# Defining $\mathbb{Z}$ in $\mathbb{Q}$ 

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#### Abstract

We show that $\mathbb{Z}$ is definable in $\mathbb{Q}$ by a universal first-order formula in the language of rings. We also present an $\forall \exists$-formula for $\mathbb{Z}$ in $\mathbb{Q}$ with just one universal quantifier. We exhibit new diophantine subsets of $\mathbb{Q}$ like the complement of the image of the norm map under a quadratic extension, and we give an elementary proof for the fact that the set of non-squares is diophantine. Finally, we show that there is no existential formula for $\mathbb{Z}$ in $\mathbb{Q}$, provided one assumes a strong variant of the Bombieri-Lang Conjecture for varieties over $\mathbb{Q}$ with many $\mathbb{Q}$-rational points 1


## $1 \mathbb{Z}$ is universally definable in $\mathbb{Q}$

Hilbert's 10th problem was to find a general algorithm for deciding, given any $n$ and any polynomial $f \in \mathbb{Z}\left[x_{1}, \ldots, x_{n}\right]$, whether or not $f$ has a zero in $\mathbb{Z}^{n}$. Building on earlier work by Martin Davis, Hilary Putnam and Julia Robinson, Yuri Matiyasevich proved in 1970 that there can be no such algorithm. In particular, the existential first-order theory $\mathrm{Th}_{\exists}(\mathbb{Z})$ of $\mathbb{Z}$ (in the language of rings $\mathcal{L}:=\{+, \cdot ; 0,1\})$ is undecidable. Hilbert's 10th problem over $\mathbb{Q}$, i.e., the question whether $\operatorname{Th}_{\exists}(\mathbb{Q})$ is decidable, is still open.

If one had an existential (or diophantine) definition of $\mathbb{Z}$ in $\mathbb{Q}$ (i.e., a definition by an existential 1st-order $\mathcal{L}$-formula), then $\operatorname{Th}_{\exists}(\mathbb{Z})$ would be interpretable in $\operatorname{Th}_{\exists}(\mathbb{Q})$, and the answer would, by Matiyasevich's Theorem, again be no. But it is still open whether $\mathbb{Z}$ is existentially definable in $\mathbb{Q}$,

[^0]and, in fact, towards the end of the paper we provide a reason why it should not (Corollary 23).

The earliest 1st-order definition of $\mathbb{Z}$ in $\mathbb{Q}$, due to Julia Robinson ( $[\mathrm{R}]$ ), can be expressed by an $\forall \exists \forall$-formula of the shape

$$
\phi(t): \forall x_{1} \forall x_{2} \exists y_{1} \ldots \exists y_{7} \forall z_{1} \ldots \forall z_{6} f\left(t ; x_{1}, x_{2} ; y_{1}, \ldots, y_{7} ; z_{1}, \ldots, z_{6}\right)=0
$$

for some $f \in \mathbb{Z}\left[t ; x_{1}, x_{2} ; y_{1}, \ldots, y_{7} ; z_{1}, \ldots, z_{6}\right]$, i.e., for any $t \in \mathbb{Q}$,

$$
t \in \mathbb{Z} \text { if and only if } \phi(t) \text { holds in } \mathbb{Q}
$$

Recently, Bjorn Poonen ([P1]) managed to find an $\forall \exists$-definition with 2 universal and 7 existential quantifiers. In this paper we present an $\forall$-definition of $\mathbb{Z}$ in $\mathbb{Q}$. To search for such a creature is motivated by the following
Observation 0. If there is an existential definition of $\mathbb{Z}$ in $\mathbb{Q}$ then there is also a universal one.

Proof: If $\mathbb{Z}$ is diophantine in $\mathbb{Q}$ then so is
$\mathbb{Q} \backslash \mathbb{Z}=\{x \in \mathbb{Q} \mid \exists m, n, a, b \in \mathbb{Z}$ with $n \neq 0, \pm 1, a m+b n=1$ and $m=x n\}$.

Theorem 1. 2 There is a positive integer $n$ and a polynomial $g \in \mathbb{Z}\left[t ; x_{1}, \ldots, x_{n}\right]$ such that, for any $t \in \mathbb{Q}$,

$$
t \in \mathbb{Z} \text { if and only if } \forall x_{1} \ldots \forall x_{n} \in \mathbb{Q} g\left(t ; x_{1}, \ldots, x_{n}\right) \neq 0
$$

If one measures logical complexity in terms of the number of changes of quantifiers then this is a definition of $\mathbb{Z}$ in $\mathbb{Q}$ of least possible complexity: there is no quantifier-free definition of $\mathbb{Z}$ in $\mathbb{Q}$.

Corollary 2. $\mathbb{Q} \backslash \mathbb{Z}$ is diophantine in $\mathbb{Q}$.
In more geometric terms, this says
Corollary 2'. There is a (not necessarily irreducible) affine variety $V$ over $\mathbb{Q}$ and a $\mathbb{Q}$-morphism $\pi: V \rightarrow \mathbb{A}^{1}$ such that the image of $V(\mathbb{Q})$ is $\mathbb{Q} \backslash \mathbb{Z}$.

Together with the undecidability of $\operatorname{Th}_{\exists}(\mathbb{Z})$, Theorem 1 immediately implies
Corollary 3. $\operatorname{Th}_{\forall \exists}(\mathbb{Q})$ is undecidable.

[^1]Here $T h_{\forall \exists}(\mathbb{Q})$ is the set of all sentences of the shape

$$
\forall x_{1} \ldots \forall x_{k} \exists y_{1} \ldots \exists y_{l} \phi\left(x_{1}, \ldots, x_{k}, y_{1}, \ldots, y_{l}\right),
$$

where $\phi$ is a quantifier-free formula in the language of rings $\mathcal{L}=\{+, \cdot ; 0,1\}$, that is, a boolean combination of polynomial equations and inequalities between polynomials in $\mathbb{Z}\left[x_{1}, \ldots, x_{k}, y_{1}, \ldots, y_{l}\right]$. Corollary 3 was proved conditionally, using a conjecture on elliptic curves, in [CZ]. Again, we can phrase this in more geometric terms:

Corollary 3'. There is no algorithm that decides on input a $\mathbb{Q}$-morphism $\pi: V \rightarrow W$ between affine $\mathbb{Q}$-varieties $V, W$ whether or not $\pi: V(\mathbb{Q}) \rightarrow$ $W(\mathbb{Q})$ is surjective.

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## 2 The proof of Theorem 1

Like all previous definitions of $\mathbb{Z}$ in $\mathbb{Q}$, we use elementary facts on quadratic forms over $\mathbb{R}$ and $\mathbb{Q}_{p}$, together with Hasse's Local-Global-Principle for quadratic forms. What is new in our approach is the use of the Quadratic Reciprocity Law (e.g., in Proposition 10 or (16) and, inspired by the model theory of local fields, the transformation of some existential formulas into universal formulas (Step 4). A technical key trick is the existential definition of the Jacobson radical of certain rings (Step 3) which makes implicit use of so-called 'rigid elements' as they occur, e.g., in [K].

## Step 1: Diophantine definition of quaternionic semi-local rings à la Poonen

The first step modifies Poonen's proof ([P1]), thus arriving at a formula for $\mathbb{Z}$ in $\mathbb{Q}$ which, like the formula in his Theorem 4.1, has $2 \forall$ 's followed by 7 $\exists$ 's, but we managed to bring down the degree of the polynomial involved from 9244 to 8 .

Definition 4. Let $\mathbb{P}$ be the set of rational primes and let $\mathbb{Q}_{\infty}:=\mathbb{R}$.
For $a, b \in \mathbb{Q}^{\times}$, let

- $H_{a, b}:=\mathbb{Q} \cdot 1 \oplus \mathbb{Q} \cdot \alpha \oplus \mathbb{Q} \cdot \beta \oplus \mathbb{Q} \cdot \alpha \beta$ be the quaternion algebra over $\mathbb{Q}$ with multiplication defined by $\alpha^{2}=a, \beta^{2}=b$ and $\alpha \beta=-\beta \alpha$,
- $\Delta_{a, b}:=\left\{p \in \mathbb{P} \cup\{\infty\} \mid H_{a, b} \otimes \mathbb{Q}_{p} \neq M_{2}\left(\mathbb{Q}_{p}\right)\right\}$ the set of primes (including $\infty$ ) where $H_{a, b}$ does not split locally $-\Delta_{a, b}$ is always finite, and $\Delta_{a, b}=\emptyset$ iff $a \in N(b)$, i.e., $a$ is in the image of the norm map $\mathbb{Q}(\sqrt{b}) \rightarrow \mathbb{Q}$,
- $S_{a, b}:=\left\{2 x_{1} \in \mathbb{Q} \mid \exists x_{2}, x_{3}, x_{4} \in \mathbb{Q}: x_{1}^{2}-a x_{2}^{2}-b x_{3}^{2}+a b x_{4}^{2}=1\right\}$ the set of traces of norm-1 elements of $H_{a, b}$, and
- $T_{a, b}:=S_{a, b}+S_{a, b}$ - note that $T_{a, b}$ is an existentially defined subset of $\mathbb{Q}$. Here we deviate from Poonen's terminology: his $T_{a, b}$ is $S_{a, b}+S_{a, b}+$ $\{0,1, \ldots, 2309\}$.

For each $p \in \mathbb{P} \cup\{\infty\}$, we can similarly define $S_{a, b}\left(\mathbb{Q}_{p}\right)$ and $T_{a, b}\left(\mathbb{Q}_{p}\right)$ by replacing $\mathbb{Q}$ by $\mathbb{Q}_{p}$.

For each $p \in \mathbb{P}$, we will denote the $p$-adic valuation on $\mathbb{Q}$ or on $\mathbb{Q}_{p}$ by $v_{p}$, and the assoicated residue map by $\phi_{p}: \mathbb{Z}_{(p)} \rightarrow \mathbb{F}_{p}$ resp. $\phi_{p}: \mathbb{Z}_{p} \rightarrow \mathbb{F}_{p}$.

An explicit criterion for checking whether or not an element $p \in \mathbb{P} \cup\{\infty\}$ belongs to $\Delta_{a, b}$, is given in the following

Observation 5. Assume $a, b \in \mathbb{Q}^{\times}$and $p \in \mathbb{P} \cup\{\infty\}$. Then $p \in \Delta_{a, b}$ if and only if
for $\mathbf{p}=\mathbf{2 :}$ After multiplying by suitable rational squares and integers $\equiv 1$ $\bmod 8$ and, possibly, swapping $a$ and $b$, the pair $(a, b)$ is one of the following:
for $2 \neq \mathbf{p} \in \mathbb{P}$ :
$v_{p}(a)$ is odd, $v_{p}(b)$ is even, and $\left(\frac{b p^{-v_{p}(b)}}{p}\right)=-1$, or
$v_{p}(a)$ is even, $v_{p}(b)$ is odd, and $\left(\frac{a p^{-v_{p}(a)}}{p}\right)=-1$ or
$v_{p}(a)$ is odd, $v_{p}(b)$ is odd, and $\left(\frac{-a b p^{-v_{p}(a b)}}{p}\right)=-1$.
for $\mathbf{p}=\infty$ : $a<0$ and $b<0$.

