T-STRUCTURES ON ELLIPTIC FIBRATIONS

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Abstract. We consider t-structures that naturally arise on elliptic fibrations. By filtering the category of coherent sheaves on an elliptic fibration using the torsion pairs corresponding to these t-structures, we prove results describing equivalences of t-structures under Fourier-Mukai transforms.

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1. Introduction

This is the first in a series of articles on elliptic fibrations. In this article, we focus on t-structures that can be constructed using the geometry of an elliptic fibration, and using a relative Fourier-Mukai transform from the derived category of coherent sheaves on the elliptic fibration. In later articles, we will study the relations between these t-structures and those that appear in the study of Bridgeland stability conditions. We will also study the different notions of stabilities associated with them.

The study of t-structures comes into various problems of active interest in algebraic geometry: for instance, explicitly in the study of Bridgeland stability conditions, and implicitly in the construction of stable sheaves.

In the study of Bridgeland stability conditions (see [Bri2, Bri3, ABL, MM, Mac, Sch, MP1, MP2, BMS]), t-structures are part of the definition for a stability condition. A Bridgeland stability condition on a smooth projective variety X is a pair $(Z, \mathcal{A})$ where $\mathcal{A}$ is the heart of a t-structure on $D(X) := D^b(\mathrm{Coh}(X))$, the bounded derived category of coherent sheaves on X, and Z a group homomorphism from the

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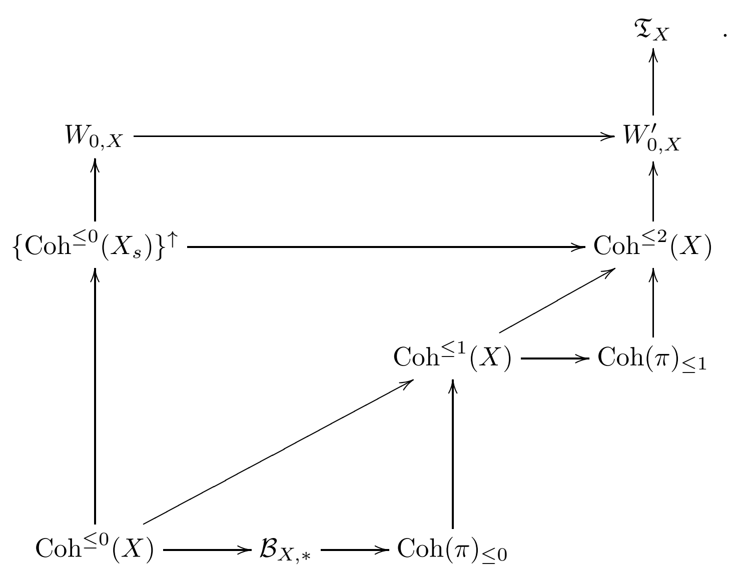
Grothendieck group $K(X)$ to $\mathbb{C}$, with $\mathcal{A}, Z$ satisfying certain properties. Therefore, undertanding t-structures on $D(X)$ has implications on the types of stability conditions that can arise, as well as on which objects in $D(X)$ can arise as stable objects.

In the construction of stable sheaves on varieties, t-structures come into play in the method of spectral construction (see [FMW] and, for instance, [BBR, CDFMR, RP, ARG]), albeit implicitly. For instance, when X admits the structure of an elliptic fibration $\pi : X \to S$, if we have a 'dual' fibration $\widehat{\pi} : Y \to S$ where Y is another elliptic fibration with a Fourier-Mukai transform $\Phi : D(Y) \to D(X)$, then the spectral construction produces stable sheaves on X of the form $\Phi(F)$ where F is a coherent sheaf on Y supported in codimension 1. In our notation in Section 3.2 below, we can view the sheaf $\Phi(F)$ as lying in the category $\Phi(\{\mathrm{Coh}^{\leq 0}(Y_s)\}^\uparrow)$, where $\{\mathrm{Coh}^{\leq 0}(Y_s)\}^\uparrow$ is contained in the heart of a t-structure that is a tilt of the standard t-structure on $D(Y)$.

In this paper, we consider various t-structures on $D(X)$ where X is a smooth projective variety that comes with an elliptic fibration $\pi : X \to S$ and a Fourier-Mukai partner $\widehat{\pi} : Y \to S$ (see Section 2 for the precise assumptions). These t-structures arise from the geometry of X itself, the geometry of the fibration π, and also from the Fourier-Mukai transform between $D(X)$ and $D(Y)$. Each of these t-structures corresponds to a torsion pair in $\mathrm{Coh}(X)$ or $\mathrm{Coh}(Y)$, which in turn is determined by the torsion class in the torsion pair.

