DIFFERENTIAL SYMMETRY BREAKING OPERATORS. I. GENERAL THEORY AND F-METHOD.

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ABSTRACT. We prove a one-to-one correspondence between differential symmetry breaking operators for equivariant vector bundles over two homogeneous spaces and certain homomorphisms for representations of two Lie algebras, in connection with branching problems of the restriction of representations.

We develop a new method (F-method) based on the *algebraic Fourier transform for generalized Verma modules*, which characterizes differential symmetry breaking operators by means of certain systems of partial differential equations.

In contrast to the setting of real flag varieties, continuous symmetry breaking operators of Hermitian symmetric spaces are proved to be differential operators in the holomorphic setting. In this case symmetry breaking operators are characterized by differential equations of second order via the F-method.

Key words and phrases: *branching laws, F-method, symmetric pair, invariant theory, Verma modules, Hermitian symmetric spaces.*

CONTENTS

1. INTRODUCTION

Let $\mathcal{W} \to Y$ and $\mathcal{V} \to X$ be two vector bundles with a smooth map $p : Y \to X$. Then we can define "differential operators" $D : C^\infty(X, \mathcal{V}) \to C^\infty(Y, \mathcal{W})$ between the spaces of smooth sections (Definition 2.1).

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Suppose that $G' \subset G$ is a pair of Lie groups acting equivariantly on $\mathcal{W} \to Y$ and $\mathcal{V} \to X$, respectively, and that p is G'-equivariant. The object of the present work is the study of G'-intertwining differential operators (*differential symmetry breaking operators*). If $\mathcal{W}$ is isomorphic to the pull-back $p^*\mathcal{V}$, then the restriction map $f \mapsto f|_Y$ is obviously a G'-intertwining operator (and a differential operator of order zero). In the general setting where there is no morphism from $p^*\mathcal{V}$ to $\mathcal{W}$, non-zero G'-intertwining differential operators may and may not exist.

Suppose that G acts transitively on X and G' acts transitively on Y. We write $X = G/H$ and $Y = G'/H'$ as homogeneous spaces. The first main result is a duality theorem that gives a one-to-one correspondence between G'-intertwining differential operators and $(\mathfrak{g}', H')$-homomorphisms for induced representations of Lie algebras (see Corollary 2.10 for the precise notation):

Theorem A. *Suppose $H' \subset H$. Then there is a natural bijection:*

$$D_{X\to Y} : \mathrm{Hom}_{(\mathfrak{g}',H')}(\mathrm{ind}_{\mathfrak{h}'}^{\mathfrak{g}'}(W^\vee), \mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)) \xrightarrow{\sim} \mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y)\,. \tag{1.1}$$

This generalizes a well-known result in the case where G and G' are the same reductive group and where X and Y are the same flag variety ([Kos74, HJ82]).

By a *branching problem* we wish to understand how a given representation of a group G behaves when restricted to a subgroup G'. For a unitary representation π of G, branching problems concern a decomposition of π into the direct integral of irreducible unitary representations of G' (branching law).

More generally, for non-unitary representations π and τ of G and G', respectively, we may consider the space $\mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$ of continuous G'-homomorphisms. The right-hand side of (1.1) concerns branching problems with respect to the restriction from G to G', whereas the left-hand side of (1.1) concerns branching laws of "generalized Verma modules".

If $\mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y)$ in (1.1) is one-dimensional, we may regard its generator as canonical up to a scalar and be tempted to find an explicit description for such a natural differential symmetry breaking operator. It should be noted that seeking explicit formulæ of intertwining operators is much more involved than finding abstract branching laws, as we may observe with the celebrated Rankin–Cohen brackets which appear as symmetry breaking operators in the decomposition of the tensor product of two holomorphic discrete series representations of $SL(2,\mathbb{R})$ (see [DP07, KP14-2] for a detailed discussion).

The condition $\dim \mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y) \le 1$ is often fulfilled when $\mathfrak{h}$ is a parabolic subalgebra of $\mathfrak{g}$ with abelian nilradical, (see [K14, Theorem 2.7]). Moreover, finding all bundles $\mathcal{W}_Y$ for which such nontrivial intertwining operators exist is a part of the initial problem, which reduces to abstract branching problems (see [KP14-2, Fact ??]).

We propose a new method to find explicit expressions for differential symmetry breaking operators appearing in this geometric setting. We call it the *F-method*, where F stands for the Fourier transform. More precisely, we consider an "algebraic Fourier transform" of generalized Verma modules, and characterize symmetry breaking operators by means of certain systems of partial differential equations. If $\mathfrak{h}$ is a parabolic subalgebra with abelian nilradical, then the system is of second order although the resulting differential symmetry breaking operators may be of any higher order. The characterization is performed by applying an algebraic Fourier transform (see Definition 3.1). A detailed recipe of the *F-method* is described in Section 4.4 relying on Theorem 4.1 and Proposition 3.11.

In general, the symmetry breaking operators between two principal series representations of real reductive Lie groups $G' \subset G$ are given by integro-differential operators in geometric models. Among them, equivariant differential operators are very special (*e.g.* [KnSt71] for $G' = G$ and [KS14] for $G' \subsetneqq G$). However, in the case where X is a Hermitian symmetric space, Y a subsymmetric space, $G' \subset G$ are the groups of biholomorphic transformations of $Y \hookrightarrow X$, respectively, we prove the following *localness and extension theorem*:

Theorem B. *Any continuous G'-homomorphism from $\mathcal{O}(X, \mathcal{V})$ to $\mathcal{O}(Y, \mathcal{W})$ is given by a holomorphic differential operator, which extends to the whole flag variety.*

See Theorem 5.3 for the precise statement. Theorem B includes the case of the tensor product of two holomorphic discrete series representations corresponding to the setting where $G \simeq G' \times G'$ and $X \simeq Y \times Y$ as a special case.

In the second part of the work [KP14-2] we apply the F-method to Hermitian symmetric spaces to find explicit formulæ of differential symmetry breaking operators in the six parabolic geometries arising from symmetric pairs of split rank one.

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Notation: $\mathbb{N} = \{0, 1, 2, \cdots\}$.

2. Differential intertwining operators

In this section we discuss equivariant differential operators between sections of homogeneous vector bundles in a more general setting than the usual. Namely, we consider vector bundles admitting a morphism between their base spaces. In this generality, we establish a natural bijection between such differential operators (*differential symmetry breaking operators*) and certain homomorphisms arising from the branching problems for infinite-dimensional representations of Lie algebras, see Theorem 2.9 (*duality theorem*).

2.1. **Differential operators between two manifolds.** We understand the notion of differential operators between two vector bundles in the usual sense when the bundles are defined over the same base space. We extend this terminology in a more general setting, where there exists a morphism between base spaces. Let $\mathcal{V} \to X$ be a vector bundle over a smooth manifold X. We write $C^\infty(X, \mathcal{V})$ for the space of smooth sections, which is endowed with the Fréchet topology of uniform convergence of sections and their derivatives of finite order on compact sets. Let $\mathcal{W} \to Y$ be another vector bundle, and $p : Y \to X$ a smooth map between the base spaces.

Definition 2.1. We say that a continuous linear map $T : C^\infty(X, \mathcal{V}) \to C^\infty(Y, \mathcal{W})$ is a *differential operator* if T satisfies

$$p\,(\operatorname{Supp} Tf) \subset \operatorname{Supp} f \qquad \text{for any } f \in C^\infty(X, \mathcal{V}). \tag{2.1}$$

We write $\operatorname{Diff}(\mathcal{V}_X, \mathcal{W}_Y)$ for the vector space of differential operators from $C^\infty(X, \mathcal{V})$ to $C^\infty(Y, \mathcal{W})$.

The condition (2.1) shows that T is a local operator in the sense that for any open subset U of X, T induces a continuous linear map:

$$T_U : C^\infty(U, \mathcal{V}|_U) \longrightarrow C^\infty\left(p^{-1}(U), \mathcal{W}|_{p^{-1}(U)}\right).$$

Remark 2.2. If $X = Y$ and p is the identity map, then the condition (2.1) is equivalent to T being a differential operator in the usual sense owing to Peetre's celebrated theorem [Pee59]. Our proof of Lemma 2.3 in this special case gives an account of this classical theorem by using the theory of distributions due to L. Schwartz [S66].

Let $\Omega_X := |\bigwedge^{\mathrm{top}} T^\vee(X)|$ be the bundle of densities. For a vector bundle $\mathcal{V} \to X$, we set $\mathcal{V}^\vee := \coprod_{x \in X} \mathcal{V}_x^\vee$ where $\mathcal{V}_x^\vee := \operatorname{Hom}_{\mathbb{C}}(\mathcal{V}_x, \mathbb{C})$, and denote by $\mathcal{V}^*$ the dualizing bundle $\mathcal{V}^\vee \otimes \Omega_X$. In what follows $\mathcal{D}'(X, \mathcal{V}^*)$ (respectively, $\mathcal{E}'(X, \mathcal{V}^*)$) denotes the space of $\mathcal{V}^*$-valued distributions (respectively, those with compact support). We shall regard distributions as generalized functions à la Gelfand rather than continuous linear forms on $C_c^\infty(X)$ or $C^\infty(X)$. In particular, we sometimes write as

$$\mathcal{E}'(X, \Omega_X) \to \mathbb{C}, \quad \omega \mapsto \int_X \omega, \tag{2.2}$$

to denote the natural pairing $\langle \omega, \mathbf{1}_X \rangle$ of ω with the constant function $\mathbf{1}_X$ on X. Composing (2.2) with the contraction on the fiber, we get a natural bilinear map

$$C^\infty(X, \mathcal{V}) \times \mathcal{E}'(X, \mathcal{V}^*) \to \mathbb{C}, \quad (f, \omega) \mapsto \langle f, \omega \rangle = \int_X f\omega. \tag{2.3}$$

Let $\mathcal{V}^* \boxtimes \mathcal{W}$ denote the tensor product bundle over $X \times Y$ of the two vector bundles $\mathcal{V}^* \to X$ and $\mathcal{W} \to Y$. Then for any continuous linear map $T : C^\infty(X, \mathcal{V}) \to C^\infty(Y, \mathcal{W})$ there exists a unique distribution $K_T \in \mathcal{D}'(X \times Y, \mathcal{V}^* \boxtimes \mathcal{W})$ such that the projection on the second factor $\mathrm{pr}_2 : X \times Y \to Y$ is proper on the support of K_T and such that

$$(Tf)(y) = \langle K_T(\cdot, y), f(\cdot) \rangle \text{ for any } f \in C^\infty(X, \mathcal{V}),$$

by the Schwartz kernel theorem.

Given a map $p : Y \to X$, we set

$$\Delta(Y) := \{(p(y), y) : y \in Y\} \subset X \times Y.$$

The following lemma characterizes differential operators by means of the distribution kernels K_T.

Lemma 2.3. *Let $p : Y \to X$ be a smooth map. A continuous operator $T : C^\infty(X, \mathcal{V}) \to C^\infty(Y, \mathcal{W})$ is a differential operator in the sense of Definition 2.1 if and only if* $\operatorname{Supp} K_T \subset \Delta(Y)$.

Proof. Suppose $\operatorname{Supp} K_T \subset \Delta(Y)$. Let $(x_o, y_o) \in \Delta(Y)$ and take a neighborhood U of $x_o = p(y_o)$ in X and a neighborhood U' of y_o in Y such that $U' \subset p^{-1}(U)$. We trivialize the bundles locally as $\mathcal{V}|_U \simeq U \times V$ and $\mathcal{W}|_{U'} \simeq U' \times W$. Let $(x_1, \cdots, x_m)$ be the coordinates in U. According to the structural theory of distributions supported on a submanifold $\Delta Y \subset X \times Y$ [S66, Chapter III, Théorème XXXVII], there exists a unique family $h_\alpha(y) \in \mathcal{D}'(U') \otimes W$ for a finite number of multi-indices $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{N}^m$, such that $\langle K_T, f \rangle \in \mathcal{D}'(U') \otimes \operatorname{Hom}_{\mathbb{C}}(V, W)$ is locally given as a finite sum

$$\sum_\alpha h_\alpha(y) \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(p(y)), \tag{2.4}$$

for every $f \in C^\infty(X, \mathcal{V})$. Hence $\langle K_T, f \rangle|_{U'} = 0$ if $f|_U = 0$. Thus T is a differential operator in the sense of Definition 2.1.

Conversely, take any $(x_o, y_o) \in \operatorname{Supp} K_T$. By the definition of the distribution kernel K_T, for any neighborhood S of x_o in X there exists $f \in C^\infty(X, \mathcal{V})$ such that $\operatorname{Supp} f \subset S$ and $(x_o, y_o) \in \operatorname{Supp} f \times \operatorname{Supp} Tf$. If T is a differential operator then by (2.1)

$$p(\operatorname{Supp} Tf) \subset \operatorname{Supp} f \subset S.$$

Since S is an arbitrary neighborhood of x_o, $p(y_o)$ must coincide with x_o. Hence $\operatorname{Supp} K_T \subset \Delta(Y)$. □

By (2.4), the terminology "differential operators" in Definition 2.1 is justified as follows:

Example 2.4. (1) *Let $p : Y \twoheadrightarrow X$ be a submersion. Choose an atlas of local coordinates $\{(x_i, z_j)\}$ on Y in such a way that $\{x_i\}$ form an atlas on X. Then, every $T \in \mathrm{Diff}(\mathcal{V}_X, \mathcal{W}_Y)$ is locally of the form*

$$\sum_{\alpha \in \mathbb{N}^{\dim X}} h_\alpha(x,z) \frac{\partial^{|\alpha|}}{\partial x^\alpha} \qquad \text{(finite sum)},$$

where $h_\alpha(x,z)$ are $\mathrm{Hom}(V,W)$*-valued smooth functions on Y.*

(2) *Let $i : Y \hookrightarrow X$ be an immersion. Choose an atlas of local coordinates $\{(y_i, z_j)\}$ on X in such a way that $\{y_i\}$ form an atlas on Y. Then, every $T \in \mathrm{Diff}(\mathcal{V}_X, \mathcal{W}_Y)$ is locally of the form*

$$\sum_{(\alpha,\beta) \in \mathbb{N}^{\dim X}} g_{\alpha\beta}(y) \frac{\partial^{|\alpha|+|\beta|}}{\partial y^\alpha \partial z^\beta} \qquad \text{(finite sum)},$$

where $g_{\alpha,\beta}(y)$ are $\mathrm{Hom}(V,W)$*-valued smooth functions on Y.*

Next, suppose that the two vector bundles $\mathcal{V} \to X$ and $\mathcal{W} \to Y$ are equivariant with respect to a given Lie group G. Then we have natural actions of G on the Fréchet spaces $C^\infty(X, \mathcal{V})$ and $C^\infty(Y, \mathcal{W})$ by translations. Denote by $\mathrm{Hom}_G(C^\infty(X,\mathcal{V}), C^\infty(Y,\mathcal{W}))$ the space of continuous G-homomorphisms. We set

$$\mathrm{Diff}_G(\mathcal{V}_X, \mathcal{W}_Y) := \mathrm{Diff}(\mathcal{V}_X, \mathcal{W}_Y) \cap \mathrm{Hom}_G(C^\infty(X,\mathcal{V}), C^\infty(Y,\mathcal{W})). \tag{2.5}$$

Example 2.5. *Suppose X and Y are both Euclidean vector spaces with an injective linear map $p : Y \hookrightarrow X$. If G contains the subgroup of all translations of Y then $\mathrm{Diff}_G(\mathcal{V}_X, \mathcal{W}_Y)$ is a subspace of the space of differential operators with constant coefficients.*

An analogous notion can be defined in the holomorphic setting. Let $\mathcal{V} \to X$ and $\mathcal{W} \to Y$ be two holomorphic vector bundles with a holomorphic map $p : Y \to X$ between the complex base manifolds X and Y. We say a differential operator $T : C^\infty(X, \mathcal{V}) \to C^\infty(Y, \mathcal{W})$ is *holomorphic* if

$$T_U(\mathcal{O}(U, \mathcal{V}|_U)) \subset \mathcal{O}(p^{-1}(U), \mathcal{W}|_{p^{-1}(U)})$$

for any open subset U of X. We denote by $\mathrm{Diff}^{\mathrm{hol}}(\mathcal{V}_X, \mathcal{W}_Y)$ the vector space of holomorphic differential operators. When a Lie group G acts biholomorphically on the two holomorphic vector bundles $\mathcal{V} \to X$ and $\mathcal{W} \to Y$, we set

$$\mathrm{Diff}_G^{\mathrm{hol}}(\mathcal{V}_X, \mathcal{W}_Y) := \mathrm{Diff}^{\mathrm{hol}}(\mathcal{V}_X, \mathcal{W}_Y) \cap \mathrm{Hom}_G(C^\infty(X,\mathcal{V}), C^\infty(Y,\mathcal{W})).$$

2.2. **Induced modules.** Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$, and $U(\mathfrak{g})$ its universal enveloping algebra. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$.

Definition 2.6. For an $\mathfrak{h}$-module V, we define the induced $U(\mathfrak{g})$-module $\mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V)$ as

$$\mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V) := U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V.$$

If $\mathfrak{h}$ is a Borel subalgebra and $\dim V = 1$, then the $\mathfrak{g}$-module $\mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V)$ is the Verma module.

For later purposes we formulate the following statement in terms of the contragredient representation $V^\vee$. Let $\mathfrak{h}'$ be another Lie subalgebra of $\mathfrak{g}$.

Proposition 2.7. *For a finite-dimensional $\mathfrak{h}'$-module W we have:*

(1) $\mathrm{Hom}_{\mathfrak{g}}(\mathrm{ind}_{\mathfrak{h}'}^{\mathfrak{g}}(W^\vee), \mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)) \simeq \mathrm{Hom}_{\mathfrak{h}'}(W^\vee, \mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee))$.

(2) *If $\mathfrak{h}' \not\subset \mathfrak{h}$, then* $\mathrm{Hom}_{\mathfrak{h}'}(W^\vee, \mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)) = \{0\}$.

Proof. The first statement is due to the functoriality of the tensor product.

For the second statement it suffices to treat the case where $\mathfrak{h}'$ is one-dimensional. Then the assumption $\mathfrak{h}' \not\subset \mathfrak{h}$ implies that $\mathfrak{h}' \cap \mathfrak{h} = \{0\}$, and therefore there is a direct sum decomposition of vector spaces:

$$\mathfrak{g} = \mathfrak{h}' + \mathfrak{q} + \mathfrak{h},$$

for some subspace $\mathfrak{q}$ in $\mathfrak{g}$. We fix a basis $X_1, \cdots, X_n$ of $\mathfrak{q}$, and define a subspace of $U(\mathfrak{g})$ by

$$U'(\mathfrak{q}) := \mathbb{C}\text{-span}\left\{X_1^{\alpha_1} \cdots X_n^{\alpha_n} : (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n\right\}.$$

Then, by the Poincaré–Birkhoff–Witt theorem we have an isomorphism of $\mathfrak{h}'$-modules:

$$\mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee) \simeq U(\mathfrak{h}') \otimes_{\mathbb{C}} U'(\mathfrak{q}) \otimes_{\mathbb{C}} V^\vee.$$

In particular, $\mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)$ is a free $U(\mathfrak{h}')$-module. Hence there does not exist a non-zero *finite-dimensional* $\mathfrak{h}'$-submodule in the $\mathfrak{g}$-module $\mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)$. □

Remark 2.8. We shall see in Theorem 2.9 that $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{h}'}^{\mathfrak{g}'}(W^\vee), \mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee))$ is equal to the dimension of the space of differential symmetry breaking operators from $C^\infty(X, \mathcal{V})$ to $C^\infty(Y, \mathcal{W})$ when H' is connected. In [KP14-2, Section **??**], we give a family of sextuples $(\mathfrak{g}, \mathfrak{g}', \mathfrak{h}, \mathfrak{h}', V, W)$ such that this dimension is one.

2.3. **Duality theorem for differential operators between two homogeneous spaces.** Let G be a real Lie group, and $\mathfrak{g}(\mathbb{R}) := \mathrm{Lie}(G)$. We denote by $U(\mathfrak{g})$ the universal enveloping algebra of the complexified Lie algebra $\mathfrak{g} := \mathfrak{g}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. Analogous notations will be applied to other Lie groups.

Let H be a closed subgroup of G. Given a finite-dimensional representation $\lambda : H \to GL_{\mathbb{C}}(V)$ we define the homogeneous vector bundle $\mathcal{V}_X \equiv \mathcal{V} := G \times_H V$ over

$X := G/H$. As a G-module, the space $C^\infty(X, \mathcal{V})$ of smooth sections is identified with the following subspace of $C^\infty(G, V) \simeq C^\infty(G) \otimes V$:

$$\begin{aligned} C^\infty(G,V)^H :=& \{f \in C^\infty(G,V) : f(gh) = \lambda(h)^{-1} f(g) \text{ for any } g \in G, h \in H\} \\ \simeq& \{F \in C^\infty(G) \otimes V : \lambda(h)F(gh) = F(g) \quad \text{for any } g \in G,\ h \in H\}. \end{aligned}$$

In dealing with a representation V of a disconnected subgroup H (*e.g.* H is a parabolic subgroup of a real reductive Lie group G), we notice that the diagonal H-action on $U(\mathfrak{g}) \otimes_{\mathbb{C}} V^\vee$ defines a representation of H on $\operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee) = U(\mathfrak{g}) \otimes_{\mathfrak{h}} V^\vee$ and thus $\operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)$ is endowed with a $(\mathfrak{g}, H)$-module structure.

Theorem 2.9 (Duality theorem)**.** *Let $H' \subset H$ be (possibly disconnected) closed subgroups of a Lie group G with Lie algebras $\mathfrak{h}' \subset \mathfrak{h}$, respectively. Suppose V and W are finite-dimensional representations of H and H', respectively. Let G' be any subgroup of G containing H', and $\mathcal{V}_X := G \times_H V$ and $\mathcal{W}_Y := G' \times_{H'} W$ be the corresponding homogeneous vector bundles. Then, there is a natural linear isomorphism:*

$$D_{X\to Y} : \operatorname{Hom}_{H'}(W^\vee, \operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)) \xrightarrow{\sim} \operatorname{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y), \tag{2.6}$$

or equivalently,

$$D_{X\to Y} : \operatorname{Hom}_{(\mathfrak{g}',H')}(\operatorname{ind}_{\mathfrak{h}'}^{\mathfrak{g}'}(W^\vee), \operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)) \xrightarrow{\sim} \operatorname{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y).$$

For $\varphi \in \operatorname{Hom}_{H'}(W^\vee, \operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee))$ and $F \in C^\infty(X, \mathcal{V}) \simeq C^\infty(G,V)^H$, $D_{X\to Y}(\varphi)F \in C^\infty(Y, \mathcal{W}) \simeq C^\infty(G', W)^{H'}$ is given by the following formula:

$$\langle D_{X\to Y}(\varphi)F, w^\vee\rangle = \sum_j \langle dR(u_j)F, v_j^\vee\rangle|_{G'} \quad \text{for } w^\vee \in W^\vee, \tag{2.7}$$

where $\varphi(w^\vee) = \sum_j u_j v_j^\vee \in \operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)$ ($u_j \in U(\mathfrak{g})$, $v_j^\vee \in V^\vee$).

When H' is connected, we can write the left-hand side of (2.6) by means of Lie algebras.

Corollary 2.10. *Suppose we are in the setting of Theorem 2.9. Assume that H' is connected. Then there is a natural linear isomorphism:*

$$D_{X\to Y} : \operatorname{Hom}_{\mathfrak{h}'}(W^\vee, \operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)) \xrightarrow{\sim} \operatorname{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y), \tag{2.8}$$

or equivalently,

$$D_{X\to Y} : \operatorname{Hom}_{\mathfrak{g}'}(\operatorname{ind}_{\mathfrak{h}'}^{\mathfrak{g}'}(W^\vee), \operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)) \xrightarrow{\sim} \operatorname{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y). \tag{2.8$'$}$$

The construction of $D_{X\to Y}$ and the fact that the formula (2.7) is well-defined will be explained in Section 2.4.

Remark 2.11. (1) Corollary 2.10 is known when $X = Y$, *i.e.* $G' = G$ and $H' = H$, especially in the setting of complex flag varieties, see *e.g.* [Kos74, HJ82].

(2) When $\mathfrak{g}'$ is a reductive subalgebra and $\mathfrak{h}'$ is a parabolic subalgebra, the existence of an $\mathfrak{h}'$-module W for which the left-hand side of (2.8) is non-zero, is closely related to the "discretely decomposability" of the $\mathfrak{g}$-module $\mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)$ when restricted to the subalgebra $\mathfrak{g}'$ ([K98, Part III], [K12]). This relationship will be used in Section 5 in proving that any continuous symmetry breaking operator in a holomorphic setting is given by a differential operator (*localness theorem*).

(3) Owing to Proposition 2.7, the left-hand side of (2.8)′ is non-zero only when $\mathfrak{h}' \subset \mathfrak{h}$. Conversely, if $H' \subset H \cap G'$, then there is a natural morphism $Y = G'/H' \to X = G/H$ and therefore "differential operators" (in the sense of Definition 2.1) from $C^\infty(X, \mathcal{V})$ to $C^\infty(Y, \mathcal{W})$ are defined.

(4) We shall consider the case where $H' = H \cap G'$ in later applications, however, Theorem 2.9 also covers the cases where the natural morphism $Y \to X$ is not injective, *i.e.* where $H' \subsetneq H \cap G'$.

An analogous result to Theorem 2.9 holds in the holomorphic setting as well. To be precise, let $G_{\mathbb{C}}$ be a complex Lie group, $G'_{\mathbb{C}}$, $H_{\mathbb{C}}$ and $H'_{\mathbb{C}}$ be closed complex subgroups such that $H'_{\mathbb{C}} \subset H_{\mathbb{C}} \cap G'_{\mathbb{C}}$. We write $\mathfrak{g}$, $\mathfrak{h}$, … for the Lie algebras of the complex Lie groups $G_{\mathbb{C}}$, $H_{\mathbb{C}}$, …, respectively. Given finite-dimensional holomorphic representations V of $H_{\mathbb{C}}$ and W of $H'_{\mathbb{C}}$, we form holomorphic vector bundles $\mathcal{V} := G_{\mathbb{C}} \times_{H_{\mathbb{C}}} V$ over $X_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$ and $\mathcal{W} := G'_{\mathbb{C}} \times_{H'_{\mathbb{C}}} W$ over $Y_{\mathbb{C}} = G'_{\mathbb{C}}/H'_{\mathbb{C}}$.

For simplicity, we assume that $H'_{\mathbb{C}}$ is connected. (This is always the case if $G'_{\mathbb{C}}$ is a connected complex reductive Lie group and $H'_{\mathbb{C}}$ is a parabolic subgroup of $G'_{\mathbb{C}}$.) Then we have:

Theorem 2.12 (Duality theorem in the holomorphic setting)**.** *There is a canonical linear isomorphism:*

$$D_{X\to Y} : \mathrm{Hom}_{\mathfrak{g}'}\left(\mathrm{ind}_{\mathfrak{h}'}^{\mathfrak{g}'}(W^\vee), \mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)\right) \xrightarrow{\sim} \mathrm{Diff}^{\mathrm{hol}}_{G'_{\mathbb{C}}}(\mathcal{V}_{X_{\mathbb{C}}}, \mathcal{W}_{Y_{\mathbb{C}}}).$$

Suppose furthermore that G, G', H and H' are real forms of the complex Lie groups $G_{\mathbb{C}}$, $G'_{\mathbb{C}}$, $H_{\mathbb{C}}$ and $H'_{\mathbb{C}}$, respectively. We regard V and W as H- and H'-modules by the restriction, and form vector bundles $\mathcal{V} = G \times_H V$ over $X = G/H$ and $\mathcal{W} = G' \times_{H'} W$ over $Y = G'/H'$.

We ask whether or not all symmetry breaking operators have holomorphic extensions. Here is a simple sufficient condition:

Corollary 2.13. *If H' is contained in the connected complexification $H'_{\mathbb{C}}$, then we have a natural bijection:*

$$\mathrm{Diff}^{\mathrm{hol}}_{G'_{\mathbb{C}}}(\mathcal{V}_{X_{\mathbb{C}}}, \mathcal{W}_{Y_{\mathbb{C}}}) \xrightarrow{\sim} \mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y).$$

Proof. Comparing Theorem 2.9 with Theorem 2.12, the proof of Corollary 2.13 reduces to the surjectivity of the inclusion

$$\mathrm{Hom}_{(\mathfrak{g}',H')}(\mathrm{ind}_{\mathfrak{h}'}^{\mathfrak{g}'}(W^\vee),\mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{h}'}^{\mathfrak{g}'}(W^\vee),\mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)). \tag{2.9}$$

We note that $\mathrm{Hom}_{(\mathfrak{g}',H'_{\mathbb{C}})}(\mathrm{ind}_{\mathfrak{h}'}^{\mathfrak{g}'}(W^\vee),\mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee))$ is a subspace of the left-hand side of (2.9) because $H' \subset H'_{\mathbb{C}}$, whereas it coincides with the right-hand side of (2.9) if $H'_{\mathbb{C}}$ is connected. Hence (2.9) is surjective. Thus Corollary 2.13 is proved. ☐

The rest of this section is devoted to the proof of Theorem 2.9. For Theorem 2.12, since the argument is parallel to that of Theorem 2.9, we omit the proof.

2.4. **Construction of $D_{X\to Y}$.** This subsection gives the definition of the linear map $D_{X\to Y}$ in Theorem 2.9.

Consider two actions dR and dL of the universal enveloping algebra $U(\mathfrak{g})$ on the space $C^\infty(G)$ of smooth complex-valued functions on G induced by the regular representation $L\times R$ of $G\times G$ on $C^\infty(G)$:

$$(dR(Z)f)(x) := \frac{d}{dt}\Big|_{t=0} f(xe^{tZ}) \quad \text{and} \quad (dL(Z)f)(x) := \frac{d}{dt}\Big|_{t=0} f(e^{-tZ}x), \tag{2.10}$$

for $Z \in \mathfrak{g}(\mathbb{R})$.

The right differentiation (2.10) defines a bilinear map

$$\Phi : C^\infty(G)\times U(\mathfrak{g}) \to C^\infty(G), \quad (F,u)\mapsto dR(u)F,$$

with the following properties

$$\Phi(L(g)F,u) = L(g)\Phi(F,u), \tag{2.11}$$

$$\Phi(F,u'u) = dR(u')\Phi(F,u), \tag{2.12}$$

for any $g\in G$ and $u,u'\in U(\mathfrak{g})$.

Combining Φ with the canonical pairing $V\times V^\vee \to \mathbb{C}$, we obtain a bilinear map

$$\Phi_V : C^\infty(G)\otimes V \times U(\mathfrak{g})\otimes_{\mathbb{C}} V^\vee \to C^\infty(G).$$

Then we have the following:

Lemma 2.14. *The map Φ_V induces a well-defined diagram of maps:*

$$\begin{array}{ccccc} C^\infty(G)\otimes V & \times & U(\mathfrak{g})\otimes_{\mathbb{C}} V^\vee & \xrightarrow{\Phi_V} & C^\infty(G) \\ \uparrow & & \downarrow & & \| \\ C^\infty(X,\mathcal{V}) & \times & \mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee) & \dashrightarrow & C^\infty(G). \end{array}$$

Proof. Denote by $\lambda^\vee$ the contragredient representation of the representation (λ,V) of H, and by $d\lambda^\vee$ the infinitesimal representation of $\mathfrak{h}$. The kernel of the natural quotient map $U(\mathfrak{g})\otimes_{\mathbb{C}} V^\vee \to \mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)$ is generated by

$$-uY\otimes v^\vee + u\otimes d\lambda^\vee(Y)v^\vee$$

with $u \in U(\mathfrak{g}), Y \in \mathfrak{h}$ and $v^\vee \in V^\vee$. Hence it suffices to show

$$\Phi_V(f, -uY \otimes v^\vee + u \otimes d\lambda^\vee(Y)v^\vee) = 0$$

for any $f \in C^\infty(X, \mathcal{V}) \simeq C^\infty(G, V)^H$.

Since $f \in C^\infty(G, V)^H$ satisfies $dR(Y)f = -d\lambda(Y)f$ for $Y \in \mathfrak{h}$, we have

$$\begin{aligned} \Phi_V(f, uY \otimes v^\vee) &= \langle dR(u)dR(Y)f, v^\vee\rangle \\ &= \langle dR(u)f, d\lambda^\vee(Y)v^\vee\rangle \\ &= \Phi_V(f, u \otimes d\lambda^\vee(Y)v^\vee\rangle. \end{aligned}$$

Thus the lemma is proved. □

Lemma 2.15. 1) *The bilinear map*

$$C^\infty(X, \mathcal{V}) \times \operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee) \to \mathbb{C}, (f, m) \mapsto \Phi_V(f, m)(e) \tag{2.13}$$

is $(\mathfrak{g}, H)$*-invariant.*

2) *If* $m \in \operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)$ *satisfies* $\Phi_V(f, m)(e) = 0$ *for all* $f \in C^\infty(X, \mathcal{V})$ *then* $m = 0$.

Proof. 1) Let $f \in C^\infty(X, \mathcal{V})$ and $m \in \operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)$. It follows from (2.11) and (2.12) that

$$\Phi_V(dL(Z)f, m) = dL(Z)\Phi_V(f, m)$$
$$\Phi_V(f, Zm) = dR(Z)\Phi_V(f, m)$$

for any $Z \in \mathfrak{g}$. Since

$$(dL(Z) + dR(Z))F(e) = 0$$

for any $F \in C^\infty(G)$, we have shown the $\mathfrak{g}$-invariance of the bilinear map (2.13):

$$\Phi_V(dL(Z)f, m)(e) + \Phi_V(f, Zm)(e) = 0.$$

The proof for the H-invariance of (2.13) is similar.

2) Take a basis $\{v_1, \cdots, v_k\}$ of V, and let $\{v_1^\vee, \cdots, v_k^\vee\}$ be the dual basis in $V^\vee$. Choose a complementary subspace $\mathfrak{q}$ of $\mathfrak{h}$ in $\mathfrak{g}$, and fix a basis $\{X_1, \cdots, X_n\}$ of $\mathfrak{q}$. Then by the Poincaré–Birkhoff–Witt theorem, we can write $m \in \operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)$ as a finite sum:

$$m = \sum_{j=1}^{k} \sum_{\alpha = (\alpha_1, \cdots, \alpha_n)} a_{\alpha, j} X_1^{\alpha_1} \cdots X_n^{\alpha_n} v_j^\vee.$$

If m is non-zero, we can find a multi-index β and j_o $(1 \le j_o \le k)$ such that $a_{\beta, j_o} \ne 0$ and that $a_{\alpha, j_0} = 0$ for any multi-index α satisfying $|\alpha| > |\beta|$ and for any j. Here $|\alpha| = \sum_{i=1}^n \alpha_i$ for $\alpha \in \mathbb{N}^n$. We take $f \in C^\infty(G, V)^H \simeq C^\infty(X, \mathcal{V})$ such that f is given in a right H-invariant neighborhood of H in G by

$$f\left(\exp\left(\sum_{i=1}^n x_i X_i\right) h\right) = x^\beta \lambda(h)^{-1} v_{j_o} \quad \text{for } x = (x_1, \cdots, x_n) \in \mathbb{R}^n \text{ and } h \in H.$$

Then $\Phi_V(f, m)(e) = a_{\beta, j_o} \beta_1! \cdots \beta_k! \ne 0$. The contraposition completes the proof. □

We regard $C^\infty(G)$ as a $G \times \mathfrak{g}$-module via the $(L \times dR)$-action. Then the space $\mathrm{Hom}_G(C^\infty(X,\mathcal{V}), C^\infty(G))$ of continuous G-homomorphisms becomes a $\mathfrak{g}$-module by the remaining dR-action on the target space. By (2.11), (2.12) and Lemma 2.14, we get the following $\mathfrak{g}$-homomorphism:

$$\mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee) \longrightarrow \mathrm{Hom}_G(C^\infty(X,\mathcal{V}), C^\infty(G)), \quad u \otimes v^\vee \mapsto (f \mapsto \langle dR(u)f, v^\vee\rangle). \tag{2.14}$$

Furthermore, it is actually a $(\mathfrak{g}, H)$-homomorphism, where the group H acts on $\mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V^\vee$ diagonally and acts on $\mathrm{Hom}_G(C^\infty(X,\mathcal{V}), C^\infty(G))$ via the R-action on $C^\infty(G)$.

Let H' be a connected closed Lie subgroup of G. Given a finite-dimensional representation W of H', we form a homogeneous vector bundle $\mathcal{W}_Z \equiv \mathcal{W} := G \times_{H'} W$ over $Z := G/H'$.

Taking the tensor product of the $(\mathfrak{g}, H)$-modules in (2.14) with the H'-module W, we get an $(H' \times (\mathfrak{g}, H))$-homomorphism:

$$\mathrm{Hom}_{\mathbb{C}}(W^\vee, \mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)) \longrightarrow \mathrm{Hom}_G(C^\infty(X,\mathcal{V}), C^\infty(G, W)).$$

Let $\Delta(H')$ be a subgroup of $H' \times H$ defined by $\{(h,h) : h \in H'\}$. Taking $\Delta(H')$-invariants, we obtain the following $\mathbb{C}$-linear map:

$$\mathrm{Hom}_{H'}(W^\vee, \mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)) \longrightarrow \mathrm{Hom}_G(C^\infty(X,\mathcal{V}), C^\infty(Z,\mathcal{W})), \quad \varphi \mapsto D_\varphi, \tag{2.15}$$

where D_φ satisfies

$$\langle D_\varphi f, w^\vee\rangle = \Phi_V(f, \varphi(w^\vee)) \tag{2.16}$$

for any $f \in C^\infty(X,\mathcal{V})$ and any $w^\vee \in W^\vee$.

Remark 2.16. If H' is connected, then we can replace $\mathrm{Hom}_{H'}$ by $\mathrm{Hom}_{\mathfrak{h}'}$ in (2.15).

Lemma 2.17. *The map* (2.15) *is injective.*

Proof. By (2.16), Lemma 2.17 is derived from the second statement of the Lemma 2.15. □

Take any subgroup G' of G containing H' and form a homogeneous vector bundle $\mathcal{W}_Y := G' \times_{H'} W$ over $Y = G'/H'$. Then, the vector bundle $\mathcal{W}_Y$ is isomorphic to the restriction $\mathcal{W}_Z|_Y$ of the vector bundle $\mathcal{W}_Z$ to the submanifold Y of the base space Z. Let

$$R_{Z\to Y} : C^\infty(Z, \mathcal{W}_Z) \to C^\infty(Y, \mathcal{W}_Y)$$

be the restriction map of sections. For $\varphi \in \mathrm{Hom}_{\mathfrak{h}'}(W^\vee, \mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee))$ we set

$$D_{X\to Y}(\varphi) := R_{Z\to Y} \circ D_\varphi. \tag{2.17}$$

Then $D_{X\to Y}(\varphi) : C^\infty(X,\mathcal{V}) \to C^\infty(Y,\mathcal{W})$ is a G'-equivariant differential operator, *i.e.* $D_{X\to Y}$ defines a linear map $\mathrm{Hom}_{\mathfrak{h}'}(W^\vee, \mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)) \to \mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y)$. Theorem 2.9

describes explicitly the image $D_{X\to Y}$ when $H' \subset H \cap G'$, namely, when the following diagram exists:

$$\begin{array}{ccccc} & & Z = G/H' & & \\ & \nearrow & & \searrow & \\ Y = G'/H' & & \longrightarrow & & X = G/H \end{array}.$$

Remark 2.18. The left-hand side of (2.8) does not depend on the choice of G'. This fact is reflected by the commutativity of the following diagram.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathfrak{h}'}(W^\vee, \mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)) & \xrightarrow{\ \sim\ } & \mathrm{Diff}_G(\mathcal{V}_X, \mathcal{W}_Z) \\ & {\scriptstyle D_{X\to Y}}\searrow^{\sim} \quad \swarrow {\scriptstyle R_{Z\to Y}} & \\ & \mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y) & \end{array} \tag{2.18}$$

2.5. **Proof of Theorem 2.9.** We have already seen in Lemma 2.17 that $D_{X\to Y}$ is injective. In order to prove the surjectivity of the linear map $D_{X\to Y}$, we realize the induced $U(\mathfrak{g})$-module $\mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)$ in the space of distributions.

We recall that $\mathcal{V}^* = \mathcal{V}^\vee \otimes \Omega_X$ is the dualizing bundle of a vector bundle $\mathcal{V}$ over X. For a closed subset S and an open subset U in X containing S, we write $\mathcal{D}'_S(U, \mathcal{V}^*)$ for the space of $\mathcal{V}^*$-valued distributions on U with support in S. Obviously, $\mathcal{D}'_S(U, \mathcal{V}^*) = \mathcal{D}'_S(X, \mathcal{V}^*)$. If S is compact, then $\mathcal{D}'_S(U, \mathcal{V}^*)$ is contained in the space $\mathcal{E}'(U, \mathcal{V}^*)$ of distributions on U with compact support, and thus coincides with $\mathcal{E}'_S(U, \mathcal{V}^*) := \mathcal{D}'_S(U, \mathcal{V}^*) \cap \mathcal{E}'(U, \mathcal{V}^*)$.

We return to the setting of Theorem 2.9, where $\mathcal{V}$ is a G-equivariant vector bundle over $X = G/H$. Then the Lie group G acts on $C^\infty(X, \mathcal{V})$ and $\mathcal{E}'(X, \mathcal{V}^*)$ by the pull-back of smooth sections and distributions, respectively. The infinitesimal action defines representations of the Lie algebra $\mathfrak{g}$ on $C^\infty(U, \mathcal{V})$ and $\mathcal{E}'_S(U, \mathcal{V}^*)$.

The "integration map" (2.2)

$$\mathcal{E}'(X, \Omega_X) \to \mathbb{C}, \qquad \omega \mapsto \int_X \omega \tag{2.19}$$

is G-invariant. Composing this with the G-invariant bilinear map (*contraction*):

$$C^\infty(X, \mathcal{V}) \times \mathcal{E}'(X, \mathcal{V}^*) \longrightarrow \mathcal{E}'(X, \Omega_X), \qquad (f, h) \mapsto \langle f, h\rangle,$$

we obtain the following G-invariant bilinear form

$$C^\infty(X, \mathcal{V}) \times \mathcal{E}'(X, \mathcal{V}^*) \longrightarrow \mathbb{C}, \qquad (f, h) \mapsto \int_X \langle f, h\rangle. \tag{2.20}$$

Similarly, we obtain the following local version:

Lemma 2.19. *Let S be a closed subset of X and U an open neighborhood of S in X. Then, we have the natural $\mathfrak{g}$-invariant bilinear form:*

$$C^\infty(U,\mathcal{V}) \times \mathcal{E}'_S(U,\mathcal{V}^*) \longrightarrow \mathbb{C}, \qquad (f,h) \mapsto \int_U \langle f,h\rangle.$$

Moreover, if $S \subset U$ are both H-invariant subsets in X, then the bilinear form is also H-invariant.

We write $o = eH \in X$ for the origin. By Lemmas 2.15 and 2.19, we have obtained two $(\mathfrak{g},H)$-invariant pairings:

$$\begin{aligned} C^\infty(X,\mathcal{V}) \times \operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee) &\longrightarrow \mathbb{C}, \qquad (f,m) \mapsto \Phi_V(f,m)(e), \\ C^\infty(X,\mathcal{V}) \times \mathcal{E}'_{\{o\}}(X,\mathcal{V}^*) &\longrightarrow \mathbb{C} \qquad (f,h) \mapsto \int_X \langle f,h\rangle. \end{aligned}$$

Let us show that there is a natural $(\mathfrak{g},H)$-isomorphism between $\operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)$ and $\mathcal{E}'_{\{o\}}(X,\mathcal{V}^*)$. In fact, it follows from Lemma 2.15 that there exists an injective $(\mathfrak{g},H)$-homomorphism

$$A : \operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee) \to \mathcal{E}_{\{o\}}(X,\mathcal{V}^*)$$

such that

$$\Phi_V(f,m)(e) = \int_X \langle f, A(m)\rangle \quad \text{for all } m \in \operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee) \text{ and } f \in C^\infty(X,\mathcal{V}).$$

For a homogeneous vector bundle $\mathcal{V} = G \times_H V$ we define a vector-valued Dirac δ-function $\delta \otimes v^\vee \in \mathcal{E}'_{\{o\}}(X,\mathcal{V}^*)$, for $v \in V^\vee$ by

$$\langle f, \delta \otimes v^\vee\rangle := \langle f(e), v^\vee\rangle \quad \text{for } f \in C^\infty(X,\mathcal{V}) \simeq C^\infty(G,V)^H. \tag{2.21}$$

By the definition of Φ_V, we have

$$\Phi_V(f, 1 \otimes v^\vee)(e) = \langle f(e), v^\vee\rangle.$$

Hence $A(1 \otimes v^\vee) = \delta \otimes v$ by (2.21). Since A is a $\mathfrak{g}$-homomorphism, we have shown that

$$A(u \otimes v^\vee) = dL(u)(\delta \otimes v^\vee), \qquad \text{for } u \in U(\mathfrak{g}), v \in V^\vee.$$

Lemma 2.20. *The $(\mathfrak{g},H)$-homomorphism*

$$A : \operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee) \longrightarrow \mathcal{E}'_{\{o\}}(X,\mathcal{V}^*), \qquad u \otimes v^\vee \mapsto dL(u)\,(\delta \otimes v^\vee)\,, \tag{2.22}$$

is bijective.

Proof. By Lemma 2.15 the map (2.22) is injective. Let us show that it is also surjective. By the structural theorem of (scalar-valued) distributions [S66, Chapter III, Théorème XXXVII], distributions supported on the singleton $\{o\}$ are obtained as a finite sum of derivatives of the Dirac's delta function. An analogous statement holds for vector-bundle valued distributions supported on $\{o\}$, as we can see by trivializing the bundle near the origin o. Choose a complementary subspace $\mathfrak{q}(\mathbb{R})$ of $\mathfrak{h}(\mathbb{R}) = \mathrm{Lie}(H)$ in $\mathfrak{g}(\mathbb{R}) = \mathrm{Lie}(G)$. Since $dL(Z)\,(Z \in \mathfrak{q}(\mathbb{R}))$ spans the tangent space $T_o(G/H) \simeq \mathfrak{q}(\mathbb{R})$, any derivative of the vector-valued Dirac's delta function is given as a linear combination of elements of the form $dL(u)(\delta \otimes v^\vee)\,(u \in U(\mathfrak{g}), v^\vee \in V^\vee)$. Thus the map (2.22) is surjective. □

Let $\mathbb{C}_{2\rho}$ denote the one-dimensional representation of H defined by

$$h \mapsto |\det(\mathrm{Ad}_{G/H}(h) : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h})|^{-1}.$$

If H is a parabolic subgroup of G with Langlands decomposition $P = MAN_+$ then the infinitesimal representation of $\mathbb{C}_{2\rho}$ is given by the sum of the roots for $\mathfrak{n}_+ = \mathrm{Lie}(N_+)$. The bundle of densities $\Omega_{G/H}$ is given as a G-equivariant line bundle,

$$\Omega_{G/H} \simeq G \times_H |\det{}^{-1} \mathrm{Ad}_{G/H}| \simeq G \times_H \mathbb{C}_{2\rho}.$$

For an H-module (λ, V), we define a "twist" of the contragredient representation $\lambda^\vee_{2\rho}$ on the dual space $V^\vee$ (or simply denoted by $V^\vee_{2\rho}$) by the formula

$$\lambda^* \equiv \lambda^\vee_{2\rho} := \lambda^\vee \otimes \mathbb{C}_{2\rho} = \lambda^\vee \otimes |\det{}^{-1} \mathrm{Ad}_{G/H}|.$$

Then the dualizing bundle $\mathcal{V}^* = \mathcal{V}^\vee \otimes \Omega_{G/H}$ of the vector bundle $\mathcal{V} = G \times_H V$ is given, as a homogeneous vector bundle, by:

$$\mathcal{V}^* \equiv \mathcal{V}^\vee_{2\rho} \simeq G \times_H V^\vee_{2\rho}. \tag{2.23}$$

Then $\mathcal{D}'(X, \mathcal{V}^*)$ is identified with

$$(\mathcal{D}'(G) \otimes V^\vee_{2\rho})^{\Delta(H)} = \{F \in \mathcal{D}'(G) \otimes V^\vee : \lambda^\vee_{2\rho}(h)F(\cdot h) = F(\cdot) \quad \text{for any } h \in H\}.$$

Now let us consider the setting of Theorem 2.9 where we have a G'-equivariant (but not necessarily injective) morphism from $Y = G'/H'$ to $X = G/H$.