Proof: This is an immediate translation of the computation of the Hilbert symbol $(a, b)_{p}$ (which is 1 or -1 depending on whether or not $p \in \Delta_{a, b}$ ) as in Theorem 1 of Ch.III in [Se]:
For finite odd $p$ and $a=p^{\alpha} u$ and $b=p^{\beta} v$ (with $u, v p$-adic units) the formula is

$$
(a, b)_{p}=(-1)^{\alpha \beta \epsilon(p)}\left(\frac{u}{p}\right)^{\beta}\left(\frac{v}{p}\right)^{\alpha},
$$

where $\epsilon(p):=\frac{p-1}{2} \bmod 2$.
For $p=2$, the formula is

$$
(a, b)_{2}=(-1)^{\epsilon(u) \epsilon(v)+\alpha \omega(v)+\beta \omega(u)},
$$

where $\omega(u):=\frac{u^{2}-1}{8} \bmod 2$.
For $p=\infty$ the statement is obvious.
Proposition 6. For any $a, b \in \mathbb{Q}^{\times}$,

$$
T_{a, b}=\bigcap_{p \in \Delta_{a, b}} \mathbb{Z}_{(p)},
$$

where $\mathbb{Z}_{(\infty)}:=\{x \in \mathbb{Q} \mid-4 \leq x \leq 4\}$.
Here and throughout the rest of the paper, we use the following
Convention: Given an empty collection of subsets of $\mathbb{Q}$, the intersection is $\mathbb{Q}$.

Proof: For each $p \in \mathbb{P}$, let

$$
U_{p}:=\left\{s \in \mathbb{F}_{p} \mid x^{2}-s x+1 \text { is irreducible over } \mathbb{F}_{p}\right\}
$$

. We shall use the following
Facts For any $a, b \in \mathbb{Q}^{\times}$and for any $p \in \mathbb{P}$ :
(a) If $p \notin \Delta_{a, b}$ then $S_{a, b}\left(\mathbb{Q}_{p}\right)=\mathbb{Q}_{p}$.
(b) If $p \in \Delta_{a, b}$ then $\phi_{p}^{-1}\left(U_{p}\right) \subseteq S_{a, b}\left(\mathbb{Q}_{p}\right) \subseteq \mathbb{Z}_{p}$.
(c) $S_{a, b}(\mathbb{R})= \begin{cases}\mathbb{R} & \text { for } a>0 \text { or } b>0 \\ {[-2,2]} & \text { for } a, b<0 .\end{cases}$
(d) If $p>11$ then $\mathbb{F}_{p}=U_{p}+U_{p}$.
(e) $S_{a, b}(\mathbb{Q})=\mathbb{Q} \cap \bigcap_{p \in \Delta_{a, b}} S_{a, b}\left(\mathbb{Q}_{p}\right)$.
(a) and (b) are [P1], Lemma 2.1, (c) is a straightforward computation, (d) is [P1], Lemma 2.3, and (e) is a special case of the Hasse-Minkowski local-global principle for representing rationals by quadratic forms.
(b) and (c) immediately give the inclusion $T_{a, b} \subseteq \bigcap_{p \in \Delta_{a, b}} \mathbb{Z}_{(p)}$.

To prove the converse inclusion $T_{a, b} \supseteq \bigcap_{p \in \Delta_{a, b}} \mathbb{Z}_{(p)}$, let us first compute $U_{p}$ for the primes $p \leq 11$ :

$$
\begin{aligned}
U_{2} & =\{1\} \\
U_{3} & =\{0\} \\
U_{5} & =\{1,4\} \\
U_{7} & =\{0,3,4\} \\
U_{11} & =\{0,1,5,6,10\} .
\end{aligned}
$$

For each $p \in \mathbb{P} \cup\{\infty\}$ define $V_{p} \subseteq \mathbb{Z}_{p}$ as follows:

$$
V_{p}= \begin{cases}\phi_{2}^{-1}\left(U_{2}\right) \cup\left(4+8 \mathbb{Z}_{2}\right) & \text { for } p=2 \\ \phi_{p}^{-1}\left(U_{p}\right) \cup\left[\left( \pm 2+p \mathbb{Z}_{p}\right) \backslash\left( \pm 2+p^{2} \mathbb{Z}_{p}\right)\right] & \text { for } 3 \leq p \leq 11 \\ \phi_{p}^{-1}\left(U_{p}\right) & \text { for } 11<p \in \mathbb{P} \\ {[-2,2]} & \text { for } p=\infty\end{cases}
$$

(We define $\mathbb{Z}_{\infty}$ to be the real interval $[-4,4] \subseteq \mathbb{R}$.)
By Fact (b), Fact (c), Observation 5together with an easy direct calculation in the cases $p=3,5,7,11$ and, for $p=2$, by the table below, one always has

$$
V_{p} \subseteq S_{a, b}\left(\mathbb{Q}_{p}\right) \text { and, for } p \neq \infty, V_{p} \text { is open. }
$$

The table for $p=2$ lists those pairs $(a, b)$ with $(a, b)_{2}=-1$ as in Observation [5 and gives, in each case,

$$
4+8 \mathbb{Z}_{2} \subseteq S_{a, b}\left(\mathbb{Q}_{2}\right)
$$

by assuming that we are given $x_{1} \in 2+8 \mathbb{Z}_{2}$ or $x_{1} \in 6+8 \mathbb{Z}_{2}$ (which is equivalent to $2 x_{1} \in 4+8 \mathbb{Z}_{2}$ ) and by specifying elements $x_{2}, x_{3}$ and $x_{4}$ which guarantee that

$$
-a x_{2}^{2}-b x_{3}^{2}+a b x_{4}^{2} \equiv_{2} 1-x_{1}^{2} \equiv_{2}-3 \bmod 8 \mathbb{Z}_{2} .
$$

Multiplying $x_{2}^{2}, x_{3}^{2}, x_{4}^{2}$ by a suitable common element from $1+8 \mathbb{Z}_{2} \subseteq\left(\mathbb{Q}_{2}^{\times}\right)^{2}$,
makes then sure that $2 x_{1} \in S_{a, b}\left(\mathbb{Q}_{2}\right)$.

| $(a, b)$ | $x_{2}$ | $x_{3}$ | $x_{4}$ |
| :---: | :---: | :---: | :---: |
| $(2,3)$ | 0 | 1 | 0 |
| $(2,5)$ | 2 | 1 | 1 |
| $(2,6)$ | 0 | 1 | $\frac{1}{2}$ |
| $(2,10)$ | 2 | 0 | $\frac{1}{2}$ |
| $(3,3)$ | 1 | 0 | 0 |
| $(3,10)$ | 1 | 0 | 0 |
| $(3,15)$ | 1 | 0 | 0 |
| $(5,6)$ | 1 | 1 | 0 |
| $(5,10)$ | 1 | 0 | 1 |
| $(5,30)$ | 1 | 1 | 0 |
| $(6,6)$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 |
| $(6,15)$ | 1 | 1 | 0 |
| $(10,30)$ | 0 | 1 | $\frac{1}{10}$ |
| $(15,15)$ | 1 | 0 | $\frac{2}{15}$ |
| $(15,30)$ | 1 | 1 | $\frac{1}{15}$ |
| $(30,30)$ | 1 | 1 | $\frac{1}{30}$ |

Fact (d) and another elementary case-by-case-check for $p \leq 11$ shows that for any $p \in \mathbb{P} \cup\{\infty\}$

$$
\mathbb{Z}_{p}=V_{p}+V_{p}
$$

Now pick $t \in \bigcap_{p \in \Delta_{a, b}} \mathbb{Z}_{(p)}$. For each $p \in \Delta_{a, b}$, there is some $s_{p} \in \mathbb{Z}_{p}$ such that $s_{p}, t-s_{p} \in V_{p}$.

If $t= \pm 4$ then, clearly, $t= \pm 2 \pm 2 \in S_{a, b}+S_{a, b}=T_{a, b}$.
If $t \neq \pm 4$ and $\infty \in \Delta_{a, b}$ we can choose $s_{\infty} \in \mathbb{Z}_{\infty}=[-4,4] \subseteq \mathbb{R}$ such that $\left.s_{\infty}, t-s_{\infty} \in\right]-2,2\left[\right.$. Now approximate the finitely many $s_{p} \in \mathbb{Z}_{p}\left(p \in \Delta_{a, b}\right)$ by a single $s \in \mathbb{Q}$ such that

$$
s-s_{p} \in \begin{cases}8 \mathbb{Z}_{2} & \text { if } p=2 \\ p^{2} \mathbb{Z}_{p} & \text { if } 3 \leq p \leq 11 \\ p \mathbb{Z}_{p} & \text { if } 11<p \in \mathbb{P} \\ ]-\epsilon, \epsilon[ & \text { if } p=\infty\end{cases}
$$

where $\epsilon=\min \left\{\left|2 \pm s_{\infty}\right|,\left|2 \pm\left(t-s_{\infty}\right)\right|\right\}$. This guarantees that for all $p \in \Delta_{a, b}$

$$
s, t-s \in V_{p} \subseteq S_{a, b}\left(\mathbb{Q}_{p}\right)
$$

and hence, by Fact (e), that $s, t-s \in S_{a, b}=S_{a, b}(\mathbb{Q})$.

One then obtains an $\forall \exists$-definition of $\mathbb{Z}$ in $\mathbb{Q}$ from the fact that

$$
\mathbb{Z}=\bigcap_{l \in \mathbb{P}} \mathbb{Z}_{(l)}=\bigcap_{a, b>0} T_{a, b}
$$

as in [P1], Theorem 4.1. With our simplified $T_{a, b}$, the formula now becomes, for any $t \in \mathbb{Q}$,

$$
t \in \mathbb{Z} \Longleftrightarrow \begin{aligned}
& \forall a, b \exists x_{1}, x_{2}, x_{3}, x_{4}, y_{2}, y_{3}, y_{4} \\
& \left(a+x_{1}^{2}+x_{2}^{2}+x_{3}^{2}+x_{4}^{2}\right) \cdot\left(b+x_{1}^{2}+x_{2}^{2}+x_{3}^{2}+x_{4}^{2}\right) . \\
& {\left[\left(x_{1}^{2}-a x_{2}^{2}-b x_{3}^{2}+a b x_{4}^{2}-1\right)^{2}+\right.} \\
& \left.+\left(\left(t-2 x_{1}\right)^{2}-4 a y_{2}^{2}-4 b y_{3}^{2}+4 a b y_{4}^{2}-4\right)^{2}\right]=0
\end{aligned}
$$

## Step 2: Towards a uniform diophantine definition of all $\mathbb{Z}_{(p)}$ 's

 in $\mathbb{Q}$We will present a diophantine definition for the local rings $\mathbb{Z}_{(p)}=\mathbb{Z}_{p} \cap \mathbb{Q}$ depending on the congruence of the prime $p$ modulo 8 , and involving $p$ (and if $p \equiv 1 \bmod 8$ an auxiliary prime $q$ ) as a parameter. However, since in any first-order definition of a subset of $\mathbb{Q}$ we can only quantify over the elements of $\mathbb{Q}$, and not, e.g., over all primes, we will allow arbitrary non-zero rational numbers $p$ and $q$ as parameters in the following definition.