When X, Y are elliptic threefolds, the torsion classes in $\mathrm{Coh}(X)$ from which we build all the other torsion classes in this article (by taking intersections and extensions) can be summarised in the following diagram, where each arrow denotes an inclusion of categories:



Here,

- $\mathrm{Coh}^{\leq d}(X)$ is the category of coherent sheaves E on X supported in dimension at most d;
- $\mathrm{Coh}(\pi)_{\leq d}$ is the category of coherent sheaves E on X such that the dimension of $\pi(\mathrm{supp}(E))$ is at most d;

- $\{\mathrm{Coh}^{\leq 0}(X)\}^{\uparrow}$ is the category of coherent sheaves E on X such that the restriction $E|_s$ to the fiber over any closed point $s \in S$ is supported in dimension 0;
- $W_{0,X}$ is the category of coherent sheaves E on X such that $\Psi(E)$ is isomorphic to a coherent sheaf on Y sitting at degree 0 (i.e. the category of the 'Ψ-WIT_0 sheaves'), with Ψ being the Fourier-Mukai trasform $D(X) \to D(Y)$ coming from the construction of Y;
- $W'_{0,X}$ is the category of coherent sheaves E on X such that, for a general closed point $s \in S$, the restriction $E|_s$ to the fiber over s is taken by Ψ_s (the base change of Ψ under the closed immersion $\{s\} \hookrightarrow S$) to a sheaf sitting at degree 0;
- $\mathfrak{T}_X$ is the extension closure of $W'_{0,X}$ and $\Phi(\{\mathrm{Coh}^{\leq 0}(Y_s)\}^{\uparrow})$.
- $\mathcal{B}_{X,*}$ is the extension closure of $\mathrm{Coh}^{\leq 0}(X)$ and a category of fiber sheaves - see (3.9).

Using these subcategories, we filter the category of coherent sheaves on X or Y into smaller pieces with distinct geometric properties, the behaviors of which under the Fourier-Mukai transform $\Psi : D(X) \to D(Y)$ are tractable. As a consequence, we obtain results that explicitly describe how t-structures on $D(X)$ and $D(Y)$ are equivalent, or related by tilts, under a relative Fourier-Mukai transform $\Psi : D(X) \to D(Y)$.

Another underlying motivation for studying the t-structures that appear in this paper is, to some extent, to mitigate the problem that stability is not always well-behaved under base change. Given a flat morphism $\pi : X \to S$ of smooth projective varieties and a torsion-free coherent sheaf E on X, there are various results on how the stability (in the sense of slope stability or Gieseker stability) of E is related to the stability of the restriction of E to the generic fiber of π - see [Bri1, Proposition 7.1], [BM, Lemma 2.1], and [ARG, Proposition 3.3]. In general, however, there seems to be few results relating the stability of E and the stability of its restriction to a special fiber.

To this end, we propose to replace the notion of stability for coherent sheaves on X by the notion of being Ψ-WIT_0. As observed in Lemma 3.6, being WIT_0 is compatible with base change when π is of relative dimension 1, and so, compared to slope stability, it gives us a better behaved notion under the Fourer-Mukai transform Ψ. More precisely, when π is an elliptic fibration and F is a sheaf supported on a fiber of π, that F is WIT_0 is equivalent to all its Harder-Narasimhan (HN) factors having strictly positive slopes [BBR, Corollary 3.29, Proposition 6.38]. Therefore, when a sheaf E on X is Ψ-WIT_0, we can restrict it to a special fiber and still understand the HN filtration after restriction. As a result, even though special cases of the torsion classes $\mathrm{Coh}^{\leq d}(X)$ and $\mathrm{Coh}(\pi)_{\leq d}$ have appeared in the author's earlier article [Lo1] (as the torsion classes $\mathcal{T}_X$ and $\mathcal{B}_X$), by taking the above perspective in this article, we obtain finer results on the behavior of these torsion classes under Fourier-Mukai transforms.