Lemma 2.21. *Suppose that G' is a subgroup of G. Then the multiplication map*

$$m : G \times G' \to G, \quad (g, g') \mapsto (g')^{-1} g,$$

induces the isomorphism:

$$m^* : (\mathcal{D}'(X, \mathcal{V}^*) \otimes W)^{\Delta(H')} \xrightarrow{\sim} \mathcal{D}'(X \times Y, \mathcal{V}^* \boxtimes \mathcal{W})^{\Delta(G')}.$$

Proof. The image of the pull-back $m^* : \mathcal{D}'(G) \to \mathcal{D}'(G \times G')$ is $\mathcal{D}'(G \times G')^{\Delta(G')}$, where G' acts diagonally from the left. Thus, considering the remaining $G \times G'$ action from the right, we take $H \times H'$-invariants with respect to the diagonal action in the $(G \times G' \times H \times H')$-isomorphism:

$$m^* \otimes \mathrm{id} \otimes \mathrm{id} : \mathcal{D}'(G) \otimes V^\vee_{2\rho} \otimes W \xrightarrow{\sim} \mathcal{D}'(G \times G')^{\Delta(G')} \otimes V^\vee_{2\rho} \otimes W,$$

and therefore we get the lemma. □

We recall from Section 2.1 that any continuous linear map $T : C^\infty(X, \mathcal{V}) \to C^\infty(Y, \mathcal{W})$ is given by a unique distribution kernel $K_T \in \mathcal{D}'(X \times Y, \mathcal{V}^* \boxtimes \mathcal{W})$. The following lemma gives a necessary and sufficient condition on the distribution K_T for the linear map T to be a G'-equivariant differential operator.

Lemma 2.22. *There is a natural linear isomorphism:*

$$\mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y) \xrightarrow{\sim} (\mathcal{D}'_{\{o\}}(X, \mathcal{V}^*) \otimes W)^{\Delta(H')}, \quad T \mapsto (m^*)^{-1}(K_T). \tag{2.24}$$

Proof. First, we show that the map (2.24) is well-defined. Suppose $T \in \mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y)$. Since K_T is uniquely determined by T, the operator T is G'-equivariant, *i.e.* $L(g) \circ T \circ L(g^{-1}) = T$ for all $g \in G'$ if and only if $K_T \in \mathcal{D}'(X \times Y, \mathcal{V}^* \boxtimes \mathcal{W})^{\Delta(G')}$. By Lemma 2.3 the distribution kernel K_T is supported on the diagonal set $\Delta(Y) = \{(p(y), y) : y \in Y\} \subset X \times Y$. Via the bijection m^* given in Lemma 2.21 we thus have

$$\mathrm{Supp}((m^*)^{-1} K_T) \subset \{o\}.$$

Hence the map (2.24) is well-defined. The injectivity of (2.24) is clear.

Conversely, take any element $k \in (\mathcal{D}'_{\{o\}}(X, \mathcal{V}^*) \otimes W)^{\Delta(H')}$. We set $K := m^*(k) \in \mathcal{D}'(X \times Y, \mathcal{V}^* \boxtimes \mathcal{W})^{\Delta(G')}$, and define a linear map

$$T : C^\infty(X, \mathcal{V}) \longrightarrow \mathcal{D}'(Y, \mathcal{W}), \quad f \mapsto \int_X f(x) K(x, \cdot).$$

Then T is G'-equivariant because K is $\Delta(G')$-invariant.

Let us show that $Tf \in C^\infty(Y, \mathcal{W})$ for any $f \in C^\infty(X, \mathcal{V})$. To see this, we take neighborhoods U, U' and U'' of $x_o = p(y_o)$ in X, y_o in Y, and e in G', respectively, such that $gU' \subset p^{-1}(U)$ for any $g \in U''$. Since the kernel K is supported on the diagonal set $\Delta(Y)$, $TF|_{U'}$ is locally of the form (2.4) as in the proof of Lemma 2.3.

Since T is G'-equivariant, we have

$$\sum_\alpha h_\alpha(y) \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(gp(y)) = \sum_\alpha h_\alpha(gy) \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(p(y)),$$

for any $y \in U', g \in U''$, and $f \in C^\infty(U) \otimes V$. By taking $f(x) = x^\alpha \otimes v$ ($\alpha \in \mathbb{N}^n$ and $v \in V$) as test functions, there are some $\varphi_{\alpha\beta} \in C^\infty(U'' \times U')$ for $|\beta| < |\alpha|$ such that

$$h_\alpha(gy) = h_\alpha(y) + \sum_{|\beta| < |\alpha|} \varphi_{\alpha\beta}(g, y) h_\beta(y).$$

Therefore we see inductively on $|\alpha|$ that $h_\alpha(y) \in C^\infty(U') \otimes \mathrm{Hom}(V, W)$ for all α because G' acts transitively on Y. Hence $Tf|_{U'} \in C^\infty(U') \otimes W$. Thus we have shown that T maps $C^\infty(X, \mathcal{V})$ into $C^\infty(Y, \mathcal{W})$.

Finally, it follows from Lemma 2.3 that T is a differential operator because $\mathrm{Supp} K \subset \Delta(Y)$. Now we have proved the lemma. $\square$

Proof of Theorem 2.9. Taking the tensor product of each term in (2.22) with the finite-dimensional representation W of H', we get a bijection between the subspaces of $\mathfrak{h}'$-invariants:

$$\operatorname{Hom}_{\mathfrak{h}'}(W^\vee, \operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)) \xrightarrow{\sim} (\mathcal{D}'_{\{o\}}(X, \mathcal{V}_X^*) \otimes W)^{\Delta(H')}.$$

Composing this with the bijection in Lemma 2.22, we obtain a bijection from $\operatorname{Hom}_{\mathfrak{h}'}(W^\vee, \operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee))$ to $\operatorname{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y)$, which is by construction nothing but $D_{X\to Y}$ in Theorem 2.9. □

3. Algebraic Fourier transform for generalized Verma modules

The duality theorem (Theorem 2.9) states that, to obtain a differential symmetry breaking operator $D \in \operatorname{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y)$, it suffices to find $\varphi \in \operatorname{Hom}_{H'}(W^\vee, \operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee))$. In Section 4, we shall present a new method (*F-method*) which characterizes the "algebraic Fourier transform" of φ as a solution to a certain system of partial differential equations.

In this section we introduce and study the "algebraic Fourier transform" of generalized Verma modules. Proposition 3.11 is particularly important to the F-method.

3.1. **Weyl algebra and algebraic Fourier transform.**

Let E be a vector space over $\mathbb{C}$. The Weyl algebra $\mathcal{D}(E)$ is the ring of holomorphic differential operators on E with polynomial coefficients.

Definition 3.1. We define the *algebraic Fourier transform* as an isomorphism of two Weyl algebras on E and its dual space $E^\vee$:

$$\mathcal{D}(E) \to \mathcal{D}(E^\vee), \qquad T \mapsto \widehat{T},$$

induced by

$$\widehat{\frac{\partial}{\partial z_j}} := -\zeta_j, \quad \widehat{z_j} := \frac{\partial}{\partial \zeta_j}, \quad 1 \le j \le n = \dim E. \tag{3.1}$$

where $(z_1, \dots, z_n)$ are coordinates on E and $(\zeta_1, \dots, \zeta_n)$ are the dual coordinates on $E^\vee$.

Example 3.2. *Let $E_z := \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}$ be the Euler operator on E. Then, by the commutation relations*

$$\frac{\partial}{\partial \zeta_i}\zeta_j - \zeta_j \frac{\partial}{\partial \zeta_i} = \delta_{ij}, \tag{3.2}$$

in the Weyl algebra $\mathcal{D}(E^\vee)$, where δ_{ij} is the Kronecker delta. Hence we have $\widehat{E_z} = -E_\zeta - n$.

The isomorphism $T \mapsto \widehat{T}$ in Definition 3.1 does not depend on the choice of coordinates. To see this, we consider the natural action of the general linear group $GL(E)$ on E, which yields automorphisms of the ring $\mathrm{Pol}(E)$ of polynomials of E and the Weyl algebra $\mathcal{D}(E)$. For $A \in GL(E)$, we set

$$\begin{aligned} A_{\#} &: \mathrm{Pol}(E) \longrightarrow \mathrm{Pol}(E), \quad F \mapsto F(A^{-1}\cdot), \\ A_{*} &: \mathcal{D}(E) \longrightarrow \mathcal{D}(E), \quad T \mapsto A_{\#} \circ T \circ A_{\#}^{-1}. \end{aligned}$$

We denote by ${}^tA \in GL(E^\vee)$ the dual map of A. Then we have

Lemma 3.3. *For any $A \in GL(E)$ and $T \in \mathcal{D}(E)$,*

$$\widehat{A_* T} = \left({}^tA^{-1}\right)_* \widehat{T}.$$

The proof is straightforward from the definition (3.1), and we omit it.

Next we consider the group homomorphism $GL(E) \longrightarrow GL(\mathrm{Pol}(E))$, $A \mapsto A_{\#}$. Taking the differential, we get a Lie algebra homomorphism $\mathrm{End}(E) \to \mathcal{D}(E)$. In the coordinates, we write $Z = {}^t(z_1, \cdots, z_n)$ and $\partial_Z = {}^t\left(\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}\right)$. Then this homomorphism amounts to

$$\Psi_E : \mathrm{End}(E) \to \mathcal{D}(E), \quad A \mapsto -{}^tZ\,{}^tA\,\partial_Z \equiv -\sum_{i,j} A_{ij} z_j \frac{\partial}{\partial z_i}. \tag{3.3}$$

Let $\sigma : \mathfrak{g} \to \mathrm{End}(E)$ be a representation of a Lie algebra $\mathfrak{g}$ on E, and $\sigma^\vee : \mathfrak{g} \to \mathrm{End}(E^\vee)$ the contragredient representation. Then the algebraic Fourier transform $T \mapsto \widehat{T}$ relates the two Lie algebra homomorphisms $\Psi_E \circ \sigma : \mathfrak{g} \to \mathcal{D}(E)$ and $\Psi_{E^\vee} \circ \sigma^\vee : \mathfrak{g} \to \mathcal{D}(E^\vee)$ as follows:

Lemma 3.4.

$$\widehat{\Psi_E \circ \sigma} = \Psi_{E^\vee} \circ \sigma^\vee + (\mathrm{Trace} \circ \sigma) \cdot \mathrm{id}_{E^\vee}.$$

Proof. In the coordinates, we write $A := \sigma(Z) \in \mathrm{End}(E) \simeq M(n, \mathbb{C})$ for $Z \in \mathfrak{g}$. Then,

$$\begin{aligned} \widehat{\Psi_E \circ \sigma(Z)} - \Psi_{E^\vee} \circ \sigma^\vee(Z) &= -\widehat{{}^tZ\,{}^tA\partial_Z} - {}^t\zeta A \partial_\zeta \\ &= {}^t\partial_\zeta\,{}^tA\zeta - {}^t\zeta A \partial_\zeta \\ &= (\mathrm{Trace}\, A)\, \mathrm{id}_{E^\vee}, \end{aligned}$$

where the last equality follows from the commutation relations (3.2). □

For actual computations that will be undertaken in a subsequent paper [KP14-2], it is convenient to give another interpretation of the algebraic Fourier transform by using real forms of E.

Definition 3.5. Fix a real form $E(\mathbb{R})$ of the complex vector space E. Let $\mathcal{E}'_{\{0\}}(E(\mathbb{R}))$ be the space of distributions on the vector space $E(\mathbb{R})$ supported at the origin 0. We define a "Fourier transform" $\mathcal{F}_c : \mathcal{E}'_{\{0\}}(E(\mathbb{R})) \to \mathrm{Pol}(E^\vee)$ by the following formula:

$$\mathcal{F}_c f(\zeta) := \langle f(\cdot), e^{\langle \cdot, \zeta\rangle}\rangle = \int_{E(\mathbb{R})} e^{\langle x,\zeta\rangle} f(x) \qquad \text{for } \zeta \in E^\vee. \tag{3.4}$$

We have used the function $e^{\langle x,\zeta\rangle}$ in (3.4) rather than $e^{-\sqrt{-1}\langle x,\zeta\rangle}$ or $e^{-\langle x,\zeta\rangle}$ which are involved in the usual Fourier transform or the Laplace transform, respectively. This convention makes later computations simpler (see Remark 4.2).

Furthermore, with our convention

$$\mathcal{F}_c(f(A\cdot)) = (\mathcal{F}_c f)({}^tA^{-1}\cdot), \tag{3.5}$$

for any $A \in GL_{\mathbb{R}}(E(\mathbb{R}))$.

The Fourier transform $\mathcal{F}_c$ induces an algebra isomorphism

$$\mathcal{F}_c : \mathcal{E}'_{\{0\}}(E(\mathbb{R})) \xrightarrow{\sim} \mathrm{Pol}(E^\vee)$$

between the polynomial algebra $\mathrm{Pol}(E^\vee)$ with unit $\mathbf{1}$, the constant function on $E^\vee$, and the convolution algebra $\mathcal{E}'_{\{0\}}(E(\mathbb{R}))$ with unit δ, the Dirac delta function. We write $\mathcal{F}_c^{-1} : \mathrm{Pol}(E^\vee) \xrightarrow{\sim} \mathcal{E}'_{\{0\}}(E(\mathbb{R}))$ for the inverse "Fourier transform":

$$\mathcal{F}_c^{-1}(\mathbf{1}) = \delta.$$

Remark 3.6. The Weyl algebra $\mathcal{D}(E)$ acts naturally on the space of distributions on $E(\mathbb{R})$, and in particular, on $\mathcal{E}'_{\{0\}}(E(\mathbb{R}))$. The algebraic Fourier transform defined in Definition 3.1 satisfies

$$\widehat{T} = \mathcal{F}_c \circ T \circ \mathcal{F}_c^{-1} \quad \text{for } T \in \mathcal{D}(E), \tag{3.6}$$

and the formula (3.6) characterizes $\widehat{T}$. To see this, we take coordinates $(x_1, \cdots, x_n)$ on $E(\mathbb{R})$, and extend them to the complex coordinates $(z_1, \cdots, z_n)$ on E and the dual ones $(\zeta_1, \cdots, \zeta_n)$ on $E^\vee$. Let $P(\zeta) = \zeta^\alpha \in \mathrm{Pol}(E^\vee)$ and $T = \sum_{\beta,\gamma} a_{\beta,\gamma} z^\beta \frac{\partial^{|\gamma|}}{\partial z^\gamma} \in \mathcal{D}(E)$. Then we have

$$\widehat{T}P = \sum_{\beta,\gamma} (-1)^{|\gamma|} a_{\beta,\gamma} \frac{\partial^{|\beta|}}{\partial z^\beta} \zeta^{\alpha+\gamma},$$

and on the other hand,

$$\begin{aligned} \mathcal{F}_c \circ T \circ \mathcal{F}_c^{-1} P &= (-1)^{|\alpha|} \mathcal{F}_c \circ T(\delta^\alpha(x)) = (-1)^{|\alpha|} \mathcal{F}_c \left(\sum_{\beta,\gamma} a_{\beta,\gamma} x^\beta \delta^{\alpha+\gamma}(x) \right) \\ &= (-1)^{|\alpha|} \sum_{\beta,\gamma} (-1)^{|\alpha|+|\gamma|} a_{\beta,\gamma} \frac{\partial^{|\beta|}}{\partial z^\beta} \zeta^{\alpha+\gamma}. \end{aligned}$$

Hence the identity (3.6) holds on $\mathrm{Pol}(E^\vee)$. Since the Weyl algebra $\mathcal{D}(E^\vee)$ acts faithfully on $\mathrm{Pol}(E^\vee)$, we have shown (3.6). In particular, the composition $\mathcal{F}_c \circ T \circ \mathcal{F}_c^{-1}$ does not depend on the choice of a real form $E(\mathbb{R})$.

3.2. **Holomorphic vector fields associated to the Gelfand–Naimark decomposition.** It is convenient to prepare some notation in the *complex* reductive Lie algebras for later purpose.

Let $\mathfrak{g}$ be a complex reductive Lie algebra, and $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}_+$ a Levi decomposition of a parabolic subalgebra. Let $G_{\mathbb{C}}$ be a connected complex Lie group with Lie algebra $\mathfrak{g}$, and $P_{\mathbb{C}} = L_{\mathbb{C}} \exp \mathfrak{n}_+$ the parabolic subgroup with Lie algebra $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}_+$. According to the Gelfand–Naimark decomposition $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{l} + \mathfrak{n}_+$ of the Lie algebra $\mathfrak{g}$, we have a diffeomorphism

$$\mathfrak{n}_- \times L_{\mathbb{C}} \times \mathfrak{n}_+ \to G_{\mathbb{C}}, \quad (Z, \ell, Y) \mapsto (\exp Z)\ell(\exp Y),$$

into an open dense subset $G_{\mathbb{C}}^{\mathrm{reg}}$ of $G_{\mathbb{C}}$. Let

$$p_\pm : G_{\mathbb{C}}^{\mathrm{reg}} \longrightarrow \mathfrak{n}_\pm, \qquad p_o : G_{\mathbb{C}}^{\mathrm{reg}} \to L_{\mathbb{C}},$$

be the projections characterized by the identity

$$\exp(p_-(g)) p_o(g) \exp(p_+(g)) = g.$$

We set

$$\alpha : \mathfrak{g} \times \mathfrak{n}_- \to \mathfrak{l}, \qquad (Y, Z) \mapsto \left.\frac{d}{dt}\right|_{t=0} p_o\left(e^{tY} e^{Z}\right), \tag{3.7}$$

$$\beta : \mathfrak{g} \times \mathfrak{n}_- \to \mathfrak{n}_-, \qquad (Y, Z) \mapsto \left.\frac{d}{dt}\right|_{t=0} p_-\left(e^{tY} e^{Z}\right). \tag{3.8}$$

$$\gamma : \mathfrak{g} \times \mathfrak{n}_- \to \mathfrak{l} + \mathfrak{n}_+, \qquad (Y, Z) \mapsto \alpha(Y, Z) + \left.\frac{d}{dt}\right|_{t=0} p_+\left(e^{tY} e^{Z}\right). \tag{3.9}$$

We regard $\beta(Y, \cdot)$ as a holomorphic vector field on $\mathfrak{n}_-$ through the following identification.

$$\mathfrak{n}_- \ni Z \mapsto \beta(Y, Z) \in \mathfrak{n}_- \simeq T_Z \mathfrak{n}_-.$$

Example 3.7. $G_{\mathbb{C}} = GL(p + q, \mathbb{C})$, $L_{\mathbb{C}} = GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$, *and* $\mathfrak{n}_- \simeq M(p, q; \mathbb{C})$.

We note that $\mathfrak{n}_-$ *is realized as upper block matrices. Then for* $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbb{C}}$, $Y = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M(p+q;\mathbb{C})$ *and* $Z \in M(p,q;\mathbb{C})$ *we have*

$$\begin{aligned} p_-(g^{-1}) &= bd^{-1}, \\ p_o(g^{-1}) &= (a - bd^{-1}c, d) && \in GL(p,\mathbb{C}) \times GL(q,\mathbb{C}), \\ \alpha(Y,Z) &= (A - ZC, CZ + D) && \in \mathfrak{gl}_p(\mathbb{C}) \oplus \mathfrak{gl}_q(\mathbb{C}), \\ \beta(Y,Z) &= AZ + B - ZCZ - ZD. \end{aligned}$$

Then $\beta(Y,\cdot)$ *is regarded as the following holomorphic vector field on* $\mathfrak{n}_- \simeq M(p,q;\mathbb{C})$ *given by*

$$\begin{aligned} \mathrm{Trace}(\beta(Y,Z)\, {}^t\partial_Z) &= \sum_{a=1}^{p}\sum_{b=1}^{q} \beta(Y,Z)_{ab} \frac{\partial}{\partial z_{ab}} \\ &= \sum_{a=1}^{p}\sum_{b=1}^{q} (\sum_{i=1}^{p} A_{ai} z_{ib} + B_{ab} - \sum_{i=1}^{p}\sum_{j=1}^{q} z_{aj} C_{ji} z_{ib} - \sum_{j=1}^{q} z_{aj} D_{jb}) \frac{\partial}{\partial z_{ab}}. \end{aligned}$$

A reductive Lie algebra $\mathfrak{g}$ is said to be *k-graded* if it admits a direct sum decomposition $\mathfrak{g} = \oplus_{j=-k}^{k} \mathfrak{g}(j)$ such that $[\mathfrak{g}(i), \mathfrak{g}(j)] \subset \mathfrak{g}(i+j)$ for all i, j. Any parabolic subalgebra $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}_+$ of $\mathfrak{g}$ is given by $\mathfrak{l} = \mathfrak{g}(0)$ and $\mathfrak{n}_+ = \oplus_{j>0} \mathfrak{g}(j)$ for some k-gradation of $\mathfrak{g}$. We then have the following estimates of coefficients of holomorphic differential operators $d\pi_\mu(Y)$.

Lemma 3.8. *According to the direct sum decomposition* $\mathfrak{g} = \oplus_{j=-k}^{k} \mathfrak{g}(j)$*, we write* $\gamma(Y,Z) = \sum_{\ell=0}^{k} \gamma_\ell$ *and* $\beta(Y,Z) = \sum_{\ell=-k}^{-1} \beta_\ell$*, where* $\gamma_\ell \in \mathfrak{g}(\ell)$ *for* $0 \le \ell \le k$ *and* $\beta_\ell \in \mathfrak{g}(\ell)$ *for* $-k \le \ell \le -1$*. Then* γ_ℓ *and* β_ℓ *are polynomials in* Z *of degree at most* $k - \ell$*.*

Proof. Since the map $N_{-,\mathbb{C}} \times P_{\mathbb{C}} \xrightarrow{\sim} G_{\mathbb{C}}^{\mathrm{reg}}$ is an analytic diffeomorphism, we have

$$e^{tY} e^{Z} = e^{Z + t\beta(Y,Z) + o(t)} e^{t\alpha(Y,Z) + o(t)} \tag{3.10}$$

for sufficiently small $t \in \mathbb{C}$, where we use the Landau symbol $o(t)$ for a $\mathfrak{g}$-valued function dominated by t when t tends to be zero. Multiplying (3.10) by e^{-Z} from the left, and taking the differential at $t = 0$, we get

$$\mathrm{Ad}(e^{-Z})Y = \gamma(Y,Z) + \frac{e^{\mathrm{ad}(Z)} - 1}{\mathrm{ad}(Z)} \beta(Y,Z),$$

because $\left.\frac{d}{dt}\right|_{t=0} e^{-Z}e^{Z+tW} = \frac{e^{\operatorname{ad}(Z)}-1}{\operatorname{ad}(Z)}W$. We note that $\operatorname{ad}(Z)$ lowers the grading of $\mathfrak{g}$, namely $\operatorname{ad}(Z)\mathfrak{g}(j) \subset \oplus_{i=-k}^{j-1}\mathfrak{g}(i)$ because $Z \in \mathfrak{n}_-$. In particular, we have

$$\sum_{j=0}^{2k} \frac{(-1)^j}{j!}\operatorname{ad}(Z)^j Y = \gamma(Y,Z) + \sum_{i=0}^{k-1} \frac{\operatorname{ad}(Z)^i}{(i+1)!}\beta(Y,Z). \tag{3.11}$$

Let $q_\ell : \mathfrak{g} \to \mathfrak{g}(\ell)$ be the projection according to the direct sum decomposition $\mathfrak{g} = \bigoplus_{j=-k}^{k} \mathfrak{g}(j)$. Suppose $\ell \geq 0$. Applying q_ℓ to (3.11), we have

$$\gamma_\ell = q_\ell\left(\sum_{j=0}^{k-\ell} \frac{(-1)^j \operatorname{ad}(Z)^j}{j!}Y\right).$$

Hence γ_ℓ is a polynomial in Z of degree at most $k-\ell$.

Suppose $\ell < 0$. Applying q_ℓ to (3.11), we get

$$\beta_\ell = q_\ell\left(\sum_{j=0}^{k-\ell} \frac{(-1)^j \operatorname{ad}(Z)^j}{j!}Y\right) - q_\ell\left(\sum_{i=0}^{j-\ell}\sum_{j=\ell}^{-1} \frac{\operatorname{ad}(Z)^i}{(i+1)!}\beta_j\right).$$

By the downward induction on ℓ, we see that β_ℓ is a polynomial in Z of degree at most $k-\ell$ for $-k \leq \ell \leq -1$. □

3.3. **Fourier transform of principal series representations.** Suppose $\mathfrak{g}$ is a complex reductive Lie algebra, $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}_+$ a parabolic subalgebra, and $\lambda : \mathfrak{p} \to \operatorname{End}_{\mathbb{C}}(V)$ a finite-dimensional representation.

We use the letter μ to denote the representation of $\mathfrak{p}$ on the dual space $V^\vee$ given by

$$\mu := \lambda^* \equiv \lambda^\vee \otimes \operatorname{Trace}(\operatorname{ad}(\cdot) : \mathfrak{n}_+ \to \mathfrak{n}_+). \tag{3.12}$$

By applying the (algebraic) Fourier transform of the Weyl algebra, we define a Lie algebra homomorphism

$$\widehat{d\pi_\mu} : \mathfrak{g} \to \mathcal{D}(\mathfrak{n}_+) \otimes \operatorname{End}_{\mathbb{C}}(V^\vee),$$

by using the complex flag variety $G_C/P_{\mathbb{C}}$ in this subsection. In Section 3.4, we relate $\widehat{d\pi_\mu}$ with the "algebraic Fourier transform" of a generalized Verma module

$$F_c : \operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee) \xrightarrow{\sim} \operatorname{Pol}(\mathfrak{n}_+) \otimes V^\vee,$$

which is defined by using a real flag variety G/P, see (3.23).

Let $G_{\mathbb{C}}$ be a connected complex reductive Lie group with complex reductive Lie algebra $\mathfrak{g}$, and $P_{\mathbb{C}} = L_{\mathbb{C}}N_{+,\mathbb{C}}$ be the parabolic subgroup with Lie algebra $\mathfrak{p}$. Let $\Omega_{X_{\mathbb{C}}}$ be the canonical line bundle of the complex generalized flag variety $X_{\mathbb{C}} = G_{\mathbb{C}}/P_{\mathbb{C}}$.

Suppose λ lifts to a holomorphic representation of $P_{\mathbb{C}}$, then so does μ. We form a $G_{\mathbb{C}}$-equivariant holomorphic vector bundle $\mathcal{V}$ and $\mathcal{V}^\vee \otimes \Omega_{X_{\mathbb{C}}}$ over $X_{\mathbb{C}}$ associated to λ and μ, respectively.

We consider the regular representation π_μ of $G_{\mathbb{C}}$ on $C^\infty(G_{\mathbb{C}}/P_{\mathbb{C}}, \mathcal{V}^\vee \otimes \Omega_{X_{\mathbb{C}}})$. The infinitesimal action will be denoted by $d\pi_\mu$, which is defined on $C^\infty(U, \mathcal{V}^\vee \otimes \Omega_{X_{\mathbb{C}}}|_U)$ for any open subset U of $G_{\mathbb{C}}/P_{\mathbb{C}}$. In particular, we take U to be the open Bruhat cell $\mathfrak{n}_- \hookrightarrow G_{\mathbb{C}}/P_{\mathbb{C}}$, $Z \mapsto \exp Z \cdot o$, where $o = eP_{\mathbb{C}} \in G_{\mathbb{C}}/P_{\mathbb{C}}$. By trivializing the holomorphic vector bundle $\mathcal{V}^\vee \otimes \Omega_{X_{\mathbb{C}}} \to G_{\mathbb{C}}/P_{\mathbb{C}}$ on it, we define a function $F \in C^\infty(\mathfrak{n}_-, V^\vee)$ for a section $f \in C^\infty(G_{\mathbb{C}}/P_{\mathbb{C}}, \mathcal{V}^\vee \otimes \Omega_{X_{\mathbb{C}}})$ by

$$F(Z) := f(\exp Z) \qquad \text{for } Z \in \mathfrak{n}_-.$$

Then the action of $\mathfrak{g}$ on $C^\infty(\mathfrak{n}_-, V^\vee)$ given by

$$\begin{aligned}(d\pi_\mu(Y)F)(Z) = &= \left.\frac{d}{dt}\right|_{t=0} f(e^{-tY}e^Z) \\ &= \mu(\gamma(Y,Z))F(Z) - (\beta(Y,\cdot)F)(Z) \quad \text{for } Y \in \mathfrak{g},\end{aligned} \tag{3.13}$$

where by a little abuse of notation μ stands for the infinitesimal action. The right-hand side of (3.13) defines a representation of Lie algebra $\mathfrak{g}$ whenever μ (or λ) is a representation of the Lie algebra $\mathfrak{p}$ without assuming that it lifts to a holomorphic representation of the complex reductive group $P_{\mathbb{C}}$.

It follows from (3.13) and Lemma 3.8 that we obtain a Lie algebra homomorphism

$$d\pi_\mu : \mathfrak{g} \to \mathcal{D}(\mathfrak{n}_-) \otimes \operatorname{End}(V^\vee), \tag{3.14}$$

for any representation λ of the Lie algebra $\mathfrak{p}$. By taking the algebraic Fourier transform on the Weyl algebra $\mathcal{D}(\mathfrak{n}_-)$ (see Definition 3.1), we get another Lie algebra homomorphism:

$$\widehat{d\pi_\mu} : \mathfrak{g} \to \mathcal{D}(\mathfrak{n}_+) \otimes \operatorname{End}(V^\vee). \tag{3.15}$$

We use the same letter π_μ to denote the "action" of $G_{\mathbb{C}}$ on $C^\infty(\mathfrak{n}_-, V^\vee)$ given as

$$(\pi_\mu(g)F)(Z) = \mu(p_o(g^{-1}\exp Z)\exp(p_+(g^{-1}\exp Z)))^{-1}F(p_-(g^{-1}\exp Z)). \tag{3.16}$$

This formula makes sense if F comes from $C^\infty(G_{\mathbb{C}}/P_{\mathbb{C}}, \mathcal{V} \otimes \Omega^\vee_{X_{\mathbb{C}}})$, or if $F \in C^\infty(\mathfrak{n}_-, V^\vee)$ and $g \in G_{\mathbb{C}}$ and $Z \in \mathfrak{n}_-$ satisfy $g^{-1}\exp Z \in G_{\mathbb{C}}^{\mathrm{reg}}$. In particular, if λ is trivial on the nilpotent radical $\mathfrak{n}_+$ for $g = m\exp W$ with $m \in L_{\mathbb{C}}$ and $W \in \mathfrak{n}_-$, and if $\mathfrak{n}_+$ is abelian we have

$$(\pi_\mu(g)F)(Z) = \mu(m)F(\operatorname{Ad}(m)^{-1}Z - W). \tag{3.17}$$

Let us analyze $d\pi_\mu(Y)$ for $Y \in \mathfrak{l} + \mathfrak{n}_+$. We begin with the case $Y \in \mathfrak{l}$. We let the Levi subgroup $L_{\mathbb{C}}$ act on $\operatorname{Pol}(\mathfrak{n}_+)$ by

$$\operatorname{Ad}_\#(l) : f(\cdot) \mapsto f(\operatorname{Ad}(l^{-1})\cdot), \qquad l \in L_{\mathbb{C}}.$$

Since this action is algebraic, the infinitesimal action defines a Lie algebra homomorphism into the Weyl algebra:

$$\mathrm{ad}_{\#} : \mathfrak{l} \to \mathcal{D}(\mathfrak{n}_+), \quad Y \mapsto \mathrm{ad}_{\#}(Y),$$

where $\mathrm{ad}_{\#}(Y)$ is a holomorphic vector field on $\mathfrak{n}_+$ given by $\mathrm{ad}_{\#}(Y)_x := \frac{d}{dt}\big|_{t=0}\mathrm{Ad}(e^{-tY})x \in T_x(\mathfrak{n}_+)$ for $x \in \mathfrak{n}_+$.

Lemma 3.9. *Let λ be a representation of the parabolic Lie algebra $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}_+$, and $\mu \equiv \lambda^*$ be as in* (3.12). *Then the following two representations of $\mathfrak{l}$ on $\mathrm{Pol}(\mathfrak{n}_+)\otimes V^\vee$ are isomorphic:*

$$\widehat{d\pi_\mu}\Big|_{\mathfrak{l}} \simeq \mathrm{ad}_{\#} \otimes \mathrm{id} + \mathrm{id} \otimes (\mu - \mathrm{Trace} \circ \mathrm{ad}\Big|_{\mathfrak{n}_-}) = \mathrm{ad}_{\#}(Y) \otimes \mathrm{id} + \mathrm{id} \otimes (-\lambda). \tag{3.18}$$

In particular, if λ lifts to a holomorphic representation of $P_{\mathbb{C}}$ then the right-hand side is the infinitesimal action of $\mathrm{Ad}_{\#} \otimes \lambda^\vee$ of $L_{\mathbb{C}}$ on $\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$.

Proof. For $Y \in \mathfrak{l}$, $X \in \mathfrak{n}_-$ we have $\gamma(Y,X) = Y$, and the formula (3.13) reduces, in $\mathcal{D}(\mathfrak{n}_-) \otimes \mathrm{End}(V^\vee)$, to

$$d\pi_\mu(Y) = \mathrm{id} \otimes \mu(Y) - \beta(Y, \cdot) \otimes \mathrm{id}.$$

We apply Lemma 3.4 to the case where (σ, E) is the adjoint representation of $\mathfrak{l}$ on $\mathfrak{n}_-$. Since $\beta(Y,\cdot) = -dL(Y)$ for $Y \in \mathfrak{l}$, we have $\Psi_{\mathfrak{n}_-} \circ \mathrm{ad} = -\beta$ on $\mathfrak{l}$, with the notation therein. Moreover, via the identification $\mathfrak{n}_-^\vee \simeq \mathfrak{n}_+$, the map $\Psi_{\mathfrak{n}_-^\vee} \circ \mathrm{ad}^\vee$ amounts to $\Psi_{\mathfrak{n}_+} \circ \mathrm{ad} = \mathrm{ad}_{\#}$. Therefore, we get

$$\begin{aligned} \widehat{d\pi_\mu}(Y) &= \mathrm{id} \otimes \mu(Y) + \Psi_{\mathfrak{n}_-^\vee} \circ \mathrm{ad}^\vee(Y) \otimes \mathrm{id} + \left(\mathrm{Trace} \circ \mathrm{ad}(Y)\Big|_{\mathfrak{n}_-}\right)\mathrm{id} \otimes \mathrm{id} \\ &= \mathrm{id} \otimes \mu(Y) + \mathrm{ad}_{\#}(Y) \otimes \mathrm{id} - \left(\mathrm{Trace} \circ \mathrm{ad}(Y)\Big|_{\mathfrak{n}_+}\right)\mathrm{id} \otimes \mathrm{id}. \end{aligned}$$

Thus, the lemma follows. □

The differential operators $\widehat{d\pi_\mu}(Y)$ with $Y \in \mathfrak{n}_+$ play a central role in the F-method. If the parabolic subalgebra $\mathfrak{p}$ is associated to a k-gradation of $\mathfrak{g}$, then these differential operators are at most of order $2k$ by Lemma 3.8. We describe their structure in the case where $k = 1$, namely $\mathfrak{n}_+$ is abelian.

Proposition 3.10. *Assume that $\mathfrak{n}_+$ is abelian. Let (λ, V) be a representation of $\mathfrak{l}$, extended trivially on $\mathfrak{n}_+$, and $\mu = \lambda^*$ be as in* (3.12). *For every $Y \in \mathfrak{n}_+$ the operator $\widehat{d\pi_\mu}(Y)$ is of the form*

$$\sum a_i^{jk}\zeta^i \frac{\partial^2}{\partial\zeta^j\partial\zeta^k} + \sum b^j \frac{\partial}{\partial\zeta^j}, \tag{3.19}$$

where a_i^{jk} and $b^j \in \mathrm{End}(V^\vee)$ are constants depending on Y.

Proof. Since $\mathfrak{n}_+$ is abelian, we can take a characteristic element H such that

$$\mathrm{Ad}(e^{sH})Y = e^s Y \quad \text{for any } Y \in \mathfrak{n}_+.$$

We set $m := e^{sH}$. Then ${}^t\mathrm{Ad}(m)^{-1} = e^{-s}\mathrm{id}$ on $\mathfrak{n}_- \simeq \mathfrak{n}_+^\vee$.

Taking the algebraic Fourier transform of the formula

$$d\pi_\mu(\mathrm{Ad}(m)Y) = \pi_\mu(m) d\pi_\mu(Y)\pi_\mu(m^{-1}),$$

where $(\pi_\mu(m)F)(Z) = \mu(m)F(\mathrm{Ad}(m^{-1})Z) = \mu(m)F(\mathrm{Ad}(m)_\# F)(Z)$ by (3.17), we get

$$\widehat{d\pi_\mu}(\mathrm{Ad}(m)Y) = \left({}^t\mathrm{Ad}(m)^{-1}\right)_* \widehat{d\pi_\mu}(Y)$$

by Lemma 3.3. Hence

$$e^s \widehat{d\pi_\mu}(Y) = (e^{-s}\mathrm{id})_* \widehat{d\pi_\mu}(Y) \tag{3.20}$$

If we write $\widehat{d\pi_\mu}(Y)$ in the form

$$\sum_{\alpha,\beta\in\mathbb{N}^n} C_{\alpha,\beta}\zeta^\alpha \frac{\partial^{|\beta|}}{\partial\zeta^\beta}$$

then (3.20) implies that $C_{\alpha,\beta} \neq 0$ only when $|\alpha| + |\beta| = -1$ because $(e^{-s}\mathrm{id})_* \frac{\partial}{\partial\zeta_j} = e^{-s}\frac{\partial}{\partial\zeta_j}$ and $(e^{-s}\mathrm{id})_*\zeta_j = e^s\zeta_j$ $(1 \le j \le n)$. As $d\pi_\mu(Y)$ is a vector field there is no term for $|\alpha| > 1$. Hence we get the expression (3.19). □

3.4. **Fourier transform on the real flag varieties.** In this subsection we define "algebraic Fourier transform" of generalized Verma modules, see (3.23):

$$F_c : \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee) \xrightarrow{\sim} \mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee.$$

As we shall prove in Proposition 3.11, the Lie algebra homomorphism $\widehat{d\pi_{\lambda^*}} : U(\mathfrak{g}) \longrightarrow \mathcal{D}(\mathfrak{n}_+) \otimes \mathrm{End}(V^\vee)$ defined in (3.15) in the previous section can be reconstructed from F_c, namely, $\widehat{d\pi_{\lambda^*}}(u)$ $(u \in U(\mathfrak{g}))$ is the operator S that is characterized by

$$SF_c(v) = F_c(u \cdot v) \quad \text{for any } v \in \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee).$$

For later purpose, we work with a real form G of $G_{\mathbb{C}}$. From now on, let G be a real semisimple Lie group, P a parabolic subgroup of G with Levi decomposition $P = LN_+$, and V a finite-dimensional representation of P.

Let LN_- be the opposite parabolic subgroup of $P = LN_+$. We write $\mathfrak{n}_+(\mathbb{R})$ and $\mathfrak{n}_-(\mathbb{R})$ for the Lie algebras of N_+ and N_-, respectively, and set $\mathfrak{n}_+ = \mathfrak{n}_+(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. The open Bruhat cell is given as the image of the following embedding

$$\iota : \mathfrak{n}_-(\mathbb{R}) \hookrightarrow G/P, \quad X \mapsto \exp(X) \cdot o,$$

where $o = eP \in G/P$.

Let $\lambda : P \to GL_{\mathbb{C}}(V)$ be a finite-dimensional representation of P, and $\mathcal{V} = G \times_P V$ the G-equivariant vector bundle over the real flag variety G/P. The pullback of the dualizing bundle $\mathcal{V}^* \equiv \mathcal{V}^\vee_{2\rho} \to G/P$ via ι is trivialized into the direct product bundle $\mathfrak{n}_-(\mathbb{R}) \times V^\vee \to \mathfrak{n}_-(\mathbb{R})$ and thus we have a linear isomorphism:

$$\iota^* : \mathcal{E}'_{\{o\}}(G/P, \mathcal{V}^\vee_{2\rho}) \xrightarrow{\sim} \mathcal{E}'_{\{0\}}(\mathfrak{n}_-(\mathbb{R})) \otimes V^\vee, \tag{3.21}$$

through which we induce the $(\mathfrak{g}, P)$-action on $\mathcal{E}'_{\{0\}}(\mathfrak{n}_-(\mathbb{R})) \otimes V^\vee$ from $\mathcal{E}'_{\{o\}}(G/P, \mathcal{V}^\vee_{2\rho})$.

The Killing form of $\mathfrak{g}$ identifies the dual space $\mathfrak{n}_-(\mathbb{R})^\vee$ with $\mathfrak{n}_+(\mathbb{R})$, and thus the Fourier transform $\mathcal{F}_c$ in (3.4) gives rise to a linear isomorphism:

$$\mathcal{F}_c \otimes \mathrm{id} : \mathcal{E}'_{\{0\}}(\mathfrak{n}_-(\mathbb{R})) \otimes V^\vee \xrightarrow{\sim} \mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee, \tag{3.22}$$

through which we induce the $(\mathfrak{g}, P)$-action further on the right-hand side.

In summary we have the following $(\mathfrak{g}, P)$-isomorphisms:

$$F_c : \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{p}}(V^\vee) \underset{(2.22)}{\xrightarrow{\sim}} \mathcal{E}'_{\{o\}}(G/P, \mathcal{V}^\vee_{2\rho}) \underset{(3.21)}{\xrightarrow{\sim}} \mathcal{E}'_{\{0\}}(\mathfrak{n}_-(\mathbb{R})) \otimes V^\vee \underset{\mathcal{F}_c\otimes\mathrm{id}}{\xrightarrow{\sim}} \mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee. \tag{3.23}$$

We say that F_c is the *algebraic Fourier transform of a generalized Verma module.*

The $(\mathfrak{g}, P)$-module structure of $\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$ is described by the following proposition.

Proposition 3.11. *Let (λ, V) be a finite-dimensional representation of P and define another representation of P on the dual space $V^\vee$ by $\mu := \lambda^* \equiv \lambda^\vee \otimes \mathbb{C}_{2\rho}$. Then,*

1) *The $\mathfrak{g}$-action on $\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$ induced by F_c in (3.23) coincides with the one given by $\widehat{d\pi_\mu}$ in (3.15).*
2) *The L action on $\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$ induced by F_c in (3.23) coincides with the one given by $\mathrm{Ad}_\# \otimes \lambda^\vee$.*

Proof. 1) Let $G_\mathbb{C}$ be a complexification of G and $P_\mathbb{C}$ the connected subgroup of $G_\mathbb{C}$ with Lie algebra $\mathfrak{p} = \mathrm{Lie}(P) \otimes_\mathbb{R} \mathbb{C}$. First we assume that λ extends to a holomorphic representation of $P_\mathbb{C}$. Then the G-equivariant vector bundle $\mathcal{V}^\vee_{2\rho}$ over $X = G/P$ is the restriction of the $G_\mathbb{C}$-equivariant holomorphic vector bundle $\mathcal{V}^\vee \otimes \Omega_{X_\mathbb{C}}$ over $X_\mathbb{C} = G_\mathbb{C}/P_\mathbb{C}$ that was introduced in the previous subsection. Therefore, the action of $Y \in \mathfrak{g}$ on $\mathcal{E}'_{\{0\}}(\mathfrak{n}_-(\mathbb{R}) \otimes V^\vee)$ induced by ι^* in (3.21) is given by the restriction of the holomorphic differential operator $d\pi_\mu(Y)$.

In turn, the action of $Y \in \mathfrak{g}$ on $\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$ induced by the isomorphism (3.22) is given by

$$(\mathcal{F}_c \otimes \mathrm{id}) \circ d\pi_\mu(Y) \circ (\mathcal{F}_c^{-1} \otimes \mathrm{id}),$$

which is equal to $\widehat{d\pi_\mu(Y)}$ by Remark 3.6.

To complete the proof in the general case we denote by $\mathrm{Hom}(P_\mathbb{C}, GL_\mathbb{C}(V))$ the set of holomorphic representations of $P_\mathbb{C}$ on V and by $\mathrm{Hom}(\mathfrak{p}, \mathrm{End}(V))$ the set of Lie algebra representations of $\mathfrak{p}$. Since the former is Zariski dense in the latter, the two $\mathfrak{g}$-actions on $\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$ coincide for all λ because both depend algebraically (actually affinely) on $\lambda \in \mathrm{Hom}(\mathfrak{p}, \mathrm{End}(V))$.

2) This statement is the analogue of Lemma 3.9 for the Lie group L. Indeed, since the group L normalizes $\mathfrak{n}_-(\mathbb{R})$ and fixes the origin 0, the isomorphism ι^* in (3.21) respects the L-action when L acts diagonally on $\mathcal{E}'_{\{0\}}(\mathfrak{n}_-(\mathbb{R})) \otimes V^\vee$. To conclude the proof we use (3.5). □

The map F_c does not depend on the choice of a real form G of $G_\mathbb{C}$ that appears in the two middle terms of (3.23). Moreover, the isomorphism $F_c : \mathrm{ind}_\mathfrak{p}^\mathfrak{g}(V^\vee) \xrightarrow{\sim} \mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$ depends only on the infinitesimal action of P on V. In fact, the following corollary follows immediately from the statement 1) of Proposition 3.11.

Corollary 3.12. *The algebraic Fourier transform of generalized Verma modules (see* (3.23)*)*

$$F_c : \mathrm{ind}_\mathfrak{p}^\mathfrak{g}(V^\vee) \xrightarrow{\sim} \mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$$

is given by

$$(u \otimes v^\vee) \mapsto \widehat{d\pi_{\lambda^*}(u)}(1 \otimes v^\vee), \quad u \in U(\mathfrak{g}),\ v^\vee \in V^\vee.$$

4. F-method

In Section 2 we have established a one-to-one correspondence between differential symmetry breaking operators for vector bundles and certain Lie algebra homomorphisms (Theorem 2.9). Using this framework our aim is to find explicit formulæ for such operators, in particular, when such operators are a priori known to be unique up to scalar. For this purpose we propose a new method, which we call the *F-method*. Its theoretical foundation is summarized in Theorem 4.1. This method becomes particularly simple when $\mathfrak{h}$ is a parabolic subalgebra with abelian nilradical. In this case we develop the F-method in more details, and give its *recipe* in Section 4.4. Some useful lemmas for actual computations for vector-valued differential operators are collected in Section 4.5.

4.1. Construction of equivariant differential operators by algebraic Fourier transform.

Let E be a finite-dimensional vector spaces over $\mathbb{C}$ and $E^\vee$ its dual space. Let $\mathrm{Diff}^{\mathrm{const}}(E)$ denote the ring of holomorphic differential operators on E with constant coefficients. We define the symbol map

$$\mathrm{Symb} : \mathrm{Diff}^{\mathrm{const}}(E) \xrightarrow{\sim} \mathrm{Pol}(E^\vee), \quad D_z \mapsto Q(\zeta)$$

by the following characterization

$$D_z e^{\langle z, \zeta \rangle} = Q(\zeta) e^{\langle z, \zeta \rangle}.$$

Then Symb is an algebra isomorphism. The differential operator on E with symbol $Q(\zeta)$ will be denoted by ∂Q_z.

By the definition of the algebraic Fourier transform (Definition 3.1) one has

$$\widehat{\partial P_z} = (-1)^\ell P(\zeta), \qquad \widehat{Q(z)} = \partial Q_\zeta \tag{4.1}$$

for any homogeneous polynomial P on $E^\vee$ of degree ℓ and any polynomial Q on E seen as a multiplication operator.

We recall from Corollary 3.12 that $\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$ is a $(\mathfrak{g}, P)$-module if V is a P-module. Note that the action of $\exp(\mathfrak{n}_+)$ $(\subset P)$ on $\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$ is not geometric, namely, it is not given by the pull-back of polynomials via the action on the base space $\mathfrak{n}_+$.

The key tool for the F-method that we explain in Section 4.4 is the following assertion. We note that the two approaches (the canonical invariant pairing (2.20)) and the algebraic Fourier transform (3.23)) give rise to the same differential operators, provided that $\mathfrak{n}_+$ is abelian:

Theorem 4.1. *Suppose that $\mathfrak{p}$ is a parabolic subalgebra $\mathfrak{g}$ and that $P = L\exp(\mathfrak{n}_+)$ is its Levi decomposition. Let P' be a closed subgroup of P such that P' has a decomposition $P' = L'\exp(\mathfrak{n}'_+)$ with $L' \subset L$ and $\mathfrak{n}'_+ \subset \mathfrak{n}_+$. Let G' be an arbitrary subgroup of G containing P'. For a representation (λ, V) of P and a representation (ν, W) of P', we form a G-equivariant vector bundle $\mathcal{V} = G \times_P V$ over $X = G/P$ and a G'-equivariant vector bundle $\mathcal{W} = G' \times_{P'} W$ over $Y = G'/P'$, respectively. Let $\mu := \lambda^*$ be as in* (3.12).

(1) *There is a natural isomorphism*

$$\begin{aligned} \mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y) &\simeq (\mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}_{\mathbb{C}}(V, W))^{L', \widehat{d\pi_\mu}(\mathfrak{n}'_+)} \\ &\simeq (\mathrm{Hom}_{L'}(V \otimes \mathrm{Pol}(\mathfrak{n}_+), W))^{\widehat{d\pi_\mu}(\mathfrak{n}'_+)}. \end{aligned} \tag{4.2}$$

Here the right-hand side of (4.2) *consists of $\psi \in \mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}_{\mathbb{C}}(V, W)$ satisfying*

$$\nu(\ell) \circ \mathrm{Ad}_{\#}(\ell)\psi \circ \lambda(\ell^{-1}) = \psi \qquad \text{for all } \ell \in L', \tag{4.3}$$

$$(\widehat{d\pi_\mu}(C) \otimes \mathrm{id}_W + \mathrm{id} \otimes \nu(C))\psi = 0 \qquad \text{for all } C \in \mathfrak{n}'_+. \tag{4.4}$$

(2) *Assume that the nilradical $\mathfrak{n}_+$ is abelian. Then the following diagram commutes:*

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathbb{C}}(W^\vee, \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) & \xrightarrow[\sim]{F_c \otimes \mathrm{id}} & \mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}_{\mathbb{C}}(V, W) & \xleftarrow[\sim]{\mathrm{Symb} \otimes \mathrm{id}} & \mathrm{Diff}^{\mathrm{const}}(\mathfrak{n}_-) \otimes \mathrm{Hom}_{\mathbb{C}}(V, W) \\ \cup & & \circlearrowleft & & \cup \\ \mathrm{Hom}_{P'}(W^\vee, \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) & & \xrightarrow[D_{X \to Y}]{\sim} & & \mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y). \end{array}$$

Remark 4.2. The convention on the Fourier transform $\mathcal{F}_c$ in Definition 3.5 makes the diagram in Theorem 4.1 commutative without additional powers of $\sqrt{-1}$.