Definition 7. For $p, q \in \mathbb{Q}^{\times}$, let

- $R_{p}^{[3]}:=T_{-1,-p}+T_{2,-p}$
- $R_{p}^{[5]}:=T_{-2,-p}+T_{2,-p}$
- $R_{p}^{[7]}:=T_{-1,-p}+T_{-2, p}$
- $R_{p, q}^{[1]}:=T_{-2 p, q}+T_{2 p, q}$

Remark 8. (a) For any $a, b, c, d \in \mathbb{Q}^{\times}$with at least one of them positive,

$$
T_{a, b}+T_{c, d}=\bigcap_{l \in \Delta_{a, b}} \mathbb{Z}_{(l)}+\bigcap_{l \in \Delta_{c, d}} \mathbb{Z}_{(l)}=\bigcap_{l \in \Delta_{a, b} \cap \Delta_{c, d}} \mathbb{Z}_{(l)}
$$

(b) The $R$ 's are are existentially defined, uniform in $p$ and $q$, so that for $k=3,5$ or 7 the sets

$$
\left\{(p, x) \in \mathbb{Q}^{\times} \times \mathbb{Q} \mid x \in R_{p}^{[k]}\right\}
$$

and the set

$$
\left\{(p, q, x) \in \mathbb{Q}^{\times} \times \mathbb{Q}^{\times} \mathbb{Q} \mid x \in R_{p, q}^{[1]}\right\}
$$

are diophantine.

Proof: (a) The first equation is from Proposition 6. For the second equation, the inclusion ' $\subseteq$ ' is obvious. For ' $\supseteq$ ', assume $x \in \bigcap_{l \in \Delta_{a, b} \cap \Delta_{c, d}} \mathbb{Z}_{(l)}$. By approximation, there is $y \in \mathbb{Q}$ such that

$$
y \in\left\{\begin{array}{cl}
x+l \mathbb{Z}_{(l)} & \text { for } l \in \Delta_{c, d} \\
\mathbb{Z}_{(l)} & \text { for } l \in \Delta_{a, b} \backslash \Delta_{c, d}
\end{array}\right.
$$

Then $y \in \bigcap_{l \in \Delta_{a, b}} \mathbb{Z}_{(l)}$ and $x-y \in \bigcap_{l \in \Delta_{c, d}} \mathbb{Z}_{(l)}$, so that $x=y+(x-y) \in$ $\bigcap_{l \in \Delta_{a, b}} \mathbb{Z}_{(l)}+\bigcap_{l \in \Delta_{c, d}} \mathbb{Z}_{(l)}$.
(b) This is immediate from definitions 4 and 7

Definition 9. (a) For $k=1,3,5$ or 7 , define $\mathbb{P}^{[k]}:=\{l \in \mathbb{P} \mid l \equiv k$ $\bmod 8\}$
(b) For $p \in \mathbb{Q}^{\times}$, define

- $\mathbb{P}(p):=\left\{l \in \mathbb{P} \mid v_{l}(p)\right.$ is odd $\}$
- $\mathbb{P}^{[k]}(p):=\mathbb{P}(p) \cap \mathbb{P}^{[k]}$, where $k=1,3,5$ or 7 .

Proposition 10. (a) $\mathbb{Z}_{(2)}=T_{3,3}+T_{2,5}$
(b) Suppose that $k=3,5$ or 7 . Then, for $p \in \mathbb{Q}^{\times}$,

$$
R_{p}^{[k]}= \begin{cases}\bigcap_{l \in \mathbb{P}^{[k]}(p)} \mathbb{Z}_{(l)} & \text { if } p \equiv k\left(\bmod 8 \mathbb{Z}_{(2)}\right) \\ \bigcap_{l \in \mathbb{P}^{[k]}(p)} \mathbb{Z}_{(l)} \text { or } \bigcap_{l \in \mathbb{P}^{[k]}(p) \cup\{2\}} \mathbb{Z}_{(l)} & \text { otherwise }\end{cases}
$$

(As before, $\bigcap_{l \in \emptyset} \mathbb{Z}_{(l)}=\mathbb{Q}$.)
In particular, if $p$ is a prime and $p \equiv k \bmod 8$ then $\mathbb{Z}_{(p)}=R_{p}^{[k]}$.
(c) For $p, q \in \mathbb{Q}^{\times}$with $p \equiv 1\left(\bmod 8 \mathbb{Z}_{(2)}\right)$ and $q \equiv 3\left(\bmod 8 \mathbb{Z}_{(2)}\right)$,

$$
R_{p, q}^{[1]}=\bigcap_{l \in \mathbb{P}(p, q)} \mathbb{Z}_{(l)}
$$

where $\mathbb{P}(p, q):=\Delta_{-2 p, q} \cap \Delta_{2 p, q}$.
In particular, if $p$ is a prime $\equiv 1 \bmod 8$ and $q$ is a prime $\equiv 3 \bmod 8$ with $\left(\frac{p}{q}\right)=-1$ then $\mathbb{Z}_{(p)}=R_{p, q}^{[1]}$.
Proof: (a) By Observation 5, $\Delta_{3,3}=\{2,3\}$ and $\Delta_{2,5}=\{2,5\}$, hence, by Remark [8)

$$
T_{3,3}+T_{2,5}=\bigcap_{l \in \Delta_{3,3} \cap \Delta_{2,5}} \mathbb{Z}_{(l)}=\mathbb{Z}_{(2)}
$$

(b) First assume $p \in \mathbb{Q}^{\times}$with $p \equiv 3\left(\bmod 8 \mathbb{Z}_{(2)}\right)$. Then, by Observation [5,

$$
\begin{aligned}
\Delta_{-1,-p} \cap \mathbb{P} & =\mathbb{P}^{[3]}(p) \cup \mathbb{P}^{[7]}(p) \\
\Delta_{2,-p} & =\mathbb{P}^{[3]}(p) \cup \mathbb{P}^{[5]}(p) \cup\{2\},
\end{aligned}
$$

so $\Delta_{-1,-p} \cap \Delta_{2,-p}=\mathbb{P}^{[3]}(p)$, and, by Remark $\mathbb{8}(\mathrm{a})$,

$$
R_{p}^{[3]}:=T_{-1,-p}+T_{2,-p}=\bigcap_{l \in \Delta_{-1,-p} \cap \Delta_{2,-p}} \mathbb{Z}_{(l)}=\bigcap_{l \in \mathbb{P}^{[3]}(p)} \mathbb{Z}_{(l)} .
$$

If $p \not \equiv 3\left(\bmod 8 \mathbb{Z}_{(2)}\right)$, the only possible additional prime is 2 (e.g. if $p \equiv 5$ ( $\left.\bmod 8 \mathbb{Z}_{(2)}\right)$ ).

If $p \equiv 5\left(\bmod 8 \mathbb{Z}_{(2)}\right)$ then, again by Observation 5

$$
\begin{aligned}
\Delta_{-2,-p} \cap \mathbb{P} & =\mathbb{P}^{[5]}(p) \cup \mathbb{P}^{[7]}(p) \\
\Delta_{2,-p} & =\mathbb{P}^{[3]}(p) \cup \mathbb{P}^{[5]}(p) \cup\{2\},
\end{aligned}
$$

so $\Delta_{-2,-p} \cap \Delta_{2,-p}=\mathbb{P}^{[5]}(p)$, and

$$
R_{p}^{[5]}:=T_{-2 p,-p}+T_{2 p,-p}=\bigcap_{l \in \Delta_{-2,-p} \cap \Delta_{2,-p}} \mathbb{Z}_{(l)}=\bigcap_{l \in \mathbb{P}^{[5]}(p)} \mathbb{Z}_{(l)}
$$

Again, the prime 2 (and no other prime) may or may not enter if $p \not \equiv 5$ ( $\left.\bmod 8 \mathbb{Z}_{(2)}\right)$.

Finally, if $p \equiv 7\left(\bmod 8 \mathbb{Z}_{(2)}\right)$ then, again by Observation 5 ,

$$
\begin{aligned}
\Delta_{-1,-p} \cap \mathbb{P} & =\mathbb{P}^{[3]}(p) \cup \mathbb{P}^{[7]}(p) \\
\Delta_{-2, p} \cap \mathbb{P} & =\mathbb{P}^{[5]}(p) \cup \mathbb{P}^{[7]}(p) \cup\{2\},
\end{aligned}
$$

so $\Delta_{-1,-p} \cap \Delta_{-2, p}=\mathbb{P}^{[7]}(p)$, and

$$
R_{p}^{[7]}:=T_{-p,-p}+T_{2 p, p}=\bigcap_{l \in \Delta_{-1,-p} \cap \Delta_{-2, p}} \mathbb{Z}_{(l)}=\bigcap_{l \in \mathbb{P}^{[7]}(p)} \mathbb{Z}_{(l)}
$$

As before, 2 may enter if $p \not \equiv 7\left(\bmod 8 \mathbb{Z}_{(2)}\right)$.
(c) The first statement is immediate from Remark $\mathbb{8}(\mathrm{a})$. For the 'in particular', assume $p$ and $q$ are primes with $p \equiv 1 \bmod 8, q \equiv 3 \bmod 8$ and $\left(\frac{p}{q}\right)=-1$. Then, by quadratic reciprocity, $\left(\frac{q}{p}\right)=-1$, and so, from Observation 5, $\Delta_{-2 p, q}=\{p\}$ and $\Delta_{2 p, q}=\{p, q\}$. Hence $R_{p, q}^{[1]}=\mathbb{Z}_{(p)}$.

## Corollary 11.

$$
\mathbb{Z}=\mathbb{Z}_{(2)} \cap \bigcap_{p, q \in \mathbb{Q}^{\times}}\left(R_{p}^{[3]} \cap R_{p}^{[5]} \cap R_{p}^{[7]} \cap R_{p, q}^{[1]}\right)
$$

Proof: By Remark [(a), all $R$ 's on the right hand side are semilocal subrings of $\mathbb{Q}$ containing $\mathbb{Z}$. On the other hand, by the 'in particular' parts of the proposition, for each prime $p$, the right hand side is contained in $\mathbb{Z}_{(p)}$; note that for $p \equiv 1 \bmod 8$ one always finds a prime $q \equiv 3 \bmod 8$ such that $q$ is congruent to a non-square $\bmod p$.

