mathcal{T}(\lambda)$ is concentrated in even or odd degree, thus its total dimension is equal to the dimension of the weight space $T(\lambda)^\nu$. The total cohomology of the stalk is graded and thus the dimension of the weight space inherits a natural grading. $\square$

Remark 4.2. In characteristic zero (where the indecomposable tilting modules and simple modules coincide) the above q-analogue of weight multiplicity is due to Lusztig [Lus83]. Lusztig shows that the q-characters of simple modules in characteristic zero are given by certain Kazhdan-Lusztig polynomials associated to the (extended) affine Weyl group. In fact, equipped with Lusztig's results, the q-characters of tilting modules can be deduced from the ordinary characters of the indecomposable tilting modules and Lustig's q-characters of simple modules in characteristic zero. Indeed, one can lift $\mathcal{E}(\lambda)$ to a parity sheaf $\mathcal{E}(\lambda, \mathbb{O})$ with coefficients in $\mathbb{O}$, a

complete local ring with residue field k. If $\mathbb{K}$ denotes the fraction field of $\mathbb{O}$ one has an isomorphism

$$\text{(4.1)} \qquad \mathcal{E}(\lambda, \mathbb{O}) \otimes_{\mathbb{O}} \mathbb{K} \cong \bigoplus \mathbf{IC}(\overline{\mathcal{G}r^{\mu}}; \mathbb{K})^{\oplus m_{\mu,\lambda}}$$

where $m_{\mu,\lambda}$ denotes the multiplicity of Δ_{μ} in a Δ-flag on $T(\lambda)$. (We use that the parity sheaves with coefficients in $\mathbb{K}$ on the affine Grassmannian are the intersection cohomology complexes and that $P(\mathcal{G}r; \mathbb{K})$ is semi-simple). The q-character of $\mathcal{E}(\lambda)$ agrees with the q-character of $\mathcal{E}(\lambda, \mathbb{O})$, which in turn agrees with that of (4.1). Hence on can deduce the q-character of $\mathcal{E}(\lambda)$ once one knows that multiplicities $m_{\mu,\lambda}$ and the q-characters in characteristic zero.

Remark 4.3. In [Bry89] R. Brylinski has shown that Lusztig's q-analogue of weight multiplicity can be interpreted in terms of a filtration on each weight space coming from the action of a principal nilpotent element. It would be interesting to find a similar interpretation for the q-character of tilting modules.

5. Appendix

Recall that a $\mathbf{G}$-module V is said to admit a good filtration if it admits a filtration with successive quotients isomorphic to the induced modules ∇_{λ}.

In the references, the Theorems 1.2 and 1.3 are formulated as follows.

Theorem 5.1. *If V and V' are $\mathbf{G}$-modules admitting a good filtration, then so is $V \otimes V'$.*

Theorem 5.2. *Let $\mathbf{L}$ be a Levi subgroup of $\mathbf{G}$. If V is a $\mathbf{G}$-module with a good filtration, then V has also a good filtration when considered as an $\mathbf{L}$-module.*

The aim of this appendix is to show that these formulations are equivalent:

Theorem 5.3. *Theorem 1.2 is equivalent to Theorem 5.1.*

Theorem 5.4. *Theorem 1.3 is equivalent to Theorem 5.2.*

Proof. Suppose Theorems 5.1 and 5.2 are true. If T and T' are tilting $\mathbf{G}$-modules, then in particular they admit good filtrations. Thus $T \otimes T'$ also admits a good filtration, as does $\mathrm{Res}^{\mathbf{G}}_{\mathbf{L}} T$. On the other hand, tensor product and $\mathrm{Res}^{\mathbf{G}}_{\mathbf{L}}$ both commute with duality. We conclude that the $T \otimes T'$ and $\mathrm{Res}^{\mathbf{G}}_{\mathbf{L}} T$ are tilting.

Conversely, suppose that Theorems 1.2 and 1.3 are true. Let V and V' be $\mathbf{G}$-modules that admit good filtrations. By Proposition 3.2, they both admit finite left resolutions by tilting modules. The tensor product of these resolutions is a finite left resolution of $V \otimes V'$ by tilting modules. Applying Proposition 3.2 again, we conclude that $V \otimes V'$ admits a good filtration.

Similarly, as $\mathrm{Res}^{\mathbf{G}}_{\mathbf{L}}$ is exact and by assumption takes tilting modules to tilting modules, $\mathrm{Res}^{\mathbf{G}}_{\mathbf{L}}$ applied to a finite left resolution of V by tilting modules is a finite left resolution of $\mathrm{Res}^{\mathbf{G}}_{\mathbf{L}} V$ by tilting modules. We conclude that $\mathrm{Res}^{\mathbf{G}}_{\mathbf{L}} V$ admits a good filtration. $\qquad\square$

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