Theorem 4.1 may be regarded as a construction of symmetry breaking operators by using the Fourier transform of generalized Verma modules.

Corollary 4.3. *Assume that* $\mathfrak{n}_+$ *is abelian and that* $P' = L' \exp(\mathfrak{n}'_+)$ *with* $L' \subset L$ *and* $\mathfrak{n}'_+ \subset \mathfrak{n}_+$. *Then the following diagram of three isomorphisms commutes.*

$$\begin{array}{ccc} & \operatorname{Hom}_{L'}(V \otimes \operatorname{Pol}(\mathfrak{n}_+), W)^{\widehat{d\pi_\mu}(\mathfrak{n}'_+)} & \\ {}^{F_c \otimes \mathrm{id}} \nearrow & & \nwarrow {}^{\mathrm{Symb} \otimes \mathrm{id}} \\ \operatorname{Hom}_{P'}(W^\vee, \operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) & \xrightarrow[D_{X \to Y}]{\sim} & \operatorname{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y) \end{array}$$

In the above corollary, $\operatorname{Hom}_{L'}(V \otimes \operatorname{Pol}(\mathfrak{n}_+), W)^{\widehat{d\pi_\mu}(\mathfrak{n}'_+)}$ consists of L'-equivariant, $\operatorname{Hom}_{\mathbb{C}}(V, W)$-valued polynomial solutions ψ on $\mathfrak{n}_+$ to a system of partial differential equations of second order, see Sections 3.3 and 4.4. Corollary 4.3 implies that, once we find such a polynomial solution ψ, we obtain a P'-submodule $W^\vee$ in $\operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$ (sometimes referred to as *singular vectors*) by $(F_c \otimes \mathrm{id})^{-1}(\psi)$, and a differential symmetry breaking operator by $(\mathrm{Symb} \otimes \mathrm{id})^{-1}(\psi)$.

We first give proofs for the first statement of Theorem 4.1 here. The proof of the second statement is postponed until the next subsection.

Proof of Theorem 4.1 (1). Combining the duality theorem (Theorem 2.9) with the algebraic Fourier transform (Corollary 3.12) we have an isomorphism

$$\operatorname{Hom}_{P'}(W^\vee, \operatorname{Pol}(\mathfrak{n}_+) \otimes V^\vee) \xrightarrow{\sim} \operatorname{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y)$$

where the P'-action on $\operatorname{Pol}(\mathfrak{n}_+) \otimes V^\vee$ is defined via the algebraic Fourier transform F_c, namely, the left-hand side consists of $\psi \in \operatorname{Hom}_{\mathbb{C}}(W^\vee, \operatorname{Pol}(\mathfrak{n}_+) \otimes V^\vee) \simeq \operatorname{Pol}(\mathfrak{n}_+) \otimes \operatorname{Hom}_{\mathbb{C}}(V, W)$ satisfying

$$\left(\widehat{d\pi_\mu}(C) \otimes \mathrm{id}_W + \mathrm{id} \otimes \nu(C)\right) \psi = 0 \quad \text{for all } C \in \mathfrak{l}' + \mathfrak{n}_+,$$

provided L' is connected. Owing to Lemma 3.9, the condition for $C \in \mathfrak{l}'$ is equivalent to that $\psi \in (\operatorname{Pol}(\mathfrak{n}_+) \otimes \operatorname{Hom}_{\mathbb{C}}(V, W))^{\mathfrak{l}'}$, where $\mathfrak{l}'$ acts on $\operatorname{Pol}(\mathfrak{n}_+) \otimes \operatorname{Hom}_{\mathbb{C}}(V, W)$ by $\mathrm{ad}_\# \otimes \mathrm{id} + \mathrm{id} \otimes (\lambda^\vee \otimes \mathrm{id} + \mathrm{id} \otimes \nu)$.

In a more general setting where we allow L' to be disconnected, by the same argument as in the proof of Lemma 3.9, we see that the P-action on $\operatorname{Pol}(\mathfrak{n}_+) \otimes V^\vee$ via the algebraic Fourier transform F_c of generalized Verma modules (Corollary 3.12) coincides with the tensor product representation $\mathrm{Ad}_\# \otimes \lambda^\vee$ when restricted to the Levi subgroup L. Thus the isomorphism (4.2) is proved. □

4.2. **Symbol map and reversing signatures.** The purpose of this section is to carefully and clearly set up relations involving various signatures in connection with the algebraic Fourier transform in a coordinates-free fashion.

Denote by $\gamma : S(E) \xrightarrow{\sim} \mathrm{Pol}(E^\vee)$ the canonical isomorphism, and define another algebra isomorphism
$$\gamma_{sgn} : S(E) \xrightarrow{\sim} \mathrm{Pol}(E^\vee),$$
by $\gamma \circ a$, where $a : S(E) \to S(E)$ denotes the automorphism of the symmetric algebra $S(E)$ induced by the linear map $X \mapsto -X$ for $X \in E$.

Now we regard E as an abelian Lie algebra over $\mathbb{C}$, and identify its enveloping algebra $U(E)$ with the symmetric algebra $S(E)$. Then, the right and left-infinitesimal actions induce two isomorphisms:
$$dR : S(E) \xrightarrow{\sim} \mathrm{Diff}^{\mathrm{const}}(E), \qquad dL : S(E) \xrightarrow{\sim} \mathrm{Diff}^{\mathrm{const}}(E).$$
By the definition of the symbol map, we get,
$$\mathrm{Symb} \circ dR = \gamma, \qquad \mathrm{Symb} \circ dL = \gamma_{sgn}.$$
On the other hand, it follows from (4.1) that
$$\widehat{dL(u)} = \gamma(u), \qquad \widehat{dR(u)} = \gamma_{sgn}(u),$$
for every $u \in S(E) \simeq U(E)$, where polynomials are regarded as multiplication operators. Hence we have proved

Lemma 4.4. *Let E be an abelian Lie algebra over $\mathbb{C}$. For any $u \in U(E)$,*
$$\mathrm{Symb} \circ dR(u) = \widehat{dL(u)}, \qquad \mathrm{Symb} \circ dL(u) = \widehat{dR(u)}.$$

4.3. **Proof of Theorem 4.1 (2).** We are ready to complete the proof of Theorem 4.1 (and Corollary 4.3).

Proof. Take an arbitrary $\varphi \in \mathrm{Hom}_{\mathbb{C}}(W^\vee, \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee))$, which may be written as a finite sum
$$\varphi = \sum_j u_j \otimes \psi_j \in U(\mathfrak{n}_-) \otimes \mathrm{Hom}_{\mathbb{C}}(V, W)$$
by the Poincaré–Birkhoff–Witt theorem $U(\mathfrak{g}) \simeq U(\mathfrak{n}_-) \otimes U(\mathfrak{p})$. Then it follows from (2.22) and (3.23) that
$$F_c\varphi = \sum_j \mathcal{F}_c\left(dL(u_j)\delta\right) \otimes \psi_j \in \mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}_{\mathbb{C}}(V, W).$$
Since $\delta = \mathcal{F}_c^{-1}(\mathbf{1})$, we get
$$F_c\varphi = \sum_j \widehat{dL(u_j)} \otimes \psi_j.$$
On the other hand, by the construction (2.18),
$$D_{X \to Y}(\varphi) = \sum_j dR(u_j) \otimes \psi_j.$$
Now we use the assumption that $\mathfrak{n}_+$ or equivalently $\mathfrak{n}_-$ is abelian. Then, in the coordinates $\mathfrak{n}_-(\mathbb{R}) \hookrightarrow G/P$ the operator $dR(u_j)$ for $u_j \in U(\mathfrak{n}_-)$ defines a constant

coefficient differential operator on $\mathfrak{n}_-$. Thus $D_{X\to Y}(\varphi)$ can be regarded as an element of $\mathrm{Diff}^{\mathrm{const}}(\mathfrak{n}_-(\mathbb{R}))\otimes \mathrm{Hom}_{\mathbb{C}}(V,W)$.

Applying the symbol map we have

$$(\mathrm{Symb}\otimes \mathrm{id})\circ D_{X\to Y}(\varphi)=\sum_j \mathrm{Symb}\circ dR(u_j)\otimes\psi_j=\sum_j \widehat{dL(u_j)}\otimes\psi_j,$$

where the last equation follows from Lemma 4.4. Thus we have proved that

$$(F_c\otimes \mathrm{id})\varphi=(\mathrm{Symb}\otimes \mathrm{id})\circ D_{X\to Y}(\varphi),$$

whence the second statement of Theorem 4.1. □

4.4. **Recipe of the F-method for abelian nilradical $\mathfrak{n}_+$.** Our goal is to find an explicit form of a differential symmetry breaking operator from $\mathcal{V}_X$ to $\mathcal{W}_Y$. Equivalently, what we call F-method provides a way to find an explicit element in the space $\mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee),\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee))\simeq \mathrm{Hom}_{\mathfrak{p}'}(W^\vee,\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee))$.

A semisimple element Z in $\mathfrak{g}$ is called *hyperbolic* if all the eigenvalues of $\mathrm{ad}(Z)$ are real. A hyperbolic element Z defines a parabolic subalgebra $\mathfrak{p}(Z)=\mathfrak{l}(Z)+\mathfrak{n}_+(Z)$, where $\mathfrak{l}(Z)$ and $\mathfrak{n}_+(Z)$ are the sum of eigenspaces of $\mathrm{ad}(Z)$ with zero and positive eigenvalues, respectively.

Let $\mathfrak{g}'$ be a reductive subalgebra in $\mathfrak{g}$, in the sense that $\mathfrak{g}'$ itself is reductive and the adjoint representation of $\mathfrak{g}'$ on $\mathfrak{g}$ is completely reducible.

Definition 4.5. A parabolic subalgebra $\mathfrak{p}$ is said to be $\mathfrak{g}'$-*compatible* if there exists a hyperbolic element $Z\in\mathfrak{g}'$ such that $\mathfrak{p}=\mathfrak{p}(Z)$.

If $\mathfrak{p}=\mathfrak{l}+\mathfrak{n}_+$ is $\mathfrak{g}'$-compatible, then $\mathfrak{p}':=\mathfrak{p}\cap\mathfrak{g}'$ becomes a parabolic subalgebra of $\mathfrak{g}'$ with the following Levi decomposition:

$$\mathfrak{p}'=\mathfrak{l}'+\mathfrak{n}'_+:=(\mathfrak{l}\cap\mathfrak{g}')+(\mathfrak{n}_+\cap\mathfrak{g}'),$$

which satisfies the assumptions of Theorem 4.1 2).

In this case the space $\mathrm{Diff}_{G'}(\mathcal{V}_X,\mathcal{W}_Y)$ of differential symmetry breaking operators is always finite-dimensional owing to Corollary 2.10 because:

$$\dim_{\mathbb{C}}\mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee),\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee))<\infty$$

for any finite-dimensional representations V and W of $\mathfrak{p}$ and $\mathfrak{p}'$, respectively [K14, Proposition 2.8].

Our assumption here is that $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}_+$ is a $\mathfrak{g}'$-compatible parabolic subalgebra of $\mathfrak{g}$ with abelian nilradical $\mathfrak{n}_+$. Based on the following diagram (see Corollary 4.3),
(4.5)

$$\begin{array}{ccc} & \left(\mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}_{\mathbb{C}}(V, W)\right)^{L', \widehat{d\pi_\mu}(\mathfrak{n}'_+)} & \\ {}^{F_c \otimes \mathrm{id}} \nearrow & & \nwarrow {}^{\mathrm{Symb} \otimes \mathrm{id}} \\ \mathrm{Hom}_{P'}(W^\vee, \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) & \underset{D_{X \to Y}}{\overset{\sim}{\longrightarrow}} & \mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y) \end{array}$$

we develop a method as follows:

Step 0. Fix a finite-dimensional representation (λ, V) of the parabolic subgroup P. It defines a G-equivariant vector bundle $\mathcal{V}_X = G \times_P V$ over $X = G/P$.

Step 1. Let $\mu := \lambda^\vee \otimes \mathbb{C}_{2\rho}$ and compute (see (3.14) and (3.15)),

$$\begin{aligned} d\pi_\mu : \mathfrak{g} &\to \mathcal{D}(\mathfrak{n}_-) \otimes \mathrm{End}(V^\vee), \\ \widehat{d\pi_\mu} : \mathfrak{g} &\to \mathcal{D}(\mathfrak{n}_+) \otimes \mathrm{End}(V^\vee). \end{aligned}$$

According to (3.13), $\widehat{d\pi_\mu}$ only depends on the infinitesimal representation λ of the parabolic subalgebra $\mathfrak{p}$.

Step 2. Find a finite-dimensional representation (ν, W) of the Lie group P' such that

$$\mathrm{Hom}_{P'}(W^\vee, \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) \neq \{0\}.$$

It defines a G'-equivariant vector bundle $\mathcal{W}_Y = G' \times_{P'} W$ over $Y = G'/P'$ such that $\mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y)$ is non-trivial.

Step 3. Consider $\psi \in \mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}_{\mathbb{C}}(V, W)$ satisfying (4.3) and (4.4). Note that the system of partial differential equations (4.4) is of second order (see Proposition 3.10).

Step 4. Take a slice S for generic $L'_{\mathbb{C}}$-orbits on $\mathfrak{n}_+$. Use invariant theory for (4.3) and consider the system of differential equations on S induced from (4.4). Find polynomials $\psi \in \mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}(V, W)$ satisfying (4.3) and (4.4) by solving those equations on S.

Step 5. Let ψ be a polynomial solution to (4.3) and (4.4) obtained in Step 4. In the diagram (4.5), $(\mathrm{Symb} \otimes \mathrm{id})^{-1}(\psi)$ gives the desired differential symmetry breaking operator in the coordinates $\mathfrak{n}_-$ of X by Theorem 4.1. In the same diagram, $(F_c \otimes \mathrm{id})^{-1}(\psi)$ gives an explicit element in $\mathrm{Hom}_{\mathfrak{p}'}(W^\vee, \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee))$ $(\simeq \mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee), \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)))$, which is sometimes referred to as a *singular vector*.

This method gives all non-trivial differential symmetry breaking operators for given data $(Y \hookrightarrow X, \mathcal{V}_X)$ by providing G'-equivariant vector bundles $\mathcal{W}_Y$ and explicit elements in $\mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y)$. In fact, Step 2 based on Theorem 2.9 gives a necessary and sufficient condition for a P'-module W to ensure that $\mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y)$ is non-zero. Steps 1, 3 and 5 based on Theorem 4.1 show that any differential symmetry breaking operator is of the form $(F_c \otimes \mathrm{id})^{-1}(\psi)$ where ψ is a polynomial solution to (4.3) and (4.4).

Actual applications of the F-method include the following cases:

1. Holomorphic discrete series representations.
2. Principal series representations of real reductive groups (Corollary 2.13).

The latter is related to questions in conformal geometry (more generally parabolic geometry), see [J09, KØSS13]. The former case includes the classical Rankin–Cohen bidifferential operators as a prototype, and it is the main object of the second part of this work [KP14-2]. The connection between these two is discussed in [KKP15].

Here we give some comments on the actual applications of the F-method when X and Y are Hermitian symmetric spaces. In Theorem 5.3 we prove that all continuous symmetry breaking operators in this case are given by holomorphic differential operators that extend to the complex flag varieties, so that the F-method for a parabolic subalgebra with abelian nilradical applies.

Furthermore, if (G, G') is a reductive symmetric pair, we know a priori that $\mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y)$ is one-dimensional for line bundles $\mathcal{V}_X$ with generic parameter [K14, Theorem 2.7]. Thus, it is natural to look for explicit formulæ for such canonical operators. In Step 2 we can use explicit branching laws (see [KP14-2, Fact **??**]) to find all W such that $\mathrm{Hom}_{\mathfrak{p}'}(W^\vee, \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee))$ is non-zero. Conversely, the differential equations in Step 3 are useful in certain cases to get a finer structure of branching laws, *e.g.*, to find the Jordan–Hölder series of the restriction for exceptional parameters λ (see [KØSS13]).

The Rankin–Cohen operators as well as Juhl's conformally covariant differential operators are recovered by the F-method as a special case where generic $L'_{\mathbb{C}}$-orbits on $\mathfrak{n}_+$ are of codimension one. The induced system of (4.4) reduces to ordinary differential equations on the one dimensional complex manifold S. In the second part of this work [KP14-2] we shall treat all the six geometries with a one-dimensional slice S.

4.5. **F-method – supplement for vector valued cases.** In order to deal with the general case where the target $\mathcal{W}_Y$ is no longer a line bundle but a vector bundle, *i.e.*, where W is an arbitrary finite-dimensional irreducible $\mathfrak{l}'$-module, we may find the condition (4.3) somewhat complicated in practice, even though it is a system of differential equations of first order. In this section we give two useful lemmas to simplify Step 3 in the recipe by reducing (4.3) to a simpler algebraic question on polynomial rings, so that we can focus on the crucial part consisting of a system of differential equations of second order (4.4). The results here will be used in [KP14-2, Sections **??** and **??**].

We fix a Borel subalgebra $\mathfrak{b}(\mathfrak{l}')$ of $\mathfrak{l}'$. Let $\chi : \mathfrak{b}(\mathfrak{l}') \to \mathbb{C}$ be a character. For an $\mathfrak{l}'$-module U, we set

$$U_\chi := \{u \in U \, : \, Zu = \chi(Z)u \quad \text{for any } Z \in \mathfrak{b}(\mathfrak{l}')\}.$$

Suppose that W is an irreducible representation of $\mathfrak{l}'$ with lowest weight $-\chi$. Then the contragredient representation $W^\vee$ has a highest weight χ. We fix a non-zero highest weight vector $w^\vee \in (W^\vee)_\chi$. Then the contraction map

$$U \otimes W \to U, \quad \psi \mapsto \langle \psi, w^\vee \rangle,$$

induces a bijection between the following two subspaces:

$$(U \otimes W)^{\mathfrak{l}'} \xrightarrow{\sim} U_\chi, \tag{4.6}$$

if U is completely reducible as an $\mathfrak{l}'$-module. By using the isomorphism (4.6), we reformulate Step 3 of the recipe for the F-method as follows:

Lemma 4.6. *Suppose we are in the setting of Section 4.4. Assume that W is an irreducible representation of the parabolic subalgebra $\mathfrak{p}'$. Let $-\chi$ be the lowest weight of W as an $\mathfrak{l}'$-module. Then we have a natural injective homomorphism*

$$\mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y) \hookrightarrow \left\{ Q \in (\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee)_\chi : \widehat{d\pi_\mu}(C) Q = 0 \quad \text{for all } C \in \mathfrak{n}'_+ \right\},$$

which is bijective if L' is connected.

Proof. Applying (4.6) to the $\mathfrak{l}'$-module $U := \mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}_{\mathbb{C}}(V, W)$, we get an isomorphism:

$$(\mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}(V, W))^{\mathfrak{l}'} \xrightarrow{\sim} (\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee)_\chi . \tag{4.7}$$

Since W is an irreducible $\mathfrak{p}'$-module, the Lie subalgebra $\mathfrak{n}'$ acts trivially on W and $\mathfrak{l}'$ acts irreducibly. In particular, the condition (4.4) amounts to

$$\left(\widehat{d\pi_\mu}(C) \otimes \mathrm{id}_W \right) \psi = 0 \quad \text{for all } C \in \mathfrak{n}'_+.$$

Therefore, the isomorphism (4.7) induces a bijection

$$\begin{aligned} &\left\{ \psi \in (\mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}(V, W))^{\mathfrak{l}'} : \psi \text{ satisfies } (4.4) \right\} \\ \xrightarrow{\sim} \quad &\left\{ Q \in (\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee)_\chi : \widehat{d\pi_\mu}(C) Q = 0 \quad \text{for all } C \in \mathfrak{n}'_+ \right\}. \end{aligned}$$

Now Lemma follows from Theorem 4.1. ∎

Since any non-zero vector in $W^\vee$ is cyclic, the next lemma explains how to recover $D_{X\to Y}(\varphi)$ from Q given in Lemma 4.6.

We assume, for simplicity, that the $\mathfrak{l}$-module (λ, V) lifts to $L_{\mathbb{C}}$, the $\mathfrak{l}'$-module (ν, W) lifts to $L'_{\mathbb{C}}$, and use the same letters to denote their liftings.

Lemma 4.7. *For any* $\varphi \in \mathrm{Hom}_{\mathfrak{p}'}(W^\vee, \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee))$*,* $\ell \in L'_{\mathbb{C}}$ *and* $w^\vee \in W^\vee$*,*

$$\langle D_{X\to Y}(\varphi), \nu^\vee(\ell) w^\vee\rangle = (\mathrm{Ad}(\ell) \otimes \lambda^\vee(\ell)) \langle D_{X\to Y}(\varphi), w^\vee\rangle . \tag{4.8}$$

Proof. We write $\varphi = \sum_j u_j \otimes \psi_j \in U(\mathfrak{n}_-) \otimes \mathrm{Hom}_{\mathbb{C}}(V, W)$. Since φ is $\mathfrak{p}'$-invariant, we have the identity:

$$\sum_j u_j \otimes \psi_j = \sum_j \mathrm{Ad}(\ell) u_j \otimes \nu(\ell) \circ \psi_j \circ \lambda(\ell^{-1}) \quad \text{for } \mathfrak{l} \in L'_{\mathbb{C}}.$$

In turn, we have

$$\begin{aligned} \langle D_{X\to Y}(\varphi), \nu^\vee(\ell) w^\vee\rangle &= \sum_j dR(\mathrm{Ad}(\ell) u_j) \otimes \langle \psi_j, w^\vee\rangle \circ \lambda(\ell^{-1}) \\ &= ((\mathrm{Ad}(\ell) \otimes \lambda^\vee(\ell)) \langle D_{X\to Y}(\varphi), w^\vee\rangle . \end{aligned}$$

Thus, we have proved Lemma. □

We notice that the right-hand side of (4.8) can be computed by using the identity in $\mathrm{Diff}^{\mathrm{const}}(\mathfrak{n}_-) \otimes V^\vee$:

$$\langle D_{X\to Y}(\varphi), w^\vee\rangle = (\mathrm{Symb}^{-1} \otimes \mathrm{id}_{V^\vee})(Q),$$

once we know the polynomial $Q = \langle \psi, w^\vee\rangle$ with $\psi = (F_c \otimes \mathrm{id})(\varphi)$ (see Theorem 4.1). In [KP14-2, Sections **??** and **??**], we find explicit formulæ for vector-bundle valued equivariant differential operators by solving equations for the polynomials Q.

5. Localness and extension theorem for symmetry breaking operators

Let $G \supset G'$ be a pair of real reductive Lie groups. In general, continuous symmetry breaking operators between two principal series representations of G and G' are not always given by differential operators. Actually, generic ones are supposed to be given by integral transforms and their meromorphic continuation, as one can see from a classification result [KS14]. In this section, however, we formulate and prove a quite remarkable phenomenon (*localness theorem*) that any continuous G'-intertwining operator between two representation spaces consisting of holomorphic sections over Hermitian symmetric spaces is given by differential operators, see Theorem 5.3. In particular, the covariant holomorphic differential operators which we shall obtain explicitly in the second part [KP14-2] of this work exhaust all continuous symmetry breaking operators.

5.1. **Formulation of the localness theorem.** Let G be a connected reductive Lie group, θ a Cartan involution, and G/K the associated Riemannian symmetric space. We write $\mathfrak{c}(\mathfrak{k})$ for the center of the complexified Lie algebra $\mathfrak{k} := \mathrm{Lie}(K) \otimes_{\mathbb{R}} \mathbb{C} \equiv \mathfrak{k}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. In order to formulate a localness theorem, we suppose that G/K is a Hermitian symmetric space. This means that there exists a characteristic element $Z \in \mathfrak{c}(\mathfrak{k})$ such that the eigenvalues of $\mathrm{ad}(Z) \in \mathrm{End}(\mathfrak{g})$ is 0 or ± 1 and that we have an eigenspace decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{n}_+ + \mathfrak{n}_-$$

of $\mathrm{ad}(Z)$ with eigenvalues 0, 1, and -1, respectively. We note that $\mathfrak{c}(\mathfrak{k})$ is one-dimensional if G is simple. With the notation of the previous sections, the complex Lie algebra $\mathfrak{k}$ plays the role of the Levi subalgebra $\mathfrak{l}$.

Let $G_{\mathbb{C}}$ be a complex reductive Lie group with Lie algebra $\mathfrak{g}$, and $P_{\mathbb{C}}$ the maximal parabolic subgroup with Lie algebra $\mathfrak{p} := \mathfrak{k} + \mathfrak{n}_+$, with abelian nilradical $\mathfrak{n}_+$. The complex structure of the homogeneous G/K is induced from the open embedding

$$G/K \subset G_{\mathbb{C}}/K_{\mathbb{C}} \exp \mathfrak{n}_+ = G_{\mathbb{C}}/P_{\mathbb{C}}.$$

Let G' be a connected reductive subgroup of G. Without loss of generality we may and do assume that G' is θ-stable. We set $K' := K \cap G'$. Our crucial assumption throughout this section is

$$Z \in \mathfrak{k}'. \tag{5.1}$$

Lemma 5.1. *If* (5.1) *holds, then the parabolic subalgebra* $\mathfrak{p}$ *is* $\mathfrak{g}'$*-compatible (see Definition 4.5), and the homogeneous space* G'/K' *is a Hermitian sub-symmetric space of* G/K *such that the embedding* $G'/K' \hookrightarrow G/K$ *is holomorphic.*

Proof. Let $G'_{\mathbb{C}}$ be the connected complex subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}' := \mathrm{Lie}(G') \otimes_{\mathbb{R}} \mathbb{C}$. Then $\mathfrak{p}' := \mathfrak{k}' + \mathfrak{n}'_+ \equiv (\mathfrak{k} \cap \mathfrak{g}') + (\mathfrak{n}_+ \cap \mathfrak{g}')$ is the sum of the eigenspaces of $\mathrm{ad}(Z)$ in $\mathfrak{g}'$ with 0 and $+1$ eigenvalues, respectively, and therefore is a parabolic subalgebra of $\mathfrak{g}'$. We set $P'_{\mathbb{C}} := P_{\mathbb{C}} \cap G'$. Then, the Riemannian symmetric space G'/K' becomes a Hermitian symmetric space, for which the complex structure is induced from the open embedding in the complex flag variety $Y_{\mathbb{C}} := G'_{\mathbb{C}}/P'_{\mathbb{C}}$:

$$\begin{array}{ccccc} & G'/K' & \hookrightarrow & G/K & \\ & {}_{\text{open}}\cap & & \cap_{\text{open}} & \\ Y_{\mathbb{C}} = & G'_{\mathbb{C}}/P'_{\mathbb{C}} & \hookrightarrow & G_{\mathbb{C}}/P_{\mathbb{C}} & = X_{\mathbb{C}}. \end{array}$$

Since $Y_{\mathbb{C}}$ is a complex submanifold of $X_{\mathbb{C}} = G_{\mathbb{C}}/P_{\mathbb{C}}$, the embedding $G'/K' \hookrightarrow G/K$ is holomorphic. □

Notice that in the setting of Lemma 5.1 the complexified Lie algebra of K' is a Levi subalgebra of the parabolic subalgebra $\mathfrak{p}'$.

Example 5.2. (1) *Let G' be a connected simple Lie group such that the associated Riemannian symmetric space G'/K' is a Hermitian symmetric space. We take a characteristic element $Z' \in \mathfrak{c}(\mathfrak{k}')$. Let $G := G' \times G'$, and we realize G' as the diagonal subgroup $\Delta(G') := \{(g,g) : g \in G'\}$ of G. Then $Z := (Z', Z') \in \mathfrak{c}(\mathfrak{k})$ satisfies* (5.1)*, yielding a holomorphic embedding $\Delta : G'/K' \hookrightarrow G/K = G'/K' \times G'/K'$.*

(2) *Let G be a connected simple Lie group such that the associated symmetric space G/K is a Hermitian symmetric space with Z a characteristic element in $\mathfrak{c}(\mathfrak{k})$. Suppose τ is an automorphism of G of finite order such that $\tau(Z) = Z$. Let G' be the identity component of the subgroup $G^\tau := \{g \in G : \tau(g) = g\}$, and $K' := G' \cap K$. Then the assumption* (5.1) *is satisfied, and G'/K' is a Hermitian sub-symmetric space of G/K. We shall focus on the case where (G, G^τ) is a symmetric pair, namely, τ is of order two in* [KP14-2] *for detailed analysis.*

Consider a finite-dimensional representation of K on a complex vector space V. We extend it to a holomorphic representation of $P_{\mathbb{C}}$ by letting the unipotent subgroup $\exp(\mathfrak{n}_+)$ act trivially, and form a holomorphic vector bundle $\mathcal{V}_{X_{\mathbb{C}}} = G_{\mathbb{C}} \times_{P_{\mathbb{C}}} V$ over $X_{\mathbb{C}} = G_{\mathbb{C}}/P_{\mathbb{C}}$. The restriction to the open set G/K defines a G-equivariant holomorphic vector bundle $\mathcal{V} := G \times_K V$. We then have a natural representation of G on the vector space $\mathcal{O}(G/K, \mathcal{V})$ of global holomorphic sections endowed with the Fréchet topology of uniform convergence on compact sets.

Likewise, given a finite-dimensional representation W of K', we form the G'-equivariant holomorphic vector bundle $\mathcal{W} = G' \times_{K'} W$ and consider the representation of G' on $\mathcal{O}(G'/K', \mathcal{W})$. By definition, it is clear that

$$\mathrm{Diff}^{\mathrm{hol}}_{G'}(\mathcal{V}_X, \mathcal{W}_Y) \subset \mathrm{Hom}_{G'}\left(\mathcal{O}\left(G/K, \mathcal{V}\right), \mathcal{O}(G'/K', \mathcal{W})\right). \tag{5.2}$$

Theorem 5.3 below shows that the two spaces do coincide.

Theorem 5.3. *Let G' be a reductive subgroup of G satisfying* (5.1)*. Let V and W be any finite-dimensional representations of K and K', respectively. Then,*

(1) (localness theorem) *any continuous G'-homomorphism from $\mathcal{O}(G/K, \mathcal{V})$ to $\mathcal{O}(G'/K', \mathcal{W})$ is given by a holomorphic differential operator, in the sense of Definition 2.1, with respect to a holomorphic map between the Hermitian symmetric spaces $G'/K' \hookrightarrow G/K$, that is,*

$$\mathrm{Diff}^{\mathrm{hol}}_{G'}(\mathcal{V}_X, \mathcal{W}_Y) = \mathrm{Hom}_{G'}\left(\mathcal{O}\left(G/K, \mathcal{V}\right), \mathcal{O}(G'/K', \mathcal{W})\right);$$

(2) (extension theorem) *any such a differential operator (or equivalently, any continuous G'-homomorphism) extends to a $G'_{\mathbb{C}}$-equivariant holomorphic differential operator with respect to a holomorphic map between the flag varieties*

$Y_{\mathbb{C}} = G'_{\mathbb{C}}/P'_{\mathbb{C}} \hookrightarrow X_{\mathbb{C}} = G_{\mathbb{C}}/P_{\mathbb{C}}$, namely, the injection

$$\mathrm{Diff}^{\mathrm{hol}}_{G'_{\mathbb{C}}}(\mathcal{V}_{X_{\mathbb{C}}}, \mathcal{W}_{Y_{\mathbb{C}}}) \hookrightarrow \mathrm{Diff}^{\mathrm{hol}}_{G'}(\mathcal{V}_X, \mathcal{W}_Y) \tag{5.3}$$

is bijective.

Remark 5.4. More generally, we may ask whether an analogous statement to Theorem 5.3 (1) holds or not if we replace $\mathcal{O}(G/K, \mathcal{V})$ and $\mathcal{O}(G'/K', \mathcal{W})$ by some other topological vector spaces having the same underlying $(\mathfrak{g}, K)$-module and $(\mathfrak{g}', K')$-module, respectively (*e.g.* the Casselman–Wallach globalization, Hilbert space globalization, etc.). This question was raised by D. Vogan in May 2014. It turns out that this generalization is also true, as we shall show in the proof of Theorem 5.3, that the natural injection

$$\mathrm{Diff}^{\mathrm{hol}}_{G'_{\mathbb{C}}}(\mathcal{V}_{X_{\mathbb{C}}}, \mathcal{W}_{Y_{\mathbb{C}}}) \hookrightarrow \mathrm{Hom}_{(\mathfrak{g}', K')}(\mathcal{O}(G/K, \mathcal{V})_{K\text{-finite}}, \mathcal{O}(G'/K', \mathcal{W})_{K'\text{-finite}}) \tag{5.4}$$

is surjective if the assumption (5.1) is satisfied.

Remark 5.5. An analogous statement for real parabolic subgroups is not true. For instance, for the pair $(G, G') = (O(n+1, 1), O(n, 1))$ there always exists a non-zero continuous G'-equivariant map from the spherical principal series representations $C^\infty(G/P, \mathcal{L}_\lambda)$ of G to the one $C^\infty(G'/P', \mathcal{L}_\nu)$ of G' for any $(\lambda, \nu) \in \mathbb{C}^2$, however, non-zero G'-equivariant *differential* operators exist if and only if $\nu - \lambda \in 2\mathbb{N}$ [KS14].

Remark 5.6. Suppose that V is a generic character of K and (G, G') is a symmetric pair. Then owing to Theorems 2.12 and 5.3 (2), $\mathrm{Diff}^{\mathrm{hol}}_{G'}(\mathcal{V}_{X_{\mathbb{C}}}, \mathcal{W}_{Y_{\mathbb{C}}})$ is at most one-dimensional for any irreducible K'-module W, and [KP14-2, Fact **??**] tells us precisely when it is non-zero.

In [KP14-2] we describe explicit formulæ of such differential operators by using the F-method (Theorem 4.1) for the six complex geometries arising from symmetric pairs of split rank one.

5.2. **Proof of the localness theorem.** Theorem 5.3 is a reflection of the theory of *discretely decomposable restrictions* (see [K94, K98]). The proof is based on a careful analysis of the following three objects:

$$(\mathfrak{g}, K)\text{-modules}, \ (\mathfrak{g}, K')\text{-modules, and } (\mathfrak{g}', K')\text{-modules.}$$

We say that a K'-module Z is *K'-admissible* if the multiplicity

$$[M : F] := \dim \mathrm{Hom}_{K'}(F, M|_{K'})$$

is finite for any $F \in \widehat{K'}$. Then, K'-admissibility is preserved by taking the tensor product with finite-dimensional representations.

We write $\mathcal{O}(G/K, \mathcal{V})_{K\text{-finite}}$ for the space of K-finite vectors of $\mathcal{O}(G/K, \mathcal{V})$, which becomes naturally a $(\mathfrak{g}, K)$-module.

Lemma 5.7. *The* $(\mathfrak{g}, K)$*-module* $\mathcal{O}(G/K, \mathcal{V})_{K\text{-finite}}$ *is* K'*-admissible if* $Z \in \mathfrak{g}'$*.*

Proof. As a K-module, we have the following isomorphism

$$\begin{aligned} \mathcal{O}(G/K, \mathcal{V})_{K-\text{finite}} &\simeq S(\mathfrak{n}_+) \otimes V \\ &\simeq \left(\bigoplus_{a \geq 0} S^a(\mathfrak{n}_+) \right) \otimes V, \end{aligned}$$

where $S^a(\mathfrak{n}_+)$ denotes the space of symmetric tensors of homogeneous degree a. Since $\exp(\mathbb{R}\sqrt{-1}Z)$ acts on $S^a(\mathfrak{n}_+)$ as the scalar $e^{\sqrt{-1}at}$ $(t \in \mathbb{R})$, the whole $S(\mathfrak{n}_+)$ is admissible as a module of the one-dimensional subgroup $\exp(\mathbb{R}\sqrt{-1}Z)$, and so is $\mathcal{O}(G/K, \mathcal{V})_{K-\text{finite}}$. Hence it is also admissible as a K'-module by [K94, Theorem 1.2]. Alternatively, the lemma follows as a special case of the general result [K94, Theorem 2.7] or [K98, II, Theorem 4.1]. □

Given a $(\mathfrak{g}, K')$-module M, we consider the contragredient representation on the dual space $M^\vee := \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$. Collecting K'-finite vectors in $M^\vee$, we get a $(\mathfrak{g}, K')$-module $(M^\vee)_{K'\text{-finite}}$.

Lemma 5.8. *Let* M *be a* K'*-admissible* $(\mathfrak{g}, K')$*-module. Then,*

(1) M *is discretely decomposable as a* $(\mathfrak{g}', K')$*-module.*
(2) *The* $(\mathfrak{g}, K')$*-module* $(M^\vee)_{K'\text{-finite}}$ *is* K'*-admissible and one has the following* K'*-isomorphism*

$$(M^\vee)_{K'\text{-finite}} \simeq \bigoplus_{F \in \widehat{K'}} [M : F]\, F^\vee.$$

For the proof we refer to [K98, Part III, Proposition 1.6].

Lemma 5.9. *Let* M *be a* K'*-admissible* $(\mathfrak{g}, K)$*-module. Then,*

$$(M^\vee)_{K\text{-finite}} = (M^\vee)_{K'\text{-finite}}.$$

Proof. There is an obvious inclusion $(M^\vee)_{K\text{-finite}} \subset (M^\vee)_{K'\text{-finite}}$. We shall prove that the multiplicities in $(M^\vee)_{K\text{-finite}}$ and $(M^\vee)_{K'\text{-finite}}$ are both finite and are the same. Indeed, M being K'-admissible, one has

$$[M : F] = \bigoplus_{E \in \widehat{K}} [M : E][E : F] < \infty.$$

Conversely, $(M^\vee)_{K'\text{-finite}} \simeq \bigoplus_{F \in \widehat{K'}} [M : F]\, F^\vee$ and thus,

$$(M^\vee)_{K\text{-finite}} \simeq \bigoplus_{E \in \widehat{K}} [M : E]\, E^\vee \simeq \bigoplus_{F \in \widehat{K'}} \left(\bigoplus_{E \in \widehat{K}} [M : E][E : F] \right) F^\vee,$$

which concludes the proof. □

The next lemma is known to experts, but for the sake of completeness, we give a proof.

Lemma 5.10. *There is a natural* $(\mathfrak{g}, K)$*-isomorphism:*

$$\left((\mathcal{O}(G/K,\mathcal{V})_{K\text{-finite}})^\vee\right)_{K\text{-finite}} \simeq \operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee).$$

Proof. As in Lemma 2.15, there is a natural non-degenerate $\mathfrak{g}$-invariant bilinear form

$$\mathcal{O}(G/K,\mathcal{V})_{K\text{-finite}} \times \operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee) \to \mathbb{C}.$$

Hence, we have an injective $(\mathfrak{g}, K)$-homomorphism $\operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee) \subset (\mathcal{O}(G/K,\mathcal{V})_{K\text{-finite}})^\vee$. Taking K-finite vectors we get the following commutative diagram of K-modules isomorphisms:

$$\begin{array}{ccc}
\operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee) & \subset & \left((\mathcal{O}(G/K,\mathcal{V})_{K\text{-finite}})^\vee\right)_{K\text{-finite}} \\
\wr\Vert & & \wr\Vert \\
S(\mathfrak{n}_-) \otimes V^\vee & \simeq & ((\operatorname{Pol}(\mathfrak{n}_-) \otimes V)^\vee)_{K\text{-finite}} .
\end{array}$$

Hence the first row is also bijective. ☐

Combining Lemmas 5.7, 5.9 and 5.10 we have shown the following key result:

Proposition 5.11. *There is a natural* $(\mathfrak{g}, K)$*-isomorphism:*

$$\left((\mathcal{O}(G/K,\mathcal{V})_{K\text{-finite}})^\vee\right)_{K'\text{-finite}} \simeq \operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee).$$

Proof of Theorem 5.3. Let $T : \mathcal{O}(G/K,\mathcal{V}) \to \mathcal{O}(G'/K',\mathcal{W})$ be a continuous G'-intertwining operator. It induces a $(\mathfrak{g}', K')$-homomorphism

$$T_K : \mathcal{O}(G/K,\mathcal{V})_{K\text{-finite}} \to \mathcal{O}(G'/K',\mathcal{W})_{K'\text{-finite}}. \tag{5.5}$$

We shall prove that any such $(\mathfrak{g}', K')$-homomorphism T_K comes from a $G'_{\mathbb{C}}$-equivariant differential operator on the flag variety.

To see this, we take the dual map (5.5), and apply Lemma 5.10 and Proposition 5.11. Then there is a $(\mathfrak{g}', K')$-homomorphism $\psi : \operatorname{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee) \to \operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$ such that the following diagram commutes:

$$\begin{array}{ccc}
\left(\mathcal{O}(G'/K',\mathcal{W})^\vee_{K'\text{-finite}}\right)_{K'\text{-finite}} & \xrightarrow{T_K^\vee} & \left(\mathcal{O}(G/K,\mathcal{V})^\vee_{K\text{-finite}}\right)_{K'\text{-finite}} \\
{\scriptstyle \text{Lemma } 5.10}\ \wr\Vert & & \wr\Vert\ {\scriptstyle \text{Proposition } 5.11} \\
\operatorname{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee) & \xrightarrow{\psi} & \operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)
\end{array}$$

The correspondence $T_K \mapsto \psi$ is one-to-one, and thus we have obtained the following natural injective map

$$\mathrm{Hom}_{(\mathfrak{g}',K')}(\mathcal{O}(G/K,\mathcal{V})_{K\text{-finite}}, \mathcal{O}(G'/K',\mathcal{W})_{K'\text{-finite}}) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee), \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)).$$

According to Theorems 2.12 and 4.1 the latter space is isomorphic to $\mathrm{Diff}_{G'}^{\mathrm{hol}}(\mathcal{V}_X, \mathcal{W}_Y)$. This shows that (5.4) is surjective.

Since the injective map (5.4) factors the two injective maps (5.2) and (5.3), both (5.2) and (5.3) are bijective. $\square$

5.3. **Automatic continuity theorem in the unitary case.** Any unitary highest weight module is realized as a subrepresentation of $\mathcal{O}(G/K,\mathcal{V})$ for some G-equivariant holomorphic vector bundle $\mathcal{V}$ over G/K. In this subsection, we prove that any continuous homomorphism between Fréchet modules $\mathcal{O}(G/K,\mathcal{V})$ and $\mathcal{O}(G'/K',\mathcal{W})$ induces a continuous homomorphism between their unitary submodules.

Definition 5.12. For a Fréchet G-module $\mathcal{F}$, we say a G-submodule $\mathcal{H}$ is a *unitary submodule* if $\mathcal{H}$ is a Hilbert space such that the inclusion map $\mathcal{H} \hookrightarrow \mathcal{F}$ is continuous and that G acts unitarily on $\mathcal{H}$.

If V is an irreducible K-module, then there exists at most one non-zero unitary submodule (up to a scaling of the inner product) of $\mathcal{O}(G/K,\mathcal{V})$. We denote by $\mathcal{H}_V^G$ the unitary submodule of $\mathcal{O}(G/K,\mathcal{V})$. The classification of irreducible K-modules V for which $\mathcal{H}_V^G \neq \{0\}$ was accomplished in [EHW83]. We shall prove that any G'-equivariant differential operator in Theorem 5.3 preserves the unitary submodules in the following sense:

Theorem 5.13. *Let G' be a reductive subgroup of G satisfying* (5.1)*. Let V and W be any irreducible finite-dimensional representations of K and K', respectively. Suppose that $T : \mathcal{O}(G/K,\mathcal{V}) \longrightarrow \mathcal{O}(G'/K',\mathcal{W})$ is a G'-equivariant differential operator such that $T\big|_{\mathcal{H}_V^G} \not\equiv 0$. Then $\mathcal{H}_W^{G'} \neq \{0\}$ and T induces a continuous G'-equivariant linear map from the Hilbert space $\mathcal{H}_V^G$ onto the Hilbert space $\mathcal{H}_W^{G'}$.*

Applying Theorems 5.3 and 5.13 to the setting of Example 5.2 (1), we have:

Example 5.14. *Any symmetry breaking operator for the tensor product of two holomorphic discrete series representations is given by a holomorphic differential operator if those representations are realized in the space of holomorphic sections for G-equivariant holomorphic vector bundles over the Hermitian symmetric space G/K. The Rankin–Cohen bidifferential operators are such operators for $G = SL(2,\mathbb{R})$.*

Remark 5.15. As we shall see in the proof, the unitary representation $\mathcal{H}_V^G$ decomposes discretely when restricted to G' if the condition (5.1) is satisfied. The unitary submodule $\mathcal{H}_W^{G'}$ occurs as a discrete summand of the restriction of the unitary representation $\mathcal{H}_V^G$ of G to the subgroup G'.

Let V be an irreducible representation of K as before. Then, there exists a unique K-submodule of $\mathcal{O}(G/K,\mathcal{V})_{K\text{-finite}} \simeq S(\mathfrak{n}_+)\otimes V$ isomorphic to V, namely $S^0(\mathfrak{n}_+)\otimes V \simeq V$.

Lemma 5.16. *Let M be a non-zero $(\mathfrak{g},K)$-submodule of $\mathcal{O}(G/K,\mathcal{V})_{K\text{-finite}}$. Then,*

1) *The module M contains V.*
2) *If M is unitarizable, then its Hilbert completion can be realized in $\mathcal{O}(G/K,\mathcal{V})$ and $M = (\mathcal{H}_V^G)_{K\text{-finite}}$.*

Proof. 1) Since any non-zero quotient of the (generalized) Verma module $\operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$ contains $V^\vee$, the first statement follows from Lemma 5.10. Alternatively, since the infinitesimal action of $\mathfrak{n}_-$ on $\mathcal{O}(G/K,\mathcal{V})_{K\text{-finite}} \simeq \operatorname{Pol}(\mathfrak{n}_-)\otimes V$ is given by directional derivatives, iterated operators of $\mathfrak{n}_-$ yield non-zero elements in V.

2) Denote by $(\pi,\overline{M})$ the unitary representation of G obtained as an (abstract) Hilbert completion of the $(\mathfrak{g},K)$-module M. We regard V as a K-submodule of M, and also of $\overline{M}$. Then the map

$$G\times\overline{M}\times V \longrightarrow \mathbb{C}, \qquad (g,w,v)\mapsto (w,\pi(g)v)_{\overline{M}},$$

induces an injective G-homomorphism $\iota:\overline{M}\longrightarrow\mathcal{O}(G/K,\mathcal{V})$. Since $\mathcal{H}_V^G$ is the unique non-zero unitary submodule, ι is an isomorphism onto $\mathcal{H}_V^G$. □

Proof of Theorem 5.13. By Lemma 5.7, the module $(\mathcal{H}_V^G)_{K\text{-finite}}$ is K'-admissible. Therefore, the unitary representation $\mathcal{H}_V^G$ decomposes into a Hilbert direct sum of irreducible unitary representations $\{U_j\}$ of G':

$$\mathcal{H}_V^G \simeq \sum_j{}^{\oplus} m_j U_j, \tag{5.6}$$

with $m_j<\infty$ for all j ([K94, Theorem 1.1]) and the underlying $(\mathfrak{g},K)$-module $(\mathcal{H}_V^G)_{K\text{-finite}}$ is isomorphic to an algebraic direct sum of irreducible and unitarizable $(\mathfrak{g}',K')$-modules

$$\left(\mathcal{H}_V^G\right)_{K\text{-finite}} = \left(\mathcal{H}_V^G\right)_{K'\text{-finite}} \simeq \bigoplus_j m_j\,(U_j)_{K'\text{-finite}}, \tag{5.7}$$

with the same multiplicities [K98, Part III]. (We remark that an analogous statement fails for the restriction $\pi|_{G'}$ of an irreducible unitary representation π of G if the branching law contains continuous spectrum).

As we saw in the proof of Theorem 5.3, the G'-equivariant differential operators T induces a $(\mathfrak{g}',K')$-homomorphism

$$T_K:\left(\mathcal{H}_V^G\right)_{K\text{-finite}} \longrightarrow \mathcal{O}(G'/K',\mathcal{W})_{K'\text{-finite}}.$$

By (5.7), $M := T_K\left((\mathcal{H}_V^G)_{K\text{-finite}}\right)$ is an algebraic direct sum of some irreducible unitarizable $(\mathfrak{g}', K')$-modules. Since $\mathcal{O}(G'/K', \mathcal{W})_{K'\text{-finite}}$ contains at most one irreducible unitarizable $(\mathfrak{g}', K')$-module, M is irreducible as a $(\mathfrak{g}', K')$-module, and we can realize its Hilbert completion as $\mathcal{H}_W^{G'}$ by Lemma 5.16.