## Step 3: An existential definition for the Jacobson radical

We will show that, for some rings $R$ occuring in Proposition 10, the Jacobson radical $J(R)$ can be defined by an existential formula. This will also give rise to new diophantine predicates in $\mathbb{Q}$.

Definition 12. For $a, b, c \in \mathbb{Q}^{\times}$we define

- $T_{a, b}^{\times}:=\left\{u \in T_{a, b} \mid \exists v \in T_{a, b}\right.$ with $\left.u v=1\right\}$
- $I_{a, b}^{c}:=c \cdot \mathbb{Q}^{2} \cdot T_{a, b}^{\times} \cap\left(1-\mathbb{Q}^{2} \cdot T_{a, b}^{\times}\right)$
- $J_{a, b}:=\left(I_{a, b}^{a}+I_{a, b}^{a}\right) \cap\left(I_{a, b}^{b}+I_{a, b}^{b}\right)$

Note that the set $\left\{(a, b, x) \in \mathbb{Q}^{\times} \times \mathbb{Q}^{\times} \times \mathbb{Q} \mid x \in J_{a, b}\right\}$ is diophantine.
Lemma 13. Assume $a, b, c \in \mathbb{Q}^{\times}$. Then
(a) $T_{a, b}^{\times}= \begin{cases}\bigcap_{l \in \Delta_{a, b}} \mathbb{Z}_{(l)}^{\times} & \text {if } \infty \notin \Delta_{a, b} \\ \left(\left[-4,-\frac{1}{4}\right] \cup\left[\frac{1}{4}, 4\right]\right) \cap \bigcap_{l \in \Delta_{a, b} \backslash\{\infty\}} \mathbb{Z}_{(l)}^{\times} & \text {if } \infty \in \Delta_{a, b}\end{cases}$
(b) $I_{a, b}^{c}=\{0\} \cup\left\{y \in \mathbb{Q}^{\times} \left\lvert\, \begin{array}{l}v_{l}(y) \text { is odd and positive for all } l \in \Delta_{a, b} \cap \mathbb{P}(c) \text { and } \\ v_{l}(y), v_{l}(1-y) \text { are even for all } l \in \Delta_{a, b} \backslash(\mathbb{P}(c) \cup\{\infty\})\end{array}\right.\right\}^{3}$
(c) $I_{a, b}^{c}+I_{a, b}^{c}=\bigcap_{l \in \Delta_{a, b} \cap \mathbb{P}(c)} l \mathbb{Z}_{(l)}$,
(d) $J_{a, b}=\bigcap_{l \in \Delta} l \mathbb{Z}_{(l)}$, where $\Delta= \begin{cases}\Delta_{a, b} \backslash\{2, \infty\} & \text { if } 2 \in \Delta_{a, b} \text { and } v_{2}(a), v_{2}(b) \text { are even } \\ \Delta_{a, b} \backslash\{\infty\} & \text { else }\end{cases}$

[^2]In particular, if $\infty \notin \Delta_{a, b}$ then $T_{a, b}^{\times}$is the group of units of the ring $T_{a, b}$ and, if also $2 \notin \Delta_{a, b}$ or at least one of $v_{2}(a), v_{2}(b)$ is odd, $J_{a, b}$ is the Jacobson radical of $T_{a, b}$.

Proof: (a) This is an immediate consequence of Proposition 6.
(b) ' $\subseteq$ ': By weak approximation,

$$
\mathbb{Q}^{2} \cdot T_{a, b}^{\times}=\{0\} \cup \bigcap_{l \in \Delta_{a, b \backslash\{\infty\}}} v_{l}^{-1}(2 \mathbb{Z})
$$

So if $y \in I_{a, b}^{c} \backslash\{0\}$ and $l \in \Delta_{a, b} \cap \mathbb{P}(c)$ then $v_{l}(y)$ is odd. On the other hand, $1-y \in \mathbb{Q}^{2} \cdot T_{a, b}^{\times}$, so $v_{l}(1-y)$ is even. By the ultrametric inequality, this is only possible when $v_{l}(y)>0$. If, on the other hand, $l \in \Delta_{a, b} \backslash(\mathbb{P}(c) \cup\{\infty\})$ then $v_{l}(y)$ and $v_{l}(1-y)$ are even.
${ }^{\prime} \supseteq$ ': Clearly, $0 \in I_{a, b}^{c}$. Now assume $y \in \mathbb{Q}^{\times}$such that, for all $l \in \Delta_{a, b} \cap \mathbb{P}(c)$, $v_{l}(y)$ is positive and odd. Then $c^{-1} y \in \bigcap_{l \in \Delta_{a, b} \cap \mathbb{P}(c)} v_{l}^{-1}(2 \mathbb{Z})$ and $1-y \in$ $\bigcap_{l \in \Delta_{a, b} \cap \mathbb{P}(c)} \mathbb{Z}_{(l)}^{\times} \subseteq \bigcap_{l \in \Delta_{a, b} \cap \mathbb{P}(c)} v_{l}^{-1}(2 \mathbb{Z})$.

If we assume that $v_{l}(y)$ and $v_{l}(1-y)$ are even for all $l \in \Delta^{\prime}:=\Delta_{a, b} \backslash$ $(\mathbb{P}(c) \cup\{\infty\})$ then both $c^{-1} y$ and $1-y$ lie in $\bigcap_{l \in \Delta^{\prime}} v_{l}^{-1}(2 \mathbb{Z})$.

So with both assumptions we see that both $c^{-1} y$ and $1-y$ lie in

$$
\bigcap_{l \in \Delta_{a, b} \backslash\{\infty\}} v_{l}^{-1}(2 \mathbb{Z}) \subseteq \mathbb{Q}^{2} \cdot T_{a, b}^{\times}
$$

(c) For any prime $l$, any $x \in \mathbb{Q}$ with $v_{l}(x)>0$ can be written as the sum of two elements of odd positive value. And any $x \in \mathbb{Q}$ can be written as the sum of two elements $y_{1}$ and $y_{2}$ such that $v_{l}\left(y_{i}\right)$ and $v_{l}\left(1-y_{i}\right)$ are both even for both $i=1,2$ : choose $y_{1}$ of even value $<\min \left\{0, v_{l}(x)\right\}$ and let $y_{2}=x-y_{1}$; then $v_{l}\left(1-y_{1}\right)=v_{l}\left(y_{1}\right)=v_{l}\left(y_{2}\right)=v_{l}\left(1-y_{2}\right)$. Hence the claim follows by approximation.
(d) By definition, $J_{a, b}=\left(I_{a, b}^{a}+I_{a, b}^{a}\right) \cap\left(I_{a, b}^{b}+I_{a, b}^{b}\right)$, so, from (c),

$$
J_{a, b}=\bigcap_{l \in \Delta_{a, b} \cap \mathbb{P}(a)} l \mathbb{Z}_{(l)} \cap \bigcap_{l \in \Delta_{a, b} \cap \mathbb{P}(b)} l \mathbb{Z}_{(l)}=\bigcap_{l \in \Delta_{a, b} \cap(\mathbb{P}(a) \cup \mathbb{P}(b))} l \mathbb{Z}_{(l)}
$$

where the second equality is, again, by weak approximation. But now, from Observation 5 ,

$$
\Delta_{a, b} \cap(\mathbb{P}(a) \cup \mathbb{P}(b))= \begin{cases}\Delta_{a, b} \backslash\{2, \infty\} & \text { if } 2 \in \Delta_{a, b} \text { and } v_{2}(a), v_{2}(b) \text { are even } \\ \Delta_{a, b} \backslash\{\infty\} & \text { else }\end{cases}
$$

Before we give the existential definition of the Jacobson radical $J(R)$ for some of the rings $R$ defined in Step 2 (Corollary 15 and Proposition 16 below) we require another easy Lemma:

Lemma 14. Let $a, b, c, d \in \mathbb{Q}^{\times}$, at least one of which positive and at least one of which with odd dyadic value. Let $\Delta:=\Delta_{a, b} \cap \Delta_{c, d}$ and let $R=\bigcap_{l \in \Delta} \mathbb{Z}_{(l)}$. Then

$$
J_{a, b}+J_{c, d}=\bigcap_{l \in \Delta} l \mathbb{Z}_{(l)}
$$

In particular, if $\Delta \neq \emptyset$ then $J_{a, b}+J_{c, d}$ is the Jacobson radical $J(R)$ of the semilocal ring $R$.

$$
\text { Proof: Let } \Delta_{a, b}^{\prime}:= \begin{cases}\Delta_{a, b} \backslash\{2, \infty\} & \text { if } 2 \in \Delta_{a, b} \text { and } v_{2}(a), v_{2}(b) \text { are even }, \\ \Delta_{a, b} \backslash\{\infty\} & \text { else }\end{cases}
$$ and similarly $\Delta_{c, d}^{\prime}$. Then, by Lemma 13(d) (for the first equality) and by weak approximation (for the second),

$$
J_{a, b}+J_{c, d}=\bigcap_{l \in \Delta_{a, b}^{\prime}} l \mathbb{Z}_{(l)}+\bigcap_{l \in \Delta_{c, d}^{\prime}} l \mathbb{Z}_{(l)}=\bigcap_{l \in \Delta_{a, b}^{\prime} \cap \Delta_{c, d}^{\prime}} l \mathbb{Z}_{(l)}
$$

By our assumption on $a, b, c, d$, however, $\Delta_{a, b} \cap \Delta_{c, d}=\Delta_{a, b}^{\prime} \cap \Delta_{c, d}^{\prime}$, which proves the first claim.

The 'in particular' follows immediately.
Now let us first turn to the rings $R_{p}^{[k]}$ for $k=3,5$ and 7 defined in Definition 7 and recall that

$$
R_{p}^{[k]}= \begin{cases}T_{-1,-p}+T_{2,-p} & \text { if } k=3 \\ T_{-2,-p}+T_{2,-p} & \text { if } k=5 \\ T_{-1,-p}+T_{-2, p} & \text { if } k=7\end{cases}
$$

Corollary 15. Define for $k=1,3,5$ and 7 ,

$$
\begin{aligned}
\Phi_{k} & :=\left\{p \in \mathbb{Q}^{>0} \mid p \equiv k\left(\bmod 8 \mathbb{Z}_{(2)}\right) \text { and } \mathbb{P}(p) \subseteq \mathbb{P}^{[1]} \cup \mathbb{P}^{[k]}\right\} \\
\Psi & :=\left\{(p, q) \in \Phi_{1} \times \Phi_{3} \mid p \in 2 \cdot\left(\mathbb{Q}^{\times}\right)^{2} \cdot\left(1+J\left(R_{q}^{[3]}\right)\right)\right\} .
\end{aligned}
$$

(a) Then $\Phi_{k}$ is diophantine in $\mathbb{Q}$.
(b) If $k=3,5$ or 7 and if $p \in \Phi_{k}$ then $\mathbb{P}^{[k]}(p) \neq \emptyset$ and

$$
\{0\} \neq J\left(R_{p}^{[k]}\right)= \begin{cases}J_{-1,-p}+J_{2,-p} & \text { if } k=3 \\ J_{-2,-p}+J_{2,-p} & \text { if } k=5 \\ J_{-1,-p}+J_{-2, p} & \text { if } k=7\end{cases}
$$

In particular, in each of the cases, the Jacobson radical is diophantine in $\mathbb{Q}$, by a formula that is uniform in $p$.
(c) $\Psi$ is diophantine in $\mathbb{Q}$.

Proof: (a) It is clear that ' $p>0$ ' is diophantine. It is also clear from Proposition 10(a) that, for $k=1,3,5$ and 7 , the property ' $p \equiv k$ ( $\left.\bmod 8 \mathbb{Z}_{(2)}\right)$ is diophantine.

Moreover, if $v_{2}(p)$ is even and $k^{\prime}=3,5$ or 7 , then, by Proposition 10(b),

$$
\mathbb{P}^{\left[k^{\prime}\right]}(p)=\emptyset \Longleftrightarrow p \in\left(\mathbb{Q}^{\times}\right)^{2} \cdot\left(R_{p}^{\left[k^{\prime}\right]}\right)^{\times}
$$

(Note that we are not assuming that $p \equiv k^{\prime}\left(\bmod 8 \mathbb{Z}_{(2)}\right.$.) So the property on the left is diophantine. But then so are

$$
\begin{aligned}
& \Phi_{1}=\left\{p \equiv 1\left(\bmod 8 \mathbb{Z}_{(2)}\right) \mid \mathbb{P}_{3}(p)=\emptyset, \mathbb{P}_{5}(p)=\emptyset \text { and } \mathbb{P}_{7}(p)=\emptyset\right\} \\
& \Phi_{3}=\left\{p \equiv 3\left(\bmod 8 \mathbb{Z}_{(2)}\right) \mid \mathbb{P}^{5}(p)=\emptyset \text { and } \mathbb{P}^{7}(p)=\emptyset\right\} \\
& \Phi_{5}=\left\{p \equiv 5\left(\bmod 8 \mathbb{Z}_{(2)}\right) \mid \mathbb{P}^{3}(p)=\emptyset \text { and } \mathbb{P}^{7}(p)=\emptyset\right\} \\
& \Phi_{7}=\left\{p \equiv 7\left(\bmod 8 \mathbb{Z}_{(2)}\right) \mid \mathbb{P}^{3}(p)=\emptyset \text { and } \mathbb{P}^{5}(p)=\emptyset\right\}
\end{aligned}
$$

(b) Assume $k=3,5$ or 7 and that $p \in \Phi_{k}$. Then $p \equiv k\left(\bmod 8 \mathbb{Z}_{(2)}\right)$ and so, by Proposition $10(\mathrm{~b}), R_{p}^{[k]}=\bigcap_{l \in \mathbb{P}^{[k]}(p)} \mathbb{Z}_{(l)}$. As $p>0$ and $p \equiv k\left(\bmod 8 \mathbb{Z}_{(2)}\right)$, $\mathbb{P}^{[k]}(p) \neq \emptyset$ and hence $J\left(R_{p}^{[k]}\right)=\bigcap_{l \in \mathbb{P}^{[k]}(p)} l \mathbb{Z}_{(l)} \neq\{0\}$. The explicit formulas now follow from Lemma 14, as the assumptions of the Lemma are satisfied in each case. (c) follows directly from (a) and (b).

The most difficult case is when $p \in \Phi_{1}$. Recall from Definition 7 and from Proposition 10(c) that, for $p, q \in \mathbb{Q}^{\times}$, we have defined $R_{p, q}^{[1]}:=T_{-2 p, q}+T_{2 p, q}$ and $\mathbb{P}(p, q):=\Delta_{-2 p, q} \cap \Delta_{2 p, q}$.

Proposition 16. (a) If $(p, q) \in \Psi$, then $\mathbb{P}(p, q) \neq \emptyset$.
(b) If $(p, q) \in \Psi$, then $J\left(R_{p, q}^{[1]}\right)=J_{-2 p, q}+J_{2 p, q}$.
(c) The set $\left\{(p, q, x) \in \mathbb{Q}^{3} \mid(p, q) \in \Psi\right.$ and $\left.x \in J\left(R_{p, q}^{[1]}\right)\right\}$ is diophantine.

Proof: (a) Assume $(p, q) \in \Psi$. Multiplying $p$ or $q$ by nonzero rational squares does not change $R_{p, q}^{[1]}$ or $J_{-2 p, q}$ or $J_{2 p, q}$, so we can assume that $p$ and $q$ are squarefree positive integers. Since $p \equiv 1\left(\bmod 8 \mathbb{Z}_{(2)}\right)$ and $q \equiv 3($ $\bmod 8 \mathbb{Z}_{(2)}$, we have, by Observation $5,(2 p, q)_{2}=-1$. By Hilbert reciprocity, there must also be an odd prime $l$ such that $(2 p, q)_{l}=-1$. By definition of $\Psi$ and, again, by Observation [5, this implies that $l \in\{1,3\}+8 \mathbb{Z}_{(2)}$ and
$l \notin \mathbb{P}^{[3]}(q)$. These two conditions imply $(-1, q)_{l}=1$. Multiplying yields $(-2 p, q)_{l}=-1$. Thus $l \in \mathbb{P}(p, q)$.
(b) is immediate from (a) and Lemma (14,
(c) follows from Corollary 15 (c), from (b) and the note preceding Lemma 13.

## Step 4: From existential to universal

Let $R$ be a semilocal subring of $\mathbb{Q}$, i.e., $R=\bigcap_{l \in \Delta} \mathbb{Z}_{(l)}$ for some finite $\Delta \subseteq \mathbb{P}$. Define

$$
\widetilde{R}:=\{x \in \mathbb{Q} \mid \neg \exists y \in J(R) \text { with } x \cdot y=1\} .
$$

Lemma 17. (a) If $J(R)$ is diophantine in $\mathbb{Q}$ then $\widetilde{R}$ is defined by a universal formula in $\mathbb{Q}$.
(b) $\widetilde{R}=\bigcup_{l \in \Delta} \mathbb{Z}_{(l)}$, provided $\Delta \neq \emptyset$, i.e., provided $R \neq \mathbb{Q}$.
(c) In particular, if $R=\mathbb{Z}_{(p)}$ for some $p \in \mathbb{P}$ then $\widetilde{R}=R$.

Proof: (a) is obvious from the definition of $\widetilde{R}$, and (c) is a special case of (b). So we only need to prove (b).

For the inclusion ' $\subseteq$ ', pick $x \in \widetilde{R}$ and assume that $x \notin \bigcup_{l \in \Delta} \mathbb{Z}_{(l)}$. Then for all $l \in \Delta, v_{l}(x)<0$, and hence $y:=x^{-1} \in \bigcap_{l \in \Delta} l \mathbb{Z}_{(l)}=J(R)$, contradicting our assumption that $x \in \widetilde{R}$.

For the converse inclusion ' $\supseteq$ ', assume $x \in \mathbb{Z}_{(l)}$ for some $l \in \Delta$. Then, for any $y \in J(R), x \cdot y \in l \mathbb{Z}_{(l)}$, so, in particular $x \cdot y \neq 1$.

Now we can give our universal definition of $\mathbb{Z}$ in $\mathbb{Q}$ :
Proposition 18. (a)

$$
\mathbb{Z}=\widetilde{\mathbb{Z}_{(2)}} \cap\left(\bigcap_{k=3,5,7} \bigcap_{p \in \Phi_{k}} \widetilde{R_{p}^{[k]}}\right) \cap \bigcap_{(p, q) \in \Psi} \widetilde{R_{p, q}^{[1]}},
$$

where $\Phi_{k}$ and $\Psi$ are the diophantine sets defined in Corollary 15 ,
(b) for any $t \in \mathbb{Q}$,

$$
\begin{aligned}
t \in \mathbb{Z} \Longleftrightarrow & t \in \widetilde{\mathbb{Z}_{(2)}} \wedge \\
& \forall p \bigwedge_{k=3,5,7}\left(t \in \widetilde{R_{p}^{[k]}} \vee p \notin \Phi_{k}\right) \wedge \\
& \forall p, q\left(t \in \widetilde{R_{p, q}^{[1]}} \vee(p, q) \notin \Psi\right)
\end{aligned}
$$

(c) (Theorem 1) There is a natural number $n$ and a polynomial $g \in$ $\mathbb{Z}\left[t ; x_{1}, \ldots, x_{n}\right]$ such that, for any $t \in \mathbb{Q}$,

$$
t \in \mathbb{Z} \text { iff } \forall x_{1} \ldots \forall x_{n} \in \mathbb{Q} g\left(t ; x_{1}, \ldots, x_{n}\right) \neq 0 .
$$

Proof: (a) The equation is valid by Proposition 10 and Lemma 17(b), (c).
(b) This is a reformulation of (a) revealing that the formula thus obtained for $\mathbb{Z}$ in $\mathbb{Q}$ is universal: the $\widetilde{R}$ 's are universal by Corollary 15, Proposition 16 and Lemma 17(a); $\Phi_{k}$ and $\Psi$ are existential by Corollary 15(a) and (c), so their negation is universal as well.
(c) This is immediate from (b).

## 3 More diophantine predicates in $\mathbb{Q}$

From the results and techniques of section 2, one obtains new diophantine predicates in $\mathbb{Q}$. They are of interest in their own right, but maybe they can also be used to show that Hilbert's 10th problem over $\mathbb{Q}$ cannot be solved, not by defining or interpreting $\mathbb{Z}$ in $\mathbb{Q}$, but, e.g., by assigning graphs to the various finite sets of primes encoded in these predicates, and using graph theoretic undecidability results. We will also use some of these new predicates for our $\forall \exists$-definition of $\mathbb{Z}$ in $\mathbb{Q}$ which uses just one universal quantifier (Corollary 21).