In view of (5.6) and (5.7), there exists a continuous G'-homomorphism between Hilbert spaces:

$$\widetilde{T} : \mathcal{H}_V^G \longrightarrow \mathcal{H}_W^{G'}$$

such that $\widetilde{T}|_{(\mathcal{H}_V^G)_{K\text{-finite}}} = T|_{(\mathcal{H}_V^G)_{K\text{-finite}}}$. Since the inclusion map $\mathcal{H}_V^G \hookrightarrow \mathcal{O}(G/K, \mathcal{V})$ and the differential operator $T : \mathcal{O}(G/K, \mathcal{V}) \longrightarrow \mathcal{O}(G'/K', \mathcal{W})$ are both continuous, we get $\widetilde{T} = T$ on $\mathcal{H}_V^G$. Hence Theorem is proved. ☐

5.4. **Orthogonal projectors.** If V is one-dimensional and (G, G') is a reductive symmetric pair satisfying (5.1), then all the multiplicities m_j in (5.6) are equal to one (see [K08]) and it becomes meaningful to describe the projector from $\mathcal{H}_V^G$ to each G'-irreducible summand. We explain briefly the relationship between the projector for the unitary representation and the symmetry breaking operator.

For this, suppose $T : \mathcal{O}(G/K, \mathcal{V}) \to O(G'/K', \mathcal{W})$ is a G'-equivariant differential operator such that $T|_{\mathcal{H}_V^G} \not\equiv 0$. By Theorem 5.13, T induces a continuous map $T : \mathcal{H}_V^G \to \mathcal{H}_W^{G'}$. Let $T^* : \mathcal{H}_W^{G'} \to \mathcal{H}_V^G$ be its the adjoint operator. Then the composition $T^*T : \mathcal{H}_V^G \to \mathcal{H}_V^G$ is a G'-intertwining operator onto the G'-irreducible summand which is isomorphic to $\mathcal{H}_W^{G'}$. Since T vanishes on the orthogonal complement to $T^*\left(\mathcal{H}_W^{G'}\right)$, it is (up to scaling) the orthogonal projector onto $\mathcal{H}_W^{G'}$.

Explicit description of such differential operators T will be the main concern of the second part [KP14-2] of this work.

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DIFFERENTIAL SYMMETRY BREAKING OPERATORS.
II. RANKIN–COHEN OPERATORS FOR SYMMETRIC PAIRS

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Abstract. Rankin–Cohen brackets are symmetry breaking operators for the tensor product of two holomorphic discrete series representations of $SL(2,\mathbb{R})$. We address a general problem to find explicit formulæ for such intertwining operators in the setting of multiplicity-free branching laws for reductive symmetric pairs.

For this purpose we use a new method (F-method) developed in [KP14-1] and based on the *algebraic Fourier transform for generalized Verma modules*. The method characterizes symmetry breaking operators by means of certain systems of partial differential equations of second order.

We discover explicit formulæ of new differential symmetry breaking operators for all the six different complex geometries arising from semisimple symmetric pairs of split rank one, and reveal an intrinsic reason why the coefficients of orthogonal polynomials appear in these operators (Rankin–Cohen type) in the three geometries and why normal derivatives are symmetry breaking operators in the other three cases. Further, we analyze a new phenomenon that the multiplicities in the branching laws of Verma modules may jump up at singular parameters.

Key words and phrases: *branching laws, Rankin–Cohen brackets, F-method, symmetric pair, invariant theory, Verma modules, Hermitian symmetric spaces, Jacobi polynomial.*

Contents

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1. Introduction

What kind of differential operators do preserve modularity? R. A. Rankin [Ra56] and H. Cohen [C75] introduced a family of differential operators transforming a given pair of modular forms into another modular form of a higher weight. Let f_1 and f_2 be holomorphic modular forms for a given arithmetic subgroup of $SL(2,\mathbb{R})$ of weight k_1 and k_2, respectively. The bidifferential operators, referred to as the *Rankin–Cohen brackets* of degree a and defined by

$$\mathcal{RC}_{k_1,k_2}^{k_3}(f_1,f_2)(z) := \sum_{\ell=0}^{a}(-1)^{\ell}\begin{pmatrix} k_1+a-1 \\ \ell \end{pmatrix}\begin{pmatrix} k_2+a-1 \\ a-\ell \end{pmatrix} f_1^{(a-\ell)}(z) f_2^{(\ell)}(z), \tag{1.1}$$

where $f^{(n)}(z) = \frac{d^n f}{dz^n}(z)$, yield holomorphic modular forms of weight $k_3 = k_1 + k_2 + 2a$ ($a = 0, 1, 2, \cdots$). (In the usual notation, these operators are written as RC_{k_1,k_2}^{a}.)

The Rankin–Cohen bidifferential operators have attracted considerable attention in recent years particularly because of their applications to various areas including

- theory of modular and quasimodular forms (special values of L-functions, the Ramanujan and Chazy differential equations, van der Pol and Niebur equalities) [CL11, MR09, Z94],
- covariant quantization [BTY07, CMZ97, CM04, OS00, DP07, P08, UU96],
- ring structures on representations spaces [DP07, Z94].

Existing methods for the $SL(2,\mathbb{R})$-case. A prototype of the Rankin–Cohen brackets was already found by P. Gordan and S. Guldenfinger [Go1887, Gu1886] in the 19th century by using recursion relations for invariant binary forms and the Cayley processes. For explicit constructions of the equivariant bidifferential operators (1.1), several different methods have been developed:

- Recurrence relations [C75, El06, HT92, P12, Z94].
- Taylor expansions of Jacobi forms [EZ85, IKO12, Ku75].
- Reproducing kernels for Hilbert spaces [PZ04, UU96, Zh10].
- Dual pair correspondence [B06, EI98].

In the first part of our work [KP14-1] we proposed yet another method (*F-method*) to find differential symmetry breaking operators in a more general setting of branching laws for infinite-dimensional representations, based on the algebraic Fourier transform of generalized Verma modules. Even in the $SL(2,\mathbb{R})$-case, the method is original and simple, and yields missing operators for singular parameters (k_1, k_2, k_3), see Corollary 9.3 for the complete classification. Moreover, the F-method leads us to discover new families of covariant differential operators for six different complex geometries beyond the $SL(2,\mathbb{R})$ case (see Table 1.1).

Branching laws for symmetric pairs. By *branching law* we mean the decomposition of an irreducible representation π of a group G when restricted to a given subgroup G'. An important and fruitful source of examples is provided by pairs of groups (G, G') such that G' is the fixed point group of an involutive automorphism τ of G, called *symmetric pairs*.

The decomposition of the tensor product of two representations is a special case of branching laws with respect to symmetric pairs (G, G'). Indeed, if $G = G_1 \times G_1$ and τ is an involutive automorphism of G given by $\tau(x,y) = (y,x)$, then $G' \simeq G_1$ and the restriction of the outer tensor product $\pi_1 \boxtimes \pi_2$ to the subgroup G' is nothing but the tensor product $\pi_1 \otimes \pi_2$ of two representations π_1 and π_2 of G_1. The Littlewood–Richardson rule for finite-dimensional representations is another classical example of branching laws with respect to the symmetric pair $(GL(p+q,\mathbb{C}), GL(p,\mathbb{C}) \times GL(q,\mathbb{C}))$. Our approach relies on recent progress in the theory of branching laws of infinite-dimensional representations for symmetric pairs even beyond completely reducible cases (see Section 9 for instance).

Rankin–Cohen operators as intertwining operators. From the view point of representation theory the Rankin–Cohen operators are intertwiners in the branching law for the tensor product of two holomorphic discrete series representations π_{k_1} and π_{k_2} of $SL(2,\mathbb{R})$. More precisely, the discrete series representation $\pi_{k_1+k_2+2a}$ ($a \in \mathbb{N}$) occurs in the following branching law [Mo80, Re79]:

$$\pi_{k_1} \otimes \pi_{k_2} \simeq {\sum_{a\in\mathbb{N}}}^{\oplus} \pi_{k_1+k_2+2a}, \tag{1.2}$$

and the operator (1.1) gives an explicit intertwining operator from $\pi_{k_1} \otimes \pi_{k_2}$ to the irreducible summand $\pi_{k_1+k_2+2a}$.

In our work [KP14-1] we developed a new method to find explicit intertwining operators for irreducible components of branching laws in a broader setting of symmetric pairs. Such operators are unique up to scalars if the representation π is a highest weight module of scalar type (or equivalently π is realized in the space of holomorphic sections of a homogeneous holomorphic line bundle over a bounded symmetric domain) and (G, G') is any symmetric pair, by the multiplicity-free theorems ([K08, K12]).

The subject of this paper is to study concrete examples where the F-method turns out to be surprisingly efficient.

Let $\mathcal{V}_X \to X$ be a homogeneous vector bundle of a Lie group G and $\mathcal{W}_Y \to Y$ a homogeneous vector bundle of G'. Then we have a natural representation π of G on the space $\Gamma(X, \mathcal{V}_X)$ of sections on X, and similarly that of G' on $\Gamma(Y, \mathcal{W}_Y)$. Assume G' is a subgroup of G. We address the following question:

Question 1. *Find explicit G'-intertwining operators from $\Gamma(X, \mathcal{V}_X)$ to $\Gamma(Y, \mathcal{W}_Y)$.*

To illustrate the nature of such operators we also refer to them as *continuous symmetry breaking operators*. They are said to be *differential symmetry breaking operators* if the operators are differential operators.

The F-method proposed in [KP14-1] provides necessary tools to give an answer to Question 1 for all symmetric pairs (G, G') of split rank one inducing a holomorphic embedding $Y \hookrightarrow X$ (see Table 2.1). We remark that the split rank one condition does not force the rank of G/G' to be equal to one (see Table 1.1 (1), (5) below).

Normal derivatives and Jacobi–type differential operators. In representation theory, taking normal derivatives with respect to an equivariant embedding $Y \hookrightarrow X$ is a standard tool to find abstract branching laws for representations that are realized on X (see [JV79] for instance).

However, we should like to emphasize that the common belief "normal derivatives with respect to $Y \hookrightarrow X$ are intertwining operators in the branching laws" is not true. Actually, it already fails for the tensor product of two holomorphic discrete series of $SL(2, \mathbb{R})$ where the Rankin–Cohen brackets are not normal derivatives with respect to the diagonal embedding $Y \hookrightarrow Y \times Y$ with Y being the Poincaré upper half plane.

We discuss when normal derivatives become intertwiners in the following six complex geometries arising from real symmetric pairs of split rank one:

(1)	$\mathbb{P}^n\mathbb{C} \hookrightarrow \mathbb{P}^n\mathbb{C} \times \mathbb{P}^n\mathbb{C}$	(4)	$\mathrm{Gr}_{p-1}(\mathbb{C}^{p+q}) \hookrightarrow \mathrm{Gr}_p(\mathbb{C}^{p+q})$
(2)	$\mathrm{LGr}(\mathbb{C}^{2n-2}) \times \mathrm{LGr}(\mathbb{C}^2) \hookrightarrow \mathrm{LGr}(\mathbb{C}^{2n})$	(5)	$\mathbb{P}^n\mathbb{C} \hookrightarrow \mathrm{Q}^{2n}\mathbb{C}$
(3)	$\mathrm{Q}^n\mathbb{C} \hookrightarrow \mathrm{Q}^{n+1}\mathbb{C}$	(6)	$\mathrm{IGr}_{n-1}(\mathbb{C}^{2n-2}) \hookrightarrow \mathrm{IGr}_n(\mathbb{C}^{2n})$

Table 1.1. Equivariant embeddings of flag varieties

Here $\mathrm{Gr}_p(\mathbb{C}^n)$ is the Grassmanian of p-planes in $\mathbb{C}^n$, $\mathrm{Q}^m\mathbb{C} := \{z \in \mathbb{P}^{m+1}\mathbb{C} : z_0^2 + \cdots + z_{m+1}^2 = 0\}$ is the complex quadric, and $\mathrm{IGr}_n(\mathbb{C}^{2n}) := \{V \subset \mathbb{C}^{2n} : \dim V = n, Q|_V \equiv 0\}$

is the Grassmanian of isotropic subspaces of $\mathbb{C}^{2n}$ equipped with a non-degenerate quadratic form Q, and $\mathrm{LGr}_n(\mathbb{C}^{2n}) := \{V \subset \mathbb{C}^{2n} : \dim V = n, \omega|_{V\times V} \equiv 0\}$ is the Grassmanian of Lagrangian subspaces of $\mathbb{C}^{2n}$ equipped with a symplectic form ω.

For $Y \hookrightarrow X$ as in Table 1.1 and any equivariant line bundle $\mathcal{L}_\lambda \to X$ with sufficiently positive λ we give a necessary and sufficient condition for normal derivatives to become intertwiners:

Theorem A. (1) *Any continuous G'-homomorphism from $\mathcal{O}(X, \mathcal{L}_\lambda)$ to $\mathcal{O}(Y, \mathcal{W})$ is given by normal derivatives with respect to the equivariant embedding $Y \hookrightarrow X$ if the embedding $Y \hookrightarrow X$ is of type (4), (5) or (6) in Table 1.1.*

(2) *None of normal derivatives of positive order is a G'-homomorphism if the embedding $Y \hookrightarrow X$ is of type (1), (2) and (3) in Table 1.1.*

See Theorem 5.3 for the precise formulation of the first statement. For the three geometries (1), (2), and (3) in Table 1.1, we construct explicitly all the continuous G'-homomorphisms which are actually holomorphic differential operators (differential symmetry breaking operators). For this, let $P_\ell^{\alpha,\beta}(x)$ be the Jacobi polynomial, and $\widetilde{C}_\ell^\alpha(x)$ the normalized Gegenbauer polynomial (see Appendix 11.3). We inflate them into polynomials of two variables by

$$P_\ell^{\alpha,\beta}(x,y) := y^\ell P_\ell^{\alpha,\beta}\left(2\frac{x}{y}+1\right) \quad \text{and} \quad \widetilde{C}_\ell^\alpha(x,y) := x^{\frac{\ell}{2}}\widetilde{C}_\ell^\alpha\left(\frac{y}{\sqrt{x}}\right).$$

In what follows, $\mathcal{L}_\lambda$ stands for a homogeneous holomorphic line bundle, and $\mathcal{W}_\lambda^a$ a homogeneous vector bundle with typical fiber $S^a(\mathbb{C}^m)$ ($m = n$ in (1); $= n-1$ in (2); m=1 in (3)) with parameter λ (see Lemma 5.5 for details). Then we prove:

Theorem B. (1) *For the symmetric pair $(U(n,1)\times U(n,1),\ U(n,1))$ the differential operator*

$$P_a^{\lambda'-1,-\lambda'-\lambda''-2a+1}\left(\sum_{i=1}^n v_i\frac{\partial}{\partial z_i}, \sum_{j=1}^n v_j\frac{\partial}{\partial z_j}\right)$$

is an intertwining operator from $\mathcal{O}(Y, \mathcal{L}_{(\lambda_1',\lambda_2')})\widehat{\otimes}\mathcal{O}(Y, \mathcal{L}_{(\lambda_1'',\lambda_2'')})$ to $\mathcal{O}(Y, \mathcal{W}^a_{(\lambda_1'+\lambda_1'',\lambda_2'+\lambda_2'')})$, for any $\lambda_1', \lambda_1'', \lambda_2', \lambda_2'' \in \mathbb{Z}$, and $a \in \mathbb{N}$. Here we set $\lambda' = \lambda_1' - \lambda_2'$ and $\lambda'' = \lambda_1'' - \lambda_2''$.

(2) *For the symmetric pair $(Sp(n,\mathbb{R}), Sp(n-1,\mathbb{R}) \times Sp(1,\mathbb{R}))$ the differential operator*

$$C_a^{\lambda-1}\left(\sum_{1\le i,j\le n-1} 2v_iv_j\frac{\partial^2}{\partial z_{ij}\partial z_{nn}}, \sum_{1\le j\le n-1} v_j\frac{\partial}{\partial z_{jn}}\right)$$

is an intertwining operator from $\mathcal{O}(X, \mathcal{L}_\lambda)$ to $\mathcal{O}(Y, \mathcal{W}_\lambda^a)$, for any $\lambda \in \mathbb{Z}$, and $a \in \mathbb{N}$.

(3) *For the symmetric pair $(SO(n,2), SO(n-1,2))$ the differential operator*

$$\widetilde{C}_a^{\lambda-\frac{n-1}{2}}\left(-\Delta^z_{\mathbb{C}^{n-1}}, \frac{\partial}{\partial z_n}\right)$$

is an intertwining operator from $\mathcal{O}(X, \mathcal{L}_\lambda)$ *to* $\mathcal{O}(Y, \mathcal{L}_{\lambda+a})$*, for any* $\lambda \in \mathbb{Z}$ *and* $a \in \mathbb{N}$*.*

See Theorems 8.1, 7.1, and 6.3 for the precise statements, respectively. By the localness theorem [KP14-1, Theorem ??], any continuous G'-homomorphisms are differential operators. Then we prove that the above operators exhaust all continuous symmetry breaking operators in (2) and (3), and for generic parameter (λ', λ'') in (1), see (8.7) for the exact condition on the parameter. The first statement of Theorem B corresponds to the decomposition of the tensor product, and gives rise to the classical Rankin–Cohen brackets in the case where $n = 1$. An analogous formula for Theorem B (3) was recently found in a completely different way by A. Juhl [J09] in the setting of conformally equivariant differential operators with respect to the embedding of Riemannian manifolds $S^{n-1} \hookrightarrow S^n$.

The proof of Theorem B is built on the F-method, which establishes in the present setting a bijection between the space

$$\mathrm{Hom}_{G'}(\mathcal{O}(X, \mathcal{L}_\lambda), \mathcal{O}(Y, \mathcal{W}_\lambda^a))$$

of symmetry breaking operators and the space of polynomial solutions to a certain ordinary differential equation, namely

$$\begin{aligned}&\mathrm{Sol}_{\mathrm{Jacobi}}(\lambda' - 1, -\lambda' - \lambda'' - 2a + 1, a) \cap \mathrm{Pol}_a[s]\\&\mathrm{Sol}_{\mathrm{Gegen}}(\lambda - 1, a) \cap \mathrm{Pol}_a[s]_{\mathrm{even}}\\&\mathrm{Sol}_{\mathrm{Gegen}}(\lambda - \frac{n-1}{2}, a) \cap \mathrm{Pol}_a[s]_{\mathrm{even}},\end{aligned}$$

for the geometries (1), (2), and (3) in Table 1.1, respectively. Here $\mathrm{Sol}_{\mathrm{Jacobi}}(\alpha, \beta, \ell) \cap \mathrm{Pol}_a[s]$ and $\mathrm{Sol}_{\mathrm{Gegen}} \cap \mathrm{Pol}_a[s]$ denote the space of polynomial solutions of degree at most a to the Jacobi differential equation (11.4) and to the Gegenbauer differential equation (11.14), respectively. (The subscript "even" stands for a parity condition (6.12).)

Surprisingly, the dimension of the space of symmetry breaking operators for the tensor product case (1) jumps up at some singular parameters. We illustrate this phenomenon by the the following result in the $\mathfrak{sl}_2$-case:

Theorem C (Theorem 9.1)**.** *The following three conditions on the parameters* $(\lambda', \lambda'', \lambda''') \in \mathbb{Z}^3$ *are equivalent:*

(i) $\dim_{\mathbb{C}} \mathrm{Hom}_{SL(2,\mathbb{R})}(\mathcal{O}(\mathcal{L}_{\lambda'}) \widehat{\otimes} \mathcal{O}(\mathcal{L}_{\lambda''}), \mathcal{O}(\mathcal{L}_{\lambda'''})) = 2.$

(ii) $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}}(\mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(-\lambda'''), \mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(-\lambda') \otimes \mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(-\lambda'')) = 2,$ *where* $\mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(-\lambda)$ *is the Verma module* $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{-\lambda}$ *of* $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$*.*

(iii) $\lambda', \lambda'' \le 0, 2 \le \lambda''', \lambda' + \lambda'' \equiv \lambda''' \bmod 2,\ -(\lambda' + \lambda'') \ge \lambda''' - 2 \ge |\lambda' - \lambda''|.$

We also prove that the analytic continuations of the Rankin–Cohen bidifferential operators $\mathcal{RC}_{\lambda', \lambda''}^{\lambda'''}$ vanish exactly at these singular parameters $(\lambda', \lambda'', \lambda''')$ in this case. Moreover, we construct explicitly *three* symmetry breaking operators in this case, and prove that any two of the three are linearly independent. Furthermore we show that each of these three symmetry breaking operators factors into two natural intertwining operators as follows:

$$\begin{array}{ccccc}
 & \left(\frac{\partial}{\partial z_1}\right)^{1-\lambda'}\otimes \mathrm{id} \nearrow & \mathcal{O}(\mathcal{L}_{2-\lambda'})\widehat{\otimes}\mathcal{O}(\mathcal{L}_{\lambda''}) & \searrow \mathcal{RC}^{\lambda'''}_{2-\lambda',\lambda''} & \\
\mathcal{O}(\mathcal{L}_{\lambda'})\widehat{\otimes}\mathcal{O}(\mathcal{L}_{\lambda''}) & \xrightarrow{\mathrm{id}\otimes\left(\frac{\partial}{\partial z_2}\right)^{1-\lambda''}} & \mathcal{O}(\mathcal{L}_{\lambda'})\widehat{\otimes}\mathcal{O}(\mathcal{L}_{2-\lambda''}) & \xrightarrow{\mathcal{RC}^{\lambda'''}_{\lambda',2-\lambda''}} & \mathcal{O}(\mathcal{L}_{\lambda'''}), \\
 & \mathcal{RC}^{2-\lambda'''}_{\lambda',\lambda''} \searrow & \mathcal{O}(\mathcal{L}_{2-\lambda'''}) & \nearrow \left(\frac{d}{dz}\right)^{\lambda'''-1} &
\end{array}$$

whereas the linear relation among the three is explicitly given by using Kummer's connection formula for Gauss hypergeometric functions via the F-method.

In Section 10 we briefly discuss some new applications of the explicit formulæ of differential symmetry breaking operators. Namely, we describe an explicit construction of the discrete spectrum of complementary series representations of $O(n+1,1)$ when restricted to $O(n,1)$ by means of the differential operator given in Theorem B (3).

In Appendix (Section 11) we collect some results on classical ordinary differential equations with focus on singular parameters for which there exist two linearly independent polynomial solutions which correspond, via the F-method, to the failure of multiplicity-one results in the branching laws.

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Notation: $\mathbb{N} = \{0, 1, 2, \cdots\}$, $\mathbb{N}_+ = \{1, 2, \cdots\}$.

2. Geometric setting: Hermitian symmetric spaces

In this section we describe the geometric setting in which Question 1 will be answered.

2.1. **Complex submanifolds in Hermitian symmetric spaces.**

Let G be a connected real reductive Lie group, θ a Cartan involution, and G/K the associated Riemannian symmetric space. We write $\mathfrak{c}(\mathfrak{k})$ for the center of the complexified Lie algebra $\mathfrak{k} := \mathrm{Lie}(K) \otimes_{\mathbb{R}} \mathbb{C} \equiv \mathfrak{k}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. We suppose that G/K is a Hermitian symmetric space. This means that there exists a characteristic element $H_o \in \mathfrak{c}(\mathfrak{k})$ such that we have an eigenspace decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{n}_+ + \mathfrak{n}_-$$

of $\mathrm{ad}(H_o)$ with eigenvalues 0, 1, and -1, respectively. We note that $\mathfrak{c}(\mathfrak{k})$ is one-dimensional if G is simple.

Let $G_{\mathbb{C}}$ be a complex reductive Lie group with Lie algebra $\mathfrak{g}$, and $P_{\mathbb{C}}$ the maximal parabolic subgroup having Lie algebra $\mathfrak{p} := \mathfrak{k} + \mathfrak{n}_+$ with abelian nilradical $\mathfrak{n}_+$. The complex structure of the homogeneous space G/K is induced from the Borel embedding

$$G/K \subset G_{\mathbb{C}}/K_{\mathbb{C}} \exp \mathfrak{n}_+ = G_{\mathbb{C}}/P_{\mathbb{C}}.$$

Let G' be a θ-stable, connected reductive subgroup of G. We set $K' := K \cap G'$ and assume

$$H_o \in \mathfrak{k}'. \tag{2.1}$$

Then the homogeneous space G'/K' carries a G'-invariant complex structure such that the embedding $G'/K' \hookrightarrow G/K$ is holomorphic by the following diagram:

$$\begin{array}{ccccc} Y = & G'/K' & \hookrightarrow & G/K = X \\ & {}_{\text{open}}\cap & & \cap_{\text{open}} \\ & G'_{\mathbb{C}}/P'_{\mathbb{C}} & \hookrightarrow & G_{\mathbb{C}}/P_{\mathbb{C}}, \end{array} \tag{2.2}$$

where $G'_{\mathbb{C}}$ and $P'_{\mathbb{C}} = K'_{\mathbb{C}} \exp \mathfrak{n}'_+$ are the connected complex subgroups of $G_{\mathbb{C}}$ with Lie algebras $\mathfrak{g}' := \mathrm{Lie}(G') \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathfrak{p}' := \mathfrak{k}' + \mathfrak{n}'_+ \equiv (\mathfrak{k} \cap \mathfrak{g}') + (\mathfrak{n}_+ \cap \mathfrak{g}')$, respectively.

Given a finite-dimensional representation of K on a complex vector space V, we extend it to a holomorphic representation of $P_{\mathbb{C}}$ by letting the unipotent subgroup $\exp(\mathfrak{n}_+)$ act trivially, and form a holomorphic vector bundle $\mathcal{V}_{G_{\mathbb{C}}/P_{\mathbb{C}}} = G_{\mathbb{C}} \times_{P_{\mathbb{C}}} V$ over $G_{\mathbb{C}}/P_{\mathbb{C}}$. The restriction to the open set G/K defines a G-equivariant holomorphic vector bundle $\mathcal{V} := G \times_K V$. We then have a natural representation of G on the vector space $\mathcal{O}(G/K, \mathcal{V})$ of global holomorphic sections.

Likewise, given a finite-dimensional representation W of K', we form the G'-equivariant holomorphic vector bundle $\mathcal{W} = G' \times_{K'} W$ and consider the representation of G' on $\mathcal{O}(G'/K', \mathcal{W})$.

Let $V^\vee$ and $W^\vee$ be the contragredient representations of V and W, respectively, and we define $\mathfrak{g}$- and $\mathfrak{g}'$-modules (*generalized Verma modules*) by

$$\begin{aligned} \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee) &:= U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V^\vee, \\ \mathrm{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee) &:= U(\mathfrak{g}') \otimes_{U(\mathfrak{p}')} W^\vee, \end{aligned}$$

where $U(\mathfrak{g})$ and $U(\mathfrak{g}')$ denote the universal enveloping algebras of the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. We endow the spaces $\mathcal{O}(G/K, \mathcal{V})$ and $\mathcal{O}(G'/K', \mathcal{W})$

with the Fréchet topology of uniform convergence on compact sets, and denote by $\mathrm{Hom}_{G'}(\cdot,\cdot)$ the space of continuous symmetry breaking operators (*i.e.* continuous G'-homomorphisms), and by $\mathrm{Diff}^{\mathrm{hol}}_{G'_{\mathbb{C}}}(\mathcal{V}_{G_{\mathbb{C}}/P_{\mathbb{C}}}, \mathcal{W}_{G'_{\mathbb{C}}/P'_{\mathbb{C}}})$ the space of $G'_{\mathbb{C}}$-equivariant holomorphic differential operators with respect to the holomorphic map $G'_{\mathbb{C}}/P'_{\mathbb{C}} \hookrightarrow G_{\mathbb{C}}/P_{\mathbb{C}}$ (see [KP14-1, Definition **??**] for the definition of differential operators between vector bundles with different base spaces). Then the localness theorem [KP14-1, Theorem **??**] and the duality theorem (*op. cit.*, Theorem **??**) assert:

Theorem 2.1. *We have the following natural isomorphisms:*

$$\begin{aligned}\mathrm{Hom}_{G'}(\mathcal{O}(G/K,\mathcal{V}),\mathcal{O}(G'/K',\mathcal{W})) &\simeq \mathrm{Diff}^{\mathrm{hol}}_{G'_{\mathbb{C}}}(\mathcal{V}_{G_{\mathbb{C}}/P_{\mathbb{C}}}, \mathcal{W}_{G'_{\mathbb{C}}/P'_{\mathbb{C}}})\\ &\simeq \mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}^{\mathfrak{g}'}_{\mathfrak{p}'}(W^\vee), \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{p}}(V^\vee)).\end{aligned}$$

2.2. **Semisimple symmetric pairs of holomorphic type and split rank.** Let τ be an involutive automorphism of a semisimple Lie group G. Without loss of generality we may and do assume that τ commutes with the Cartan involution θ of G. We define a θ-stable subgroup by

$$G^\tau := \{g \in G : \tau g = g\}.$$

Then the homogeneous space G/G^τ carries a G-invariant pseudo-Riemannian structure g induced from the Killing form of $\mathfrak{g}(\mathbb{R}) = \mathrm{Lie}(G)$, and becomes an affine symmetric space with respect to the Levi-Civita connection. We use the same letters τ and θ to denote the differentials and also their complex linear extensions. We set $\mathfrak{g}(\mathbb{R})^\tau := \{Y \in \mathfrak{g}(\mathbb{R}) : \tau Y = Y\}$, the Lie algebra of G^τ. The pair $(\mathfrak{g}(\mathbb{R}), \mathfrak{g}(\mathbb{R})^\tau)$ is said to be a semisimple symmetric pair. It is *irreducible* if $\mathfrak{g}(\mathbb{R})$ is simple or is a direct sum of two copies of a simple Lie algebra $\mathfrak{g}'(\mathbb{R})$ with $\mathfrak{g}(\mathbb{R})^\tau \simeq \mathfrak{g}'(\mathbb{R})$. Then any semisimple symmetric pair is isomorphic to a direct sum of irreducible ones.

Definition 2.2. Geometrically, the *split rank* of the semisimple symmetric space G/G^τ is the dimension of a maximal flat, totally geodesic submanifold B in G/G^τ such that the restriction of g to B is positive definite. Algebraically, it is the dimension of a maximal abelian subspace of $\mathfrak{g}(\mathbb{R})^{-\tau,-\theta} := \{Y \in \mathfrak{g}(\mathbb{R}) : \tau Y = \theta Y = -Y\}$. The dimension is independent of the choice of the data, and the geometric and algebraic definitions coincide. We denote it by $\mathrm{rank}_{\mathbb{R}}\, G/G^\tau$.

The automorphism $\tau\theta$ is also an involution because $\tau\theta = \theta\tau$. Since

$$\mathfrak{g}(\mathbb{R})^{\tau\theta,-\theta} := \{Y \in \mathfrak{g}(\mathbb{R}) : \tau\theta Y = Y, \theta Y = -Y\}$$

coincides with $\mathfrak{g}(\mathbb{R})^{-\tau,-\theta}$, we have $\mathrm{rank}_{\mathbb{R}} G/G^\tau = \mathrm{rank}_{\mathbb{R}} G^{\tau\theta}$, the split rank of the reductive Lie group $G^{\tau\theta}$.

Suppose now that G/K is a Hermitian symmetric space with a characteristic element H_o as in Section 2.1.

	$\mathfrak{g}(\mathbb{R})$	$\mathfrak{g}(\mathbb{R})^\tau$	$\mathfrak{g}(\mathbb{R})^{\tau\theta}$
1	$\mathfrak{su}(n,1)\oplus\mathfrak{su}(n,1)$	$\mathfrak{su}(n,1)$	$\mathfrak{su}(n,1)$
2	$\mathfrak{sp}(n+1,\mathbb{R})$	$\mathfrak{sp}(n,\mathbb{R})\oplus\mathfrak{sp}(1,\mathbb{R})$	$\mathfrak{u}(1,n)$
3	$\mathfrak{so}(n,2)$	$\mathfrak{so}(n-1,2)$	$\mathfrak{so}(n-1)\oplus\mathfrak{so}(1,2)$
4	$\mathfrak{su}(p,q)$	$\mathfrak{s}(\mathfrak{u}(1)\oplus\mathfrak{u}(p-1,q))$	$\mathfrak{s}(\mathfrak{u}(1,q)\oplus\mathfrak{u}(p-1))$
5	$\mathfrak{so}(2,2n)$	$\mathfrak{u}(1,n)$	$\mathfrak{u}(1,n)$
6	$\mathfrak{so}^*(2n)$	$\mathfrak{so}(2)\oplus\mathfrak{so}^*(2n-2)$	$\mathfrak{u}(1,n-1)$

TABLE 2.1. Split rank one irreducible symmetric pairs of holomorphic type

Definition 2.3. An irreducible symmetric pair $(\mathfrak{g}(\mathbb{R}),\mathfrak{g}(\mathbb{R})^\tau)$ (or (G,G^τ)) is said to be of *holomorphic type* (with respect to the complex structure on G/K defined by the characteristic element H_o) if $\tau(H_o)=H_o$, namely $H_o\in\mathfrak{k}^\tau$.

If (G,G^τ) is of holomorphic type, then G^τ/K^τ carries a G^τ-invariant complex structure such that the embedding $G^\tau/K^\tau\hookrightarrow G/K$ is holomorphic.

Among irreducible symmetric pairs $(\mathfrak{g}(\mathbb{R}),\mathfrak{g}(\mathbb{R})^\tau)$ of holomorphic type Table 2.1 gives the infinitesimal classification of those of split rank one.

The pairs of flag varieties (see (2.2)) associated with the six pairs (G,G^τ) in Table 2.1 correspond to the six complex parabolic geometries given in Table 1.1.

3. F-method in holomorphic setting

In this section we reformulate the recipe of the F-method ([KP14-1, Section **??**]) in the holomorphic setting, that is, in the setting of Section 2.1 where G'/K' is a complex submanifold of the Hermitian symmetric space G/K.

3.1. **F-method for Hermitian symmetric spaces.** The algebraic Fourier transform on a vector space E is an isomorphism of the Weyl algebras of holomorphic differential operators with polynomial coefficients on a complex vector spaces E and its dual space $E^\vee$:

$$\mathcal{D}(E)\to\mathcal{D}(E^\vee),\qquad T\mapsto\widehat{T}$$

induced by

$$\widehat{\frac{\partial}{\partial z_j}}:=-\zeta_j,\quad \widehat{z_j}:=\frac{\partial}{\partial\zeta_j},\quad 1\le j\le n=\dim E,\tag{3.1}$$

where $(z_1,\ldots,z_n)$ are coordinates on E and $(\zeta_1,\ldots,\zeta_n)$ are the dual coordinates on $E^\vee$.

Let $G_\mathbb{C}$ be a connected complex reductive Lie group with Lie algebra $\mathfrak{g}$ and $P_\mathbb{C}=K_\mathbb{C}N_{+,\mathbb{C}}$ be a parabolic subgroup with Lie algebra $\mathfrak{p}=\mathfrak{k}+\mathfrak{n}_+$. Let λ be a holomorphic representation of $K_{\mathbb{C}}$ on V. We extend it to $P_{\mathbb{C}}$ by letting $N_{+,\mathbb{C}} = \exp(\mathfrak{n}_+)$ act trivially, and form a $G_{\mathbb{C}}$-equivariant holomorphic vector bundle $\mathcal{V} = G_{\mathbb{C}} \times_{P_{\mathbb{C}}} V$ over $G_{\mathbb{C}}/P_{\mathbb{C}}$. Let $\mathbb{C}_{2\rho}$ be the holomorphic character defined by $p \mapsto \det(\mathrm{Ad}(p) : \mathfrak{p} \to \mathfrak{p})$, and define a twist of the contragredient representation $(\lambda^\vee, V^\vee)$ of $P_{\mathbb{C}}$ by $\lambda^* := \lambda^\vee \otimes \mathbb{C}_{2\rho}$. We set $\mathcal{V}^* \equiv \mathcal{V}^\vee_{2\rho} := G_{\mathbb{C}} \times_{P_{\mathbb{C}}} (V^\vee \otimes \mathbb{C}_{2\rho})$, which is isomorphic to the tensor bundle of the dual bundle $\mathcal{V}^\vee$ and the canonical line bundle of $G_{\mathbb{C}}/P_{\mathbb{C}}$. We shall apply the algebraic Fourier transform to the infinitesimal representation $d\pi_{\lambda^*}$ of $\mathfrak{g}$ on $\mathcal{O}(G_{\mathbb{C}}/P_{\mathbb{C}}, \mathcal{V}^*)$ as follows.

We recall that the Gelfand–Naimark decomposition $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{k} + \mathfrak{n}_+$ induces a diffeomorphism

$$\mathfrak{n}_- \times K_{\mathbb{C}} \times \mathfrak{n}_+ \to G_{\mathbb{C}}, \quad (X, \ell, Y) \mapsto (\exp X)\ell(\exp Y),$$

into an open dense subset, denoted by $G_{\mathbb{C}}^{\mathrm{reg}}$, of $G_{\mathbb{C}}$. Let $p_\pm : G_{\mathbb{C}}^{\mathrm{reg}} \to \mathfrak{n}_\pm, p_o : G_{\mathbb{C}}^{\mathrm{reg}} \to K_{\mathbb{C}}$, be the projections characterized by the identity

$$\exp(p_-(g))p_o(g)\exp(p_+(g)) = g.$$

Furthermore, we introduce the following maps:

$$\alpha : \mathfrak{g} \times \mathfrak{n}_- \to \mathfrak{k}, \qquad (Y, Z) \mapsto \left.\frac{d}{dt}\right|_{t=0} p_o\left(e^{tY} e^Z\right), \tag{3.2}$$

$$\beta : \mathfrak{g} \times \mathfrak{n}_- \to \mathfrak{n}_-, \qquad (Y, Z) \mapsto \left.\frac{d}{dt}\right|_{t=0} p_-\left(e^{tY} e^Z\right). \tag{3.3}$$

For $F \in \mathcal{O}(\mathfrak{n}_-, V^\vee) \simeq \mathcal{O}(\mathfrak{n}_-) \otimes V^\vee$, we set $f : G_{\mathbb{C}}^{\mathrm{reg}} \longrightarrow V^\vee$ by $f(\exp Zp) = \lambda^*(p)^{-1} F(Z)$ for $Z \in \mathfrak{n}_-$ and $p \in P_{\mathbb{C}}$. Then the infinitesimal action of $\mathfrak{g}$ on $\mathcal{O}(\mathfrak{n}_-, V^\vee)$ is given by

$$\begin{aligned}(d\pi_{\lambda^*}(Y)F)(Z) &= \left.\frac{d}{dt}\right|_{t=0} f(e^{-tY} e^Z) \\ &= \lambda^*(\alpha(Y, Z))F(Z) - (\beta(Y, \cdot)F)(Z) \quad \text{for } Y \in \mathfrak{g},\end{aligned} \tag{3.4}$$

where we use the same letter λ^* to denote the infinitesimal action of $\mathfrak{p}$ on $V^\vee$. This action yields a Lie algebra homomorphism

$$d\pi_{\lambda^*} : \mathfrak{g} \to \mathcal{D}(\mathfrak{n}_-) \otimes \mathrm{End}(V^\vee). \tag{3.5}$$

In turn, we get another Lie algebra homomorphism by the algebraic Fourier transform on the Weyl algebra $\mathcal{D}(\mathfrak{n}_-)$:

$$\widehat{d\pi_{\lambda^*}} : \mathfrak{g} \to \mathcal{D}(\mathfrak{n}_+) \otimes \mathrm{End}(V^\vee), \tag{3.6}$$

where we identify $\mathfrak{n}_-^\vee$ with $\mathfrak{n}_+$ by a $\mathfrak{g}$-invariant non-degenerate bilinear form on $\mathfrak{g}$ (*e.g.* the Killing form).

Theorem 3.1 (F-method for Hermitian symmetric spaces)**.** *Suppose we are in the setting of Section 2.1.*

(1) *We have the following commutative diagram of three isomorphisms:*

(3.7)

$$\begin{array}{ccc} & \mathrm{Hom}_{K'}(V, \mathrm{Pol}(\mathfrak{n}_+) \otimes W)^{\widehat{d\pi_{\lambda^*}}(\mathfrak{n}'_+)} & \\ {}^{F_c \otimes \mathrm{id}} \nearrow & & \nwarrow {}^{\mathrm{Symb} \otimes \mathrm{id}} \\ \mathrm{Hom}_{\mathfrak{p}'}(W^\vee, \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) & \xrightarrow[D_{X \to Y}]{\sim} & \mathrm{Hom}_{G'}(\mathcal{O}(X, \mathcal{V}), \mathcal{O}(Y, \mathcal{W})). \end{array}$$

(2) *Let $\mathfrak{b}(\mathfrak{k}')$ be a Borel subalgebra of $\mathfrak{k}'$, and assume that W is the irreducible representation of K' with lowest weight $-\chi$. Then we have the following isomorphism:*

$$\mathrm{Hom}_{K'}(V, \mathrm{Pol}(\mathfrak{n}_+) \otimes W)^{\widehat{d\pi_{\lambda^*}}(\mathfrak{n}'_+)} \xrightarrow{\sim} \{P \in \mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee : P \,\text{satisfies}\,(3.8)\,\text{and}\,(3.9)\}$$

$$ZP = \chi(Z)P, \qquad \text{for all } Z \in \mathfrak{b}(\mathfrak{k}'). \tag{3.8}$$

$$\widehat{d\pi_{\lambda^*}}(C)P = 0, \qquad \text{for all } C \in \mathfrak{n}'_+. \tag{3.9}$$

Proof. 1) The first statement follows from Theorem 2.1 and [KP14-1, Corollary **??**].

2) Via the linear isomorphism $\mathrm{Hom}_{\mathbb{C}}(V, \mathrm{Pol}(\mathfrak{n}_+) \otimes W) \simeq \mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}_{\mathbb{C}}(V, W)$, we have an isomorphism

$$\begin{aligned} &\mathrm{Hom}_{K'}(V, \mathrm{Pol}(\mathfrak{n}_+) \otimes W)^{\widehat{d\pi_{\lambda^*}}(\mathfrak{n}'_+)} \\ \simeq\ &\{\psi \in \mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}_{\mathbb{C}}(V, W) : \psi \text{ satisfies } (3.10)\,\text{and } (3.11)\}, \end{aligned}$$

$$\nu(\ell) \circ \mathrm{Ad}_\sharp(\ell)\psi \circ \lambda(\ell^{-1}) = \psi \qquad \text{for all}\, \ell \in K', \tag{3.10}$$

$$(\widehat{d\pi_{\lambda^*}}(C) \otimes \mathrm{id}_W)\psi = 0 \qquad \text{for all}\, C \in \mathfrak{n}'_+, \tag{3.11}$$

where $\mathrm{Ad}_\sharp(\ell) : \mathrm{Pol}(\mathfrak{n}_+) \to \mathrm{Pol}(\mathfrak{n}_+), \varphi \mapsto \varphi \circ \mathrm{Ad}(\ell)^{-1}$.

On the other hand, if $-\chi$ is the lowest weight of the irreducible representation W of K', we have an isomorphism

$$\mathrm{Hom}_{K'}(V, \mathrm{Pol}(\mathfrak{n}_+) \otimes W) \simeq (\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee)_\chi, \tag{3.12}$$

where

$$\left(\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee\right)_\chi := \{P \in \mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee : P \text{ satisfies } (3.8)\}.$$

Therefore, Theorem 3.1 (2) is deduced from Theorem 3.1 (1) and from the following natural isomorphism:

$$\{\psi \,\text{satisfying}\,(3.10)\,\text{and}\,(3.11)\} \xrightarrow{\sim} \{P \,\text{satisfying}\,(3.8)\,\text{and}\,(3.9)\}.$$

See also [*op. cit.*, Lemma **??**]. □

The F-method (see [*op.cit.*, Section **??**]) in this setting consists of the following five steps:

Step 0. Fix a finite-dimensional representation (λ, V) of the maximal compact subgroup K. Form a G-equivariant holomorphic vector bundle $\mathcal{V}_X \equiv \mathcal{V} = G \times_K V$ on $X = G/K$.

Step 1. Extend λ to a representation of the Lie algebra $\mathfrak{p} = \mathfrak{k} + \mathfrak{n}_+$ by letting $\mathfrak{n}_+$ act trivially, and define another representation $\lambda^* := \lambda^\vee \otimes \mathbb{C}_{2\rho}$ of $\mathfrak{p}$ on $V^\vee$. Compute $d\pi_{\lambda^*}$ and $\widehat{d\pi_{\lambda^*}}$.

Step 2. Find a finite-dimensional representation (ν, W) of the Lie group K' such that

$$\operatorname{Hom}_{\mathfrak{g}'}(\operatorname{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee), \operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) \neq \{0\},$$

or equivalently,

$$\operatorname{Hom}_{\mathfrak{k}'}(W^\vee, \operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) \neq \{0\}.$$

Form a G'-equivariant holomorphic vector bundle $\mathcal{W}_Y \equiv \mathcal{W} = G' \times_{K'} W$ on $Y = G'/K'$. According to the duality theorem [KP14-1, Theorem **??**] the space of differential symmetry breaking operators $\operatorname{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y)$ is then nontrivial.

Step 3. Write down the condition on $\operatorname{Hom}_{K'}(V, \operatorname{Pol}(\mathfrak{n}_+) \otimes W)^{\widehat{d\pi_{\lambda^*}}(\mathfrak{n}'_+)}$, namely, the space of $\psi \in \operatorname{Pol}(\mathfrak{n}_+) \otimes \operatorname{Hom}_{\mathbb{C}}(V, W)$ satisfying (3.10) and (3.11) or equivalently $P \in \operatorname{Pol}(\mathfrak{n}_+) \otimes V^\vee$ satisfying (3.8) and (3.9).

Step 4. Use the invariant theory and give a simple description of

$$\operatorname{Hom}_{K'}(V, \operatorname{Pol}(\mathfrak{n}_+) \otimes W) \simeq \left(\operatorname{Pol}(\mathfrak{n}_+) \otimes V^\vee\right)_\chi, \quad \psi \leftrightarrow P$$

by means of "regular functions $g(s)$ on a slice" S for generic $K'_{\mathbb{C}}$-orbits on $\mathfrak{n}_+$. Induce differential equations for $g(s)$ on S from (3.11) (or equivalently (3.9)). Concrete computations are based on the technique of the T-saturation of differential operators, see Section 3.2. Solve the differential equations of $g(s)$.

Step 5. Transfer a solution g obtained in Step 4 into a polynomial solution ψ to (3.10) and (3.11). In the diagram (3.7), $(\operatorname{Symb} \otimes \operatorname{id})^{-1}(\psi)$ gives the desired differential symmetry breaking operator in the coordinates $\mathfrak{n}_-$ of X. As a byproduct, obtain an explicit K'-type $W^\vee$ annihilated by $\mathfrak{n}'_+$ in $\operatorname{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(V^\vee)$ (sometimes referred to as *singular vectors*) as $(F_c \otimes \operatorname{id})^{-1}(\psi)$.

We shall give some comments on Steps 3 and 4 in Sections 3.3 and 3.2 respectively. For Step 2, there are two approaches: one is to use (abstract) branching laws for the restriction of $\operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$ to the subalgebra $\mathfrak{g}'$ (*e.g.* Fact 4.2) or the restriction of $\mathcal{O}(G/K, \mathcal{V})$ to the subgroup G' (*e.g.* Fact 4.3). The other one is to apply the F-method and reduce it to a question of solving differential equations of second order. The former approach works well for generic parameters. We shall see that the latter approach is efficient for singular parameters in our setting (Theorems 6.1, 7.1 and 8.1, see also [KØSS13]).

3.2. **T-saturation of differential operators.** In order to implement Step 4, our idea is to introduce saturated differential operators as follows. For simplicity consider the case when $\dim_{\mathbb{C}} V = 1$. Then $\mathrm{Hom}_{K'}(V, \mathrm{Pol}(\mathfrak{n}_+) \otimes W)$ is identified with a subspace of $\mathrm{Pol}(\mathfrak{n}_+)$ via the isomorphism (3.12). Let $\mathbb{C}(\mathfrak{n}_+)$ denote the field of rational functions on $\mathfrak{n}_+$. Suppose that we have a morphism $T : \mathbb{C}[S] \longrightarrow \mathbb{C}(\mathfrak{n}_+)$ such that T induces an isomorphism

$$T : \Gamma(S) \overset{\sim}{\to} \mathrm{Hom}_{K'}(V, \mathrm{Pol}(\mathfrak{n}_+) \otimes W)$$

for some algebraic variety S ("slice" of a generic $K'_{\mathbb{C}}$-orbit on $\mathfrak{n}_+$), and for some appropriate function space $\Gamma(S)$ (*e.g.* $\Gamma(S) = \mathrm{Pol}_a[t]_{\mathrm{even}}$, see (6.12)). In the special case where V and W are the trivial one-dimensional representations of K and K', respectively, we may take $S = \mathfrak{n}_+ // K'_{\mathbb{C}}$ (geometric quotient) and T is the natural morphism $\mathbb{C}[S] \overset{\sim}{\longrightarrow} \mathbb{C}[\mathfrak{n}_+]^{K'_{\mathbb{C}}}$.

Definition 3.2. A differential operator R on $\mathfrak{n}_+$ with rational coefficients is T-*saturated* if there exists an operator D such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}[S] & \xrightarrow{T} & \mathbb{C}(\mathfrak{n}_+) \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle R} \\ \mathbb{C}[S] & \xrightarrow{T} & \mathbb{C}(\mathfrak{n}_+). \end{array}$$

Such an operator D is unique (if exists), and we denote it by $T^\sharp R$. Then we have

$$T^\sharp(R_1 \cdot R_2) = T^\sharp(R_1) T^\sharp(R_2) \qquad (3.13)$$

whenever it makes sense.