Before listing the new diophantine predicates we shall first prove the following

Lemma 19. Assume $p \in \Phi_{1}$ and defind

$$
R_{p}^{[1]}:=\left\{x \in \mathbb{Q} \mid \exists q \text { with }(p, q) \in \Psi, q \in\left(R_{p, q}^{[1]}\right)^{\times} \text {and } x \in R_{p, q}^{[1]}\right\} .
$$

Then $R_{p}^{[1]}$ is diophantine in $\mathbb{Q}$ and $R_{p}^{[1]}=\bigcup_{l \in \mathbb{P}(p)} \mathbb{Z}_{(l)}$ (which is $\emptyset$ if $\mathbb{P}(p)=\emptyset$, i.e., if $p \in \mathbb{Q}^{2}$ ).

In particular, if $p$ is a prime $\equiv 1 \bmod 8$ then $R_{p}^{[1]}=\mathbb{Z}_{(p)}$.
Proof: That $R_{p}^{[1]}$ is diophantine in $\mathbb{Q}$ is immediate from Corollary 15 ,
Assuming $(p, q) \in \Psi$, the condition ' $q \in\left(R_{p, q}^{[1]}\right)^{\times}$' implies that, in the terminology of Proposition $10(\mathrm{c}), \mathbb{P}(p, q) \subseteq \mathbb{P}(p)$ : Suppose $q \in\left(R_{p, q}^{[1]}\right)^{\times}$and

[^3]$l \in \mathbb{P}(p, q)$. Then $v_{l}(q)=0$, so $l \notin \mathbb{P}(q)$. Hence, by the definition of $\mathbb{P}(p, q)$, $l \in \mathbb{P}(p)$ (note that, by Observation 5, $2 \notin \mathbb{P}(p, q)$, as $\left.(p, q) \in \Phi_{1} \times \Phi_{3}\right)$.

This yields the last inclusion in

The first equality is by definition, the second by Proposition 10(c) using that, from Proposition 16(a), $\mathbb{P}(p, q) \neq \emptyset$.

Conversely, suppose $l \in \mathbb{P}(p)$ and $x \in \mathbb{Z}_{(l)}$. Choose a prime $q \equiv 3 \bmod 8$ with $\left(\frac{l}{q}\right)=-1$ and with $\left(\frac{l^{\prime}}{q}\right)=1$ for each $l^{\prime} \in \mathbb{P}(p) \backslash\{l\}$.

Then $\left(\frac{p q^{-v_{q}(p)}}{q}\right)=-1$, so $(p, q) \in \Psi$ : note that $v_{q}(p)$ is even since $p \in \Phi_{1}$, so $\phi_{q}\left(p q^{-v_{q}(p)}\right)$ is a non-square in $\mathbb{F}_{q}$, i.e., $\in 2 \cdot\left(\mathbb{F}_{q}^{\times}\right)^{2}$; hence $p \in$ $2 \cdot\left(\mathbb{Q}^{\times}\right)^{2}\left(1+q \mathbb{Z}_{(q)}\right)$.

Therefore, by Proposition 10(c) and the Quadratic Reciprocity Law, $\mathbb{P}(p, q)=\{l\}:$ Clearly $l \in \mathbb{P}(p, q)$ as $l \in \mathbb{P}(p)$ and $\left(\frac{q}{l}\right)=\left(\frac{l}{q}\right)=-1$ $\left(p \in \Phi_{1}\right.$, so $\left.l \equiv 1 \bmod 8\right)$; and for any $l^{\prime} \in \mathbb{P}(p) \backslash\{l\},\left(\frac{l^{\prime}}{q}\right)=\left(\frac{q}{l^{\prime}}\right)=1$, so $l^{\prime} \notin \mathbb{P}(p, q)$; finally $\mathbb{P}(q)=\{q\}$, but $\left(\frac{2 p q^{-v_{q}(2 p)}}{q}\right)=1$, hence $q \notin \mathbb{P}(p, q)$.

Thus $x \in \mathbb{Z}_{(l)}=R_{p, q}^{[1]}$ and $q \in\left(R_{p, q}^{[1]}\right)^{\times}$.
Proposition 20. For $x, y \in \mathbb{Q}^{\times}$, the following properties are diophantine:
(a) for fixed $k \in\{3,5,7\}$, the property that $x, y \in \Phi_{k}$ and $\mathbb{P}^{[k]}(x) \cap \mathbb{P}^{[k]}(y)=$ $\emptyset$
(b) $x \notin \mathbb{Q}^{2}$
(c) for fixed $k \in\{1,3,5,7\}$, the property that $x \equiv k\left(\bmod 8 \mathbb{Z}_{(2)}\right)$ and $x \notin \Phi_{k}$
(d) for fixed $k \in\{3,5,7\}$, the property that $\mathbb{P}^{[k]}(x)=\emptyset$
(e) $x \notin N(y)$, where $N(y)$ is the image of the norm $\mathbb{Q}(\sqrt{y}) \rightarrow \mathbb{Q}$

Proof: (a) By Corollary 15(a), $\Phi_{k}$ is diophantine. By Corollary 15)(b), for any $x \in \Phi_{k}, \mathbb{P}^{[k]}(x) \neq \emptyset$ and hence $J\left(R_{x}^{[k]}\right)$ is diophantine. Now let $x, y \in \Phi_{k}$ and recall that, by Proposition [10, $R_{x}^{[k]}=\bigcap_{l \in \mathbb{P}^{[k]}(x)} \mathbb{Z}_{(l)}$, and likewise for $R_{y}^{[k]}$. So we have the equivalence

$$
\mathbb{P}^{[k]}(x) \cap \mathbb{P}^{[k]}(y)=\emptyset \Longleftrightarrow 1 \in J\left(R_{x}^{[k]}\right)+J\left(R_{y}^{[k]}\right)
$$

which then proves (a).
(b) The property that ' $v_{2}(x)$ is odd' is diophantine: $v_{2}(x)$ is odd if and only if $x=2 y z^{2}$ for some $y \in \mathbb{Z}_{(2)}^{\times}$and some $z \in \mathbb{Q}^{\times}$. As the property ' $x<0$ ' is diophantine as well, by Corollary 15(a) and (b), it suffices to show

$$
x \notin \mathbb{Q}^{2} \Longleftrightarrow\left\{\begin{array}{l}
x<0 \text { or } v_{2}(x) \text { is odd or } \\
\exists p \in \Phi_{3} \text { with } x \in 2 \cdot\left(\mathbb{Q}^{\times}\right)^{2} \cdot\left(1+J\left(R_{p}^{[3]}\right)\right)
\end{array}\right.
$$

$' \Rightarrow$ ': Assume that $x \notin \mathbb{Q}^{2}$, that $x>0$ and that $v_{2}(x)$ is even. Multiplying $x$ by a nonzero rational square does not change the truth of either side of the implication, so we may assume that $x=p_{1} \cdots p_{r}$ for distinct odd primes $p_{1}, \ldots, p_{r}$ where $r \geq 1$.

$$
\text { Choose } a_{1} \in \mathbb{Z} \text { with }\left(\frac{a_{1}}{p_{1}}\right)=\left\{\begin{array}{cl}
-1 & \text { if } p_{1} \equiv 1 \bmod 4 \\
1 & \text { if } p_{1} \equiv 3 \bmod 4
\end{array}\right.
$$

and, for $i>1$,

$$
\text { choose } a_{i} \in \mathbb{Z} \text { with }\left(\frac{a_{i}}{p_{i}}\right)= \begin{cases}1 & \text { if } p_{i} \equiv 1 \bmod 4 \\ -1 & \text { if } p_{i} \equiv 3 \bmod 4\end{cases}
$$

Finally, choose a prime $p \equiv 3 \bmod 8$ with $p \equiv a_{i} \bmod p_{i}(i=1, \ldots, r)$. Then, by the Quadratic Reciprocity Law, $\left(\frac{x}{p}\right)=-1$.

Clearly, $p \in \Phi_{3}$. By Lemma 10(b), $R_{p}^{[3]}=\mathbb{Z}_{(p)}$. Hence $x \in 2 \cdot\left(\mathbb{Q}^{\times}\right)^{2}$. $\left(1+J\left(R_{p}^{[3]}\right)\right)$, as $\left(\frac{2}{p}\right)=-1$.
' $\Leftarrow$ ': If $x<0$ or $v_{2}(x)$ is odd then clearly $x \notin \mathbb{Q}^{2}$.
If $x \in 2 \cdot\left(\mathbb{Q}^{\times}\right)^{2} \cdot\left(1+J\left(R_{p}^{[3]}\right)\right)$ for some $p \in \Phi_{3}$ then $\mathbb{P}^{[3]}(p) \neq \emptyset$, and for any $l \in \mathbb{P}^{[3]}(p)$ one has that $v_{l}(x)$ is even and $\left(\frac{x l^{-v_{l}(x)}}{l}\right)=\left(\frac{2}{l}\right)=-1$. Hence $x \notin \mathbb{Q}^{2}$.
(c) By Proposition 10(a), $x \equiv k\left(\bmod 8 \mathbb{Z}_{(2)}\right)$ is diophantine. First assume $x \equiv 1\left(\bmod 8 \mathbb{Z}_{(2)}\right)$. Then $x \notin \Phi_{1}$ if and only if $x \leq 0$ or $x>0$ and, for some $k \in\{3,5,7\}, \mathbb{P}^{[k]}(x) \neq \emptyset$.

This last condition can be expressed diophantinely by distinguishing the cases whether the number of $k \in\{3,5,7\}$ with $\mathbb{P}^{[k]}(x) \neq \emptyset$ is 1,2 or 3 .

If it is 1 , say $\mathbb{P}^{[k]}(x) \neq \emptyset$, then $\sharp \mathbb{P}^{[k]}(x)$ must be even (in order to get $\left.x \equiv 1\left(\bmod 8 \mathbb{Z}_{(2)}\right)\right)$, so we can choose $p \in \mathbb{P}^{[k]}(x)$ and let

$$
y_{k}:=p^{v_{p}(x)} \text { and } y_{k}^{\prime}:=\prod_{l \in \mathbb{P}^{[1]}(x)} l^{v_{l}(x)} . \prod_{l \in \mathbb{P}^{[k]}(x) \backslash\{p\}} l^{v_{l}(x)} .
$$

Then $y_{k}, y_{k}^{\prime} \in \Phi_{k}, \mathbb{P}^{[k]}\left(y_{k}\right) \cap \mathbb{P}^{[k]}\left(y_{k}^{\prime}\right)=\emptyset$ and $x=y_{k} \cdot y_{k}^{\prime}$. By (a), the condition that there exist such $y_{k}, y_{k}^{\prime}$ is diophantine, and, when satisfied, it implies $x \notin \Phi_{1}$.

If $\left\{k \in\{3,5,7\} \mid \mathbb{P}^{[k]}(x) \neq \emptyset\right\}=\left\{k_{1}, k_{2}\right\}$ for distinct $k_{1}, k_{2}$ then both $\sharp \mathbb{P}^{\left[k_{1}\right]}(x)$ and $\sharp \mathbb{P}^{\left[k_{2}\right]}(x)$ must be even, again, and so one constructs similarly $y_{1}, y_{1}^{\prime} \in \Phi_{k_{1}}$ and $y_{2}, y_{2}^{\prime} \in \Phi_{k_{2}}$ with $\mathbb{P}^{\left[k_{i}\right]}\left(y_{i}\right) \cap \mathbb{P}^{\left[k_{i}\right]}\left(y_{i}^{\prime}\right)=\emptyset$ for $i=1,2$ such that $x=y_{1} \cdot y_{1}^{\prime} \cdot y_{2} \cdot y_{2}^{\prime}$.

If $\mathbb{P}^{[k]}(x) \neq \emptyset$ for all three $k \in\{3,5,7\}$ then either all three sets have an even number of elements or all three have an odd number of elements, and in either case it is clear how to proceed along the same lines.

Now assume $x \equiv 3\left(\bmod 8 \mathbb{Z}_{(2)}\right)$. Then $x \notin \Phi_{3}$ if and only if $x \leq 0$ or $x>$ 0 and $\mathbb{P}^{[5]}(x) \neq \emptyset$ or $\mathbb{P}^{[7]}(x) \neq \emptyset$. Here the last condition is diophantine again, distinguishing the cases whether the number of $k \in\{5,7\}$ with $\mathbb{P}^{[k]}(x) \neq \emptyset$ is 1 or 2 etc.

It is clear how similar existential formulas can be written down for ' $x \notin$ $\Phi_{5}$ ' and ' $x \notin \Phi_{7}$ '.
(d) $\mathbb{P}^{[3]}(x)=\emptyset$ if and only if, modulo a nonzero rational square factor, $x$ or $-x$ or $2 x$ or $-2 x$ is a product of primes in $\bigcup_{k=1,5,7} \mathbb{P}^{[k]}$. Note that for a fixed $k \in\{1,5,7\}$, each product of primes in $\mathbb{P}^{[k]}$ can be expressed as a product of one or two factors of elements in $\Phi_{k}$. Hence $\mathbb{P}^{[3]}(x)=\emptyset$ if and only if

$$
\begin{gathered}
\exists y_{1}, \ldots, y_{8}, z \\
\left(y_{1}, y_{2} \in \Phi_{1} \wedge y_{3}, y_{4} \in \Phi_{5} \wedge y_{5}, y_{6} \in \Phi_{7} \wedge y_{7}=-1 \wedge y_{8}=2\right. \\
\left.\wedge \bigvee_{I \subseteq\{1, \ldots, 8\}} x=z^{2} \prod_{i \in I} y_{i}\right)
\end{gathered}
$$

And, again, similar formulas hold for $k=5$ and $k=7$.
(e) $x \notin N(y)$ iff

$$
\begin{aligned}
& (x<0 \wedge y<0) \\
& \vee \bigvee_{k=3,5,7} \exists p \in \Phi_{k} \text { with } \\
& \left(\left(x \in p \cdot\left(\mathbb{Q}^{\times}\right)^{2} \cdot\left(R_{p}^{[k]}\right)^{\times}\right) \wedge\left(y \text { or }-x y \in a_{k} \cdot\left(\mathbb{Q}^{\times}\right)^{2} \cdot\left(1+J\left(R_{p}^{[k]}\right)\right)\right)\right. \\
& \left.\vee\left(y \in p \cdot\left(\mathbb{Q}^{\times}\right)^{2} \cdot\left(R_{p}^{[k]}\right)^{\times}\right) \wedge\left(x \text { or }-x y \in a_{k} \cdot\left(\mathbb{Q}^{\times}\right)^{2} \cdot\left(1+J\left(R_{p}^{[k]}\right)\right)\right)\right) \\
& \vee \exists(p, q) \in \Psi \text { with } q \in\left(R_{p, q}^{[1]}\right)^{\times} \text {and } \\
& \left(\left(x \in p \cdot\left(\mathbb{Q}^{\times}\right)^{2} \cdot\left(R_{p, q}^{[1]}\right)^{\times}\right) \wedge\left(y \text { or }-x y \in q \cdot\left(\mathbb{Q}^{\times}\right)^{2} \cdot\left(1+J\left(R_{p, q}^{[1]}\right)\right)\right)\right. \\
& \left.\vee\left(y \in p \cdot\left(\mathbb{Q}^{\times}\right)^{2} \cdot\left(R_{p, q}^{[1]}\right)^{\times}\right) \wedge\left(x \text { or }-x y \in q \cdot\left(\mathbb{Q}^{\times}\right)^{2} \cdot\left(1+J\left(R_{p, q}^{[1]}\right)\right)\right)\right)
\end{aligned}
$$

where $a_{3}=a_{5}=2$ and $a_{7}=-1$.
This uses Observation5(b) and (c), Corollary 15(b) and (c), the previous parts and the local-global principle for norms.

The first line says that $x \notin N(y)$ over $\mathbb{R}$.
Lines 2-4 say that $x \notin N(y)$ over $\mathbb{Q}_{l}$ for some non-empty set of primes $l \equiv 3,5$ or $7 \bmod 8$ : Fix $k \in\{3,5,7\}$. By Corollary 15(b), $p \in \Phi_{k}$ implies that $\mathbb{P}^{[k]}(p) \neq \emptyset$. We claim that

$$
(x, y)_{l}=-1 \text { for some } l \in \mathbb{P}^{[k]} \Longleftrightarrow \exists p \in \Phi_{k} \text { with }(\ldots)
$$

where (...) is the bracket is line 3 and 4 .
$' \Rightarrow$ ': Assume $l \in \mathbb{P}^{[k]}$ with $(x, y)_{l}=-1$. Let $p=l$. Then $R_{p}^{[k]}=\mathbb{Z}_{l}$ and ' $(\ldots)$ ' says that $v_{l}(x)$ is odd and $y l^{-v_{l}(y)}$ or $-x y l^{-v_{l}(x y)}$ is a quadratic nonresidue $\bmod l$ or the same with $x$ and $y$ swapped. By Observation 5, this is equivalent to $(x, y)_{l}=-1$, so it holds by our assumption.
' $\Leftarrow$ ': Suppose $p \in \Phi_{k}$ satisfies ' $(\ldots)$ '. Then $\mathbb{P}^{[k]}(p) \neq \emptyset$ and, for any $l \in$ $\mathbb{P}^{[k]}(p), v_{l}(x)$ is odd and, by the choice of $a_{k}$, either $y l^{-v_{l}(y)}$ or $-x y l^{-v_{l}(x y)}$ is a quadratic non-residues $\bmod l$ or the same with $x$ and $y$ swapped, so $(x, y)_{l}=-1$.

Lines 5-7 say that $x \notin N(y)$ over $\mathbb{Q}_{l}$ for some non-empty set of primes $l \equiv 1 \bmod 8$. As in the proof of Lemma 19, the condition ' $q \in\left(R_{p, q}^{[1]}\right)^{\times}$, makes sure that, in the terminology of Proposition 10(c), $\mathbb{P}(p, q) \cap \mathbb{P}(q)=\emptyset$, so $\mathbb{P}(p, q) \subseteq \mathbb{P}(p)$. And, by Proposition 16(a), $\mathbb{P}(p, q) \neq \emptyset$. Line 6 and 7 then say that $x \notin N(y)$ over $\mathbb{Q}_{l}$ for any $l \in \mathbb{P}(p, q)$. Note that the role of $a_{k}$ in lines 3 and 4 of being a quadratic non-residue $\bmod l$ for all $l \in \mathbb{P}^{[k]}$ is here taken by $q$ which is a quadratic non-residue for all $l \in \mathbb{P}^{[1]}(p)$ with $(p, q) \in \Psi$.

We could disregard the prime $p=2$, as ' $x \notin N(y)$ ' either happens nowhere locally, or at least at two primes in $\mathbb{P} \cup\{\infty\}$.

The result in (b) was also obtained in [P2] - using a deep result on Châtelet surfaces from [CSS] - our proof is elementary.

Let us also mention that (b) follows from (e): $x \notin \mathbb{Q}^{2} \Leftrightarrow \exists y x \notin N(y)$ (and we did not use (b) in order to prove (e)).

We close this section by showing that there is an $\forall \exists$-definition of $\mathbb{Z}$ in $\mathbb{Q}$ with just one universal quantifier:

Corollary 21. For all $t \in \mathbb{Q}, t \in \mathbb{Z}$ if and only if
$\forall p\left(t \in \mathbb{Z}_{(2)} \wedge\left\{\begin{array}{l}\left(p \in \mathbb{Q}^{2} \cdot\left(2+4 \mathbb{Z}_{(2)}\right)\right) \\ \vee \bigvee_{k=1,3,5,7}\left\{\begin{array}{l}\left(p \neq 0 \wedge p \in \mathbb{Q}^{2} \cdot\left(k+8 \mathbb{Z}_{(2)}\right)\right) \\ \wedge\left(\left(p \notin \Phi_{k}\right) \vee p \in \mathbb{Q}^{2} \vee\left(p \in \Phi_{k} \backslash \mathbb{Q}^{2} \wedge t \in R_{p}^{[k]}\right)\right)\end{array}\right)\end{array}\right.\right.$
Proof: The equivalence holds by Proposition 10(a) and (b) and by Lemma 19 ,

That the resulting formula is of the shape $\forall \exists$ with just one universal quantifier ' $\forall p$ ' follows from Proposition 10, Corollary 15, Lemma 19 and Proposition 20. Note that, under the assumption ' $p \in \mathbb{Q}^{2} \cdot\left(k+8 \mathbb{Z}_{(2)}\right)$ ', the property ' $p \notin \Phi_{k}$ ' is equivalent to ' $p \notin \mathbb{Z}_{(2)}^{\times}$or $\left(p \in k+\mathbb{Z}_{(2)}\right.$ and $\left.p \notin \Phi_{k}\right)$ ' which is diophantine by Proposition 20(c). And ' $p \notin \mathbb{Q}^{2}$ ' is diophantine by 20(b).

## 4 Why $\mathbb{Z}$ should not be diophantine in $\mathbb{Q}$

In this section we show that $\mathbb{Z}$ is not diophantine in $\mathbb{Q}$, provided one believes in a certain version of what one may (arguably) call 'the Bombieri-Lang Conjecture' on varieties with many rational points.

The version of this conjecture in the special case of varieties over $\mathbb{Q}$ on which our result is based is the following (mainly after section F.5.2 of [HS]):

Bombieri-Lang Conjecture Let $V$ be an absolutely irreducible affine or projective positive-dimensional variety over $\mathbb{Q}$ such that $V(\mathbb{Q})$ is Zariski dense in $V$. Then so is

$$
\bigcup_{\phi: A \longrightarrow V} \phi(A(\mathbb{Q})),
$$

where the $\phi: A \rightarrow V$ run through all non-constant $\mathbb{Q}$-rational maps from positive-dimensional abelian varieties $A$ defined over $\mathbb{Q}$ to $V$.