Proposition 3.3. *Let $C_1, \cdots, C_k$ be a basis of $\mathfrak{n}'_+$. Suppose there exist non-zero $Q_j \in \mathbb{C}(\mathfrak{n}_+)$ such that $Q_j \widehat{d\pi_{\lambda^*}}(C_j)$ is T-saturated ($1 \le j \le k$) and set $D_j := T^\sharp(Q_j \widehat{d\pi_{\lambda^*}}(C_j))$. Then T induces a bijection*

$$\begin{aligned} & \{g \in \Gamma[S] : D_j g = 0, \ (1 \le j \le k)\} \\ \overset{\sim}{\to} \ & \{\psi \in \mathrm{Hom}_{K'}(V, \mathrm{Pol}(\mathfrak{n}_+) \otimes W) : \psi \,\text{satisfies (3.10) and (3.11)}\} \\ \simeq \ & \left\{P \in (\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee)_\chi : P \,\text{satisfies (3.9)}\right\}. \end{aligned}$$

We shall use this idea in Sections 6-8 where S is one-dimensional and D_j are ordinary differential operators. We note that $D_j g = 0$ $(1 \le \forall j \le k)$ is equivalent to a single equation $D_i g = 0$ if K' acts irreducibly on $\mathfrak{n}'_+$.

3.3. **Complement for the F-method in vector-valued cases and highest weight varieties.** If the target $\mathcal{W}_Y$ is no longer a line bundle but a vector bundle, *i.e.*, if W is an arbitrary finite-dimensional, irreducible $\mathfrak{k}'$-module, we recall two supplementary ingredients of Step 3 in the recipe by reducing (3.10) to a simpler algebraic question on polynomial rings, so that we can focus on the crucial part consisting of a system of differential equations of second order (3.11). This construction is based on the notion of *highest weight variety* of the fiber W and is summarized in the following two lemmas (see [KP14-1, Lemmas **??** and **??**].

We fix a Borel subalgebra $\mathfrak{b}(\mathfrak{k}')$ of $\mathfrak{k}'$. Let $\chi : \mathfrak{b}(\mathfrak{k}') \to \mathbb{C}$ be a character. For a $\mathfrak{k}'$-module U, we set

$$U_\chi := \{u \in U \,:\, Zu = \chi(Z)u \quad \text{for any } Z \in \mathfrak{b}(\mathfrak{k}')\}.$$

Suppose that W is the irreducible representation of $\mathfrak{k}'$ with lowest weight $-\chi$. Then the contragredient representation $W^\vee$ has a highest weight χ. We fix a non-zero highest weight vector $w^\vee \in (W^\vee)_\chi$. Then the contraction map

$$U \otimes W \to U, \quad \psi \mapsto \langle \psi, w^\vee \rangle,$$

induces a bijection between the following two subspaces:

$$(U \otimes W)^{\mathfrak{k}'} \xrightarrow{\sim} U_\chi, \tag{3.14}$$

if U is completely reducible as a $\mathfrak{k}'$-module. By using the isomorphism (3.14), we reformulate Step 3 of the recipe for the F-method as follows:

Lemma 3.4. *Assume that W is an irreducible representation of the parabolic subalgebra $\mathfrak{p}'$. Let $-\chi$ be the lowest weight of W as a $\mathfrak{k}'$-module. Then we have a natural injective homomorphism*

$$\mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y) \hookrightarrow \left\{ Q \in (\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee)_\chi : \widehat{d\pi_\mu}(C)Q = 0 \quad \text{for all } C \in \mathfrak{n}'_+ \right\},$$

which is bijective if K' is connected.

See [KP14-1, Lemma **??**] for the proof.

Since any non-zero vector in $W^\vee$ is cyclic, the next lemma explains how to recover $D_{X\to Y}(\varphi)$ from Q given in Lemma 3.4.

We assume, for simplicity, that the $\mathfrak{k}$-module (λ, V) lifts to $K_{\mathbb{C}}$, the $\mathfrak{k}'$-module (ν, W) lifts to $K'_{\mathbb{C}}$, and use the same letters to denote their liftings.

Lemma 3.5. *For any $\varphi \in \mathrm{Hom}_{\mathfrak{p}'}(W^\vee, \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee))$, $k \in K'_{\mathbb{C}}$ and $w^\vee \in W^\vee$,*

$$\langle D_{X\to Y}(\varphi), \nu^\vee(k) w^\vee \rangle = (\mathrm{Ad}(k) \otimes \lambda^\vee(k)) \langle D_{X\to Y}(\varphi), w^\vee \rangle. \tag{3.15}$$

See [KP14-1, Lemma **??**] for the proof.

4. Branching laws and Hermitian symmetric spaces

The existence, respectively the uniqueness (up to scaling) of differential symmetry breaking operators from $\mathcal{V}_X$ to $\mathcal{W}_Y$ are subject to the conditions

$$\dim \mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y) \geq 1, \text{ respectively } \leq 1. \tag{4.1}$$

So we need to find the geometric settings (i.e. the pair $Y \subset X$ of generalized flag varieties and two homogeneous vector bundles $\mathcal{V}_X \to X$ and $\mathcal{W}_Y \to Y$) satisfying (4.1). This is the main ingredient of Step 2 in the recipe of the F-method, and thanks to [KP14-1, Theorem **??**], the existence and uniqueness are equivalent to the following question concerning (abstract) branching laws: Given a $\mathfrak{p}$-module V, find all finite-dimensional $\mathfrak{p}'$-modules W such that $\dim \mathrm{Hom}_{\mathfrak{p}'}(W^\vee, \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) = 1$, and equivalently,

$$\dim \mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee), \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) = 1. \tag{4.2}$$

This section briefly reviews what is known on this question (see Fact 4.2).

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $\mathfrak{j}$ a Cartan subalgebra of $\mathfrak{g}$. We fix a positive root system $\Delta^+ \equiv \Delta^+(\mathfrak{g}, \mathfrak{j})$, write ρ for half the sum of positive roots, $\alpha^\vee$ for the coroot for $\alpha \in \Delta$, and $\mathfrak{g}_\alpha$ for the root space. Define a Borel subalgebra $\mathfrak{b} = \mathfrak{j} + \mathfrak{n}$ with nilradical $\mathfrak{n} := \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$.

The BGG category $\mathcal{O}$ is defined as the full subcategory of $\mathfrak{g}$-modules whose objects are finitely generated, $\mathfrak{j}$-semisimple and locally $\mathfrak{n}$-finite [BGG76].

As in the previous sections, fix a standard parabolic subalgebra $\mathfrak{p}$ with Levi decomposition $\mathfrak{p} = \mathfrak{k} + \mathfrak{n}_+$ such that the Levi factor $\mathfrak{k}$ contains $\mathfrak{j}$. We set $\Delta^+(\mathfrak{k}) := \Delta^+ \cap \Delta(\mathfrak{k}, \mathfrak{j})$. The parabolic BGG category $\mathcal{O}^{\mathfrak{p}}$ is defined as the full subcategory of $\mathcal{O}$ whose objects are locally $\mathfrak{k}$-finite.

We define

$$\Lambda^+(\mathfrak{k}) := \{\lambda \in \mathfrak{j}^* : \langle \lambda, \alpha^\vee \rangle \in \mathbb{N} \text{ for any } \alpha \in \Delta^+(\mathfrak{k})\},$$

the set of linear forms λ on $\mathfrak{j}$ whose restrictions to $\mathfrak{j} \cap [\mathfrak{k}, \mathfrak{k}]$ are dominant integral. We write V_λ for the finite-dimensional simple $\mathfrak{k}$-module with highest weight λ, regard it as a $\mathfrak{p}$-module by letting $\mathfrak{n}_+$ act trivially, and consider the generalized Verma module

$$\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) \equiv \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V_\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V_\lambda.$$

Then $\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) \in \mathcal{O}^{\mathfrak{p}}$ and any simple object in $\mathcal{O}^{\mathfrak{p}}$ is the quotient of some generalized Verma module. If

$$\langle \lambda, \alpha^\vee \rangle = 0 \qquad \text{for all} \qquad \alpha \in \Delta(\mathfrak{k}), \tag{4.3}$$

then V_λ is one-dimensional, to be denoted also by $\mathbb{C}_\lambda$. In this case we say $\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is of *scalar type*.

Let $\tau \in \mathrm{Aut}(\mathfrak{g})$ be an involutive automorphism of the Lie algebra $\mathfrak{g}$. We write

$$\mathfrak{g}^{\pm\tau} := \{v \in \mathfrak{g} : \tau v = \pm v\}$$

for the ±1 eigenspaces of τ, respectively. We say that $(\mathfrak{g}, \mathfrak{g}')$ is a symmetric pair if $\mathfrak{g}' = \mathfrak{g}^\tau$ for some τ.

For a general choice of τ and $\mathfrak{p}$, the space considered in (4.2) may be reduced to zero for all $\mathfrak{p}'$-modules W. Suppose $V \equiv V_\lambda$ with $\lambda \in \Lambda^+(\mathfrak{k})$ generic. Then a necessary and sufficient condition for the existence of W such that the left-hand side of (4.2) is non-zero is given by the geometric requirement on the generalized flag variety $G_{\mathbb{C}}/P_{\mathbb{C}}$, namely, the set $G^\tau_{\mathbb{C}} P_{\mathbb{C}}$ is closed in $G_{\mathbb{C}}$, see [K12, Proposition 3.8].

Consider now the case where the nilradical $\mathfrak{n}_+$ of $\mathfrak{p}$ is abelian. Then, the following result holds :

Fact 4.1 ([K12])**.** *If the nilradical $\mathfrak{n}_+$ of $\mathfrak{p}$ is abelian, then for any symmetric pair $(\mathfrak{g}, \mathfrak{g}^\tau)$ the restriction of a generalized Verma module of scalar type $\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{p}}(-\lambda)|_{\iota(\mathfrak{g}^\tau)}$ is multiplicity-free for any embedding $\iota : \mathfrak{g}^\tau \to \mathfrak{g}$ such that $\iota(G^\tau_{\mathbb{C}})P_{\mathbb{C}}$ is closed in $G_{\mathbb{C}}$ and for any sufficiently positive λ.*

A combinatorial description of the branching law is given as follows. Suppose that $\mathfrak{p}$ is $\mathfrak{g}^\tau$-compatible (see [KP14-1, Definition **??**]). Then the involution τ stabilizes $\mathfrak{k}$ and $\mathfrak{n}_+$, respectively, the nilradical $\mathfrak{n}_+$ decomposes into a direct sum of eigenspaces $\mathfrak{n}_+ = \mathfrak{n}^\tau_+ + \mathfrak{n}^{-\tau}_+$ and $G^\tau_{\mathbb{C}} P_{\mathbb{C}}$ is closed in $G_{\mathbb{C}}$. Fix a Cartan subalgebra $\mathfrak{j}$ of $\mathfrak{k}$ such that $\mathfrak{j}^\tau := \mathfrak{j} \cap \mathfrak{g}^\tau$ is a Cartan subalgebra of $\mathfrak{k}^\tau$.

We define $\theta \in \mathrm{End}(\mathfrak{g})$ by $\theta|_{\mathfrak{k}} = \mathrm{id}$ and $\theta|_{\mathfrak{n}_+ + \mathfrak{n}_-} = -\mathrm{id}$. Then θ is an involution commuting with τ. Moreover it is an automorphism if $\mathfrak{n}_+$ is abelian. The reductive subalgebra $\mathfrak{g}^{\tau\theta} = \mathfrak{k}^\tau + \mathfrak{n}^{-\tau}_- + \mathfrak{n}^{-\tau}_+$ decomposes into simple or abelian ideals $\bigoplus_i \mathfrak{g}^{\tau\theta}_i$, and we write the decomposition of $\mathfrak{n}^{-\tau}_+$ as $\mathfrak{n}^{-\tau}_+ = \bigoplus_i \mathfrak{n}^{-\tau}_{+,i}$ correspondingly. Each $\mathfrak{n}^{-\tau}_{+,i}$ is a $\mathfrak{j}^\tau$-module, and we denote by $\Delta(\mathfrak{n}^{-\tau}_{+,i}, \mathfrak{j}^\tau)$ the set of weights of $\mathfrak{n}^{-\tau}_{+,i}$ with respect to $\mathfrak{j}^\tau$. The roots α and β are said to be *strongly orthogonal* if neither $\alpha + \beta$ nor $\alpha - \beta$ is a root. We take a maximal set of strongly orthogonal roots $\{\nu^{(i)}_1, \cdots, \nu^{(i)}_{k_i}\}$ in $\Delta(\mathfrak{n}^{-\tau}_{+,i}, \mathfrak{j}^\tau)$ inductively as follows:

1) $\nu^{(i)}_1$ is the highest root of $\Delta(\mathfrak{n}^{-\tau}_{+,i}, \mathfrak{j}^\tau)$.
2) $\nu^{(i)}_{j+1}$ is the highest root among the elements in $\Delta(\mathfrak{n}^{-\tau}_{+,i}, \mathfrak{j}^\tau)$ that are strongly orthogonal to $\nu^{(i)}_1, \cdots, \nu^{(i)}_j$ $(1 \le j \le k_i - 1)$.

We define the following subset of $\mathbb{N}^k$ $(k = \sum k_i)$ by

$$A^+ := \prod_i A_i, \quad A_i := \{(a^{(i)}_j)_{1 \le j \le k_i} \in \mathbb{N}^{k_i} : a^{(i)}_1 \ge \cdots \ge a^{(i)}_{k_i} \ge 0\}. \tag{4.4}$$

Introduce the following positivity condition:

$$\langle \lambda - \rho_{\mathfrak{g}}, \alpha \rangle > 0 \quad \text{for any } \alpha \in \Delta(\mathfrak{n}_+, \mathfrak{j}). \tag{4.5}$$

Fact 4.2 ([K08])**.** *Suppose $\mathfrak{p}$ is $\mathfrak{g}^\tau$-compatible, and λ satisfies* (4.3) *and* (4.5)*. Then the generalized Verma module $\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{p}}(-\lambda)$ decomposes into a multiplicity-free direct sum* *of irreducible* $\mathfrak{g}^\tau$*-modules :*

$$\operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(-\lambda)|_{\mathfrak{g}^\tau} \simeq \bigoplus_{(a_j^{(i)})\in A^+} \operatorname{ind}_{\mathfrak{p}^\tau}^{\mathfrak{g}^\tau}(-\lambda|_{\mathfrak{j}^\tau} - \sum_i \sum_{j=1}^{k_i} a_j^{(i)} \nu_j^{(i)}). \tag{4.6}$$

In particular, for a simple $\mathfrak{p}^\tau$*-module* W *(namely, a simple* $\mathfrak{k}^\tau$*-module with trivial action of* $\mathfrak{n}^\tau$*),*

$$\dim \operatorname{Hom}_{\mathfrak{g}^\tau}(\operatorname{ind}_{\mathfrak{p}^\tau}^{\mathfrak{g}^\tau}(W^\vee), \operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_{-\lambda})) = 1$$

if and only if the highest weight of the $\mathfrak{k}^\tau$*-module* W *is of the form* $\lambda|_{\mathfrak{j}^\tau} + \sum_i \sum_{j=1}^{k_i} a_j^{(i)} \nu_j^{(i)}$ *for some* $(a_j^{(i)}) \in A^+$*.*

Notice that when τ is a Cartan involution, G^τ is compact and $\mathfrak{g}^\tau = \mathfrak{p}^\tau$. In this case, the formula (4.6) is due to L. K. Hua [H63] (classical case), B. Kostant (unpublished), and W. Schmid [Sch69]. In general G^τ is non-compact, and we need to consider infinite-dimensional irreducible representations of G^τ when we consider the branching law $G \downarrow G^\tau$.

In remaining Sections 5, 6, 7 and 8 we construct a family of equivariant differential operators for all symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\tau)$ with G^τ non-compact and $k = 1$ (in particular, $\Delta(\mathfrak{n}_{+,i}^{-\tau}, \mathfrak{j}^\tau)$ is empty for all but one i).

In conclusion, we recall the corresponding branching laws in the category of unitary representations, which are the dual of the formulæ in Fact 4.2. We denote by $\mathcal{H}^2(M, \mathcal{V})$ the Hilbert space of square integrable holomorphic sections of the Hermitian vector bundle $\mathcal{V}$ over a Hermitian manifold M. If the positivity condition (4.5) holds, then $\mathcal{H}^2(G/K, \mathcal{L}_\lambda) \neq \{0\}$, and G acts unitarily and irreducibly on it.

Given $\underline{a} = (a_j^{(i)}) \in A^+$ $(\subset \mathbb{N}^k)$, we write $\mathcal{W}_\lambda^{\underline{a}}$ for the G^τ-equivariant holomorphic vector bundle over G^τ/K^τ associated to the irreducible representation $\mathcal{W}_\lambda^{\underline{a}}$ of $\mathfrak{k}^\tau$ with highest weight $\lambda|_{\mathfrak{j}^\tau} + \sum_i \sum_{j=1}^{k_i} a_j^{(i)} \nu_j^{(i)}$.

Fact 4.3 ([K08])**.** *If the positivity condition* (4.5) *is satisfied, then* $\mathcal{H}^2(G^\tau/K^\tau, \mathcal{W}_\lambda^{\underline{a}})$ *is non-zero and* G^τ *acts on it irreducibly and unitarily for any* $\underline{a} \in A^+$*. Moreover, the branching law for the restriction* $G \downarrow G^\tau$ *is given by*

$$\mathcal{H}^2(G/K, \mathcal{L}_\lambda) \simeq \sum_{\underline{a}\in A^+}^{\oplus} \mathcal{H}^2(G^\tau/K^\tau, \mathcal{W}_\lambda^{\underline{a}}) \quad \text{(Hilbert direct sum)}. \tag{4.7}$$

5. Normal derivatives versus intertwining operators

Let G'/K' be a subsymmetric space of the Hermitian symmetric space G/K as in Section 2.1. Consider the Taylor expansion of any holomorphic function (section) on G/K with respect to the normal direction. Then the coefficients give rise to holomorphic sections of a family of vector bundles over the submanifold G'/K'. This idea was used earlier by Jakobsen and Vergne [JV79], and by the first author [K08] for filtered modules to find *abstract* branching laws.

However, it should be noted that normal derivatives do not always give rise to symmetry breaking operators. In this section we clarify the reason in the general setting, and then give a classification of all irreducible symmetric pairs $(\mathfrak{g}(\mathbb{R}), \mathfrak{g}(\mathbb{R})^\tau)$ of split rank one for which it happens.

5.1. **Normal derivatives and the Borel embedding.** Suppose $E = E' \oplus E''$ is a direct sum of complex vector spaces. Let $\mathcal{V}_E := E \times V$ and $\mathcal{W}_{E'} := E' \times W$ be direct product vector bundles over E and E', respectively. Clearly, we have isomorphisms $\mathcal{O}(E, \mathcal{V}_E) \simeq \mathcal{O}(E) \otimes V$, and $\mathcal{O}(E', \mathcal{W}_{E'}) \simeq \mathcal{O}(E') \otimes W$.

Take coordinates $y = (y_1, \cdots, y_p)$ in E' and $z = (z_1, \cdots, z_n)$ in E''. The subspace E' is given by the condition $z = 0$ in $E = \{(y,z) : y \in E', z \in E''\}$. A holomorphic differential operator $\widetilde{T} : \mathcal{O}(E) \otimes V \longrightarrow \mathcal{O}(E') \otimes W$, $f(y,z) \mapsto (\widetilde{T} f)(y)$ is said to be a *normal derivative with respect to the decomposition* $E = E' \oplus E''$ if it is of the form

$$\left(\widetilde{T} f\right)(y) = \sum_{\alpha \in \mathbb{N}^q} T_\alpha(y) \left(\left. \frac{\partial^{|\alpha|} f(y,z)}{\partial z^\alpha} \right|_{z=0} \right), \tag{5.1}$$

for some $T_\alpha \in \mathcal{O}(E') \otimes \mathrm{Hom}_{\mathbb{C}}(V, W)$.

We write $\mathcal{N}\mathrm{Diff}^{\mathrm{hol}}(\mathcal{V}_E, \mathcal{W}_{E'})$ for the space of (holomorphic) normal derivatives. This notion depends on the direct sum decomposition $E = E' \oplus E''$.

Since the commutative groups $E \supset E'$ act on the direct product bundles $\mathcal{V}_E$ and $\mathcal{W}_{E'}$, respectively, we can consider symmetry breaking operators in this abelian setting, namely, E'-equivariant normal derivatives, which amount to the condition that $T_\alpha(y)$ in (5.1) is a differential operator with constant coefficients for every $\alpha \in \mathbb{N}^q$. We denote $\mathcal{N}\mathrm{Diff}^{\mathrm{const}}(\mathcal{V}_E, \mathcal{W}_{E'})$ the subspace of $\mathcal{N}\mathrm{Diff}^{\mathrm{hol}}(\mathcal{V}_E, \mathcal{W}_{E'})$ consisting of those operators.

Thus we have seen the following:

Lemma 5.1. *There is a natural isomorphism:*

$$\mathrm{Hom}_{\mathbb{C}}(V, W) \otimes S(E'') \xrightarrow{\sim} \mathcal{N}\mathrm{Diff}^{\mathrm{const}}(\mathcal{V}_E, \mathcal{W}_{E'}).$$

Suppose we are in the setting of Section 2.1. We apply the concept of normal derivatives to the subsymmetric space G'/K' in the Hermitian symmetric space G/K. Let $\mathcal{V}$ be a homogeneous vector bundle over $X = G/K$ associated with a finite-dimensional representation V of K. Similarly, let $\mathcal{W}$ be a homogeneous vector bundle over the subsymmetric space $Y = G'/K'$ associated with a finite-dimensional representation W of K'.

By using the Killing form, we take a complementary subspace $\mathfrak{g}''$ of $\mathfrak{g}'$ in $\mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$ is a direct sum of G'-modules. We set $\mathfrak{n}_-'' := \mathfrak{n}_- \cap \mathfrak{g}''$. Since the characteristic element $H_o \in \mathfrak{g}'$ (see (2.1)), we have a direct sum decomposition of K'-modules:

$$\mathfrak{n}_- = \mathfrak{n}'_- \oplus \mathfrak{n}''_-. \tag{5.2}$$

Accordingly, we can consider the space $\mathcal{N}\mathrm{Diff}^{\mathrm{hol}}(\mathcal{V}_{\mathfrak{n}_-}, \mathcal{W}_{\mathfrak{n}'_-})$ of holomorphic normal derivatives with respect to (5.2).

We write $\mathcal{N}\mathrm{Diff}^{\mathrm{hol}}(\mathcal{V}_X, \mathcal{W}_Y)$ and $\mathcal{N}\mathrm{Diff}^{\mathrm{const}}(\mathcal{V}_X, \mathcal{W}_Y)$ for the images of $\mathcal{N}\mathrm{Diff}^{\mathrm{hol}}(\mathcal{V}_{\mathfrak{n}_-}, \mathcal{W}_{\mathfrak{n}'_-})$ and $\mathcal{N}\mathrm{Diff}^{\mathrm{const}}(\mathcal{V}_{\mathfrak{n}_-}, \mathcal{W}_{\mathfrak{n}'_-})$, respectively, under the natural injective map:

$$\mathrm{Diff}^{\mathrm{hol}}(\mathcal{V}_{\mathfrak{n}_-}, \mathcal{W}_{\mathfrak{n}'_-}) \hookrightarrow \mathrm{Diff}^{\mathrm{hol}}(\mathcal{V}_X, \mathcal{W}_Y)$$

induced by the following map:

$$\begin{array}{ccc} \mathcal{O}(\mathfrak{n}_-, V) & \longrightarrow & \mathcal{O}(\mathfrak{n}'_-, W) \\ {\scriptstyle \text{restriction}} \downarrow & & \downarrow {\scriptstyle \text{restriction}} \\ \mathcal{O}(G/K, \mathcal{V}) & \dashrightarrow & \mathcal{O}(G'/K', \mathcal{W}). \end{array} \tag{5.3}$$

Since the trivialization of the vector bundle $G_{\mathbb{C}} \times_{P_{\mathbb{C}}} V$

$$\begin{array}{ccccc} \mathfrak{n}_- \times V & \hookrightarrow & G_{\mathbb{C}} \times_{P_{\mathbb{C}}} V & \hookleftarrow & \mathcal{V}_X \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{n}_- & \hookrightarrow & G_{\mathbb{C}}/P_{\mathbb{C}} & \hookleftarrow & X = G/K \end{array}$$

is $K_{\mathbb{C}}$-equivariant, Lemma 5.1 implies:

Proposition 5.2. *There is a natural isomorphism:*

$$\mathrm{Hom}_{K'}(V, S(\mathfrak{n}''_-) \otimes W) \xrightarrow{\sim} \mathcal{N}\mathrm{Diff}^{\mathrm{const}}_{K'}(\mathcal{V}_X, \mathcal{W}_Y).$$

We study whether or not the following two subspaces

- $\mathcal{N}\mathrm{Diff}_{K'}(\mathcal{V}_X, \mathcal{W}_Y)$ of K'-equivariant normal derivatives and
- $\mathrm{Hom}_{G'}(\mathcal{O}(\mathcal{V}_X), \mathcal{O}(\mathcal{W}_Y))$ of symmetry breaking operators

coincide in $\mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(\mathcal{V}_X), \mathcal{O}(\mathcal{W}_Y))$. Owing to Theorem 3.1 and Proposition 5.2, it reduces to an algebraic problem to compare

- $\mathrm{Hom}_{K'}(V, S(\mathfrak{n}''_-) \otimes W)$ and
- $\mathrm{Hom}_{K'}(V, \mathrm{Pol}(\mathfrak{n}_+) \otimes W)^{\widehat{d\pi_{\lambda^*}}(\mathfrak{n}'_+)}$

in $\mathrm{Hom}_{\mathbb{C}}(V, \mathrm{Pol}(\mathfrak{n}_+) \otimes W) \simeq \mathrm{Hom}_{\mathbb{C}}(V, S(\mathfrak{n}_-) \otimes W)$. We shall see in the next subsection that they actually coincide for the three families of symmetric pairs out of the six listed in Table 2.1.

5.2. **When are normal derivatives intertwining operators?** Let $\dim V = 1$, and we write as before $\mathcal{L}_\lambda$ for the homogeneous line bundle over $X = G/K$ associated to the character $\mathbb{C}_\lambda$ of K.

Theorem 5.3. *Suppose* $(\mathfrak{g}(\mathbb{R}), \mathfrak{g}(\mathbb{R})^\tau)$ *is a split rank one irreducible symmetric pair of holomorphic type (see Definition 2.3). Then, the following three conditions on the pair* $(\mathfrak{g}(\mathbb{R}), \mathfrak{g}(\mathbb{R})^\tau)$ *are equivalent:*

(i) *For any* λ *satisfying the positivity condition* (4.5) *and for any irreducible* K^τ*-module* W*, all continuous* G^τ*-homomorphisms*

$$\mathcal{O}(X, \mathcal{L}_\lambda) \longrightarrow \mathcal{O}(Y, \mathcal{W}),$$

are given by normal derivatives with respect to the decomposition $\mathfrak{n}_- = \mathfrak{n}_-^\tau \oplus \mathfrak{n}_-^{-\tau}$.

(ii) *For some* λ *satisfying* (4.5) *and for some irreducible* K^τ*-module* W*, there exists a non-trivial* G^τ*-intertwining operator*

$$\mathcal{O}(X, \mathcal{L}_\lambda) \longrightarrow \mathcal{O}(Y, \mathcal{W})$$

which is given by normal derivatives of positive order.

(iii) *The symmetric pair* $(\mathfrak{g}(\mathbb{R}), \mathfrak{g}(\mathbb{R})^\tau)$ *is isomorphic to one of* $(\mathfrak{su}(p,q), \mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(p-1,q)))$, $(\mathfrak{so}(2,2n), \mathfrak{u}(1,n))$ *or* $(\mathfrak{so}^*(2n), \mathfrak{so}(2) \oplus \mathfrak{so}^*(2n-2))$.

Notice that the geometric nature of embeddings $Y \hookrightarrow X$ mentioned in the condition (iii) corresponds to the following inclusions of flag varieties:

$$\begin{aligned} \mathrm{Gr}_{p-1}(\mathbb{C}^{p+q}) &\hookrightarrow \mathrm{Gr}_p(\mathbb{C}^{p+q}); \\ \mathbb{P}^n\mathbb{C} &\hookrightarrow Q^{2n}\mathbb{C}; \\ \mathrm{IGr}_{n-1}(\mathbb{C}^{2n-2}) &\hookrightarrow \mathrm{IGr}_n(\mathbb{C}^{2n}), \end{aligned}$$

where $\mathrm{Gr}_p(\mathbb{C}^k) := \{V \subset \mathbb{C}^k : \dim V = p\}$ is the complex Grassmanian, $Q^m\mathbb{C} := \{z \in \mathbb{P}^{m+1}\mathbb{C} : z_0^2 + \cdots + z_{m+1}^2 = 0\}$ is the complex quadric and $\mathrm{IGr}_n(\mathbb{C}^{2n}) := \{V \subset \mathbb{C}^{2n} : \dim V = n, Q\big|_V \equiv 0\}$ is the isotropic Grassmanian for $\mathbb{C}^{2n}$ equipped with a non-degenerate quadratic form Q.

5.3. **Outline of the proof of Theorem 5.3.** The implication (i)⇒(ii) is obvious. On the other hand, for split rank one symmetric spaces there are three other cases (i.e., (1), (2) and (3) in Table 2.1) where the G^τ-intertwining operators are not given by normal derivatives. In Sections 6, 7 and 8 we construct them explicitly. This will conclude the implication (ii)⇒(iii). For the rest of this section we shall give a proof for the implication (iii)⇒(i).

Consider a homomorphism: $T : W^\vee \longrightarrow S(\mathfrak{n}_-^{-\tau}) \otimes V^\vee$. We regard $S(\mathfrak{n}_-^{-\tau}) \otimes V^\vee$ as a subspace of $\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$ on which the Lie algebra $\mathfrak{g}$ acts by $\widehat{d\pi_{\lambda^*}}$, see (3.6). If T is a K^τ-homomorphism, the differential operator $\widetilde{T} : \mathcal{O}(G/K, \mathcal{V}_X) \to \mathcal{O}(G^\tau/K^\tau, \mathcal{W}_Y)$

is K^τ-equivariant. The following statement gives a sufficient condition for $\widetilde{T}$ to be G^τ-equivariant.

Proposition 5.4. *The normal derivative* $\widetilde{T} \in \mathcal{N}\mathrm{Diff}^{\mathrm{const}}(\mathcal{V}_X, \mathcal{W}_Y)$ *induces a* G^τ-*equivariant differential operator from* $\mathcal{V}_X$ *to* $\mathcal{W}_Y$ *if and only if* T *is a* K^τ-*homomorphism and* $T(W^\vee)$ *is contained in* $(\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee)^{\widehat{d\pi_{\lambda^*}}(\mathfrak{n}_+^\tau)}$.

Proof. The proof is a direct consequence of the F-method. Indeed, by Theorem 3.1, $\widetilde{T} \in \mathcal{N}\mathrm{Diff}^{\mathrm{const}}(\mathcal{V}_X, \mathcal{W}_Y) \subset \mathrm{Diff}^{\mathrm{const}}(\mathfrak{n}_-) \otimes \mathrm{Hom}_{\mathbb{C}}(V, W)$ is a G^τ-equivariant differential operator if and only if $(\mathrm{Symb} \otimes \mathrm{id})(\widetilde{T}) \in (\mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}(V, W))^{\widehat{d\pi_{\lambda^*}}(\mathfrak{p}^\tau)}$ where

$$\begin{aligned} &(\mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}(V, W))^{\widehat{d\pi_{\lambda^*}}(\mathfrak{p}^\tau)} \\ = \; &(\mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}(V, W))^{\widehat{d\pi_{\lambda^*}}(\mathfrak{k}^\tau)} \cap (\mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}(V, W))^{\widehat{d\pi_{\lambda^*}}(\mathfrak{n}_+^\tau)}. \end{aligned}$$

Furthermore, by Theorem 3.1, for $\widetilde{T} \in \mathcal{N}\mathrm{Diff}^{\mathrm{const}}(\mathcal{V}_X, \mathcal{W}_Y)$, we have $(\mathrm{Symb} \otimes \mathrm{id})(\widetilde{T}) \in (\mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}(V, W))^{\widehat{d\pi_{\lambda^*}}(\mathfrak{k}^\tau)}$ if and only if $T \in \mathrm{Hom}_{\mathfrak{k}^\tau}(W^\vee, S(\mathfrak{n}_-^{-\tau}) \otimes V^\vee)$, as $(\mathrm{Symb} \otimes \mathrm{id})(\widetilde{T}) = (F_c \otimes \mathrm{id})(T)$. Hence the statement is proved. $\square$

Lemma 5.5. *Suppose* $(\mathfrak{g}(\mathbb{R}), \mathfrak{g}(\mathbb{R})^\tau)$ *is a split rank one irreducible symmetric pair of holomorphic type and* λ *satisfying* (4.3) *and* (4.5). *For* $a \in \mathbb{N}$ *we define a* K^τ-*module:*

$$W_\lambda^a := S^a(\mathfrak{n}_-^{-\tau}) \otimes \mathbb{C}_\lambda. \tag{5.4}$$

(1) *The module* W_λ^a *is irreducible for any* $a \in \mathbb{N}$.
(2) *If for an irreducible* K^τ-*module* W *there exists a non-zero continuous* G^τ-*homomorphism* $\mathcal{O}(G/K, \mathcal{L}_\lambda) \to \mathcal{O}(G^\tau/K^\tau, \mathcal{W})$, *then the module* W *is isomorphic to* W_λ^a *for some* $a \in \mathbb{N}$.
(3) *Assume that*

$$\mathrm{Hom}_{\mathfrak{k}^\tau}(S^a(\mathfrak{n}_-^{-\tau}), S^{a_1}(\mathfrak{n}_-^\tau) \otimes S^{a-a_1}(\mathfrak{n}_-^{-\tau})) = \{0\} \quad \text{for any } 1 \le a_1 \le a. \tag{5.5}$$

Then, the normal derivative $\widetilde{T}$ *corresponding to the natural inclusion* $T : (W_\lambda^a)^\vee \to S(\mathfrak{n}_-^{-\tau}) \otimes (\mathbb{C}_\lambda)^\vee$ *is a* G^τ-*equivariant differential operator.*

Proof. If $\mathrm{rank}_{\mathbb{R}}\, G/G^\tau = 1$, then the non-compact part of $\mathfrak{g}(\mathbb{R})^{\tau\theta}$ is isomorphic to $\mathfrak{su}(1, n)$ for some n. Thus the first statement follows from the observation that $S^a(\mathbb{C}^n)$ is an irreducible $\mathfrak{gl}_n(\mathbb{C})$-module for any $a \in \mathbb{N}$ because the action of $\mathfrak{k}^\tau$ on $\mathfrak{n}_+^{-\tau}$ corresponds to the natural action of $\mathfrak{gl}_n(\mathbb{C})$ on $\mathbb{C}^n$.

The second statement is due to the localness theorem [KP14-1, Theorem **??**] for $k = \mathrm{rank}_{\mathbb{R}}\, G/G^\tau = 1$.

To show the third statement, observe that we have the following natural inclusions $A \subset B \supset C$, where

$$A := \mathrm{Pol}^a(\mathfrak{n}_+^{-\tau}) \otimes \mathbb{C}_\lambda^\vee,\; B := \mathrm{Pol}^a(\mathfrak{n}_+) \otimes \mathbb{C}_\lambda^\vee,\; C := (\mathrm{Pol}^a(\mathfrak{n}_+) \otimes \mathbb{C}_\lambda^\vee)^{\widehat{d\pi_{\lambda^*}}(\mathfrak{n}_+^\tau)}.$$

Therefore

$$\operatorname{Hom}_{\mathfrak{k}^\tau}((W^a_\lambda)^\vee, A) \hookrightarrow \operatorname{Hom}_{\mathfrak{k}^\tau}((W^a_\lambda)^\vee, B) \hookleftarrow \operatorname{Hom}_{\mathfrak{k}^\tau}((W^a_\lambda)^\vee, C).$$

By Proposition 5.2 and Theorem 3.1, we have

$$\mathcal{N}\mathrm{Diff}^{\mathrm{const}}_{K^\tau}(\mathcal{V}_X, \mathcal{W}_Y) \hookrightarrow \operatorname{Hom}_{\mathfrak{k}^\tau}((W^a_\lambda)^\vee, B) \hookleftarrow \operatorname{Hom}_{G'}(\mathcal{O}(X,\mathcal{V}), \mathcal{O}(Y,\mathcal{W})).$$

Since $\mathrm{Pol}^a(\mathfrak{n}_+) \simeq \bigoplus_{a_1=0}^{a} \mathrm{Pol}^{a_1}(\mathfrak{n}_+^\tau) \otimes \mathrm{Pol}^{a-a_1}(\mathfrak{n}_+^{-\tau})$, the assumption (5.5) implies that $\operatorname{Hom}_{\mathfrak{k}^\tau}((W^a_\lambda)^\vee, A) \xrightarrow{\sim} \operatorname{Hom}_{\mathfrak{k}^\tau}((W^a_\lambda)^\vee, B)$, and therefore the first inclusion is an isomorphism. Moreover, since A is isomorphic to the irreducible $\mathfrak{k}^\tau$-module $(W^a_\lambda)^\vee$, the first term is one-dimensional by Schur's lemma. The last one is also one-dimensional according to the multiplicity-one decomposition given in Fact 4.2. Therefore, all the three terms coincide.

Hence the canonical isomorphism $T : (W^a_\lambda)^\vee \to S(\mathfrak{n}_-^{-\tau}) \otimes (\mathbb{C}_\lambda)^\vee$ satisfies the assumption of Proposition 5.4. Thus Lemma follows. □

Remark 5.6. The highest weight vectors of the generalized Verma module $\operatorname{ind}^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{C}^\vee_\lambda)$ with respect to $\mathfrak{p}^\tau$ have a particularly simple form if the condition (5.5) is satisfied. In fact, by Poincaré–Birkhoff–Witt theorem $\operatorname{ind}^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{C}^\vee_\lambda)$ is isomorphic, as a $\mathfrak{k}$-module, to $S(\mathfrak{n}_-) \otimes \mathbb{C}^\vee_\lambda$, when $\mathfrak{n}_-$ is abelian. Under the assumption (5.5) we thus have

$$\left(\operatorname{ind}^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{C}^\vee_\lambda)\right)^{\mathfrak{n}_+^\tau} \simeq \bigoplus_{a=0}^{\infty} S^a(\mathfrak{n}_-^{-\tau}) \otimes \mathbb{C}^\vee_\lambda.$$

This formula is an algebraic explanation of the fact that G^τ-equivariant operators are given by normal derivatives in this setting.

In order to conclude the proof of Theorem 5.3 we have to show that in all cases mentioned in (iii) the condition (5.5) is fulfilled. It will be done in the next subsection.

5.4. **An application of the classical branching rules.** In what follows, we shall verify the condition (5.5) for the last three cases (4), (5) and (6) in Table 2.1 by using some classical branching rules of irreducible representations of $\mathfrak{gl}_m(\mathbb{C})$.

Denote by $F(\mathfrak{gl}_m(\mathbb{C}), \mu)$ the finite dimensional irreducible $\mathfrak{gl}_m(\mathbb{C})$-module with highest weight μ. For example, the natural representation of the Lie algebra $\mathfrak{gl}_m(\mathbb{C})$ on $\mathbb{C}^m$ corresponds to $F(\mathfrak{gl}_m(\mathbb{C}), (1,0,\dots,0))$ and its contragredient representation on $(\mathbb{C}^m)^\vee$ to $F(\mathfrak{gl}_m(\mathbb{C}), (0,0,\dots,0,-1))$, while the action of $\mathfrak{gl}_m(\mathbb{C})$ on the space of symmetric matrices $\mathrm{Sym}(m,\mathbb{C}) \simeq S^2(\mathbb{C}^m)$ given by $C \mapsto XC\,{}^tX$ for $X \in \mathfrak{gl}_m(\mathbb{C})$ and $C \in Sym(m,\mathbb{C})$ corresponds to $F(\mathfrak{gl}_m(\mathbb{C}), (2,0,\dots,0))$. More generally, the action of $\mathfrak{gl}_m(\mathbb{C})$ on the space of i-th symmetric tensors is no longer irreducible and decomposes as follows:

$$\begin{aligned} S^i(\mathrm{Sym}(m,\mathbb{C})) &\simeq S^i\left(S^2(\mathbb{C}^m)\right) \\ &\simeq \bigoplus_{\substack{i_1\geq\cdots\geq i_m\geq 0\\ i_1+\cdots+i_m=i}} F(\mathfrak{gl}_m(\mathbb{C}),\,(2i_1,2i_2,\dots,2i_m)). \end{aligned} \tag{5.6}$$

In turn, classical Pieri's rule gives the following irreducible decomposition for the tensor product of such modules:

$$S^i\left(S^2(\mathbb{C}^m)\right)\otimes S^k(\mathbb{C}^m)\simeq \bigoplus_{\substack{i_1\geq\cdots\geq i_m\geq 0,\\ i_1+\cdots+i_m=i}}\ \bigoplus_{\substack{\ell_1\geq 2i_1\geq\cdots\geq\ell_m\geq 2i_m,\\ \sum_{r=1}^m(\ell_r-2i_r)=k}} F(\mathfrak{gl}_m(\mathbb{C}),\,(\ell_1,\dots,\ell_m)).$$

Remark 5.7. The summand of the form $F(\mathfrak{gl}_m(\mathbb{C}),\,(\ell,0,\dots,0))$ occurs in the right-hand side if and only if $i_2=\cdots=i_m=0$, hence $i_1=i$ and $\ell-2i=k$. This remark will be used in Section 7.

Example 5.8. *Let $G=U(p,q)$, $G^\tau=U(1)\times U(p-1,q)$ and $\mathfrak{k}^\tau=\mathfrak{k}^\tau(\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C}\simeq\mathfrak{gl}_1(\mathbb{C})\oplus\mathfrak{gl}_{p-1}(\mathbb{C})\oplus\mathfrak{gl}_q(\mathbb{C})$. Then, the decomposition $\mathfrak{n}_-=\mathfrak{n}_-^\tau\oplus\mathfrak{n}_-^{-\tau}$ as a $\mathfrak{k}^\tau$-module amounts to*

$$(\mathbb{C}^p)^\vee\boxtimes\mathbb{C}^q\simeq(\mathbb{C}\boxtimes(\mathbb{C}^{p-1})^\vee\boxtimes\mathbb{C}^q)\oplus(\mathbb{C}_{-1}\boxtimes\mathbb{C}\boxtimes\mathbb{C}^q),$$

where $\boxtimes$ stands for the outer tensor product representation. Therefore, for $a=a_1+a_2$,

$$\begin{aligned} &\mathrm{Hom}_{\mathfrak{k}^\tau}(S^a(\mathfrak{n}_-^{-\tau}),S^{a_1}(\mathfrak{n}_-^\tau)\otimes S^{a_2}(\mathfrak{n}_-^{-\tau})) \\ \simeq\ &\mathrm{Hom}_{\mathfrak{gl}_1(\mathbb{C})}(\mathbb{C}_{-a},\mathbb{C}_{-a_2})\otimes\mathrm{Hom}_{\mathfrak{gl}_{p-1}(\mathbb{C})}(\mathbb{C},S^{a_1}((\mathbb{C}^{p-1})^\vee))\otimes\mathrm{Hom}_{\mathfrak{gl}_q(\mathbb{C})}(S^a(\mathbb{C}^q),S^{a_2}(\mathbb{C}^q)) \end{aligned}$$

is not reduced to zero if and only if $a_1=0$ and $a_2=a$. Thus, the condition (5.5) is satisfied.

Example 5.9. *Let $G=SO(2,2n)$, $G^\tau=U(1,n)$ and $\mathfrak{k}^\tau=\mathfrak{gl}_1(\mathbb{C})\oplus\mathfrak{gl}_n(\mathbb{C})$. Then the decomposition $\mathfrak{n}_-=\mathfrak{n}_-^\tau\oplus\mathfrak{n}_-^{-\tau}$ as a $\mathfrak{k}^\tau$-module amounts to*

$$\mathbb{C}_{-1}\boxtimes\mathbb{C}^{2n}\simeq(\mathbb{C}_{-1}\boxtimes\mathbb{C}^n)\oplus(\mathbb{C}_{-1}\boxtimes(\mathbb{C}^n)^\vee).$$

Therefore, for $a=a_1+a_2$, we have

$$\begin{aligned} &\mathrm{Hom}_{\mathfrak{k}^\tau}(S^a(\mathfrak{n}_-^{-\tau}),S^{a_1}(\mathfrak{n}_-^\tau)\otimes S^{a_2}(\mathfrak{n}_-^{-\tau})) \\ \simeq\ &\mathrm{Hom}_{\mathfrak{gl}_1(\mathbb{C})}(\mathbb{C}_{-a},\mathbb{C}_{-a_1-a_2})\otimes\mathrm{Hom}_{\mathfrak{gl}_n(\mathbb{C})}(S^a((\mathbb{C}^n)^\vee),S^{a_1}(\mathbb{C}^n)\otimes S^{a_2}((\mathbb{C}^n)^\vee)) \\ \simeq\ &\bigoplus_{b=0}^{\min(a_1,a_2)}\mathrm{Hom}_{\mathfrak{gl}_n(\mathbb{C})}(F(\mathfrak{gl}_n(\mathbb{C}),(0,\cdots,0,-a)),F(\mathfrak{gl}_n(\mathbb{C}),(a_1-b,0,\cdots,0,-a_2+b))), \end{aligned}$$

where the second isomorphism follows from Pieri's rule. Thus, the left-hand side is not reduced to zero if and only if $a_1=0$ and $a_2=a$. Hence, the condition (5.5) is satisfied.

Example 5.10. *Let* $G = SO^*(2n)$, $G^\tau = SO^*(2n-2) \times SO(2)$ *and* $\mathfrak{k}^\tau = \mathfrak{gl}_{n-1}(\mathbb{C}) \oplus \mathfrak{gl}_1(\mathbb{C})$. *In this case, the decomposition* $\mathfrak{n}_- = \mathfrak{n}_-^\tau \oplus \mathfrak{n}_-^{-\tau}$ *as a* $\mathfrak{k}^\tau$*-module amounts to*

$$(\mathrm{Alt}(\mathbb{C}^{n-1})^\vee \boxtimes \mathbf{1}) \oplus ((\mathbb{C}^{n-1})^\vee \boxtimes \mathbb{C}_{-1}).$$

Therefore, for $a = a_1 + a_2$

$$\begin{aligned} & \mathrm{Hom}_{\mathfrak{k}^\tau}(S^a(\mathfrak{n}_-^{-\tau}), S^{a_1}(\mathfrak{n}_-^\tau) \otimes S^{a_2}(\mathfrak{n}_-^{-\tau})) \\ \simeq\ & \mathrm{Hom}_{\mathfrak{gl}_{n-1}(\mathbb{C})}(S^a((\mathbb{C}^{n-1})^\vee), S^{a_1}(\mathrm{Alt}(\mathbb{C}^{n-1})^\vee) \otimes S^{a_2}((\mathbb{C}^{n-1})^\vee)) \otimes \mathrm{Hom}_{\mathfrak{gl}_1(\mathbb{C})}(\mathbb{C}_{-a}, \mathbb{C}_{-a_2}). \end{aligned}$$

In view of the $\mathfrak{gl}_1(\mathbb{C})$*-action on the right-hand side, it is non-zero only if* $a_2 = a$ *(and therefore* $a_1 = 0$*). Thus the condition* (5.5) *is satisfied.*

Hence we have verified the assumption (5.5) for all the three symmetric pairs $(\mathfrak{g}(\mathbb{R}), \mathfrak{g}(\mathbb{R})^\tau)$ corresponding to the three complex geometries (4), (5) and (6) in Table 1.1, and have proved the implication (iii) ⇒ (i) in Theorem 5.3 by Lemma 5.5 (3).

6. Symmetry breaking operators for the restriction $SO(n,2) \downarrow SO(n-1,2)$

Let $n \geq 3$. In what follows, we realize the indefinite orthogonal group $SO(n,2)$ in a slightly non-standard way, namely, use a non-degenerate quadratic form on $\mathbb{C}^{n+2}$ defined by

$$\widetilde{Q}(w) := w_0^2 + \cdots + w_n^2 - w_{n+1}^2 \quad \text{for } w = (w_0, \cdots, w_{n+1}) \in \mathbb{C}^{n+2},$$

and restrict it to a certain real form $E(\mathbb{R})$ (see (6.3) below) of $\mathbb{C}^{n+2}$. (The restriction to the standard real form $\mathbb{R}^{n+2}$ yields conformally covariant differential operators corresponding to another pair of real forms $(SO(n+1,1), SO(n,1))$, see Remark 6.13.)