Lemma 22. Assume the Bombieri-Lang Conjecture as above. Let $f \in$ $\mathbb{Q}\left[x_{1}, \ldots, x_{n+1}\right] \backslash \mathbb{Q}\left[x_{1}, \ldots, x_{n}\right]$ be absolutely irreducible and let Let $V=$ $V(f) \subseteq \mathbb{A}^{n+1}$ be the affine hypersurface defined over $\mathbb{Q}$ by $f$. Assume that $V(\mathbb{Q})$ is Zariski dense in $V$. Let $\pi: \mathbb{A}^{n+1} \rightarrow \mathbb{A}^{1}$ be the projection onto the first coordinate. Then $V(\mathbb{Q}) \cap \pi^{-1}(\mathbb{Q} \backslash \mathbb{Z})$ is also Zariski dense in $V$.
(For $n=1$ the Lemma holds unconditionally, by Siegel's Theorem.)
Proof: Choose any $g \in \mathbb{Q}\left[x_{1}, \ldots, x_{n}\right] \backslash\{0\}$. By the Bombieri-Lang Conjecture there are an abelian variety $A$ and a rational map $\phi: A \rightarrow V$, both defined over $\mathbb{Q}$, such that $\phi(A(\mathbb{Q})) \backslash V(g)(\mathbb{Q})$ is infinite (considering $V(g)$ as subset of $\mathbb{A}^{n+1}$ ). By possibly composing $\phi$ with another rational map (from the left), we may assume that $\pi(\phi(A(\mathbb{Q})) \backslash V(g)(\mathbb{Q}))$ is infinite, and that the pole divisor $D$ of $\pi \circ \phi$ is ample. By Corollary 6.2 in [F], there are only finitely many $P \in A(\mathbb{Q}) \backslash D(\mathbb{Q})$ with $\pi(\phi(P)) \in \mathbb{Z}$ (cf. the remarks following Theorem 1 in $[\mathrm{Si}])$. This implies that $(V(\mathbb{Q}) \backslash V(g)(\mathbb{Q})) \cap \pi^{-1}(\mathbb{Q} \backslash \mathbb{Z}) \neq \emptyset$. Since $g$ was arbitrary this shows that $V(\mathbb{Q}) \cap \pi^{-1}(\mathbb{Q} \backslash \mathbb{Z})$ is Zariski dense in $V$.

Corollary 23. Assume the Bombieri-Lang Conjecture as stated above. Then there is no infinite subset of $\mathbb{Z}$ existentially definable in $\mathbb{Q}$. In particular, $\mathbb{Z}$ is not diophantine in $\mathbb{Q}$.

Proof: If $\mathbb{Z}$ contains an infinite subset that is diophantine over $\mathbb{Q}$ then there is a hypersurface $W$ in $\mathbb{A}^{n+1}$ such that $\pi(W(\mathbb{Q})$ ) is infinite (where $\pi$ is as in Lemma 22). Replace $W$ by the Zariski closure $\bar{W}$ of $W(\mathbb{Q})$; this ensures that the irreducible components $V$ of $\bar{W}$ are geometrically irreducible (given by absolutely irreducible polynomials). For at least one such $V$ the set $\pi(V(\mathbb{Q}))$ is infinite (and still contained in $\mathbb{Z}$ : note that $W(\mathbb{Q})=\bar{W}(\mathbb{Q})$ ). This contradicts Lemma 22.

Let us conclude with a collection of closure properties for pairs of models of $\operatorname{Th}(\mathbb{Q})$ (in the ring language), one a substructure of the other, which might have a bearing on the final (unconditional) answer to the question whether or not $\mathbb{Z}$ is diophantine in $\mathbb{Q}$.

Proposition 24. Let $\mathbb{Q}^{\star}, \mathbb{Q}^{\star \star}$ be models of $\operatorname{Th}(\mathbb{Q})$ (i.e. elementary extensions of $\mathbb{Q}$ ) with $\mathbb{Q}^{\star} \subseteq \mathbb{Q}^{\star \star}$, and let $\mathbb{Z}^{\star}$ and $\mathbb{Z}^{\star \star}$ be their rings of integers. Then
(a) $\mathbb{Z}^{\star \star} \cap \mathbb{Q}^{\star} \subseteq \mathbb{Z}^{\star}$.
(b) $\mathbb{Z}^{\star \star} \cap \mathbb{Q}^{\star}$ is integrally closed in $\mathbb{Q}^{\star}$.
(c) $\left(\mathbb{Q}^{\star \star}\right)^{2} \cap \mathbb{Q}^{\star}=\left(\mathbb{Q}^{\star}\right)^{2}$, i.e. $\mathbb{Q}^{\star}$ is quadratically closed in $\mathbb{Q}^{\star \star}$.
(d) If $\mathbb{Z}$ is diophantine in $\mathbb{Q}$ then $\mathbb{Z}^{\star \star} \cap \mathbb{Q}^{\star}=\mathbb{Z}^{\star}$ and $\mathbb{Q}^{\star}$ is algebraically closed in $\mathbb{Q}^{\star \star}$.
(e) $\mathbb{Q}$ is not model complete, i.e., there are $\mathbb{Q}^{\star}$ and $\mathbb{Q}^{\star \star}$ such that $\mathbb{Q}^{\star}$ is not existentially closed in $\mathbb{Q}^{\star \star}$.

Proof: (a) is an immediate consequence of our universal definition of $\mathbb{Z}$ in $\mathbb{Q}$. The very same definition holds for $\mathbb{Z}^{\star}$ in $\mathbb{Q}^{\star}$ and for $\mathbb{Z}^{\star \star}$ in $\mathbb{Q}^{\star \star}$ (it is part of $\operatorname{Th}(\mathbb{Q})$ that all definitions of $\mathbb{Z}$ in $\mathbb{Q}$ are equivalent). So if this universal formula holds for $x \in \mathbb{Z}^{\star \star} \cap \mathbb{Q}^{\star}$ in $\mathbb{Q}^{\star \star}$ it also holds in $\mathbb{Q}^{\star}$, i.e. $x \in \mathbb{Z}^{\star}$.
(b) is true because $\mathbb{Z}^{\star \star}$ is integrally closed in $\mathbb{Q}^{\star \star}$.
(c) follows from the fact that both being a square and, by Proposition 20(b), not being a square are diophantine in $\mathbb{Q}$.
(d) If $\mathbb{Z}$ is diophantine in $\mathbb{Q}$ then $\mathbb{Z}^{\star \star} \cap \mathbb{Q}^{\star} \supseteq \mathbb{Z}^{\star}$ and hence equality holds, by (a).

To show that then also $\mathbb{Q}^{\star}$ is algebraically closed in $\mathbb{Q}^{\star \star}$, let us observe that, for each $n \in \mathbb{N}$,

$$
A_{n}:=\left\{\left(a_{0}, \ldots, a_{n-1}\right) \in \mathbb{Z}^{n} \mid \exists x \in \mathbb{Z} \text { with } x^{n}+a_{n-1} x^{n-1}+\ldots+a_{0}=0\right\}
$$

is decidable: zeros of polynomials in one variable are bounded in terms of their coefficients, so one only has to check finitely many $x \in \mathbb{Z}$. In particular, by (for short) Matiyasevich's Theorem, there is an $\exists$-formula $\phi\left(t_{0}, \ldots, t_{n-1}\right)$ such that

$$
\mathbb{Z} \models \forall t_{0} \ldots t_{n-1}\left(\left\{\forall x\left[x^{n}+t_{n-1} x^{n-1}+\ldots+t_{0} \neq 0\right]\right\} \leftrightarrow \phi\left(t_{0}, \ldots, t_{n-1}\right)\right) .
$$

Since both $A_{n}$ and its complement in $\mathbb{Z}^{n}$ are diophantine in $\mathbb{Z}$, the same holds in $\mathbb{Q}$, by our assumption of $\mathbb{Z}$ being diophantine in $\mathbb{Q}$, i.e. $A_{n}^{\star \star} \cap\left(\mathbb{Q}^{\star}\right)^{n}=A_{n}^{\star}$. As any finite extension of $\mathbb{Q}^{\star}$ is generated by an integral primitive element this implies that $\mathbb{Q}^{\star}$ is relatively algebraically closed in $\mathbb{Q}^{\star \star}$.
(e) Choose a recursivley enumerable subset $A \subseteq \mathbb{Z}$ which is not decidable. Then $B:=\mathbb{Z} \backslash A$ is definable in $\mathbb{Z}$, and hence in $\mathbb{Q}$. If $B$ were diophantine in $\mathbb{Q}$ it would be recursively enumerable. But then $A$ would be decidable: contradiction.

So not every definable subset of $\mathbb{Q}$ is diophantine in $\mathbb{Q}$, and hence $\mathbb{Q}$ is not model complete. Or, in other words, there are models $\mathbb{Q}^{\star}, \mathbb{Q}^{\star \star}$ of $\operatorname{Th}(\mathbb{Q})$ with $\mathbb{Q}^{\star} \subseteq \mathbb{Q}^{\star \star}$ where $\mathbb{Q}^{\star}$ is not existentially closed in $\mathbb{Q}^{\star \star}$.

We are confident that with similar methods as used in this paper one can show for an arbitrary prime $p$ that the unary predicate ' $x \notin \mathbb{Q}^{p}$ ' is also
diophantine. This would imply that, in the setting of the Proposition, $\mathbb{Q}^{\star}$ is always radically closed in $\mathbb{Q}^{\star \star}$. However, we have no bias towards an answer (let alone an answer) to the following (unconditional)
Question 25. For $\mathbb{Q}^{\star} \equiv \mathbb{Q}^{\star \star} \equiv \mathbb{Q}$ with $\mathbb{Q}^{\star} \subseteq \mathbb{Q}^{\star \star}$, is $\mathbb{Q}^{\star}$ always algebraically closed in $\mathbb{Q}^{\star \star}$ ?

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[^1]:    ${ }^{2}$ In the meantime, Theorem 1 has been generalized to arbitrary number fields $K$ : the ring of integers of $K$ is universally definable in $K$ ([Pa]).

[^2]:    ${ }^{3}$ Here we adopt the convention that $\infty$ is even (to include the case that $y=1$ which can only happen when $\Delta_{a, b} \cap \mathbb{P}(c)=\emptyset$, a case that will never be used later).

[^3]:    ${ }^{4}$ We hope the notation $R_{p}^{[1]}$ is not too confusing as the definition is different from that of $R_{p}^{[k]}$ for $k=3,5$ or 7 . The crucial property, however, the 'in particualr', gives the same result as in Corollary 10 (b).