Let $G_\mathbb{C}$ be the complex special orthogonal group $SO(\mathbb{C}^{n+2}, \widetilde{Q})$ with respect to the quadratic form $\widetilde{Q}$. Then $G_\mathbb{C}$ acts transitively on the isotropic cone

$$\Xi_\mathbb{C} := \{w \in \mathbb{C}^{n+2} \setminus \{0\} : \widetilde{Q}(w) = 0\},$$

and also on the complex quadric

$$\mathrm{Q}^n\mathbb{C} := \Xi_\mathbb{C}/\mathbb{C}^* \subset \mathbb{P}^{n+1}\mathbb{C}$$

by $w \to g \cdot [w] := [gw]$ for $w \in \mathbb{C}^{n+1} \setminus \{0\}$. Let $w_o = {}^t(1, 0, \cdots, 0, 1) \in \Xi_\mathbb{C}$, and $P_\mathbb{C}$ be the stabilizer of the base point $[w_o] = [1 : 0 : \cdots : 0 : 1] \in Q^n\mathbb{C}$, which is a maximal parabolic subgroup of $G_\mathbb{C}$. Then we have an isomorphism $Q^n\mathbb{C} \simeq G_\mathbb{C}/P_\mathbb{C}$. We define an embedding

$$\iota : \mathbb{C}^n \to \Xi_\mathbb{C}, \quad z \mapsto {}^t(1 - Q_n(z), 2z_1, \cdots, 2z_n, 1 + Q_n(z)), \tag{6.1}$$

where $Q_n(z) := \sum_{j=1}^n z_j^2$ for $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$. Then we get coordinates on $Q^n\mathbb{C}$ by

$$\mathbb{C}^n \hookrightarrow \mathrm{Q}^n\mathbb{C}, \quad z \mapsto [\iota(z)] \tag{6.2}$$

which define the open Bruhat cell (see (6.7) below).

The quadratic form $\widetilde{Q}$ is of signature $(n, 2)$ when restricted to the real vector space

$$E(\mathbb{R}) := \sqrt{-1}\mathbb{R}e_0 + \sum_{j=1}^{n+1} \mathbb{R}e_j, \tag{6.3}$$

where $\{e_j : 0 \le j \le n+1\}$ is the standard basis in $\mathbb{C}^{n+2}$. Thus we have an isomorphism:

$$SO(\mathbb{C}^{n+2}, \widetilde{Q}) \cap GL_{\mathbb{R}}(E(\mathbb{R})) \simeq SO(n, 2).$$

Let G be its identity component $SO_o(n, 2)$. Then the G-orbit through the base point $[w_o]$ in $Q^n\mathbb{C}$ is still contained in $\mathbb{C}^n$, and is identified with the Lie ball $X := \{z \in \mathbb{C}^n : |z\,{}^tz|^2 + 1 - 2\overline{z}\,{}^tz > 0, |z\,{}^tz| < 1\} \simeq G/K$ which is the bounded Hermitian symmetric domain of type IV in the É. Cartan classification.

Let τ be the involution of $GL(n+1, \mathbb{C})$ by conjugation by $\operatorname{diag}(1, \ldots, 1, -1, 1)$. It leaves G invariant, and we denote by G' the identity component of the fixed point group G^τ. The group $G' = SO_o(n-1, 2)$ acts on the subsymmetric domain

$$Y := X \cap \{z_n = 0\}.$$

Then $Y \simeq G'/K' = SO_o(n-1, 2)/SO(n-1) \times SO(2)$ a subsymmetric space of X of complex codimension one.

We take $H_o := E_{0,n+1} + E_{n+1,0}$.Then H_o is a characteristic element as in Section 2.1. For $\lambda \in \mathbb{Z}$ we define a character of $\mathfrak{c}(\mathfrak{k})$ by $tH_o \mapsto \lambda t$, and lift it to a character $\mathbb{C}_\lambda$ of K. Let $\mathcal{L}_\lambda$ be the G-equivariant holomorphic line bundle $G \times_K \mathbb{C}_\lambda$. The holomorphic line bundle $\mathcal{L}_\lambda \to X$ is trivialized by using the open Bruhat cell, and the representation of G on $\mathcal{O}(X, \mathcal{L}_\lambda)$ is identified with the multiplier representation $\pi_\lambda \equiv \pi_\lambda^G$ of the same group on $\mathcal{O}(X)$ given by

$$F(z) \mapsto (\pi_\lambda(g)F)(z) = J(g^{-1}, z)^{-\lambda} F(g^{-1} \cdot z), \tag{6.4}$$

where we define a map $J : G \times X \to \mathbb{C}^*$ by

$$J(g, z) := \frac{1}{2}{}^t w_o g \iota(z), \quad \text{for } g \in G \text{ and } z \in X.$$

Since $H_o \in \mathfrak{k}'$ (see (2.1)), we can also define a G'-equivariant holomorphic line bundle $\mathcal{L}_\nu = G' \times_{K'} \mathbb{C}_\nu$ over $Y = G'/K'$ for $\nu \in \mathbb{Z}$.

Let $\widetilde{G}$ be the universal covering group of $G = SO_o(n, 2)$. Then for any $\lambda \in \mathbb{C}$ one can define a $\widetilde{G}$-equivariant holomorphic line bundle $\mathcal{L}_\lambda = \widetilde{G} \times_{\widetilde{K}} \mathbb{C}_\lambda$ over $X = G/K \simeq \widetilde{G}/\widetilde{K}$, and a representation of the same group on $\mathcal{O}(X, \mathcal{L}_\lambda)$. Similarly, for $\nu \in \mathbb{C}$, the universal covering group $\widetilde{G}'$ of $G' = SO_o(n-1, 2)$ acts on $\mathcal{O}(Y, \mathcal{L}_\nu)$.

Here is a complete classification of symmetry breaking operators from $\mathcal{O}(X,\mathcal{L}_\lambda)$ to $\mathcal{O}(Y,\mathcal{L}_\nu)$ with respect to the symmetric pair $\widetilde{G} \supset \widetilde{G}'$:

Theorem 6.1. *Let $n \geq 3$ and $\widetilde{G}'$ be the universal covering group of $SO_o(n-1,2)$. Suppose $\lambda, \nu \in \mathbb{C}$. Then the following three conditions on the parameters $(\lambda,\nu) \in \mathbb{C}^2$ are equivalent:*

(i) $\mathrm{Hom}_{\widetilde{G}'}(\mathcal{O}(X,\mathcal{L}_\lambda), \mathcal{O}(Y,\mathcal{L}_\nu)) \neq \{0\}$.
(ii) $\dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{G}'}(\mathcal{O}(X,\mathcal{L}_\lambda), \mathcal{O}(Y,\mathcal{L}_\nu)) = 1$.
(iii) $\nu - \lambda \in \mathbb{N}$.

Remark 6.2. The equivalence (i)⇔(ii) in Theorem 6.1 is not true for singular parameters (λ,ν) in the case of $n = 2$. This situation will be treated carefully in Section 9. In fact, the symmetric pair $(SO_o(2,2), SO_o(2,1))$ is locally isomorphic to the pair $(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}), \Delta(SL(2,\mathbb{R}))$ modulo the center. We note that $n = 2$ in Theorem 6.1 corresponds to $\lambda' = \lambda''$ in Theorem 9.1.

Let $\widetilde{C}_\ell^\alpha(x)$ be the renormalized Gegenbauer polynomial (see Appendix 11.3). We inflate it to a polynomial of two variables x and y:

$$\widetilde{C}_\ell^\alpha(x,y) := x^{\frac{\ell}{2}} \widetilde{C}_\ell^\alpha\left(\frac{y}{\sqrt{x}}\right) = \sum_{k=0}^{[\frac{\ell}{2}]} (-1)^k \frac{\Gamma(\ell-k+\alpha)}{\Gamma\left(\alpha+\left[\frac{\ell+1}{2}\right]\right)\Gamma(k+1)\Gamma(\ell-2k+1)} (2y)^{\ell-2k} x^k. \tag{6.5}$$

For instance, $\widetilde{C}_0^\alpha(x,y) = 1$, $\widetilde{C}_1^\alpha(x,y) = 2y$, $\widetilde{C}_2^\alpha(x,y) = 2(\alpha+1)y^2 - x$, etc. Notice that $\widetilde{C}_\ell^\alpha(x^2,y)$ is a homogeneous polynomial of x and y of degree ℓ.

Theorem 6.3. *Retain the setting of Theorem 6.1. Let $a := \nu - \lambda \in \mathbb{N}$. Then the differential operator from $\mathcal{O}(X)$ to $\mathcal{O}(Y)$ defined by*

$$D_{X\to Y,a} := \widetilde{C}_a^{\lambda-\frac{n-1}{2}}\left(-\Delta_{\mathbb{C}^{n-1}}^z, \frac{\partial}{\partial z_n}\right) \tag{6.6}$$

intertwines the restriction $\pi_\lambda^{\widetilde{G}}\big|_{\widetilde{G}'}$ with $\pi_{\lambda+a}^{\widetilde{G}'}$ (see (6.4)). Here $\Delta_{\mathbb{C}^m}^z := \sum_{k=1}^m \frac{\partial^2}{\partial z_k^2}$ denotes the holomorphic Laplacian on $\mathbb{C}^m$ in the coordinates $(z_1,\cdots,z_m)$.

It follows from Theorems 6.1 and 6.3 that any symmetry breaking operator from $\mathcal{O}(X,\mathcal{L}_\lambda)$ to $\mathcal{O}(Y,\mathcal{L}_{\lambda+a})$ is proportional to $D_{X\to Y,a}$ for any $\lambda \in \mathbb{C}$ and $a \in \mathbb{N}$.

Remark 6.4. If $\lambda \in \mathbb{R}$ and $\lambda > n-1$, then $\mathcal{H}^2(X,\mathcal{L}_\lambda) := \mathcal{O}(X,\mathcal{L}_\lambda) \cap L^2(X,\mathcal{L}_\lambda)$ is a non-zero Hilbert space on which $\widetilde{G}$ acts unitarily and irreducibly, giving a holomorphic discrete series representation of $\widetilde{G}$ modulo the center. By [KP14-1, Theorem **??**]

the same statement as Theorems 6.1 and 6.3 remains true for symmetry breaking operators between the unitary representations $\mathcal{H}^2(X, \mathcal{L}_\lambda)$ and $\mathcal{H}^2(Y, \mathcal{L}_{\lambda+a})$.

In order to prove Theorems 6.1 and 6.3 we apply the F-method (see Section 3.1). The Lie algebra $\mathfrak{g} = \mathfrak{so}(\mathbb{C}^{n+2}, \widetilde{Q})$ has a direct sum decomposition

$$\mathfrak{g} = \mathfrak{n}_- + \mathfrak{k} + \mathfrak{n}_+$$

of $-1, 0$, and 1 eigenspaces of $\mathrm{ad}(H_o)$, respectively. Then the maximal parabolic subgroup $P_\mathbb{C}$ has a Levi decomposition $P_\mathbb{C} = K_\mathbb{C} N_{+,\mathbb{C}}$, where $N_{+,\mathbb{C}} = \exp \mathfrak{n}_+$.

As Step 1 of the F-method we define a standard basis of $\mathfrak{n}_+ \simeq \mathbb{C}^n$ by

$$C_j \quad := \quad E_{j,0} - E_{j,n+1} - E_{0,j} - E_{n+1,j} \quad (1 \leq j \leq n),$$

and similarly a standard basis of $\mathfrak{n}_- \simeq \mathbb{C}^n$ by

$$\overline{C}_j \quad := \quad E_{j,0} + E_{j,n+1} - E_{0,j} + E_{n+1,j} \quad (1 \leq j \leq n).$$

Then the decomposition $\mathfrak{n}_+ = \mathfrak{n}_+^\tau \oplus \mathfrak{n}_+{-\tau}$ is given by

$$\mathfrak{n}_+ = \sum_{j=1}^{n-1} \mathbb{C} C_j \oplus \mathbb{C} C_n.$$

Let $Z = \sum_{i=1}^n z_i \overline{C}_i \in \mathfrak{n}_-$ and $Y = \sum_{j=1}^n y_j C_j \in \mathfrak{n}_+$. By a simple computation we have

$$\exp(Z) \cdot w_o = \iota(z) \in \mathbb{C}^{n+2}, \tag{6.7}$$

the open Bruhat cell is given by (6.2). Moreover, by using

$$\exp(tY)\exp(Z) w_o = \iota(z) - 2t \begin{pmatrix} (y,z) \\ Q(z) y \\ (y,z) \end{pmatrix} + o(t),$$

we obtain formulæ of the maps (3.2) and (3.3), as

$$\begin{aligned} \alpha(Y,Z) \quad &= \quad -2(z,y) H_o \quad \mathrm{mod}\, \mathfrak{so}(n, \mathbb{C}); \\ \beta(Y,Z) \quad &= \quad 2(z,y) E_z - Q_n(z) \sum_{j=1}^n y_j \frac{\partial}{\partial z_j}, \end{aligned}$$

where we regard $\beta(Y, \cdot)$ as a holomorphic vector field on $\mathfrak{n}_-$ and recall that $E_z := \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}$, $Q_n(z) = z_1^2 + \cdots + z_n^2$ and $(z,y) = z_1 y_1 + \cdots + z_n y_n$.

Then the infinitesimal action $d\pi_{\lambda^*}(C_j)$ with

$$\lambda^* = \lambda^\vee \otimes \mathbb{C}_{2\rho} = -\lambda + n,$$

is given by

$$d\pi_{\lambda^*}(C_j) = 2(\lambda - n) z_j - 2 z_j E_z + Q_n(z) \frac{\partial}{\partial z_j}. \tag{6.8}$$

Lemma 6.5. *For* $C \in \mathbb{C}^n \simeq \mathfrak{n}_+$ *and* $\zeta \in \mathbb{C}^n \simeq \mathfrak{n}_-$ *one has,*

$$\widehat{d\pi_{\lambda^*}}(C_j) \;=\; 2\lambda \frac{\partial}{\partial \zeta_j} + 2E_\zeta \frac{\partial}{\partial \zeta_j} - \zeta_j \Delta^{\zeta}_{\mathbb{C}^n}, \qquad 1 \le j \le n,$$

where $E_\zeta := \sum_{i=1}^n \zeta_i \frac{\partial}{\partial \zeta_i}$ *and* $\Delta^{\zeta}_{\mathbb{C}^n} = \frac{\partial^2}{\partial \zeta_1^2} + \cdots \frac{\partial^2}{\partial \zeta_n^2}$.

Proof. According to Definition 3.1 we have $\widehat{z_j} = \frac{\partial}{\partial \zeta_j}$ and hence $\widehat{E_z} = -E_\zeta - n$. On the other hand, using the commutation relations of the Weyl algebra (see *e.g.* [KP14-1, (**??**)]) we get

$$\Delta^{\zeta}_{\mathbb{C}^n} \zeta_j = \zeta_j \Delta^{\zeta}_{\mathbb{C}^n} + 2\frac{\partial}{\partial \zeta_j}, \qquad \frac{\partial}{\partial \zeta_j} E_\zeta = E_\zeta \frac{\partial}{\partial \zeta_j} + \frac{\partial}{\partial \zeta_j}.$$

Thus the above formula for the algebraic Fourier transform $\widehat{d\pi_{\lambda^*}}(C_j)$ of the differential operator (6.8) follows. ☐

For Step 2 we apply Lemma 5.5 (2) and get the following.

Proposition 6.6. *Assume* $\lambda > n-1$. *If*

$$\mathrm{Hom}_{G'}(\mathcal{O}(G/K, \mathcal{L}_\lambda), \mathcal{O}(G'/K', \mathcal{W})) \neq \{0\}$$

for an irreducible representation W *of* K', *then* W *must be one-dimensional and of the form*

$$W^a_\lambda := S^a(\mathfrak{n}_-^{-\tau}) \otimes \mathbb{C}_\lambda \simeq \mathrm{Pol}^a(\mathfrak{n}_+^{-\tau}) \otimes \mathbb{C}_\lambda \tag{6.9}$$

for some $a \in \mathbb{N}$.

We denote by ν the action of K' on W^a_λ. In our setting where $\dim V = \dim W^a_\lambda = 1$ we write $\zeta = (\zeta', \zeta_n) \in \mathbb{C}^n$ with $\zeta' = (\zeta_1, \ldots, \zeta_{n-1}) \in \mathbb{C}^{n-1}$, and identify an element of $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}_\lambda, \mathrm{Pol}(\mathfrak{n}_+) \otimes W^a_\lambda)$ with a polynomial $\psi(\zeta)$ of n variables. Then, for Step 3, the condition (3.10) implies that $\psi(\zeta)$ is homogeneous of degree a and the condition (3.11) amounts to the system of differential equations:

$$\widehat{d\pi_{\lambda^*}}(C_j)\psi = \left(2\lambda \frac{\partial}{\partial \zeta_j} + 2E_\zeta \frac{\partial}{\partial \zeta_j} - \zeta_j \Delta^{\zeta}_{\mathbb{C}^n}\right)\psi = 0, \qquad 1 \le j \le n-1$$

by Lemma 6.5.

To be prepared for Step 4, observe that the $K'_{\mathbb{C}}$-action on $\mathfrak{n}_- = \mathfrak{n}_-^{\tau} \oplus \mathfrak{n}_-^{-\tau}$ is identified with the action of $SO(n-1, \mathbb{C}) \times SO(2, \mathbb{C})$ on $\mathbb{C}^n$ given as

$$\mathbb{C}^n \boxtimes \mathbb{C}_{-1} \simeq (\mathbb{C}^{n-1} \boxtimes \mathbb{C}_{-1}) \oplus (\mathbb{C} \boxtimes \mathbb{C}_{-1}).$$

Then generic $K'_{\mathbb{C}}$-orbits are of codimension one in $\mathfrak{n}_-$, and the $K'_{\mathbb{C}}$-orbit space in $\{\zeta \in \mathbb{C}^n : Q_{n-1}(\zeta') \neq 0\}$ has coordinates $\dfrac{\zeta_n^2}{Q_{n-1}(\zeta')}$.

For $a \in \mathbb{N}$, we introduce an operator T_a by

$$(T_a g)(\zeta) := Q_{n-1}(\zeta')^{\frac{a}{2}} g\left(\frac{\zeta_n}{\sqrt{Q_{n-1}(\zeta')}}\right), \tag{6.10}$$

for $g \in \mathbb{C}[t]$. We note that $T_a g$ is a (multi-valued) meromorphic function of $\zeta_1, \dots, \zeta_n$.

We set

$$\mathrm{Pol}_a[t] := \mathbb{C}\text{-span}\left\langle t^{a-i} : 0 \le i \le a \right\rangle, \tag{6.11}$$

$$\mathrm{Pol}_a[t]_{\mathrm{even}} := \mathbb{C}\text{-span}\left\langle t^{a-2j} : 0 \le j \le \left[\frac{a}{2}\right] \right\rangle. \tag{6.12}$$

Then $(T_a g)(\zeta)$ is a homogeneous polynomial of degree a if $g \in \mathrm{Pol}_a[t]_{\mathrm{even}}$.

Remark 6.7. In this section we have assumed $n \ge 3$, and therefore $Q_{n-1}(\zeta')^{\frac{1}{2}} = (\zeta_1^2 + \cdots + \zeta_{n-1}^2)^{\frac{1}{2}}$ is not a polynomial and the parity condition in (6.12) is necessary. However, for $n = 2$, $T_a g$ is a polynomial for $g \in \mathrm{Pol}_a[t]$ as we can take a branch as $Q_1(\zeta')^{\frac{1}{2}} = \zeta_1$.

The first half of Step 4 is summarized in the following lemma:

Lemma 6.8. *For $n \ge 3$ we have,*

$$\mathrm{Hom}_{\mathfrak{k}'}(\mathbb{C}_\lambda, \mathrm{Pol}(\mathfrak{n}_+) \otimes \mathbb{C}_\nu) \simeq \begin{cases} \{0\} & \text{if } \nu - \lambda \notin \mathbb{N}, \\ T_{\nu-\lambda}(\mathrm{Pol}_{\nu-\lambda}[t]_{\mathrm{even}}) & \text{if } \nu - \lambda \in \mathbb{N}. \end{cases}$$

Proof. As modules of $\mathfrak{k}' = \mathfrak{so}(n-1,\mathbb{C}) \oplus \mathfrak{so}(2,\mathbb{C})$, we have the following isomorphisms:

$$\mathrm{Pol}(\mathfrak{n}_+) \simeq S(\mathfrak{n}_-) \simeq \bigoplus_{a_1, a_2 \in \mathbb{N}} S^{a_1}(\mathfrak{n}_-^{\tau}) \otimes S^{a_2}(\mathfrak{n}_-^{-\tau}) \simeq \bigoplus_{a=0}^{\infty} \bigoplus_{a_1=0}^{a} S^{a_1}(\mathbb{C}^{n-1}) \boxtimes \mathbb{C}_{-a}.$$

Therefore

$$\mathrm{Hom}_{\mathfrak{k}'}(\mathbb{C}_\lambda, \mathrm{Pol}(\mathfrak{n}_+) \otimes \mathbb{C}_\nu) \simeq \bigoplus_{a=0}^{\infty} \bigoplus_{a_1=0}^{a} \left(S^{a_1}(\mathbb{C}^{n-1})\right)^{SO(n-1,\mathbb{C})} \boxtimes \left(\mathbb{C}_{\nu-a-\lambda}\right)^{SO(2,\mathbb{C})}.$$

The right-hand side is non-zero only when $\nu - \lambda \in \mathbb{N}$. In this case the summand is non-trivial only when $a = \nu - \lambda$. On the other hand, since $n \ge 3$, we have

$$S^{a_1}(\mathbb{C}^{n-1})^{SO(n-1,\mathbb{C})} \simeq \begin{cases} \mathbb{C} Q_{n-1}(\zeta')^{\frac{a_1}{2}} & \text{if } a_1 \in 2\mathbb{N}, \\ 0 & \text{if } a_1 \notin 2\mathbb{N}. \end{cases}$$

Hence the lemma follows. $\square$

To implement the second part of Step 4 we apply Proposition 3.3 to the map (6.10). For this we collect some formulæ for saturated differential operators that we shall use later.

Lemma 6.9. *For every $0 \leq j \leq n-1$ one has:*

$$T_a^\sharp\left(\zeta_j E_{\zeta'} - Q_{n-1}(\zeta')\frac{\partial}{\partial\zeta_j}\right) = 0, \tag{6.13}$$

$$T_a^\sharp\left((a-1)\zeta_n - E_\zeta \frac{\partial}{\partial\zeta_j}\right) = 0. \tag{6.14}$$

Proof. The proof of both statements is straightforward from the definition of T_a. □

Lemma 6.10. *Let T_a be the operator defined in* (6.10). *We write $\zeta' = (\zeta_1, \cdots, \zeta_{n-1})$ and $\vartheta_t := t\frac{d}{dt}$. One then has:*

(1) $T_a^\sharp(E_{\zeta'}) = a - \vartheta_t$.
(2) $T_a^\sharp\left(\frac{Q_{n-1}(\zeta')}{\zeta_j}\frac{\partial}{\partial\zeta_j}\right) = a - \vartheta_t$, $(1 \leq j \leq n-1)$.
(3) $T_a^\sharp\left(\frac{Q_{n-1}(\zeta')}{\zeta_j}E_\zeta\frac{\partial}{\partial\zeta_j}\right) = (a-1)(a-\vartheta_t)$, $(1 \leq j \leq n-1)$.
(4) $T_a^\sharp(\zeta_n^2 \Delta^\zeta_{\mathbb{C}^{n-1}}) = t^2(\vartheta_t - a)(\vartheta_t - n - a + 3)$.
(5) $T_a^\sharp(Q_{n-1}(\zeta')\Delta^\zeta_{\mathbb{C}^{n-1}}) = (\vartheta_t - a)(\vartheta_t - n - a + 3)$.
(6) $T_a^\sharp(Q_{n-1}(\zeta')\frac{\partial^2}{\partial\zeta_n^2}) = t^{-2}(\vartheta_t^2 - \vartheta_t)$.
(7) $T_a^\sharp(\zeta_n\frac{\partial}{\partial\zeta_n}) = \vartheta_t$.
(8) $T_a^\sharp(\zeta_n^2\frac{\partial^2}{\partial\zeta_n^2}) = \vartheta_t^2 - \vartheta_t$.

Proof. Notice first that the identity (1) is equivalent to (2) according to (6.13) and that the identity (3) may be deduced from (1) or (2) by (6.14). Furthermore, identities (4) and (5) on the one hand and (6) and (8) on the other are equivalent according to the definition of the T-saturation as $t = \frac{\zeta_n}{\sqrt{Q_{n-1}(\zeta')}}$.

Thus, it would be enough to show the identities (1), (4), (7) and (8). We give a proof for the first statement, and the remaining cases can be treated in a similar way. Let $1 \leq j \leq n-1$. Then

$$\begin{aligned}
&\left(T_a^\sharp(E_{\zeta'})g\right)(t) = \sum_{j=1}^{n-1}\zeta_j\frac{\partial}{\partial\zeta_j}\left(Q_{n-1}(\zeta')^{\frac{a}{2}}g\left(\frac{\zeta_n}{\sqrt{Q_{n-1}(\zeta')}}\right)\right)\\
= \; & aQ_{n-1}(\zeta')^{\frac{a}{2}-1}g\left(\frac{\zeta_n}{\sqrt{Q_{n-1}(\zeta')}}\right)\sum_{j=1}^{n-1}\zeta_j^2 - Q_{n-1}(\zeta')^{\frac{a}{2}}g'\left(\frac{\zeta_n}{\sqrt{Q_{n-1}(\zeta')}}\right)\sum_{j=1}^{n-1}\frac{\zeta_j^2\zeta_n}{\sqrt{Q_{n-1}^3(\zeta')}}\\
= \; & aQ_{n-1}(\zeta')^{\frac{a}{2}}g\left(\frac{\zeta_n}{\sqrt{Q_{n-1}(\zeta')}}\right) - \frac{\zeta_n}{\sqrt{Q_{n-1}(\zeta')}}Q_{n-1}(\zeta')^{\frac{a}{2}}g'\left(\frac{\zeta_n}{\sqrt{Q_{n-1}(\zeta')}}\right)\\
= \; & \left(a - t\frac{d}{dt}\right)g(t).
\end{aligned}$$

□

For the second half of Step 4 we apply the idea of T-saturated differential operators (see Definition 3.2). Although the differential operator $\widehat{d\pi_{\lambda^*}}(C_j)$ itself is not T_a-saturated, we shall see that $Q_j\widehat{d\pi_{\lambda^*}}(C_j)$ is T_a-saturated if we set $Q_j = \zeta_j^{-1}Q_{n-1}(\zeta')$. In the following lemma, we note that the right-hand side is independent of j.

Lemma 6.11. *The T_a-saturation of the differential operators $\widehat{d\pi_{\lambda^*}}(C_j)$ with $C_j \in \mathfrak{n}_+^\tau$ is given for any $1 \le j \le n-1$ by*

$$T_a^\sharp\left(\frac{Q_{n-1}(\zeta')}{\zeta_j}\widehat{d\pi_{\lambda^*}}(C_j)\right) = \frac{-1}{t^2}\left((1+t^2)\vartheta_t^2 - (1-(2\lambda-n+1)t^2)\vartheta_t - a(a+2\lambda-n+1)t^2\right).$$

Proof. Suppose $1 \le j \le n-1$. Applying (2), (3) and (5), (6) of Lemma 6.10, respectively, we have following identities:

$$\begin{aligned}
T_a^\sharp\left(\frac{Q_{n-1}(\zeta')}{\zeta_j}\frac{\partial}{\partial\zeta_j}\right) &= a-\theta_t,\\
T_a^\sharp\left(\frac{Q_{n-1}(\zeta')}{\zeta_j}E_\zeta\frac{\partial}{\partial\zeta_j}\right) &= (a-1)(a-\vartheta_t),\\
T_a^\sharp\left(\frac{Q_{n-1}(\zeta')}{\zeta_j}\zeta_j\Delta_{\mathbb{C}^n}^\zeta\right) &= T_a^\sharp\left(Q_{n-1}(\zeta')\left(\Delta_{\mathbb{C}^{n-1}}^\zeta + \frac{\partial^2}{\partial\zeta_n^2}\right)\right)\\
&= (\vartheta_t-a)(\vartheta_t-n+3-a)+t^{-2}(\vartheta_t^2-\vartheta_t).
\end{aligned}$$

We recall from Lemma 6.5 that $\widehat{d\pi_{\lambda^*}}(C_j) = 2\lambda\frac{\partial}{\partial\zeta_j} + 2E_\zeta\frac{\partial}{\partial\zeta_j} - \zeta_j\Delta_{\mathbb{C}^n}^\zeta$. Summing up these terms we get the lemma. □

Proposition 6.12. *Let $a \in \mathbb{N}$, and T_a be as in* (6.10). *The polynomial $\psi(\zeta) = (T_a g)(\zeta)$ of n variables satisfies the system of partial differential equations* (3.11) *if and only if $g(t)$ satisfies the following single ordinary differential equation:*

$$\left((1-s^2)\vartheta_s^2 - (1+(2\lambda-n+1)s^2)\vartheta_s + a(a+2\lambda-n+1)s^2\right)g(-\sqrt{-1}s) = 0, \tag{6.15}$$

or equivalently, $g(t)$ is proportional to the normalized Gegenbauer polynomial $\widetilde{C}_a^{\lambda-\frac{n-1}{2}}(\sqrt{-1}t)$. (For the Gegenbauer polynomial, see Section 11.3.)

Proof. The statement follows from Lemma 6.11 after the change of variable $t = -\sqrt{-1}s$. □

We have carried out the crucial part of the F-method. Let us complete the proof of Theorems 6.1 and 6.3.

Proof of Theorems 6.1 and 6.3. By the general result of the F-method (see Theorem 3.1), the symbol map of differential operators gives an isomorphism

$$\operatorname{Hom}_{\widetilde{G}'}(\mathcal{O}(X,\mathcal{L}_\lambda),\mathcal{O}(Y,\mathcal{L}_\nu)) \overset{\text{Symb}}{\underset{\sim}{\to}} \operatorname{Hom}_{\mathfrak{k}'}(\mathbb{C}_\lambda, \operatorname{Pol}(\mathfrak{n}_+)\otimes\mathbb{C}_\nu)^{\widehat{d\pi_{\lambda^*}}(\mathfrak{n}'_+)}.$$

By Lemma 6.8, the right-hand side is reduced to zero if $\nu - \lambda \notin \mathbb{N}$. From now on, we assume $a := \nu - \lambda \in \mathbb{N}$, and identify the right-hand side with a subspace of $\mathrm{Pol}(\mathfrak{n}_+)$. Then it follows from Lemma 6.8 and Proposition (6.12) that the bijections

$$\begin{array}{ccccc} \mathrm{Pol}_a[s]_{\mathrm{even}} & \overset{T_a}{\underset{\sim}{\longrightarrow}} & \mathrm{Pol}_a[t]_{\mathrm{even}} & \overset{\sim}{\longrightarrow} & \mathrm{Hom}_{\mathfrak{k}'}(\mathbb{C}_\lambda, \mathrm{Pol}(\mathfrak{n}_+) \otimes \mathbb{C}_\nu) \\ h(s) & \mapsto & g(t) = h(\sqrt{-1}t) & \mapsto & Q_{n-1}(\zeta')^{\frac{a}{2}} g\left(\dfrac{\zeta_n}{\sqrt{Q_{n-1}(\zeta')}}\right) \end{array}$$

induces an isomorphism

$$\mathrm{Sol}_{\mathrm{Gegen}}\left(\lambda - \frac{n-1}{2}, a\right) \cap \mathrm{Pol}_a[s]_{\mathrm{even}} \overset{\sim}{\longrightarrow} \mathrm{Hom}_{\mathfrak{k}'}(\mathbb{C}_\lambda, \mathrm{Pol}(\mathfrak{n}_+) \otimes \mathbb{C}_\nu)^{\widehat{d\pi_{\lambda^*}}(\mathfrak{n}'_+)}.$$

Since the left-hand side is always one-dimensional (see Theorem 11.4 in Appendix), the first statement follows.

Furthermore, since $\mathrm{Sol}_{\mathrm{Gegen}}\left(\lambda - \frac{n-1}{2}, a\right) \cap \mathrm{Pol}_a[s]_{\mathrm{even}}$ is spanned by $\widetilde{C}_a^{\lambda - \frac{n-1}{2}}(s)$ by Theorem (11.4) (2), the space $\mathrm{Hom}_{\widetilde{G}'}(\mathcal{O}(X, \mathcal{L}_\lambda), \mathcal{O}(Y, \mathcal{L}_\nu))$ is spanned by

$$\mathrm{Symb}^{-1} \circ T_a \widetilde{C}_a^{\lambda - \frac{n-1}{2}}(\sqrt{-1}t) = (-1)^{-\frac{a}{2}} \widetilde{C}_a^{\lambda - \frac{n-1}{2}}\left(-\Delta^z_{\mathbb{C}^{n-1}}, \frac{\partial}{\partial z_n}\right).$$

Hence Theorems 6.1 and 6.3 are proved. □

Remark 6.13. Theorem 6.3 is a "holomorphic version" of the conformally covariant operator considered by A. Juhl [J09] in the setting $S^{n-1} \hookrightarrow S^n$, with equivariant actions of the pair of groups $SO(n,1) \subset SO(n+1,1)$, respectively. Our proof based on the F-method is much shorter than the original proof in [J09, Chapter 6] that relies on combinatorial argument using recurrence relations of the coefficients of differential operators. The F-method gives a conceptual explanation for the appearance of Gegenbauer polynomials in Theorem 6.3. The relationship of symmetry breaking operators between real flag varieties (*e.g.* [J09, KØSS13]) and the holomorphic setting is illustrated by an SL_2-example in [KKP15].

7. Symmetry breaking operators for the restriction $Sp(n,\mathbb{R}) \downarrow Sp(n-1,\mathbb{R}) \times Sp(1,\mathbb{R})$

Let $n \geq 2$. In what follows, we realize the real symplectic group $G = Sp(n,\mathbb{R})$ as a subgroup of the indefinite unitary group $U(n,n)$, so that we can directly apply the computation of $d\pi_{\lambda^*}(C)$ ($C \in \mathfrak{n}_+$) in [KP14-1, Example **??**].

Let $G_{\mathbb{C}}$ be the complex symplectic group $Sp(n,\mathbb{C})$ which preserves the standard symplectic form ω defined on $\mathbb{C}^{2n}$ by

$$\omega(u,v) := {}^t u J_n v, \quad \text{for } u, v \in \mathbb{C}^{2n},$$

where $J_n := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. Let $E(\mathbb{R}) := \left\{ \begin{pmatrix} z \\ \bar{z} \end{pmatrix} : z \in \mathbb{C}^n \right\}$ be a totally real vector subspace of $\mathbb{C}^{2n}$, and we set

$$G := GL_{\mathbb{R}}(E(\mathbb{R})) \cap Sp(n, \mathbb{C}) \simeq Sp(n, \mathbb{R}).$$

Then the Lie algebra $\mathfrak{g}(\mathbb{R}) \simeq \mathfrak{sp}(n, \mathbb{R})$ of G is given by

$$\mathfrak{g}(\mathbb{R}) = \mathfrak{gl}_{\mathbb{R}}(E(\mathbb{R})) \cap \mathfrak{sp}(n, \mathbb{C}) = \left\{ \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} : A = -{}^t\overline{A}, B \in \mathrm{Sym}(n, \mathbb{C}) \right\},$$

where we recall that $\mathrm{Sym}(n, \mathbb{C})$ is the space of complex symmetric matrices.

Let $H_n := \{Z \in \mathrm{Sym}(n, \mathbb{C}) : \|Z\|_{\mathrm{op}} < 1\}$ be the bounded symmetric domain of type CI in the É. Cartan classification, where $\|Z\|_{\mathrm{op}}$ denotes the operator norm of $Z \in \mathrm{End}(\mathbb{C}^n)$. The Lie group $G = Sp(n, \mathbb{R})$ acts biholomorphically on H_n by

$$g \cdot Z = (aZ + b)(cZ + d)^{-1} \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \, Z \in H_n.$$

The isotropy subgroup K of G at the origin 0 is identified with $U(n)$ by the isomorphism:

$$K \xrightarrow{\sim} U(n), \quad \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \mapsto A.$$

We write $\widetilde{G}$ for the universal covering of G, and $\widetilde{K}$ for the connected subgroup with Lie algebra $\mathfrak{k}(\mathbb{R})$.

Let G' be the subgroup of $G = Sp(n, \mathbb{R})$ that preserves the direct sum decomposition $E(\mathbb{R}) \simeq \mathbb{R}^{2n} = \mathbb{R}^{2n-2} \oplus \mathbb{R}^2$ in the standard coordinates. Then G' is isomorphic to the connected group $Sp(n-1, \mathbb{R}) \times Sp(1, \mathbb{R})$. The pair (G, G') is a symmetric pair as G' is the fixed point subgroup of an involution τ of G defined by

$$\tau(g) = \begin{pmatrix} I_{n-1,1} & 0 \\ 0 & I_{n-1,1} \end{pmatrix} g \begin{pmatrix} I_{n-1,1} & 0 \\ 0 & I_{n-1,1} \end{pmatrix},$$

where $I_{n-1,1} = \mathrm{diag}(1, \cdots, 1, -1)$.

We set $X := H_n \simeq G/K$ and $Y := X \cap \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a \in \mathrm{Sym}(n-1, \mathbb{C}), d \in \mathbb{C} \right\} \simeq H_{n-1} \times H_1 \simeq G'/K'$. The symmetric pair (G, G') is of holomorphic type, and the embedding of the complex manifold $Y \hookrightarrow X$ is G'-equivariant.

Let $\mathfrak{j}$ be the standard Cartan subalgebra $\sum_{i=1}^n \mathbb{C}(E_{ii} - E_{n+i,n+i})$ of $\mathfrak{k}$, and $\{e_1, \cdots, e_n\}$ the standard basis. Then $\mathfrak{j}$ is a Cartan subalgebra of $\mathfrak{g}$ and we choose $\Delta^+(\mathfrak{k}, \mathfrak{j}) = \{e_i - e_j : 1 \le i < j \le n\}$ and $\Delta(\mathfrak{n}_+, \mathfrak{j}) = \{-(e_i + e_j) : 1 \le i \le j \le n\}$ so that $\rho_{\mathfrak{g}} = (-1, -2, \cdots, -n)$.

Then we have the following decomposition of the Lie algebra

$$\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C}) = \mathfrak{n}_- + \mathfrak{k} + \mathfrak{n}_+, \quad \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \mapsto (B, A, C)$$

with $B = {}^tB$ and $C = {}^tC$. Here we have chosen a realization of $\mathfrak{n}_+$ in the *lower* triangular matrices. Accordingly, we adopt the following notation for characters of $\mathfrak{k} \simeq \mathfrak{gl}_n(\mathbb{C})$: for $\lambda \in \mathbb{C}$ the character $\mathbb{C}_\lambda$ of $\mathfrak{k}$ is defined by:

$$\mathfrak{k} \longrightarrow \mathbb{C}, \quad \begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix} \mapsto -\lambda \operatorname{Trace} A.$$

Its restriction to $\mathfrak{j}$ is given by $(-\lambda, \cdots, -\lambda) \in \mathfrak{j}^\vee \simeq \mathbb{C}^n$.

For $\lambda \in \mathbb{C}$, the character $\mathbb{C}_\lambda$ lifts to $\widetilde{K}$ and defines a $\widetilde{G}$-equivariant holomorphic line bundle $\mathcal{L}_\lambda$ over $X = \widetilde{G}/\widetilde{K} \simeq G/K$. It descends to a G-equivariant bundle if $\lambda \in \mathbb{Z}$. In our parametrization, $\mathcal{L}_{n+1}$ is the canonical line bundle of $X = G/K$, namely, $\mathbb{C}_{2\rho} = \mathbb{C}_{n+1}$.

We shall construct differential symmetry breaking operators from $\mathcal{O}(X, \mathcal{L}_\lambda)$ to $\mathcal{O}(Y, \mathcal{W}_Y)$ where $\mathcal{W}_Y$ is a G'-equivariant holomorphic vector bundle over Y. Unlike in the previous section, we have to deal with vector bundles rather than line bundles because, by Proposition 7.4 below, there exists a non-trivial G'-intertwining operator from $\mathcal{O}(X, \mathcal{L}_\lambda)$ to $\mathcal{O}(Y, \mathcal{W}_Y)$ only if $\dim W > 1$ for generic λ except for the case when $\mathcal{W}_Y = \mathcal{L}_\lambda|_Y$ or $n = 2$.

More precisely, such an irreducible representation W of $\mathfrak{k}' \simeq \mathfrak{gl}_{n-1}(\mathbb{C}) \oplus \mathfrak{gl}_1(\mathbb{C})$ must be isomorphic to

$$W_\lambda^a = F(\mathfrak{gl}_{n-1}(\mathbb{C}), (-\lambda, \cdots, -\lambda, -\lambda - a)) \boxtimes F(\mathfrak{gl}_1(\mathbb{C}), (-\lambda - a)e_n), \tag{7.1}$$

for some $a \in \mathbb{N}$. This is a representation of $K' = GL(n-1, \mathbb{C}) \times GL(1, \mathbb{C})$ on the space $\operatorname{Pol}^a[v_1, \cdots, v_{n-1}]$ of homogeneous polynomials of degree a on $\mathbb{C}^{n-1}$ twisted by the one-dimensional representation $(\det_{n-1})^{-\lambda}(\det_1)^{-\lambda-a}$ of K' where $\det_k A$ denotes the determinant of $A \in M(k, \mathbb{C})$.

In order to give a concrete model for the natural action of G on $\mathcal{O}(X, \mathcal{V})$ consider an irreducible representation ν of $U(m)$ with highest weight $(\nu_1, \cdots, \nu_m)$ acting on a finite-dimensional complex vector space W. We extend it into a holomorphic representation denoted by the same letter ν of $GL(m, \mathbb{C})$ on W. Then the holomorphic vector bundle $\mathcal{W} = Sp(m, \mathbb{R}) \times_{U(m)} W$ over H_m is trivialized using the open Bruhat cell, and the regular representation of $Sp(m, \mathbb{R})$ on $\mathcal{O}(H_m, \mathcal{W})$ is identified with the multiplier representation of the same group on $\mathcal{O}(H_m) \otimes W$ given by

$$\left(\pi^{Sp(m,\mathbb{R})}_{(\nu_1, \cdots, \nu_m)}(g) F\right)(Z) = \nu\left({}^t(cZ + d)\right) F\left((aZ + b)(cZ + d)^{-1}\right),$$

for $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(m,\mathbb{R})$, $Z \in H_m$. For $\lambda \in \mathbb{Z}$, the one-dimensional representation $\mathbb{C}_\lambda$ of K has a highest weight $(-\lambda,\cdots,-\lambda)$ and we shall simply write $\pi_\lambda^{Sp(m,\mathbb{R})}$ for the representation $\pi_{(-\lambda,\cdots,-\lambda)}^{Sp(m,\mathbb{R})}$ of $Sp(m,\mathbb{R})$ on $\mathcal{O}(H_m)$ given by

$$\left(\pi_\lambda^{Sp(m,\mathbb{R})}(g)F\right)(Z) = \det(cZ+d)^{-\lambda} F\left((aZ+b)(cZ+d)^{-1}\right),$$

for $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(m,\mathbb{R})$, $Z \in H_m$. For $\lambda \in \mathbb{C}$, it gives a representation of $\widetilde{Sp(m,\mathbb{R})}$ on the same space $\mathcal{O}(H_m)$. Similarly, for $a \in \mathbb{N}$, we denote by $\pi_{\lambda,a}^{Sp(m,\mathbb{R})}$ the representation $\pi_{(0,\cdots,0,-a)+(-\lambda,\cdots,-\lambda)}^{Sp(m,\mathbb{R})}$ of the same group on $\mathcal{O}(H_m) \otimes \mathrm{Pol}^a[v_1,\cdots,v_m]$.

The representation W_λ^a may be realized on the space $\mathrm{Pol}^a[v_1,\cdots,v_{n-1}]$ where $(v_1,\cdots,v_{n-1})$ are the standard coordinates on $\mathfrak{n}_-^{-\tau} \simeq \mathbb{C}^{n-1}$. Hence, the differential symmetry breaking operators can be thought of as elements of $\mathbb{C}\left[\frac{\partial}{\partial z_{ij}}\right] \otimes \mathrm{Pol}^a[v_1,\ldots,v_{n-1}]$, where z_{ij} $(1 \le i,j \le n)$ are the standard coordinates on $\mathfrak{n}_- \simeq \mathrm{Sym}(n,\mathbb{C})$.

Theorem 7.1. *Let $n \ge 2$. Suppose $\lambda \in \mathbb{C}$ and $a \in \mathbb{N}$.*

(1) *The vector space*

$$\mathrm{Hom}_{\widetilde{Sp(n-1,\mathbb{R})\times Sp(1,\mathbb{R})}}(\mathcal{O}(H_n,\mathcal{L}_\lambda), \mathcal{O}(H_{n-1}\times H_1, \mathcal{W}_\lambda^a))$$

is one-dimensional.

(2) *The vector-valued differential operator from $\mathcal{O}(X)$ to $\mathcal{O}(Y)\otimes W$ defined by* (7.2)

$$D_{X\to Y,a} := \widetilde{C}_a^{\lambda-1}\left(\sum_{1\le i,j\le n-1} 2v_iv_j \frac{\partial^2}{\partial z_{ij}\partial z_{nn}}, \sum_{1\le j\le n-1} v_j\frac{\partial}{\partial z_{jn}}\right) \in \mathbb{C}\left[\frac{\partial}{\partial z_{ij}}\right] \otimes \mathrm{Pol}^a[v_1,\cdots,v_{n-1}]$$

intertwines the restriction $\pi_\lambda^{Sp(n,\mathbb{R})}\Big|_{Sp(n-1,\mathbb{R})\times Sp(1,\mathbb{R})}$ and $\pi_{\lambda,a}^{Sp(n-1,\mathbb{R})} \boxtimes \pi_{\lambda+a}^{Sp(1,\mathbb{R})}$. Here the polynomial $\widetilde{C}_a^{\lambda-1}(x,y)$ is the inflated normalized Gegenbauer polynomial defined in (6.5).

It follows from Theorem 7.1 that any symmetry breaking operator from $\mathcal{O}(X,\mathcal{L}_\lambda)$ to $\mathcal{O}(Y,\mathcal{W}_\lambda^a)$ is proportional to $D_{X\to Y,a}$.

Remark 7.2. If $\lambda > n$ then $\mathcal{H}^2(X,\mathcal{L}_\lambda) := \mathcal{O}(X,\mathcal{L}_\lambda) \cap L^2(X,\mathcal{L}_\lambda)$ is a non-zero Hilbert space on which G acts unitarily and irreducibly. Then, $\mathcal{H}^2(Y,\mathcal{W}_\lambda^a) := \mathcal{O}(Y,\mathcal{W}_\lambda^a) \cap L^2(Y,\mathcal{W}_\lambda^a) \ne \{0\}$ for any $a \in \mathbb{N}$, and the same statements as in Theorem 7.1 remain true for symmetry breaking operators between the representation spaces $\mathcal{H}^2(X,\mathcal{L}_\lambda)$ and $\mathcal{H}^2(Y,\mathcal{W}_\lambda^a)$.

In order to prove Theorem 7.1 we apply the F-method. Its Step 1 is given by

Lemma 7.3. *For $\lambda \in \mathbb{C}$, we set $\lambda^* = \lambda^\vee \otimes \mathbb{C}_{2\rho} = -\lambda + n + 1$. For $C \in Sym(n,\mathbb{C}) \simeq \mathfrak{n}_+$ and $Z \in Sym(n,\mathbb{C}) \simeq \mathfrak{n}_-$ we have*

$$\begin{aligned} d\pi_{\lambda^*}(C) &= (-\lambda+n+1)\,\mathrm{Trace}(CZ) + \sum_{i\le j}\sum_{k,\ell} C_{k\ell} z_{ik} z_{j\ell} \frac{\partial}{\partial z_{ij}}, \\ \widehat{d\pi_{\lambda^*}}(C) &= -\lambda \sum_{i\le j} C_{ij} \frac{\partial}{\partial \zeta_{ij}} - \frac{1}{2}\left(\sum_{i\le k, j\le \ell} C_{k\ell}\zeta_{ij} \frac{\partial^2}{\partial\zeta_{ik}\partial\zeta_{j\ell}} + \sum_{i\ge k, j\ge \ell} C_{k\ell}\zeta_{ij} \frac{\partial^2}{\partial\zeta_{ik}\partial\zeta_{j\ell}} \right). \end{aligned}$$

Proof. We embed the group $Sp(n,\mathbb{R})$ into $U(n,n)$ and apply the results of [KP14-1, Example **??**] with $p = q = n$. Thus, the first statement follows from the formula (3.4).

We consider a bilinear form

$$\mathfrak{n}_+ \times \mathfrak{n}_- \to \mathbb{C}, \qquad (C,Z) \mapsto \mathrm{Trace}(C\,{}^tZ),$$

where $\mathfrak{n}_+ \simeq \mathrm{Sym}(n,\mathbb{C}) \simeq \mathfrak{n}_-$. Recall that ζ_{ij} with $1 \le i \le j \le n$ are the coordinates on $\mathfrak{n}_+ \simeq \mathrm{Sym}(n,\mathbb{C})$. However, it is convenient for the computations below to allow us to use $\frac{\partial}{\partial\zeta_{ij}}$ $(i > j)$ for the same meaning with $\frac{\partial}{\partial\zeta_{ji}}$. Then

$$\widehat{z_{ij}} = \frac{1}{2}(1+\delta_{ij})\frac{\partial}{\partial\zeta_{ij}}, \qquad \widehat{\frac{\partial}{\partial z_{ij}}} = (\delta_{ij}-2)\zeta_{ij}.$$

Thus the algebraic Fourier transform of the first term of $d\pi_{\lambda^*}(C)$ amounts to

$$(\mathrm{Trace}(CZ))\widehat{\ } = \frac{1}{2}\sum_{i,j} C_{ij}(1+\delta_{ij})\frac{\partial}{\partial\zeta_{ij}} = \sum_{i\le j} C_{ij}\frac{\partial}{\partial\zeta_{ij}},$$

whereas that of the second term of $d\pi_{\lambda^*}(C)$ amounts to

$$\begin{aligned} &\left(\sum_{i\le j}\sum_{k,\ell} C_{k\ell} z_{ik} z_{j\ell}\frac{\partial}{\partial z_{ij}}\right)\widehat{\ } = -(n+1)\sum_{i\le j} C_{ij}\frac{\partial}{\partial\zeta_{ij}} - \frac{1}{4}\sum_{i,j,k,l} C_{kl}(1+\delta_{ik})(1+\delta_{jl})\zeta_{ij}\frac{\partial^2}{\partial\zeta_{ik}\partial\zeta_{j\ell}} \\ &= -(n+1)\sum_{i\le j} C_{ij}\frac{\partial}{\partial\zeta_{ij}} - \frac{1}{2}\left(\sum_{i\le k, j\le\ell} C_{k\ell}\zeta_{ij}\frac{\partial^2}{\partial\zeta_{ik}\partial\zeta_{j\ell}} + \sum_{i\ge k, j\ge\ell} C_{k\ell}\zeta_{ij}\frac{\partial^2}{\partial\zeta_{ik}\partial\zeta_{j\ell}}\right). \end{aligned}$$

Hence the formula for $\widehat{d\pi_{\lambda^*}}(C)$ follows. □

The condition (4.5) amounts to $\langle(-\lambda+1,\cdots,-\lambda+n), -(e_i+e_j)\rangle > 0$ for any $1 \le i \le j \le n$, namely $\lambda > n$.

For the Step 2 we apply Lemma 5.5.

Proposition 7.4. *Assume $\lambda > n$. If*

$$\mathrm{Hom}_{G'}(\mathcal{O}(G/K,\mathcal{L}_\lambda), \mathcal{O}(G'/K',\mathcal{W})) \ne \{0\}$$

for an irreducible representation W of K', then W is of the form

$$W = W^a_\lambda = S^a(\mathfrak{n}_+^{-\tau}) \otimes (-\lambda\,\mathrm{Trace}_n),$$

for some $a \in \mathbb{N}$ *see* (7.1).

From now on, we aim to construct (differential) symmetry breaking operators from $\mathcal{O}(X, \mathcal{L}_\lambda)$ to $\mathcal{O}(Y, \mathcal{W})$ in the case $W = W_\lambda^a$.

Define a Borel subalgebra $\mathfrak{b}(\mathfrak{k}')$ corresponding to the positive root system $\Delta^+(\mathfrak{k}', \mathfrak{j}) := \Delta^+(\mathfrak{k}, \mathfrak{j}) \cap \Delta(\mathfrak{k}', \mathfrak{j})$.

For Step 3 we apply Lemma 3.4 and we get:

Lemma 7.5. *Let* W_λ^a *be the irreducible* $\mathfrak{k}'$*-module defined in* (7.1).

(1) *The highest weight of* $(W_\lambda^a)^\vee$ *is given by*

$$\chi = (a, 0, \dots, 0; a) + (\lambda, \dots, \lambda; \lambda).$$

(2) *For the* $\mathfrak{k}$*-module* $\operatorname{Pol}(\mathfrak{n}_+) \otimes \mathbb{C}_\lambda^\vee$, *the* χ*-weight space for* $\mathfrak{b}(\mathfrak{k}')$ *is given by:*

$$(\operatorname{Pol}(\mathfrak{n}_+) \otimes \mathbb{C}_\lambda^\vee)_\chi \simeq \bigoplus_{2j+k=a} \mathbb{C}\zeta_{11}^j \zeta_{1n}^k \zeta_{nn}^j, \tag{7.3}$$

where we identify $\operatorname{Pol}(\mathfrak{n}_+) \otimes \mathbb{C}_\lambda^\vee$ *with* $\operatorname{Pol}(\mathfrak{n}_+)$ *as vector spaces.*

Proof. The statement (1) is clear from the definition of W_λ^a given in (7.1). Notice that in our convention $\Delta(\mathfrak{n}_-)$ is given as $\Delta(\mathfrak{n}_-) = \{e_i + e_j : 1 \le i \le j \le n\}$. Thus $\mathfrak{n}_-$ decomposes into irreducible representations of $\mathfrak{k}'$ as

$$\begin{aligned} \mathfrak{n}_- &\simeq (\operatorname{Sym}(n-1), \mathbb{C}) \boxtimes \mathbb{C}) \oplus (\mathbb{C} \boxtimes \mathbb{C}_2) \oplus (\mathbb{C}^{n-1} \boxtimes \mathbb{C}_1) \\ &\simeq (F(\mathfrak{gl}_{n-1}, 2e_1) \boxtimes F(\mathfrak{gl}_1, 0)) \oplus (F(\mathfrak{gl}_{n-1}, 0) \boxtimes F(\mathfrak{gl}_1, 2e_n)) \\ &\quad \oplus (F(\mathfrak{gl}_{n-1}, e_1) \boxtimes F(\mathfrak{gl}_1, e_n)). \end{aligned} \tag{7.4}$$

Accordingly we get an isomorphism of $\mathfrak{k}'$-modules:

$$\operatorname{Pol}(\mathfrak{n}_+) \simeq S(\mathfrak{n}_-) \simeq \bigoplus_{i,j,k} \left(S^i(\operatorname{Sym}(n-1), \mathbb{C})) \otimes S^k(\mathbb{C}^{n-1})\right) \boxtimes \mathbb{C}_{2j+k}. \tag{7.5}$$

Since ζ_{11}, ζ_{nn} and ζ_{1n} are highest weight vectors in the $\mathfrak{k}'$-module $\mathfrak{n}_-$ with respect to $\Delta^+(\mathfrak{k}')$ (see (7.4)), so is any monomial $\zeta_{11}^i \zeta_{nn}^j \zeta_{1n}^k$ in the $\mathfrak{k}'$-module $S(\mathfrak{n}_-) \simeq \operatorname{Pol}(\mathfrak{n}_+)$ of weight $(2i+k)e_1 + (k+2j)e_n$.

According to the irreducible decomposition (7.5) and Remark 5.7, it follows that the right-hand side of (7.3) exhausts all highest weight vectors in $\operatorname{Pol}(\mathfrak{n}_+)$ of weight $a(e_1 + e_n)$. Thus, taking into account the $\mathfrak{k}'$-action on $\mathbb{C}_\lambda^\vee \simeq \lambda \operatorname{Trace}_n$, we get Lemma. □

As Step 4, we reduce the system of differential equations (3.11), *i.e.* $\widehat{d\pi_{\lambda^*}}(C)\psi = 0$, to an ordinary differential equation. For this, we identify $\operatorname{Pol}(\mathfrak{n}_+) \otimes V^\vee$ with the space of polynomials in ζ on $\mathfrak{n}_+ \simeq \operatorname{Sym}(n, \mathbb{C})$. For a polynomial $g(t) \in \operatorname{Pol}_a[t]_{\text{even}}$ (see (6.12)) we set

$$(T_a g)(\zeta) := (\sqrt{2\zeta_{11}\zeta_{nn}})^a g\left(\frac{\zeta_{1n}}{\sqrt{2\zeta_{11}\zeta_{nn}}}\right).$$

Proposition 7.6. *Let χ be as in Lemma 7.5 (1).*

(1) $T_a : \mathrm{Pol}_a[t]_{\mathrm{even}} \xrightarrow{\sim} (\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee)_\chi$.

(2) *The map T_a induces an isomorphism*

$$\mathrm{Sol}_{\mathrm{Gegen}}(\lambda-1,a) \cap \mathrm{Pol}_a[t]_{\mathrm{even}} \xrightarrow{\sim} (\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee)_\chi^{\widehat{d\pi_{\lambda^*}}(\mathfrak{n}'_+)}.$$

(3) *Any polynomial $\psi(\zeta) \equiv \psi(\zeta_{ij})$ in the right-hand side of (7.3) is given by*

$$\psi(\zeta) = (T_a g)(\zeta) := (\sqrt{2\zeta_{11}\zeta_{nn}})^a g\left(\frac{\zeta_{1n}}{\sqrt{2\zeta_{11}\zeta_{nn}}}\right), \tag{7.6}$$

for some $g(t) \in \mathrm{Pol}_a[t]_{\mathrm{even}}$.

(4) *The polynomial $\psi(\zeta)$ on $\mathrm{Sym}(n,\mathbb{C})$ satisfies the system of partial differential equations $\widehat{d\pi_{\lambda^*}}(C)\psi = 0$ for any $C \in \mathfrak{n}'_+$ if and only if $g(t)$ satisfies the Gegenbauer differential equation*

$$\left((1-t^2)\vartheta_t^2 - (1+2(\lambda-1)t^2)\vartheta_t + a(a+2(\lambda-1))t^2\right) g(t) = 0, \tag{7.7}$$

where we denote $\vartheta_t = t\frac{d}{dt}$ as before.

Proof. The first two statements follow from Theorem 3.1, Proposition 3.3 and Lemma 3.4. The third statement is clear from (7.3). The proof of the last assertion is similar to the one of Lemma 6.11 and uses the following identities for T_a-saturated differential operators:

$$T_a^\sharp \vartheta_{\zeta_{11}} = T_a^\sharp \vartheta_{\zeta_{nn}} = \frac{1}{2}(a - \vartheta_t), \quad T_a^\sharp \vartheta_{\zeta_{1n}} = \vartheta_t,$$

where $\vartheta_{\zeta_{ij}} = \zeta_{ij}\frac{\partial}{\partial \zeta_{ij}}$. □

We are ready to complete the proof of Theorem 7.1.

Proof of Theorem 7.1. By the general result of the F-method (see Theorem 2.1) and owing to Proposition 3.3 and Lemma 3.4, we have the following isomorphism

$$\mathrm{Sol}_{\mathrm{Gegen}}(\lambda-1,a) \cap \mathrm{Pol}_a[t]_{even} \simeq \mathrm{Hom}_{\widetilde{G}'}(\mathcal{O}(X,\mathcal{L}_\lambda), \mathcal{O}(Y,\mathcal{W}_\lambda^a)).$$

Hence, the uniqueness of the G'-intertwining operator amounts to the fact that the Gegenbauer differential equation has a unique polynomial solution up to a scalar multiple (see Theorem 11.4 (2) in Appendix).

Let us prove that $D_{X\to Y,a}$ defined in (7.2) belongs to $\mathrm{Diff}_{G'}(\mathcal{L}_\lambda, \mathcal{W}_\lambda^a)$. Using the F-method we have proved that if $D \in \mathrm{Diff}_{G'}(\mathcal{L}_\lambda, \mathcal{W}_\lambda^a)$ and $w^\vee$ is a highest weight vector in $(W_\lambda^a)^\vee$, then $\langle D, w^\vee\rangle$ is of the form $(\mathrm{Symb}^{-1} \otimes \mathrm{id})T_a g$, where $g(t)$ is a polynomial satisfying (7.7). Hence $g(t)$ is, up to a scalar multiple, the Gegenbauer polynomial $\widetilde{C}_a^{\lambda-1}(t)$. In turn, $(T_a g)(\zeta) = \widetilde{C}_a^{\lambda-1}(2\zeta_{11}\zeta_{nn}, \zeta_{1n})$ up to a scalar.

Thus, in order to show $D_{X\to Y,a} \in \mathrm{Diff}_{G'}(\mathcal{L}_\lambda, \mathcal{W}^a_\lambda)$ it is sufficient to verify for all $\ell \in K'_{\mathbb{C}}$:

$$(\mathrm{Symb}\otimes \mathrm{id})\langle D_{X\to Y,a}, \nu^\vee(\ell^{-1})w^\vee\rangle = (\mathrm{Ad}_\sharp(\ell^{-1})\otimes \lambda^\vee(\ell^{-1}))(T_a g), \tag{7.8}$$

by Lemma 3.5 and by the observation that every non-zero $w^\vee \in W^\vee$ is cyclic. The left-hand side of (7.8) amounts to

$$\begin{aligned}
&\left\langle \widetilde{C}_a^{\lambda-1}\left(\sum_{1\le i,j\le n-1} 2v_iv_j\zeta_{ij}\zeta_{nn}, \sum_{1\le j\le n-1} v_j\zeta_{jn}\right), \nu^\vee(\ell^{-1})w^\vee\right\rangle \\
= \ &(\det \ell)^{-\lambda}\left\langle \widetilde{C}_a^{\lambda-1}\left(\sum_{1\le i,j\le n-1} 2(\ell v)_i(\ell v)_j\zeta_{ij}\zeta_{nn}, \sum_{1\le j\le n-1} (\ell v)_j\zeta_{jn}\right), w^\vee\right\rangle,
\end{aligned}$$

where $v = {}^t(v_1,\dots,v_{n-1})$ stands for the column vector. Since $\langle Q(v), w^\vee\rangle$ gives the coefficients of v_1^a in the polynomial $Q(v)$, it is equal to

$$\begin{aligned}
&(\det \ell)^{-\lambda}\widetilde{C}_a^{\lambda-1}\left(\sum_{1\le i,j\le n-1} 2\ell_{i1}\ell_{j1}\zeta_{ij}\zeta_{nn}, \sum_{1\le j\le n-1} \ell_{j1}\zeta_{jn}\right) \\
= \ &(\det \ell)^{-\lambda}\widetilde{C}_a^{\lambda-1}\left(\sum_{1\le i,j\le n-1} 2({}^t\ell\zeta\ell)_{11}\zeta_{nn}, \sum_{1\le j\le n-1} ({}^t\ell\zeta)_{1n}\right).
\end{aligned}$$

On the other hand, the action of $\mathrm{Ad}(\ell^{-1})$ on $\mathrm{Pol}(\mathfrak{n}_+)$ is generated by

$$\zeta_{ij} \mapsto ({}^t\ell\zeta\ell)_{ij}, \quad \zeta_{in} \mapsto ({}^t\ell\zeta)_{in}.$$

Hence, the right-hand side of (7.8) amounts to

$$(\det \ell)^{-\lambda}\widetilde{C}_a^{\lambda-1}\left(\sum_{1\le i,j\le n-1} 2({}^t\ell\zeta\ell)_{11}\zeta_{nn}, \sum_{1\le j\le n-1} ({}^t\ell\zeta)_{1n}\right),$$

whence the equality (7.8).

For the existence, we know that $\mathrm{Hom}_{G'}(\mathcal{O}(G/K,\mathcal{L}_\lambda), \mathcal{O}(G'/K', \mathcal{W}^a_\lambda)) \neq \{0\}$for $\lambda > n$ by Theorem 2.1 and the branching law given by Fact 4.2. In this case, it is given by the differential operator (7.2) by the F-method. The same formula defines a non-zero differential operator which depends holomorphically on $\lambda \in \mathbb{C}$. Since the actions of $\widetilde{G}$ on $\mathcal{O}(G/K,\mathcal{L}_\lambda)$ and that of $\widetilde{G}'$ on $\mathcal{O}(G'/K', \mathcal{W}^a_\lambda)$ can be realized on H_n and $H_{n-1}\times H_1$, respectively, by operators depending holomorphically on $\lambda \in \mathbb{C}$, the differential operator (7.2) respects the $\widetilde{G}'$ for all $\lambda \in \mathbb{C}$ by holomorphic continuation. □

8. Symmetry breaking operators for the tensor product representations of $U(n,1)$

In this section we discuss a higher dimensional generalization of the Rankin–Cohen bidifferential operators by considering the symmetric pair $(G'\times G', G')$ with $G' = U(n,1)$. First we fix some notations. Let $U(n,1)$ be the Lie group of all matrices preserving the standard Hermitian form of signature $(n,1)$ on $\mathbb{C}^{n+1}$ given by $I_{n,1} = \mathrm{diag}(1,\cdots,1,-1) \in GL(n+1,\mathbb{C})$.

Let D be the unit ball $\{Z \in \mathbb{C}^n : \|Z\| < 1\}$, where $\|Z\|^2 := \sum_{j=1}^n |z_j|^2$ for $Z = (z_1,\cdots,z_n)$. It is the Hermitian symmetric domain of type AIII in $\mathbb{C}^n$ in É. Cartan classification. Then the Lie group $U(n,1)$ acts biholomorphically on D by

$$g\cdot Z = (aZ+b)(cZ+d)^{-1} \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(n,1),\ Z \in D,$$

and the isotropy subgroup at the origin is isomorphic to $U(n)\times U(1)$. Since $cZ+d \in GL(1,\mathbb{C})$, we identify $cZ+d$ as a non-zero complex number and write $\frac{aZ+b}{cZ+d}$ instead of $(aZ+b)(cZ+d)^{-1}$ from now on.

We adapt the same convention as in [KP14-1, Example **??**] with $p=n$ and $q=1$. In particular, we use the decomposition of the Lie algebra

$$\mathrm{Lie}(U(n,1))\otimes_{\mathbb{R}}\mathbb{C} \simeq \mathfrak{gl}_{n+1}(\mathbb{C}) = \mathfrak{n}'_- + \mathfrak{k}' + \mathfrak{n}'_+, \quad \begin{pmatrix} A & B \\ C & d \end{pmatrix} \mapsto (B,(A,d),C).$$

Given a representation $\nu = \nu_1 \boxtimes \nu_2$ of $U(n)\times U(1)$ on a finite-dimensional complex vector space W, we extend it to a holomorphic representation, denoted by the same letter $\nu = \nu_1\boxtimes\nu_2$, of $GL(n,\mathbb{C})\times GL(1,\mathbb{C})$ on W. Then the holomorphic vector bundle $\mathcal{W} = U(n,1)\times_{U(n)\times U(1)} W$ over D is trivialized by using the open Bruhat cell $\mathfrak{n}'_- \simeq \mathbb{C}^n$, and the regular representation of $U(n,1)$ on $\mathcal{O}(D,\mathcal{W})$ is identified with the multiplier representation π_W of the same group on $\mathcal{O}(D)\otimes W$ given by

$$(8.1)\qquad (\pi_W(g)F)(Z) := \nu_1\left(a - \frac{(aZ+b)c}{cZ+d}\right)^{-1} \nu_2(cZ+d)^{-1} F\left(\frac{aZ+b}{cZ+d}\right),$$

for $F \in \mathcal{O}(D)\otimes W, g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(n,1)$ and $Z \in D$. We note that $cZ+d \neq 0$.

For $\lambda_1,\lambda_2 \in \mathbb{C}$, the map

$$(8.2)\qquad \mathfrak{gl}_n(\mathbb{C})\oplus\mathfrak{gl}_1(\mathbb{C}) \to \mathbb{C}, (A,d) \mapsto -\lambda_1 \,\mathrm{Trace}\, A - \lambda_2 d$$

is a one-dimensional representation of the Lie algebra $\mathfrak{k}'$, which we denote by $\mathbb{C}_{(\lambda_1,\lambda_2)}$. The negative signature in (8.2) is chosen according to our realization of $\mathfrak{n}_+$ in the lower triangular matrices. For integral values of λ_1 and λ_2 the character $\mathbb{C}_{(\lambda_1,\lambda_2)}$ lifts to $U(n)\times U(1)$. The restriction of the one-dimensional representation (8.2) to the Cartan subalgebra $\bigoplus_{i=1}^{n+1}\mathbb{C}E_{ii}$ is given by $(-\lambda_1,\cdots,-\lambda_1;-\lambda_2)$ in the dual basis $\{e_1,\cdots,e_{n+1}\}$.

For $\lambda_1,\lambda_2 \in \mathbb{Z}$, we form a $U(n,1)$-equivariant holomorphic line bundle $\mathcal{L}_{\lambda_1,\lambda_2} = U(n,1)\times_{U(n)\times U(1)}\mathbb{C}_{(\lambda_1,\lambda_2)}$ over D. By (8.1), the representation of $U(n,1)$ on $\mathcal{O}(D,\mathcal{L}_{\lambda_1,\lambda_2})$

is identified with the multiplier representation, denoted simply by $\pi_{\lambda_1,\lambda_2}$, of $U(n,1)$ on $\mathcal{O}(D)$ given by

$$(\pi_{\lambda_1,\lambda_2}(g)F)(Z) = (cZ+d)^{-\lambda_1+\lambda_2}(\det g)^{-\lambda_1} F\left(\frac{aZ+b}{cZ+d}\right).$$

In our normalization, the canonical bundle of D is given by $\mathcal{L}_{(1,-n)}$ associated with $\mathbb{C}_{2\rho} = \mathrm{Trace}(\mathrm{ad}(\cdot) : \mathfrak{n}_+ \to \mathfrak{n}_+) \simeq \mathbb{C}_{(1,-n)}$ with the notation of (8.2), and the dualizing bundle of $\mathcal{L}_{\lambda_1,\lambda_2}$ is given as

$$\mathcal{L}^*_{\lambda_1,\lambda_2} = \mathcal{L}^\vee_{\lambda_1,\lambda_2} \otimes \mathbb{C}_{2\rho} \simeq \mathcal{L}_{-\lambda_1+1,-\lambda_2-n}, \tag{8.3}$$

associated with

$$\mathbb{C}^*_{(\lambda_1,\lambda_2)} = \mathbb{C}_{(-\lambda_1,-\lambda_2)} \otimes \mathbb{C}_{2\rho} \simeq \mathbb{C}_{(-\lambda_1+1,-\lambda_2-n)}.$$

Now we consider the setting of symmetry breaking operators for the tensor product representations. We set $X := D \times D$ and $Y := \Delta(D)$. Thus, we have the following diagram:

$$\begin{array}{ccccccc} X = D\times D & \subset & \mathbb{C}^n \times \mathbb{C}^n & \simeq & \mathfrak{n}_- & \subset & \mathbb{P}^n\mathbb{C}\times\mathbb{P}^n\mathbb{C} \\ \cup & & \cup & & \cup & & \cup \\ Y = \Delta(D) & \subset & \Delta(\mathbb{C}^n) & \simeq & \mathfrak{n}'_- & \subset & \Delta(\mathbb{P}^n\mathbb{C}). \end{array}$$

We also set

$$G := U(n,1) \times U(n,1),$$

and let τ be the involution of G acting by $\tau : (g,h) \mapsto (h,g)$. Then the fixed point subgroup G^τ is isomorphic to $\Delta(U(n,1))$. Its identity component G' coincides with G^τ which is already connected. We consider the symmetric pair of holomorphic type (G,G').

According to the branching law in Fact 4.3, for $(\lambda'_1,\lambda'_2,\lambda''_1,\lambda''_2) \in \mathbb{Z}^4$ with $\lambda'_1-\lambda'_2 > n$ and $\lambda''_1 - \lambda''_2 > n$, there exists a non-trivial G'-intertwining operator $D_{X\to Y}(\varphi)$ from $\mathcal{O}(X, \mathcal{L}_{(\lambda'_1,\lambda'_2)} \boxtimes \mathcal{L}_{(\lambda''_1,\lambda''_2)})$ to $\mathcal{O}(Y,\mathcal{W}_Y)$ if and only the irreducible representation W of $U(n)\times U(1)$ has the highest weight $(-\lambda_1,\cdots,-\lambda_1,-\lambda_1-a;-\lambda_2+a)$ for some $a\in\mathbb{N}$. We denote it by $W^a_{(\lambda_1,\lambda_2)}$ and realize on the space $\mathrm{Pol}^a[v_1,\cdots,v_n]$ of homogeneous polynomials of degree a where $(v_1,\ldots,v_n)$ are the standard coordinates on $\mathfrak{n}_-^{-\tau} \simeq \mathbb{C}^n$. Then the vector-valued differential symmetry breaking operators can be thought of as elements of

$$\mathbb{C}\left[\frac{\partial}{\partial z'_1},\ldots,\frac{\partial}{\partial z'_n},\frac{\partial}{\partial z''_1},\ldots,\frac{\partial}{\partial z''_n}\right] \otimes \mathrm{Pol}^a[v_1,\ldots,v_n], \tag{8.4}$$

where z'_i, z''_j $(1\le i,j\le n)$ are the standard coordinates on $\mathfrak{n}_- \simeq \mathbb{C}^n \oplus \mathbb{C}^n$.

Let $P^{\alpha,\beta}_\ell(t)$ be the Jacobi polynomial defined by

$$P^{\alpha,\beta}_\ell(t) = \frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha+\beta+\ell+1)} \sum_{m=0}^{\ell} \binom{\ell}{m} \frac{\Gamma(\alpha+\beta+\ell+m+1)}{\ell!\,\Gamma(\alpha+m+1)} \left(\frac{t-1}{2}\right)^m, \tag{8.5}$$

see Appendix 11.2 for more details. We inflate it to a homogeneous polynomial of two variables x and y by

$$P_\ell^{\alpha,\beta}(x,y) := y^\ell P_\ell^{\alpha,\beta}\left(2\frac{x}{y}+1\right). \tag{8.6}$$

For instance, $P_0^{\alpha,\beta}(x,y) = 1$, $P_1^{\alpha,\beta}(x,y) = (2+\alpha+\beta)x + (\alpha+1)y$, etc.

We write $\widetilde{U(n,1)}$ for the universal covering of the group $U(n,1)$. Then we can define a $\widetilde{U(n,1)}$-equivariant holomorphic line bundle $\mathcal{L}_{(\lambda_1,\lambda_2)}$ over D for all $\lambda_1, \lambda_2 \in \mathbb{C}$, as well as a representation of $\widetilde{U(n,1)}$ on $\mathcal{O}(D, \mathcal{L}_{(\lambda_1,\lambda_2)})$.

We denote by $\widehat{\otimes}$ the completion of the tensor product of two nuclear spaces.

Theorem 8.1. *Suppose that $\lambda_1', \lambda_2', \lambda_1'', \lambda_2'' \in \mathbb{C}$ and $a \in \mathbb{N}$. We set $\lambda' := \lambda_1' - \lambda_2'$ and $\lambda'' := \lambda_1'' - \lambda_2''$.*

(1) *The dimension of the vector space*

$$\mathrm{Hom}_{\widetilde{U(n,1)}}\left(\mathcal{O}(D, \mathcal{L}_{(\lambda_1',\lambda_2')}) \widehat{\otimes} \mathcal{O}(D, \mathcal{L}_{(\lambda_1'',\lambda_2'')}), \mathcal{O}(D, \mathcal{W}^a_{(\lambda_1'+\lambda_1'', \lambda_2'+\lambda_2'')})\right)$$

is either one or two. It is equal to two if and only if

$$\lambda', \lambda'' \in \{-1, -2, \cdots\} \quad \text{and} \quad a \geq \lambda' + \lambda'' + 2a - 1 \geq |\lambda' - \lambda''|. \tag{8.7}$$

(2) *The vector-valued differential operator from $\mathcal{O}(D \times D)$ to $\mathcal{O}(D) \otimes \mathrm{Pol}^a[v_1, \cdots, v_n]$ defined by*

$$D_{X\to Y,a} := P_a^{\lambda'-1, -\lambda'-\lambda''-2a+1}\left(\sum_{i=1}^n v_i \frac{\partial}{\partial z_i}, \sum_{j=1}^n v_j \frac{\partial}{\partial z_j}\right) \tag{8.8}$$

intertwines $\pi_{\lambda_1',\lambda_2'} \boxtimes \pi_{\lambda_1'',\lambda_2''}\big|_{G'}$ and π_W, where $W \simeq W^a_{\lambda_1'+\lambda_1'', \lambda_2'+\lambda_2''}$.

(3) *If the triple (λ', λ'', a) satisfies (8.7), then $D_{X\to Y,a} = 0$. Otherwise, any symmetry breaking operator is proportional to $D_{X\to Y,a}$.*

Remark 8.2.

(1) The representation theoretic interpretation of the condition (8.7) will be clarified in Section 9 in the case $n = 1$, where we construct three symmetry breaking operators for singular parameters satisfying (8.7) and discuss their linear relations.

(2) The fiber of the vector bundle $\mathcal{W}^a_{(\lambda_1,\lambda_2)}$ is isomorphic to the space $S^a(\mathbb{C}^n)$ of symmetric tensors of degree a. It is a line bundle if and only if $a = 0$ or $n = 1$. In the case $n = 1$, the formula (8.8) reduces to the classical Rankin–Cohen bidifferential operators (see (1.1)) with an appropriate choice of spectral parameters, namely, for $a := \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$, the following identity holds:

$$\mathcal{RC}^{\lambda'''}_{\lambda',\lambda''} = (-1)^a P_a^{\lambda'-1, 1-\lambda'''}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)\Big|_{z_1=z_2=z}. \tag{8.9}$$

Remark 8.3. (1) If $\lambda_1', \lambda_2', \lambda_1'', \lambda_2'' \in \mathbb{Z}$ and $a \in \mathbb{N}$, then the linear groups G and G' act equivariantly on the two bundles $\mathcal{L}_{(\lambda_1',\lambda_2')} \boxtimes \mathcal{L}_{(\lambda_1'',\lambda_2'')} \to D \times D$ and $\mathcal{W}^a_{(\lambda_1,\lambda_2)} \to D$, respectively.

(2) If $\lambda', \lambda'' > n$, then analogous statements as in Theorem 8.1 remain true for continuous G'-homomorphisms between the Hilbert spaces $\mathcal{H}^2\left(X, \mathcal{L}_{(\lambda_1',\lambda_2')} \otimes \mathcal{L}_{(\lambda_1'',\lambda_2'')})\right)$ and $\mathcal{H}^2\left(Y, \mathcal{W}^a_{(\lambda_1'+\lambda_1'',\lambda_2'+\lambda_2'')}\right)$.

(3) Similar statements hold for continuous G'-homomorphisms between the Casselman–Wallach globalizations by the localness theorem [KP14-1, Theorem **??**].

In order to prove Theorem 8.1, we apply again the F-method. Its Step 1 is given by

Lemma 8.4. *For $(\lambda_1', \lambda_2') \in \mathbb{C}^2$, we set $(\mu_1', \mu_2') := (-\lambda_1' + 1, -\lambda_2' - n)$ and likewise we define (μ_1'', μ_2'') from $(\lambda_1'', \lambda_2'')$. Let $C := C' + C'' = (c_1', \ldots, c_n') + (c_1'', \ldots, c_n'') \in \mathfrak{n}_+ \simeq \mathbb{C}^n \oplus \mathbb{C}^n$. Then*

$$\begin{aligned} d\pi_{\mu_1',\mu_2'}(C') \oplus d\pi_{\mu_1'',\mu_2''}(C'') &= \sum_{i=1}^n c_i' z_i' (E_{z'} - \lambda' + n + 1) + \sum_{j=1}^n c_j'' z_j'' (E_{z''} - \lambda'' + n + 1), \\ \widehat{d\pi}_{\mu_1',\mu_2'}(C') \oplus \widehat{d\pi}_{\mu_1'',\mu_2''}(C'') &= -\left(\lambda' \sum_{i=1}^n c_i' \frac{\partial}{\partial \zeta_i'} + \sum_{i,j=1}^n c_i' \zeta_j' \frac{\partial^2}{\partial \zeta_i' \partial \zeta_j'}\right) \\ &\quad -\left(\lambda'' \sum_{j=1}^n c_j'' \frac{\partial}{\partial \zeta_j''} + \sum_{i,j=1}^n c_i'' \zeta_j'' \frac{\partial^2}{\partial \zeta_i'' \partial \zeta_j''}\right). \end{aligned}$$

For the Step 2 we apply Lemma 5.5.

Proposition 8.5. *Assume $\lambda' = \lambda_1' - \lambda_2' > n$ and $\lambda'' = \lambda_1'' - \lambda_2'' > n$. If*

$$\operatorname{Hom}_{G'}(\mathcal{O}(G/K, \mathcal{L}_{(\lambda_1',\lambda_2')} \otimes \mathcal{L}_{(\lambda_1'',\lambda_2'')}, \mathcal{O}(G'/K', \mathcal{W})) \neq \{0\}$$

for an irreducible representation W of K', then W is of the form

$$\begin{aligned} W &= W^a_{(\lambda_1'+\lambda_1'',\lambda_2'+\lambda_2'')} = S^a(\mathfrak{n}_+^{-\tau}) \otimes \mathbb{C}_{(\lambda_1'+\lambda_1'',\lambda_2'+\lambda_2'')} \\ &\simeq \left(S^a\left((\mathbb{C}^n)^\vee\right) \otimes (-\lambda_1 \operatorname{Trace}_n)\right) \boxtimes F(\mathfrak{gl}_1, (-\lambda_2 + a)e_{n+1}) \end{aligned} \tag{8.10}$$

for some $a \in \mathbb{N}$.

For Step 3 we apply Lemma 3.4 and we get:

Lemma 8.6. *Suppose $\lambda_1', \lambda_2', \lambda_1'', \lambda_2'' \in \mathbb{C}$ and $a \in \mathbb{N}$. Let V be the one-dimensional representation $\mathbb{C}_{(\lambda_1',\lambda_2')} \boxtimes \mathbb{C}_{(\lambda_1'',\lambda_2'')}$ of $\mathfrak{k}$, and W the irreducible representation of $\mathfrak{k}' \simeq \mathfrak{gl}_n(\mathbb{C}) \oplus \mathfrak{gl}_1(\mathbb{C})$ defined in* (8.10)*.*

(1) *The highest weight of the contragredient representation $W^\vee$ with respect to the standard Borel subalgebra $\mathfrak{b}(\mathfrak{k}')$ of $\mathfrak{k}'$ is given by*

$$\chi = (a, 0, \cdots, 0; -a) + (\lambda_1' + \lambda_1'', \cdots, \lambda_1' + \lambda_1'; \lambda_2' + \lambda_2'').$$

(2) *We regard the* $\mathfrak{k}$*-module* $\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$ *as a* $\mathfrak{b}(\mathfrak{k}')$*-module. Then the* χ*-weight space is given by*

$$(\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee)_\chi \simeq \bigoplus_{i+j=a} \mathbb{C}(\zeta_1')^i(\zeta_1'')^j, \tag{8.11}$$

where we identify $\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$ *with* $\mathrm{Pol}(\mathfrak{n}_+)$ *as vector spaces.*

Proof. 1) Since the highest weight of W is given by

$$(-\lambda_1' - \lambda_1'', \cdots, -\lambda_1' - \lambda_1''; -\lambda_2' - \lambda_2'') + (0, \cdots, 0, -a; a),$$

see (7.1), the first statement is clear.

2) The Lie algebra $\mathfrak{k}' \simeq \mathfrak{gl}_n(\mathbb{C}) \oplus \mathfrak{gl}_1(\mathbb{C})$ acts on $\mathfrak{n}_+ \simeq \mathbb{C}^n \oplus \mathbb{C}^n$ as the direct sum of two copies of irreducible representations

$$F(\mathfrak{gl}_n(\mathbb{C}), (0, \cdots, 0; -1)) \boxtimes F(\mathfrak{gl}_1(\mathbb{C}), 1),$$

and thus one has the following irreducible decomposition

$$\begin{aligned}
\mathrm{Pol}(\mathfrak{n}_+) &\simeq \bigoplus_{i,j} \mathrm{Pol}^i(\mathbb{C}^n) \otimes \mathrm{Pol}^j(\mathbb{C}^n) \\
&\simeq \bigoplus_{i,j} (F(\mathfrak{gl}_n(\mathbb{C}), (i, 0, \cdots, 0)) \otimes F(\mathfrak{gl}_n(\mathbb{C}), (j, 0, \cdots, 0))) \boxtimes F(\mathfrak{gl}_1(\mathbb{C}), -(i+j)) \\
&\simeq \bigoplus_{i,j} \bigoplus_{\underline{s}} F(\mathfrak{gl}_n(\mathbb{C}), (s_1, s_2, 0, \cdots, 0)) \otimes F(\mathfrak{gl}_1(\mathbb{C}), -(i+j)),
\end{aligned}$$

where the sum in the last line is taken over all $\underline{s} = (s_1, s_2, 0, \cdots, 0) \in \mathbb{N}^n$ satisfying $s_1 \geq s_2 \geq 0$, and $i + j \geq s_1 \geq \max(i, j)$ and $s_1 + s_2 = i + j$. In particular, the weight χ occurs a highest weight in $\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$, or equivalently, the one-dimensional $\mathfrak{b}(\mathfrak{k}')$-module $(a, 0, \cdots, 0; -a)$ occurs in $\mathrm{Pol}(\mathfrak{n}_+)$, if and only if $i + j = a$ and $s_2 = 0$. In this case the weight vectors are the monomials $(\zeta_1')^i(\zeta_1'')^j$. Lemma follows. □

As Step 4, we reduce the system of differential equations (3.9) to an ordinary differential equation. For this, we recall from (6.11) that $\mathrm{Pol}_a[t]$ is the space of polynomials in one variable t of degree at most a. We identify $\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$ with the space of polynomials in (ζ', ζ'') on $\mathfrak{n}_+ \simeq \mathbb{C}^n \oplus \mathbb{C}^n$. For $g \in \mathrm{Pol}_a[t]$ we set

$$(T_a g)(\zeta', \zeta'') := (\zeta_1'')^a g\left(\frac{\zeta_1'}{\zeta_1''}\right).$$

Proposition 8.7. *Let* χ *be the character of* $\mathfrak{b}(\mathfrak{k}')$ *given in Lemma 8.6.*

(1) *The map* T_a *induces an isomorphism* $T_a : \mathrm{Pol}_a[t] \xrightarrow{\sim} (\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee)_\chi$.

(2) *The polynomial* $T_a g$ *satisfies the system of partial differential equations* (3.9) *if and only if the polynomial* $g(t)$ *solves the single ordinary differential equation*

$$\left((t + t^2)\frac{d^2}{dt^2} + (\lambda' - (\lambda'' - 2a + 2)t)\frac{d}{dt} + a(\lambda'' + a - 1)\right) g(t) = 0. \tag{8.12}$$

For the proof of Proposition 8.7 we use the following identities for T_a-saturated operators whose verification is similar to the one for Lemma 6.10.

Lemma 8.8. *One has:*

(1) $T_a^\sharp\left(\zeta_1''\frac{\partial}{\partial\zeta_1'}\right)=\frac{d}{dt}$.

(2) $T_a^\sharp\left(\zeta_1'\zeta_1''\frac{\partial^2}{\partial(\zeta_1')^2}\right)=t\frac{d^2}{dt^2}$.

(3) $T_a^\sharp\left(\zeta_1''\frac{\partial}{\partial\zeta_1''}\right)=a-t\frac{d}{dt}$.

(4) $T_a^\sharp\left((\zeta_1'')^2\frac{\partial^2}{\partial(\zeta_1'')^2}\right)=a(a-1)-2(a-1)t\frac{d}{dt}+t^2\frac{d^2}{dt^2}$.

Proof of Proposition 8.7. The general condition (3.9) of the F-method amounts to the following differential equation:

$$\left(\lambda'\frac{\partial}{\partial\zeta_i'}+\zeta_i'\frac{\partial^2}{\partial(\zeta_i')^2}+\lambda''\frac{\partial}{\partial\zeta_i''}+\zeta_i''\frac{\partial^2}{\partial(\zeta_i'')^2}\right)\psi(\zeta',\zeta'')=0, \tag{8.13}$$

for $C_i=(\underbrace{0,\dots,0}_{i-1},1,0,\cdots,0)+(\underbrace{0,\dots,0}_{i-1},1,0,\cdots,0)\in\Delta(\mathfrak{n}_+)\simeq\mathfrak{n}_+'\simeq\mathbb{C}^n$ $(1\le i\le n)$. Applying this to $\psi=T_a g$, and using Lemma 8.8, we obtain the differential equation (8.12) for $g(t)$. □

We give a proof of Theorem 8.1 below. Note that the proof requires some general argument on the Jacobi polynomials, which is summarized in Appendix, namely, Section 11.2. We naturally quote necessary facts from the section, although they are discussed later.

Proof of Theorem 8.1. We set

$$h(s):=g\left(\frac{s-1}{2}\right).$$

Then $g(t)\in\mathrm{Pol}_a[t]$ if and only if $h(s)\in\mathrm{Pol}_a[s]$, and $g(t)$ satisfies (8.12) if and only if $h(s)$ satisfies

$$\left((1-s^2)\frac{d^2}{ds^2}+(\beta-\alpha-(\alpha+\beta+2)s)\frac{d}{ds}+a(a+\alpha+\beta+1)\right)h(s)=0, \tag{8.14}$$

where $\alpha:=\lambda'-1$ and $\beta:=-\lambda'-\lambda''-2a+1$. Thus, combining with Theorem 3.1, we have shown the following bijection

$$\begin{aligned}&\mathrm{Hom}_{\widetilde{U(n,1)}}\left(\mathcal{O}(D,\mathcal{L}_{(\lambda_1',\lambda_2')})\widehat{\otimes}\mathcal{O}(D,\mathcal{L}_{(\lambda_1'',\lambda_2'')}),\mathcal{O}(D,\mathcal{W}^a_{(\lambda_1'+\lambda_1'',\lambda_2'+\lambda_2'')})\right)\\ \simeq\ &\mathrm{Sol}_{\mathrm{Jacobi}}(\lambda'-1,-\lambda'-\lambda''-2a+1,a)\cap\mathrm{Pol}_a[s],\end{aligned} \tag{8.15}$$

where $\mathrm{Sol}_{\mathrm{Jacobi}}(\alpha,\beta,\ell)\cap\mathrm{Pol}_a[s]$ denotes the space of polynomials of degree at most a satisfying the Jacobi differential equation (11.4).

By the bijection (8.15) the first statement is reduced to Theorem 11.2 in Appendix on the dimension of polynomial solutions to the Jacobi differential equation.

Since the Jacobi polynomial $P_a^{\lambda'-1,-\lambda'-\lambda''-2a+1}(s)$ belongs to the right-hand side of (8.15), it follows from Theorem 3.1 (2) and Lemma 3.5 that $D_{X\to Y,a}$ is a symmetry breaking operator. The last statement follows from the fact that Jacobi polynomial $P_a^{\lambda'-1,-\lambda'-\lambda''-2a+1}(t)$ is identically zero as a polynomial of t if and only if the triple (λ',λ'',a) satisfies (8.7), by Theorem 11.2 (1) in Appendix. □

Remark 8.9. In all the three cases we have reduced a system of partial differential equations to a single ordinary differential equation in Step 4 of the F-method. The latter equation has regular singularities at $t = \pm 1$ and ∞. We describe the corresponding singularities via the map T_a as follows:

(1) The singularities of the differential equation (6.15) correspond to the varieties given by $\zeta_n = 0$ and $Q_{n-1}(\zeta') = 0$.
(2) The singularities of the differential equation (7.7) correspond to the varieties given by $\zeta_{1n} = 0$ and $\det \begin{vmatrix} \zeta_{11} & \zeta_{1n} \\ \zeta_{1n} & \zeta_{nn} \end{vmatrix} = 0$.
(3) The singularities of the differential equation (8.14) correspond to the varieties given by $\zeta_1' = 0$ and $\zeta_1' = \pm\zeta_1''$.

9. Higher multiplicity phenomenon for singular parameter

It is well-known that the branching law for the tensor product of two holomorphic discrete series representations of $SL(2,\mathbb{R})\,(\simeq SU(1,1))$ is multiplicity free. More generally, the branching laws for holomorphic discrete series representations of scalar type in the setting of reductive symmetric pairs remain multiplicity free for positive parameters [K08], as well as their counterpart for generalized Verma modules for *generic* parameters [K12]. However, we discover that such multiplicity one results may fail for singular parameters. In this section, we examine why and how it happens in the example of $SL(2,\mathbb{R})$. We shall see that the F-method reduces it to the question of finding polynomial solutions to the Gauss hypergeometric equation with all the parameters being negative integers. We give a complete answer to this question in Appendix.

9.1. **Multiplicity two results for singular parameters.** From now on, we consider the setting of the previous section for $n = 1$, and let $G = SU(1,1)$ rather than $U(1,1)$.

For $\lambda \in \mathbb{Z}$, we write $\mathcal{L}_\lambda$ for the G-equivariant holomorphic line bundle over the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$, where $\lambda = \lambda_1 - \lambda_2$ in the notations of the previous section. Using the Bruhat decomposition, we trivialize the line bindle $\mathcal{L}_\lambda$ and identify the regular representation of G on $\mathcal{O}(D,\mathcal{L}_\lambda)$ with the following multiplier representation on $\mathcal{O}(D)$:

$$(\pi_\lambda(g)F)(z) = (cz+d)^{-\lambda} F\left(\frac{az+b}{cz+d}\right), \quad \text{for} \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } F \in \mathcal{O}(D).$$

For $\lambda \in \mathbb{C}$, we extend π_λ to a representation of the universal covering group $\widetilde{G} = \widetilde{SU(1,1)}$.

We write $\operatorname{ind}_{\mathfrak{b}}^{\mathfrak{g}}(\nu)$ for the Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\nu$ of the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$. In our parametrization, if $\lambda = 1-k$ ($k \in \mathbb{N}$), then the k-dimensional irreducible representation occurs as a subrepresentation of $(\pi_\lambda, \mathcal{O}(D))$ and as a quotient of $\operatorname{ind}_{\mathfrak{b}}^{\mathfrak{g}}(-\lambda)$.

We consider symmetry breaking operators from the tensor product representation $\mathcal{O}(\mathcal{L}_{\lambda'}) \widehat{\otimes} \mathcal{O}(\mathcal{L}_{\lambda''})$ to $\mathcal{O}(\mathcal{L}_{\lambda'''})$, where $\widehat{\otimes}$ denotes the completion of the tensor product of two nuclear spaces. As we saw in (1.1), the Rankin–Cohen bidifferential operator $\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''}$ is an example of such an operator when $\lambda''' - \lambda' - \lambda'' \in 2\mathbb{N}$ (see also Example 9.9 below).

For $(\lambda',\lambda'',\lambda''') \in \mathbb{C}^3$, we set

$$\begin{aligned} H(\lambda',\lambda'',\lambda''') &:= \operatorname{Hom}_{\widetilde{G}}(\mathcal{O}(\mathcal{L}_{\lambda'})\widehat{\otimes}\mathcal{O}(\mathcal{L}_{\lambda''}), \mathcal{O}(\mathcal{L}_{\lambda'''})) \\ &= \operatorname{Diff}_{\widetilde{G}}(\mathcal{O}(\mathcal{L}_{\lambda'})\widehat{\otimes}\mathcal{O}(\mathcal{L}_{\lambda''}), \mathcal{O}(\mathcal{L}_{\lambda'''})) \\ &\simeq \operatorname{Hom}_{\mathfrak{g}}(\operatorname{ind}_{\mathfrak{b}}^{\mathfrak{g}}(-\lambda'''), \operatorname{ind}_{\mathfrak{b}}^{\mathfrak{g}}(-\lambda') \otimes \operatorname{ind}_{\mathfrak{b}}^{\mathfrak{g}}(-\lambda'')), \end{aligned}$$

where the second equality and the third isomorphism follow from Theorem 2.1. The general theory (see Fact 4.2) shows that $H(\lambda',\lambda'',\lambda''')$ is generically equal to 0 or 1. Here is a precise dimension formula:

Theorem 9.1. *The vector space $H(\lambda',\lambda'',\lambda''')$ is finite dimensional for any $(\lambda',\lambda'',\lambda''') \in \mathbb{C}^3$. More precisely,*

(1) $\dim_{\mathbb{C}} H(\lambda',\lambda'',\lambda''') \in \{0,1,2\}$.
(2) $H(\lambda',\lambda'',\lambda''') \neq \{0\}$ *if and only if*

$$\lambda''' - \lambda' - \lambda'' \in 2\mathbb{N}. \tag{9.1}$$

(3) *Suppose* (9.1) *is satisfied. Then the following three conditions are equivalent:*
  (i) $\dim_{\mathbb{C}} H(\lambda',\lambda'',\lambda''') = 2$.
  (ii)

$$\lambda',\lambda'',\lambda''' \in \mathbb{Z}, \quad 2 \geq \lambda' + \lambda'' + \lambda''', \quad \text{and} \quad \lambda''' \geq |\lambda' - \lambda''| + 2. \tag{9.2}$$

  (iii) $\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''} = 0$.

Next, let us give an explicit basis of $H(\lambda', \lambda'', \lambda''')$. For this consider the polynomials of one variable $\widetilde{g}_j$ ($j = 1, 2, 3$) which will be defined in Lemma 11.3 with

$$\alpha = \lambda' - 1,\ \beta = 1 - \lambda''', \quad \text{and} \quad \ell = \frac{1}{2}(-\lambda' - \lambda'' + \lambda''').$$

We inflate $\widetilde{g}_j$ into homogeneous polynomials of degree ℓ of two variables by

$$G_j(x, y) := (-y)^\ell \widetilde{g}_j\left(1 + \frac{2x}{y}\right),$$

and set

$$D_j := \operatorname{Rest}_{z_1 = z_2 = z} \circ G_j\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right),$$

for $j = 1, 2, 3$.

Theorem 9.2. *Suppose the conditions* (9.1) *and* (9.2) *hold.*

(1) *The operators D_j ($j = 1, 2, 3$) are G-homomorphisms from $\mathcal{O}(\mathcal{L}_{\lambda'}) \widehat{\otimes} \mathcal{O}(\mathcal{L}_{\lambda''})$ to $\mathcal{O}(\mathcal{L}_{\lambda'''})$.*

(2) *$1 - \lambda', 1 - \lambda''$ and $1 - \lambda''' \in \mathbb{N}_+$, and the operators D_j ($j = 1, 2, 3$) factorize into two natural intertwining operators as follows:*

$$D_1 = \mathcal{RC}^{\lambda'''}_{2-\lambda', \lambda''} \circ \left(\left(\frac{\partial}{\partial z_1}\right)^{1-\lambda'} \otimes \mathrm{id}\right),$$

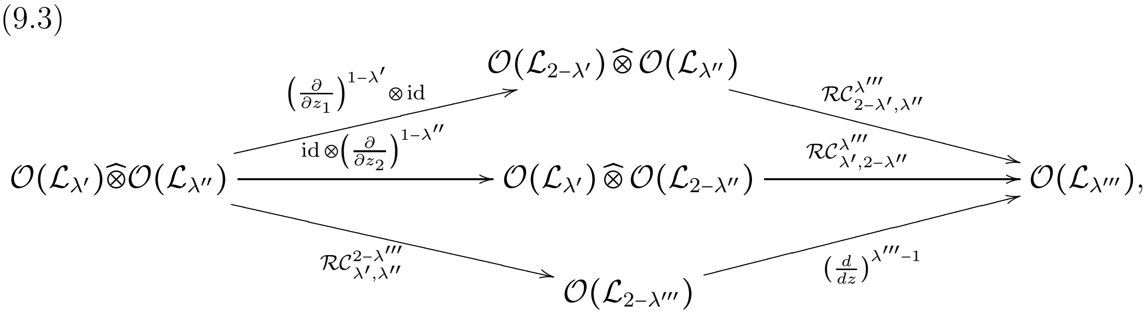
$$D_2 = \mathcal{RC}^{\lambda'''}_{\lambda', 2-\lambda''} \circ \left(\mathrm{id} \otimes \left(\frac{\partial}{\partial z_2}\right)^{1-\lambda''}\right),$$

$$D_3 = \left(\frac{d}{dz}\right)^{\lambda''' - 1} \circ \mathcal{RC}^{2-\lambda'''}_{\lambda', \lambda''}.$$

(3) *The following linear relation holds:*

$$D_1 - D_2 + (-1)^{\lambda'} D_3 = 0.$$

The factorizations in Theorem 9.2 are illustrated by the following diagram:
(9.3)



To summarize we consider the following three cases.

Case 0. $\lambda''' - \lambda' - \lambda'' \notin 2\mathbb{N}$.
Case 1. $\lambda''' - \lambda' - \lambda'' \in 2\mathbb{N}$ but the condition (9.2) is not fulfilled.
Case 2. $\lambda''' - \lambda' - \lambda'' \in 2\mathbb{N}$ and the condition (9.2) is satisfied.

Corollary 9.3.

$$H(\lambda', \lambda'', \lambda''') = \begin{cases} \{0\} & \text{Case } 0, \\ \mathbb{C} \cdot \mathcal{RC}_{\lambda', \lambda''}^{\lambda'''} & \text{Case } 1, \\ \mathbb{C}\langle D_1, D_2 \rangle = \mathbb{C}\langle D_1, D_3 \rangle = \mathbb{C}\langle D_2, D_3 \rangle & \text{Case } 2. \end{cases}$$

The rest of this section is devoted to the proof of Theorems 9.1 and 9.2.

9.2. **Application of the F-method.** For $\alpha, \beta \in \mathbb{C}$, and $\ell \in \mathbb{N}$, we denote by $\mathrm{Sol}_{\mathrm{Jacobi}}(\alpha, \beta, \ell) \cap \mathrm{Pol}_\ell[t]$ the space of polynomials $g(t)$ of degree at most ℓ satisfying the Jacobi differential equation (see Appendix 11.2):

$$(1 - t^2) g''(t) + (\beta - \alpha - (\alpha + \beta + 2)t) g'(t) + \ell(\ell + \alpha + \beta + 1) g(t) = 0.$$

Lemma 9.4. *Suppose* $(\lambda', \lambda'', \lambda''') \in \mathbb{C}^3$. *Then,*

(1) $H(\lambda', \lambda'', \lambda''') = \{0\}$ *if* $\lambda''' - \lambda' - \lambda'' \notin 2\mathbb{N}$.
(2) *Suppose* $\lambda''' - \lambda' - \lambda'' \in 2\mathbb{N}$. *Then the F-method gives a bijection*

$$H(\lambda', \lambda'', \lambda''') \xrightarrow{\sim} \mathrm{Sol}_{\mathrm{Jacobi}}(\alpha, \beta, \ell) \cap \mathrm{Pol}_\ell[t],$$

with $\alpha = \lambda' - 1$, $\beta = 1 - \lambda'''$, *and* $\ell = \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$.

Proof. By Step 3 of the F-method, the symbol map induces a bijection between $H(\lambda', \lambda'', \lambda''')$ and the space of polynomials $\psi(\zeta_1, \zeta_2)$ of two variables satisfying the following two conditions

- $\psi(\zeta_1, \zeta_2)$ is homogeneous of degree $\frac{1}{2}(\lambda''' - \lambda' - \lambda'')$,
- $\left(\lambda' \frac{\partial}{\partial \zeta_1} + \zeta_1 \frac{\partial^2}{\partial \zeta_1^2}\right) \psi = \left(\lambda'' \frac{\partial}{\partial \zeta_2} + \zeta_2 \frac{\partial^2}{\partial \zeta_2^2}\right) \psi = 0,$

corresponding to (3.10) and (3.11), respectively. Hence the first statement follows.

The second statement follows from Step 4 of the F-method, namely, Proposition 8.7 with $n = 1$ shows that there is a correspondence between $\psi(\zeta_1, \zeta_2)$ and $g(t) \in \mathrm{Sol}_{\mathrm{Jacobi}}(\alpha, \beta, \ell) \cap \mathrm{Pol}_\ell[t]$ with α, β and ℓ as above given by

$$\psi(\zeta_1, \zeta_2) = \zeta_2^\ell \, g\left(\frac{2\zeta_1}{\zeta_2} + 1\right).$$

□

We consider the transformation $(\lambda', \lambda'', \lambda''') \mapsto (\alpha, \beta, \ell)$ given by

$$\alpha := \lambda' - 1, \quad \beta := 1 - \lambda''', \quad \ell := \frac{1}{2}(\lambda''' - \lambda' - \lambda''). \tag{9.4}$$

For $\ell \in \mathbb{N}$, we define a finite set by

$$\Lambda_\ell := \{(\alpha, \beta) \in \mathbb{Z}^2 : \alpha + \ell \geq 0, \beta + \ell \geq 0, \alpha + \beta \leq -(\ell + 1)\}. \tag{9.5}$$

We note that $\Lambda_\ell \in (-\mathbb{N}_+) \times (-\mathbb{N}_+)$ and $\#\Lambda_\ell = \frac{1}{2}\ell(\ell + 1)$.

Lemma 9.5. *Suppose α, β, ℓ are given by (9.4). Then $\ell \in \mathbb{N}$ and $(\alpha, \beta) \in \Lambda_\ell$ if and only if $(\lambda', \lambda'', \lambda''') \in \mathbb{C}^3$ satisfies the following two conditions:*

$$\lambda', \lambda'', \lambda''' \in \mathbb{Z}, \quad \lambda' + \lambda'' \equiv \lambda''' \bmod 2, \tag{9.6}$$

$$-(\lambda' + \lambda'') \geq \lambda''' - 2 \geq |\lambda' - \lambda''|. \tag{9.7}$$

Since the proof is elementary and follows from the definition, we omit it. Note that the conditions (9.6) and (9.7) imply that

$$\lambda' \leq 0, \quad \lambda'' \geq 0, \quad \text{and} \quad 2 \leq \lambda''',$$

which are equivalent to $\alpha \leq -1$, $\alpha + \beta + 2\ell \geq 0$, and $\beta \leq -1$, respectively.

Proof of Theorem 9.1. By Lemma 9.4, the proof is reduced to the computation of the dimension of $\mathrm{Sol}_{\mathrm{Jacobi}}(\alpha, \beta, \ell) \cap \mathrm{Pol}_\ell[t]$.

1) Since the Jacobi differential equation is of second order, the space of its polynomial solutions is at most two-dimensional.

2) If $\ell = \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$, then Theorem 11.1 (1) shows that $\dim \mathrm{Sol}_{\mathrm{Jacobi}}(\alpha, \beta, \ell) \cap \mathrm{Pol}_\ell[t] \geq 1$ for any $\alpha, \beta \in \mathbb{C}$.

3) The equivalence follows from Theorem 11.2 (1) in light of Lemma 9.5. □

9.3. **Factorization of symmetry breaking operators.** We have seen in Theorem 9.1 that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{G}}(\mathcal{O}(\mathcal{L}_{\lambda'}) \widehat{\otimes} \mathcal{O}(\mathcal{L}_{\lambda''}), \mathcal{O}(\mathcal{L}_{\lambda'''})) = 2,$$

when $(\lambda', \lambda'', \lambda''')$ satisfies (9.6) and (9.7). In this subsection, we show that the other three symmetry breaking operators in the diagram (9.3) are unique up to scalars. To be precise, we prove the following.

Proposition 9.6. *Suppose $(\lambda', \lambda'', \lambda''')$ satisfies (9.6) and (9.7). Then*

$$\begin{aligned} & \dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{G}}(\mathcal{O}(\mathcal{L}_{2-\lambda'}) \widehat{\otimes} \mathcal{O}(\mathcal{L}_{\lambda''}), \mathcal{O}(\mathcal{L}_{\lambda'''})) \\ = \ & \dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{G}}(\mathcal{O}(\mathcal{L}_{\lambda'}) \widehat{\otimes} \mathcal{O}(\mathcal{L}_{2-\lambda''}), \mathcal{O}(\mathcal{L}_{\lambda'''})) \\ = \ & \dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{G}}(\mathcal{O}(\mathcal{L}_{\lambda'}) \widehat{\otimes} \mathcal{O}(\mathcal{L}_{\lambda''}), \mathcal{O}(\mathcal{L}_{2-\lambda'''})) = 1. \end{aligned}$$

Proof. The transformation $(\lambda', \lambda'', \lambda''') \mapsto (\alpha, \beta, \ell)$ given by (9.4) yields

$$\begin{aligned} (2 - \lambda', \lambda'', \lambda''') & \mapsto (-\alpha, \beta, \alpha + \ell), \\ (\lambda', 2 - \lambda'', \lambda''') & \mapsto (\alpha, \beta, -\alpha - \beta - \ell - 1), \\ (\lambda', \lambda'', 2 - \lambda''') & \mapsto (\alpha, -\beta, \beta + \ell). \end{aligned}$$

Moreover, if $(\alpha, \beta) \in \Lambda_\ell$ for some $\ell \in \mathbb{N}$, then

(1) $\alpha + \ell \in \mathbb{N}$ and $(-\alpha, \beta) \notin \Lambda_{\alpha+\ell}$,
(2) $-\alpha - \beta - \ell - 1 \in \mathbb{N}$ and $(\alpha, \beta) \notin \Lambda_{-\alpha-\beta-\ell-1}$,
(3) $\beta + \ell \in \mathbb{N}$ and $(\alpha, -\beta) \notin \Lambda_{\beta+\ell}$.

Then the proposition follows from Lemma 9.4 (2) and Theorem 11.2 (1). $\square$

9.4. **Differential intertwining operators for SL_2.** Obviously, both the F-method and the localness theorem hold in the case when $G = G'$, for which symmetry breaking operators are usual intertwining operators, and have been extensively studied. Lemma 9.7 below is well-known, but we illustrate its proof by using the F-method. The operators $\left(\frac{d}{dz}\right)^k$ are used for the factorization of D_j $(j = 1, 2, 3)$ in Theorem 9.2.

For $(\lambda, \nu) \in \mathbb{C}^2$, we set

$$\begin{aligned} H(\lambda,\nu) &:= \mathrm{Hom}_{\widetilde{G}}(\mathcal{O}(\mathcal{L}_\lambda), \mathcal{O}(\mathcal{L}_\nu)) \\ &= \mathrm{Diff}_{\widetilde{G}}(\mathcal{O}(\mathcal{L}_\lambda), \mathcal{O}(\mathcal{L}_\nu)) \\ &\simeq \mathrm{Hom}_{\mathfrak{g}}(\mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(-\nu), \mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(-\lambda)). \end{aligned}$$

Lemma 9.7.

(1) $\dim_{\mathbb{C}} H(\lambda,\nu) \leq 1$, *and the equality holds if and only if* $\lambda = \nu$ *or* $(\lambda,\nu) = (1-k, 1+k)$ *for some* $k \in \mathbb{N}$.
(2) *If* $(\lambda,\nu) = (1-k, 1+k)$ *for some* $k \in \mathbb{N}$, *then*

$$H(\lambda,\nu) = \mathbb{C}\left(\frac{d}{dz}\right)^k.$$

Proof. By the F-method, we have the following bijection between $H(\lambda,\nu)$ and the space of polynomials $g(t)$ of one variable satisfying the following two conditions

- $g(t)$ is a monomial of degree $\frac{1}{2}(\nu - \lambda)$, *i.e.* $g(t) = C\, t^{\frac{\nu-\lambda}{2}}$ for some $C \in \mathbb{C}$,
- $\left(\lambda \frac{d}{dt} + \frac{d^2}{dt^2}\right) g(t) = 0$,

according to (3.10) and (3.11).

The first condition forces $\nu - \lambda$ to be in $2\mathbb{N}$ in order to have $H(\lambda,\nu)$ not reduced to zero, whereas the second one implies $(\nu-\lambda)(\lambda+\nu-2) = 0$. Hence either $\lambda = \nu$ or $(\lambda,\nu) = (1+k, 1-k)$ for some $k \in \mathbb{N}$. In the latter case, $g(t) = Ct^k$ for some $k \in \mathbb{N}$, which yields $\left(\frac{d}{dz}\right)^k$ as a $\widetilde{G}$-intertwining operator from $\mathcal{O}(\mathcal{L}_\lambda)$ to $\mathcal{O}(\mathcal{L}_\nu)$. $\square$

9.5. **Construction of homogeneous polynomials by inflation.** In order to analyze symmetry breaking operators in the setting when the Rankin–Cohen bidifferential operators $\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''}$ vanish identically, we introduce the following notation.

For a polynomial $g(s)$ of degree *at most* ℓ, we set a polynomial of two variables

$$(\mathcal{I}_\ell g)(x,y) = (-y)^\ell g\left(-\frac{x}{y}\right).$$

The proof of factorization of symmetry breaking operators will be reduced to the following elementary factorization of homogeneous polynomials $(\mathcal{I}_\ell g)(x,y)$. The following observation follows immediately from the definition.

Lemma 9.8.

(1) *Suppose $g_1(s)$ is of the form $g_1(s) = s^m h_1(s)$ for some polynomial $h_1(s)$ of degree $\ell - m$, then*

$$(\mathcal{I}_\ell g_1)(x,y) = (-x)^m (\mathcal{I}_{\ell-m} h_1)(x,y).$$

(2) *Suppose $g_2(s)$ is a polynomial of degree $\ell - m$, then*

$$(\mathcal{I}_\ell g_2)(x,y) = (-y)^m (\mathcal{I}_{\ell-m} g_2)(x,y).$$

(3) *Suppose $g_3(s)$ is a polynomial of the form $g_3(s) = (1-s)^m h_3(s)$ for some polynomial $h_3(s)$ of degree $\ell - m$, then*

$$(\mathcal{I}_\ell g_3)(x,y) = (-1)^m (x+y)^m (\mathcal{I}_{\ell-m} h_3)(x,y).$$

Suppose $\ell = \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$. Then it follows from the proof of Lemma 9.4 that the inverse of the following bijection

$$H(\lambda',\lambda'',\lambda''') \overset{\sim}{\to} \mathrm{Sol}_{\mathrm{Jacobi}}(\lambda'-1, 1-\lambda''', \ell) \cap \mathrm{Pol}_\ell[t], \quad D \mapsto g \tag{9.8}$$

is given (up to multiplication by $(-1)^\ell$) by

$$D = \mathrm{Rest}_{z_1=z_2=z} \circ (\mathcal{I}_\ell g(1-2s)) \left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right).$$

Example 9.9. *The Rankin–Cohen bidifferential operator* (1.1) *is given for $(\lambda',\lambda'',\lambda''') \in \mathbb{C}^3$ with $\ell := \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$ by*

$$\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''} = \mathrm{Rest}_{z_1=z_2=z} \circ \left(\mathcal{I}_\ell P_\ell^{\lambda'-1,1-\lambda'''}(1-2s)\right) \left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right). \tag{9.9}$$

Proof of Theorem 9.2. 1) Since $\widetilde{g}_j \in \mathrm{Sol}_{\mathrm{Jacobi}}(\lambda'-1, 1-\lambda''', \ell) \cap \mathrm{Pol}_\ell[t]$ with $\ell = \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$ by Theorem 11.2 in the Appendix, we have $D_j \in H(\lambda',\lambda'',\lambda''')$ by (9.8).

2) Combining Lemmas 9.8 and 11.3 we have the following identities of the homogeneous polynomials $G_j(x,y)$:

$$\begin{aligned}
G_1(x,y) &= (-x)^{-\alpha} \left(\mathcal{I}_{\alpha+\ell} P_{\alpha+\ell}^{-\alpha,\beta}(1-2s)\right)(x,y), \\
G_2(x,y) &= (-y)^{-\beta} \left(\mathcal{I}_{\beta+\ell} P_{-\alpha-\beta-\ell-1}^{\alpha,\beta}(1-2s)\right)(x,y), \\
G_3(x,y) &= (-x-y)^{-\beta} \left(\mathcal{I}_{\beta+\ell} P_{\beta+\ell}^{\alpha,-\beta}(1-2s)\right)(x,y).
\end{aligned}$$

The first two identities yield the factorization of D_1 and D_2, and the last one yields the factorization of G_3 in light of the formula:

$$\mathrm{Rest}_{z_1=z_2=z} \circ \left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}\right)^j = \left(\frac{d}{dz}\right)^j \circ \mathrm{Rest}_{z_1=z_2=z}, \quad \text{for all } j \in \mathbb{N}.$$

3) The identity is reduced to the linear relations among the polynomials $\widetilde{g}_j(s)$ ($j = 1,2,3$) (see Lemma 11.3) which are obtained by Kummer's connection formula for the Gauss hypergeometric function at the regular singularities $s = 0$ ($\widetilde{g}_1(s)$ and $\widetilde{g}_2(s)$) and $s = 1$ ($\widetilde{g}_3(s)$). Hence Theorem 9.2 is proved. $\square$

10. An application of differential symmetry breaking operators

10.1. **Remark on the discrete spectrum of the branching rule for complementary series for $O(n+1,1) \downarrow O(n,1)$.** B. Kostant proved in [Kos69] the existence of the "long" complementary series representations of $SO(n,1)$ and $SU(n,1)$. In general, branching problems for the complementary series are more involved than the ones for principal series representations because the Mackey machinery does not apply.

In this section we explain briefly how the differential operators $D_{X\to Y,a}$ ($a \in \mathbb{N}$) given in Theorem 6.3 explicitly characterize discrete summands in the branching laws of the complementary series representations of $O(n+1,1)$ when restricted to the subgroup $O(n,1)$.

For this we first observe that $G'_{\mathbb{C}}$-equivariant holomorphic differential operators $D_{X\to Y,a}$ associated to the embedding of complex flag varieties $G_{\mathbb{C}}/P_{\mathbb{C}} \hookleftarrow G'_{\mathbb{C}}/P'_{\mathbb{C}}$ induce $G_{\mathbb{R}}$-equivariant differential operators associated to the embedding of the real flag varieties $G_{\mathbb{R}}/P_{\mathbb{R}} \hookleftarrow G'_{\mathbb{R}}/P'_{\mathbb{R}}$ for any pair $(G_{\mathbb{R}}, G'_{\mathbb{R}})$ of real forms of $(G_{\mathbb{C}}, G'_{\mathbb{C}})$ as far as $(P_{\mathbb{C}}, P'_{\mathbb{C}})$ have real forms $(P_{\mathbb{R}}, P'_{\mathbb{R}})$ in $(G_{\mathbb{R}}, G'_{\mathbb{R}})$.

In particular, for the pair $(G,G') = (SO_o(n,2), SO_o(n-1,2))$ and $(G_{\mathbb{R}}, G'_{\mathbb{R}}) := (SO_o(n+1,1), SO_o(n,1))$ whose complexifications are the same, we see that G-equivariant holomorphic differential operators $D_{X\to Y,a} : \mathcal{O}(G/K, \mathcal{L}_\lambda) \to \mathcal{O}(G'/K', \mathcal{L}_{\lambda+a})$ induce a $G'_{\mathbb{R}}$-equivariant differential operators

$$D_{X_{\mathbb{R}}\to Y_{\mathbb{R}},a} : C^\infty(G_{\mathbb{R}}/P_{\mathbb{R}}, \mathcal{L}_\lambda) \to C^\infty(G'_{\mathbb{R}}/P'_{\mathbb{R}}, \mathcal{L}_{\lambda+a}), \tag{10.1}$$

for two spherical principal series representations of $G_{\mathbb{R}}$ and $G'_{\mathbb{R}}$, owing to [KP14-1, Theorem **??** (2)] (extension theorem). In our parametrization, for $0 < \lambda < n$, there is a complementary series $\mathcal{H}_\lambda$ that contains $C^\infty(G_{\mathbb{R}}/P_{\mathbb{R}}, \mathcal{L}_\lambda)$ as a dense subset.

We define a family of Hilbert spaces $L^2(\mathbb{R}^n)_s$ with parameter $s \in \mathbb{R}$ by

$$L^2(\mathbb{R}^n)_s := L^2(\mathbb{R}^n, (\xi_1^2 + \cdots + \xi_n^2)^{\frac{s}{2}} d\xi_1 \cdots d\xi_n).$$

Then, for $0 < \lambda < n$, the Euclidean Fourier transform $\mathcal{F}_{\mathbb{R}^n}$ on the N-picture gives a unitary isomorphism

$$\mathcal{F}_{\mathbb{R}^n} : \mathcal{H}_{n-\lambda} \xrightarrow{\sim} L^2(\mathbb{R}^n)_{2\lambda-n}.$$

Correspondingly to the explicit formula

$$D_{X_{\mathbb{R}}\to Y_{\mathbb{R}},a} = \widetilde{C}_a^{\lambda-\frac{n-1}{2}} \left(-\Delta_{\mathbb{C}^{n-1}}, \frac{\partial}{\partial z_n} \right)$$

that was established in Theorem 6.3, we see that the multiplication of the inflated Gegenbauer polynomial $\widetilde{C}_a^{\lambda-\frac{n-1}{2}}(|\xi|^2, \xi_n)$ (see (6.5)) yields an explicit construction of discrete summands of the branching law for the restriction of complementary series as follows:

Proposition 10.1. *Suppose $a \in \mathbb{N}$ and $0 < \lambda < \frac{n-1}{2} - a$. For $\xi = (\xi_1, \cdots, \xi_{n-1}) \in \mathbb{R}^{n-1}$, we set $|\xi| := (\xi_1^2 + \cdots + \xi_{n-1}^2)^{\frac{1}{2}}$. Then,*

$$L^2(\mathbb{R}^{n-1})_{2(\lambda+a)-n-1} \hookrightarrow L^2(\mathbb{R}^n)_{2\lambda-n}, \quad v(\xi) \mapsto C_a^{\lambda-\frac{n-1}{2}}\left(|\xi|^2, \xi_n\right) v(\xi)$$

is an isometric and $G'_{\mathbb{R}}$-intertwining map from the complementary series of $G'_{\mathbb{R}} = SO_o(n,1)$ to that of $G_{\mathbb{R}} = SO_o(n+1,1)$.

See [KS13, Chapter 15] for the proof that (10.1) implies the proposition in the case $a \in 2\mathbb{N}$ (with both $G_{\mathbb{R}}$ and $G'_{\mathbb{R}}$ replaced by disconnected groups $O(n+1,1)$ and $O(n,1)$, respectively).

11. Appendix: Jacobi polynomials and Gegenbauer polynomials

11.1. Polynomial solutions to the hypergeometric differential equation.

In this subsection we discuss polynomial solutions to the Gauss hypergeometric differential equation

$$\left(z(1-z)\frac{d^2}{dz^2} - (c - (a+b+1)z)\frac{d}{dz} - ab\right)u(z) = 0. \tag{11.1}$$

For $c \notin -\mathbb{N}$, the hypergeometric series

$${}_2F_1(a,b;c;z) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j j!} z^j \tag{11.2}$$

is a non-zero solution to (11.1). It is easy to see from (11.2) that ${}_2F_1(a,b;c;z)$ is a polynomial if and only if $a \in -\mathbb{N}$ or $b \in -\mathbb{N}$.

Furthermore, we may ask if there exist two linearly independent polynomial solutions to (11.1). In fact, this never happens when $c \notin -\mathbb{N}$. More precisely, we have the following:

Theorem 11.1. *Suppose $a, b, c \in \mathbb{C}$.*

(1) *The following two conditions are equivalent.*
  (i) *There exists a non-zero polynomial solution to* (11.1).
  (ii) *$a \in -\mathbb{N}$ or $b \in -\mathbb{N}$.*
(2) *The following two conditions are equivalent.*
  (iii) *There exist two linearly independent polynomial solutions to* (11.1).
  (iv) *$a, b, c \in -\mathbb{N}$ and either* (iv-a) *or* (iv-b) *holds:*
    (iv-a) *$a \geq c > b$,*

(iv-b) $b \geq c > a$.

In this case the two linearly independent polynomial solutions are of degree $-a$ *and* $-b$.

Proof. (1) We have already discussed the case where $c \notin -\mathbb{N}$. Suppose now that $c \in -\mathbb{N}$. Since $1 - c > 0$, we have linearly independent solutions to (11.1) near $z = 0$ as follows

$$\begin{aligned} h_1(z) &= z^{1-c} {}_2F_1(a - c + 1, b - c + 1; 2 - c; z), \\ h_2(z) &= g(z) + (\mathrm{Res}_{\gamma=c}\, {}_2F_1(a, b; \gamma; z)) \log z, \end{aligned}$$

where $g(z)$ is a holomorphic function near $z = 0$ satisfying $g(0) = 1$. We divide the proof into two cases depending on whether $\mathrm{Res}_{\gamma=c}\, {}_2F_1(a, b; \gamma; z) = 0$ or not.

Case 1. Assume $\mathrm{Res}_{\gamma=c}\, {}_2F_1(a, b; \gamma; z) = 0$. In view of the residue formula

$$\mathrm{Res}_{\gamma=c}\, {}_2F_1(a, b; \gamma; z) = \frac{(-1)^c (a)_{1-c} (b)_{1-c}}{(-c)!(1-c)!} z^{1-c} {}_2F_1(a + 1 - c, b + 1 - c; 2 - c; z)$$

this expression vanishes if and only if $(a)_{1-c}(b)_{1-c} = 0$, namely

$$-\mathbb{N} \ni a \geq c \quad \text{or} \quad -\mathbb{N} \ni b \geq c.$$

In this case ${}_2F_1(a, b; \gamma; z)$ is holomorphic in γ near $\gamma = c$, and

$$\lim_{\gamma \to c} {}_2F_1(a, b; \gamma; z) = \sum_{j=0}^{L} \frac{(a)_j (b)_j}{(c)_j j!} z^j,$$

where $L = -a$ or $-b$, is a polynomial solution to (11.1).

Case 2. Assume $\mathrm{Res}_{\gamma=c}\, {}_2F_1(a, b; \gamma; z) \neq 0$. Since the logarithmic term does not vanish, there exists a non-zero polynomial solution to (11.1) if and only if $h_1(z)$ is a polynomial, or equivalently,

$$a - c + 1 \in -\mathbb{N} \quad \text{or} \quad b - c + 1 \in -\mathbb{N},$$

namely,

$$-\mathbb{N} \ni a < c \quad \text{or} \quad -\mathbb{N} \ni b < c.$$

Combining Case 1 and Case 2, we conclude the equivalence of (i) and (ii) in (1) for $c \in -\mathbb{N}$.

(2) We recall that the differential equation (11.1) has regular singularities at $z = 0, 1$, and ∞, and its characteristic exponents are indicated in the Riemann scheme

$$P \begin{Bmatrix} z = 0 & 1 & \infty & \\ 0 & 0 & a & ; z \\ 1 - c & c - a - b & b & \end{Bmatrix}.$$

(iii)⇒(iv). Suppose (iii) holds. Since the space of local solutions to (11.1) is two dimensional, any solution must be a polynomial. This forces the characteristic exponents to satisfy the following conditions:

$$1-c, c-a-b \in \mathbb{N}, \quad \text{and} \quad a, b \in \mathbb{N}.$$

Furthermore, the condition (iii) shows that there is no local solution which involves a non-zero logarithmic term near each regular singularity point, which in particular implies that the two characteristic exponents at $z = 0, 1$ or ∞ cannot coincide. Hence we get

$$1-c \neq 0,\ c-a-b \neq 0, \quad \text{and} \quad a \neq b.$$

Thus we have shown that the condition (iii) implies

$$a, b, c \in -\mathbb{N}. \tag{11.3}$$

From now we assume $c \in -\mathbb{N}$. As in the proof of (1), the condition (iii) implies that $\operatorname{Res}_{\gamma=c}\, {}_2F_1(a,b;\gamma;z) = 0$, and $h_1(z)$ is a polynomial. The latter conditions amount to

$$-\mathbb{N} \ni a \geq c \quad \text{or} \quad -\mathbb{N} \ni b \geq c,$$
$$-\mathbb{N} \ni a < c \quad \text{or} \quad -\mathbb{N} \ni b < c,$$

respectively. Equivalently, we have either $a \geq c > b$ or $b \geq c > a$ under the condition that $a, b, c \in -\mathbb{N}$ (see (11.3)). Hence the implication (iii)⇒(iv) is proved.

(iv)→(iii). Conversely, suppose (iv) holds. Then as we saw in the proof of (1), $h_1(z)$ and

$$\lim_{\gamma \to c} {}_2F_1(a,b;\gamma;z) = \sum_{j=0}^{\min(-a,-b)} \frac{(a)_j (b)_j}{(c)_j j!} z^j$$

are both polynomial solutions to (11.1), corresponding to the characteristic exponents $1-c$ and 0, respectively. Thus they are linearly independent, and we have completed the proof of the equivalence of (iii) and (iv). □

11.2. **Jacobi polynomials.** In this subsection, we discuss polynomial solutions to the Jacobi differential equation with emphasis on singular parameters where the corresponding Jacobi polynomial $P_\ell^{\alpha,\beta}(t)$ vanishes. In particular, we give a criterion for the space of polynomial solutions to be two-dimensional, and find its explicit basis.

First we quickly review the classical facts on Jacobi polynomials. Suppose $\alpha, \beta \in \mathbb{C}$ and $\ell \in \mathbb{N}$. The Jacobi differential equation

$$\left((1-t^2)\frac{d^2}{dt^2} + (\beta - \alpha - (\alpha+\beta+2)t)\frac{d}{dt} + \ell(\ell+\alpha+\beta+1)\right) y = 0 \tag{11.4}$$

is a particular case of the Gauss hypergeometric equation (11.1), and has at least one non-zero polynomial solution by Theorem 11.1 (1).

The Jacobi polynomial $P_\ell^{\alpha,\beta}(t)$ is the normalized polynomial solution to (11.4) that is subject to the Rodrigues formula

$$(1-t)^\alpha(1+t)^\beta P_\ell^{\alpha,\beta}(t)=\frac{(-1)^\ell}{2^\ell \ell!}\left(\frac{d}{dt}\right)^\ell\left((1-t)^{\ell+\alpha}(1+t)^{\ell+\beta}\right),$$

from which we have

$$P_\ell^{\beta,\alpha}(-t)=(-1)^\ell P_\ell^{\alpha,\beta}(t). \tag{11.5}$$

The Jacobi polynomial $P_\ell^{\alpha,\beta}(t)$ is generically non-zero (see Theorem 11.2 below for a precise condition) and is a polynomial of degree ℓ satisfying $P_\ell^{\alpha,\beta}(1)=\frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha+1)\ell!}$. Explicitly, for $\alpha\notin-\mathbb{N}_+$,

$$\begin{aligned} P_\ell^{\alpha,\beta}(t) &= \frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha+1)\ell!}\,{}_2F_1\left(-\ell,\alpha+\beta+\ell+1;\alpha+1;\frac{1-t}{2}\right)\\ &= \frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha+\beta+\ell+1)}\sum_{m=0}^{\ell}\binom{\ell}{m}\frac{\Gamma(\alpha+\beta+\ell+m+1)}{\Gamma(\alpha+m+1)\ell!}\left(\frac{t-1}{2}\right)^m. \end{aligned} \tag{11.6}$$

Here are the first three Jacobi polynomials.

- $P_0^{\alpha,\beta}(t)=1$.
- $P_1^{\alpha,\beta}(t)=\frac{1}{2}(\alpha-\beta+(2+\alpha+\beta)t)$.
- $P_2^{\alpha,\beta}(t)=\frac{1}{2}(1+\alpha)(2+\alpha)+\frac{1}{2}(2+\alpha)(3+\alpha+\beta)(t-1)+\frac{1}{8}(3+\alpha+\beta)(4+\alpha+\beta)(t-1)^2$.

If $\alpha>-1$ and $\beta>-1$, then the Jacobi polynomials $P_\ell^{\alpha,\beta}(t)$ ($\ell\in\mathbb{N}$) form an orthogonal basis in $L^2([-1,1],(1-t)^\alpha(1+t)^\beta dt)$.

When $\alpha=\beta$ these polynomials yield Gegenbauer polynomials (see the next section for more details), and they further reduce to Legendre polynomials in the case when $\alpha=\beta=0$.

Theorem 11.2. *Suppose $\ell\in\mathbb{N}$. We recall from* (9.5) *that $\Lambda_\ell\subset(-\mathbb{N})^2$ is a finite set of the cardinality $\frac{1}{2}\ell(\ell+1)$.*

(1) *The following three conditions on $(\alpha,\beta)\in\mathbb{C}^2$ are equivalent:*
  (i) *The Jacobi polynomial $P_\ell^{\alpha,\beta}(t)$ is equal to zero as a polynomial of t.*
  (ii) *There exist two linearly independent polynomial solutions to* (11.4) *of degree less than or equal to ℓ, namely,*

$$\dim_{\mathbb{C}}(\mathrm{Sol}_{\mathrm{Jacobi}}(\alpha,\beta,\ell)\cap\mathrm{Pol}_\ell[t])=2.$$

  (iii) $(\alpha,\beta)\in\Lambda_\ell$.

(2) *If one of (therefore any of) the equivalent conditions (i)-(iii) is satisfied, then*

$$\lim_{\varepsilon\to0}{}_2F_1(-\ell,\alpha+\beta+1;\alpha+\varepsilon+1;z) \tag{11.7}$$

exists and is a polynomial in z, which we denote by ${}_2F_1(-\ell,\alpha+\beta+1;\alpha+1;z)$. Then any two of the following three polynomials

$$g_1(z) := z^{-\alpha}{}_2F_1(-\alpha-\ell,\beta+\ell+1;1-\alpha;z), \tag{11.8}$$

$$g_2(z) := {}_2F_1(-\ell,\alpha+\beta+\ell+1;\alpha+1;z), \tag{11.9}$$

$$g_3(z) := (1-z)^{-\beta}{}_2F_1(-\beta-\ell,\alpha+\ell+1;1-\beta;1-z), \tag{11.10}$$

with $z=\frac{1}{2}(1-t)$ are linearly independent polynomial solutions to (11.4) of degree ℓ, $-(\alpha+\beta+\ell+1)$, and ℓ, respectively. In particular, any polynomial solution is of degree at most ℓ.

Proof. (1). (i)⇔(iii). By the expression

$$P_\ell^{\alpha,\beta}(t)=\sum_{j=0}^{\ell}\frac{(\alpha+j+1)_{\ell-j}(\alpha+\beta+\ell+1)_j}{j!(\ell-j)!}\left(\frac{t-1}{2}\right)^j,$$

one has $P_\ell^{\alpha,\beta}(t)\equiv 0$ as a polynomial of t if and only if

$$\underbrace{(\alpha+j+1)\cdots(\alpha+\ell)}_{\ell-j}\underbrace{(\alpha+\beta+\ell+1)\cdots(\alpha+\beta+\ell+j)}_{j}=0,\text{for all } j\ (0\le j\le\ell). \tag{11.11}$$

The condition (11.11) implies $\alpha\in\{-1,\cdots,-\ell\}$ by taking $j=0$. Conversely, if $\alpha\in\{-1,\cdots,-\ell\}$, then $(\alpha+j+1)\cdots(\alpha+\ell)=0$ for all j $(0\le j\le\ell)$, and therefore (11.11) is equivalent to $(\alpha+\beta+\ell+1)\cdots(\alpha+\beta+\ell+j)=0$ with $j=1-\alpha$, namely, $\alpha+\beta+\ell+1\le 0\le\beta+\ell+1$. Hence the equivalence of (i) and (iii) is proved.

(ii)⇔(iii). We recall from Theorem 11.1 that if the condition (iii), or equivalently (iv), is satisfied, then there are two linearly independent polynomial solutions to (11.1) of degrees $-a$ and $-b$, respectively. Applying Theorem 11.1 (2) with

$$a=-\ell,\quad b=\alpha+\beta+\ell+1,\quad\text{and}\quad c=1+\alpha,$$

we see that the condition on the degree of polynomials in (ii) corresponds to the condition $-a\ge -b$, which excludes (iv-b) in Theorem 11.1, and therefore, the condition (ii) is equivalent to

$$-\ell,\alpha+\beta+\ell+1,1+\alpha\in-\mathbb{N},\quad \alpha+\beta+\ell+1\ge 1+\alpha>-\ell,$$

which is nothing but $(\alpha,\beta)\in\Lambda_\ell$.

(2). Suppose $(\alpha,\beta)\in\Lambda_\ell$ for some $\ell\in\mathbb{N}$.

Since $-\alpha-\ell\in-\mathbb{N}$ and $\beta+\ell+1, 1-\alpha\notin-\mathbb{N}$, the polynomial $g_1(z)$ is of degree $-\alpha+(\alpha+\ell)=\ell$.

Secondly, the expression $-(\alpha+\beta+\ell+1)$ defines a non-negative integer smaller than $-\ell$ and we have:

$${}_2F_1(-\ell,\alpha+\beta+1;\alpha+\varepsilon+1;z)=\sum_{j=0}^{-(\alpha+\beta+\ell+1)}\frac{(-\ell)_j(\alpha+\beta+\ell+1)_j}{(\alpha+\varepsilon+1)_j j!}z^j.$$

Since $\alpha + j \le -(\beta + \ell + 1) < 0$ for all j with $0 \le j \le -(\alpha + \beta + \ell + 1)$, the denominator in each summand does not vanish at $\varepsilon = 0$, and therefore, $g_2(z)$ is well-defined and is a polynomial of degree $-(\alpha + \beta + \ell + 1)$.

Thirdly, since $-\beta - \ell \in -\mathbb{N}$ and $\alpha + \ell + 1, 1 - \beta \in \mathbb{N}_+$, the function ${}_2F_1(-\beta - \ell, \alpha + \ell + 1; 1 - \beta; 1 - z)$ is a polynomial of homogeneous degree $\ell + \beta$, and thus $g_3(z)$ is a polynomial of degree ℓ.

Moreover, since $g_j(z)$ $(j = 1, 2, 3)$ are local solutions to

$$\left(z(1-z)\frac{d^2}{dz^2} - ((\alpha+1) - (\alpha+\beta+2)z)\frac{d}{dz} + \ell(\alpha+\beta+\ell+1)\right)u(z) = 0 \tag{11.12}$$

near zero depending meromorphically on parameters $(\alpha, \beta) \in \mathbb{C}^2$, and since they do not admit poles at any point of Λ_ℓ, they are actually solutions to (11.12). Since $g_1(0) = 0$ and $g_2(0) = 1$, these functions are linearly independent.

Finally, we apply Kummer's connection formula (see [EMOT53, 2.9 (4.3)])

$$\begin{aligned} &(1-z)^{c-a-b}{}_2F_1(c-a, c-b; c-a-b+1; 1-z) \\ = \ & \frac{\Gamma(c-1)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)} z^{1-c}{}_2F_1(a+1-c, b+1-c; 2-c; z) \\ + \ & \frac{\Gamma(1-c)\Gamma(c-a-b+1)}{\Gamma(1-a)\Gamma(1-b)} {}_2F_1(a, b; c; z) \end{aligned}$$

with

$$a = -\ell, \quad b = \alpha + \beta + \ell + 1, \quad c = 1 + \alpha + \varepsilon,$$

and taking the limit $\varepsilon \to 0$, we obtain

$$g_3(z) = (-1)^{\alpha+\beta+\ell}\frac{(-\beta)!(\beta+\ell)!}{(-\alpha)!(\alpha+\ell)!}\, g_1(z) + \frac{(-\alpha-1)!(-\beta)!}{l!(-\alpha-\beta-\ell-1)!}\, g_2(z). \tag{11.13}$$

Since the scalars of this linear combination are non-zero, both pairs $\{g_1(z), g_3(z)\}$ and $\{g_2(z), g_3(z)\}$ are linearly independent. □

To end this subsection, we express $g_j(z)$ $(j = 1, 2, 3)$ in terms of the Jacobi polynomials. As a byproduct, we also give an identity among the Jacobi polynomials when $(\alpha, \beta) \in \Lambda_\ell$, or equivalently, when $P_\ell^{\alpha,\beta}(t) \equiv 0$ (Theorem 11.2).

Lemma 11.3. *Suppose* $(\alpha,\beta)\in\Lambda_\ell$. *Then,*

(1) $$\widetilde{g}_1(z) := \binom{\ell}{-\alpha}\cdot g_1(z) = z^{-\alpha}P_{\ell+\alpha}^{-\alpha,\beta}(1-2z);$$
$$\widetilde{g}_2(z) := (-1)^{-\ell-\alpha-\beta-1}\binom{-\alpha-1}{\ell+\beta}\cdot g_2(z) = P_{-\ell-\alpha-\beta-1}^{\alpha,\beta}(1-2z);$$
$$\widetilde{g}_3(z) := (-1)^{\beta+\ell}\binom{\ell}{-\beta}\cdot g_3(z) = (1-z)^{-\beta}P_{\ell+\beta}^{\alpha,-\beta}(1-2z).$$

(2) $$(-1)^\alpha\widetilde{g}_3(z) = \widetilde{g}_1(z)-\widetilde{g}_2(z), \quad \text{namely,}$$
$$P_{-\ell-\alpha-\beta-1}^{\alpha,\beta}(t) = (-1)^{\alpha+1}\left(\frac{1+t}{2}\right)^{-\beta}P_{\beta+\ell}^{\alpha,-\beta}(t)+\left(\frac{1-t}{2}\right)^{-\alpha}P_{\alpha+\ell}^{-\alpha,\beta}(t).$$

Proof. 1) The first and third formulæ follow from the equation (11.6) and the identity $\Gamma(\lambda)\Gamma(1-\lambda)=\frac{\lambda}{\sin\pi\lambda}$. The second one is more subtle because $g_2(z)$ is defined as the limit of the Gauss hypergeometric function in a specific direction (see (11.7)). Taking this into account, we deduce the second formula from (11.6).

2) The second identity follows directly from the first statement and (11.13). □

11.3. **Gegenbauer Polynomials.** Let $\vartheta_t := t\frac{t}{dt}$. For $\alpha\in\mathbb{C}$ and $\ell\in\mathbb{N}$, the Gegenbauer differential equation

$$\left((1-t^2)\frac{d^2}{dt^2}-(2\alpha+1)t\frac{d}{dt}+\ell(\ell+2\alpha)\right)y=0$$

or, equivalently,

$$\left((1-t^2)\vartheta_t^2-(1+2\alpha t^2)\vartheta_t+\ell(\ell+2\alpha)t^2\right)y=0 \tag{11.14}$$

is a particular case of the Jacobi differential equation (11.4) where (α,β) are set to be $(\alpha-\frac{1}{2},\alpha-\frac{1}{2})$, and has at least one non-zero polynomial solution owing to Theorem 11.1 (1). The Gegenbauer (or ultraspherical) polynomial $C_\ell^\alpha(t)$ is a solution to (11.14) given by the following formula:

$$\begin{aligned} C_\ell^\alpha(t) &= \frac{\Gamma(\ell+2\alpha)}{\Gamma(2\alpha)\Gamma(\ell+1)}{}_2F_1\left(-\ell,\ell+2\alpha;\alpha+\frac{1}{2};\frac{1-t}{2}\right)\\ &= \sum_{k=0}^{[\frac{\ell}{2}]}(-1)^k\frac{\Gamma(\ell-k+\alpha)}{\Gamma(\alpha)\Gamma(k+1)\Gamma(\ell-2k+1)}(2t)^{\ell-2k}.\end{aligned}$$

It is a specialization of the Jacobi polynomial

$$C_\ell^\alpha(t) = \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\ell+2\alpha)}{\Gamma(2\alpha)\Gamma(\ell+\alpha+\frac{1}{2})}P_\ell^{\alpha-\frac{1}{2},\alpha-\frac{1}{2}}(t). \tag{11.15}$$

The Gegenbauer polynomial $C_\ell^\alpha(t)$ is a polynomial of degree ℓ. Here are the first five Gegenbauer polynomials.

- $C_0^\alpha(t) = 1$.
- $C_1^\alpha(t) = 2\alpha t$.
- $C_2^\alpha(t) = -\alpha(1-2(\alpha+1)t^2)$.
- $C_3^\alpha(t) = -2\alpha(\alpha+1)(t-\frac{2}{3}(\alpha+2)t^3)$.
- $C_4^\alpha(t) = \frac{1}{2}\alpha(\alpha+1)(1-4(\alpha+2)t^2+\frac{4}{3}(\alpha+2)(\alpha+3)t^4)$.

We note that $C_\ell^\alpha(t) \equiv 0$ if $\ell \geq 1$ and $\alpha = 0, -1, -2, \cdots, -\left[\frac{\ell-1}{2}\right]$. Slightly differently from the usual notation in the literature, we renormalize the Gegenbauer polynomial by

$$\widetilde{C}_\ell^\alpha(t) := \frac{\Gamma(\alpha)}{\Gamma\left(\alpha+\left[\frac{\ell+1}{2}\right]\right)} C_\ell^\alpha(t). \tag{11.16}$$

Then $\widetilde{C}_\ell^\alpha(t)$ is a non-zero solution to (11.14) for all $\alpha \in \mathbb{C}$ and $\ell \in \mathbb{N}$.

As in the case of the Jacobi differential equation, there are some exceptional parameters (α, ℓ) for which the Gegenbauer differential equation (11.14) has two linearly independent polynomial solutions. For this we denote by

$$\mathrm{Sol}_{\mathrm{Gegen}}(\alpha, \ell) \cap \mathrm{Pol}[t]$$

the space of polynomial solutions to (11.14), and consider its subspace $\mathrm{Sol}_{\mathrm{Gegen}}(\alpha, \ell) \cap \mathrm{Pol}_\ell[t]_{\mathrm{even}}$ where $\mathrm{Pol}_\ell[t]_{\mathrm{even}} = \mathbb{C}\text{-span}\left\langle t^{\ell-2j} : 0 \leq j \leq \left[\frac{\ell}{2}\right]\right\rangle$. Then we have the following:

Theorem 11.4. *(1) Suppose $\ell \in \mathbb{N}$ and $\alpha \in \mathbb{C}$. Then*

$$\dim_{\mathbb{C}}(\mathrm{Sol}_{\mathrm{Gegen}}(\alpha, \ell) \cap \mathrm{Pol}[t]) = 2$$

if and only if (α, ℓ) satisfies

$$\alpha \in \mathbb{Z} + \frac{1}{2} \quad \text{and} \quad 1 - 2\ell \leq 2\alpha \leq -\ell. \tag{11.17}$$

(2) For any $\ell \in \mathbb{N}$ and any $\alpha \in \mathbb{C}$, the space $\mathrm{Sol}_{\mathrm{Gegen}}(\alpha, \ell) \cap \mathrm{Pol}_\ell[t]_{\mathrm{even}}$ is one-dimensional, and is spanned by $\widetilde{C}_\ell^\alpha(t)$.

Proof. (1) The first statement follows immediately from Theorem 11.2 by replacing (α, β) with $(\alpha - \frac{1}{2}, \alpha - \frac{1}{2})$.

(2) Clearly, $\widetilde{C}_\ell^\alpha(t) \in \mathrm{Sol}_{\mathrm{Gegen}}(\alpha, \ell) \cap \mathrm{Pol}_\ell[t]_{\mathrm{even}}$ for all $\alpha \in \mathbb{C}$ and $\ell \in \mathbb{N}$. Hence it suffices to show that another solution (see Theorem 11.2 and (11.7))

$${}_2F_1\left(-\ell, 2\alpha+\ell; \alpha+\frac{1}{2}; \frac{1-t}{2}\right) \notin \mathrm{Pol}_\ell[t]_{\mathrm{even}}$$

when α satisfies (11.17). Indeed ${}_2F_1\left(-\ell, 2\alpha+\ell; \alpha+\frac{1}{2}; \frac{1-t}{2}\right)$ is a polynomial in t whose top term is a non-zero multiple of $t^{-(2\alpha+\ell)}$, but $-(2\alpha+\ell) \not\equiv \ell \bmod 2$ because $\alpha \in \mathbb{Z}+\frac{1}{2}$. Hence Theorem is proved. $\square$

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