1.1. **Main Results.** Let us write $\mathfrak{C}_X$ to denote the heart of the t-structure on $D(X)$ given by

$$D^{\leq 0}_{\mathfrak{C}} := \{E \in D(X) : H^0(E) \in \mathfrak{T}_X, H^i(E) = 0 \,\forall\, i > 0\},$$
$$D^{\geq 0}_{\mathfrak{C}} := \{E \in D(X) : H^{-1}(E) \in \mathfrak{T}_X^{\circ}, H^i(E) = 0 \,\forall\, i < -1\},$$

where $\mathfrak{T}_X^\circ$ is the torsion-free class in $\mathrm{Coh}(X)$ corresponding to the torsion class $\mathfrak{T}_X$, and write $\mathfrak{D}_Y$ to denote the heart of the t-structure on $D(Y)$ given by

$$D_{\mathfrak{D}}^{\leq 0} := \{E \in D(Y) : H^0(E) \in \mathcal{B}_{Y,*}, H^i(E) = 0 \,\forall\, i > 0\},$$
$$D_{\mathfrak{D}}^{\geq 0} := \{E \in D(Y) : H^{-1}(E) \in \mathcal{B}_{Y,*}^\circ, H^i(E) = 0 \,\forall\, i < -1\},$$

where $\mathcal{B}_{Y,*}^\circ$ is the torsion-free class in $\mathrm{Coh}(Y)$ corresponding to the torsion class $\mathcal{B}_{Y,*}$. That is,

$$\mathfrak{C}_X = \langle \mathfrak{T}_X^\circ[1], \mathfrak{T}_X \rangle \quad \text{and} \quad \mathfrak{D}_Y = \langle \mathcal{B}_{Y,*}^\circ[1], \mathcal{B}_{Y,*} \rangle.$$

Let us also write Λ to denote the composition of the Fourier-Mukai functor $\Psi(-)$ with the derived dual functor $-^\vee$, i.e. $\Lambda(-) = (\Psi(-))^\vee$. Then we have:

Theorem 4.12. *When X is a smooth elliptic surface, the functor Λ induces an equivalence between the t-structure $(D_{\mathfrak{C}}^{\leq 0}, D_{\mathfrak{C}}^{\geq 0})$ on $D(X)$, and the t-structure $(D_{\mathfrak{D}}^{\leq 0}, D_{\mathfrak{D}}^{\geq 0})$ on $D(Y)$. Equivalently, Λ induces an equivalence of hearts*

$$\mathfrak{C}_X \xrightarrow{\sim} \mathfrak{D}_Y[-1]. \tag{1.1}$$

Theorem 4.12 for elliptic surfaces can be considered as a special case of a result of Yoshioka's [Yos3, Proposition 3.3.5] - see the end of Section 4.1, including Lemma 4.13, for a precise explanation of this. Yoshioka's result is more general in the case of elliptic surfaces, as it is stated for tilts of categories of perverse sheaves, as opposed to tilts of categories of coherent sheaves. The result [Yos3, Proposition 3.3.5] was obtained by Yoshioka as part of an argument towards showing an isomorphism between moduli spaces of twisted stable sheaves on elliptic fibrations that are Fourier-Mukai partners (see [Yos3, Proposition 3.4.4] or [Yos1, Theorem 3.15]).

When X and Y are elliptic threefolds, the functor Λ no longer induces an equivalence between the t-structures $(D_{\mathfrak{C}}^{\leq 0}, D_{\mathfrak{C}}^{\geq 0})$ and $(D_{\mathfrak{D}}^{\leq 0}, D_{\mathfrak{D}}^{\geq 0})$. Instead, the hearts of these two t-structures differ by a tilt (up to a shift):

Theorem 4.26. *Suppose X is a smooth elliptic threefold. Then the heart $\Lambda(\mathfrak{C}_X)$ differs from the heart $\mathfrak{D}_Y[-2]$ by one tilt.*

We prove Theorem 4.26 by explicitly studying the effects of Λ on various subcategories of $\mathrm{Coh}(X)$, which arise from a nested sequence of Serre subcategories of $\mathrm{Coh}(X)$ - see Theorem 4.24 and its proof.

The implications of our results on moduli spaces will be addressed in a later paper. The relationships between the t-structures considered in this paper, as well as other t-structures that come up in the study of Bridgeland stability conditions, will also be explored in a later article.

Some of the t-structures considered in this article were already considered in [Lo2] and [CL], which grew out of an attempt to understand the results in [Bri1] and [BM].

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2. Notation

The setup considered in the rest of this article (except for Section 3.1, where the setup is slightly more general) is as follows: we will assume that we have a pair of morphisms of smooth projective varieties $\pi : X \to S$ and $\hat{\pi} : Y \to S$ that satisfies the following conditions:

(i) There is a pair of relative integral functors that are quasi-inverse to each other, up to a shift (so that they are necessarily equivalences):

$$\Psi : D^b(X) \xrightarrow{\sim} D^b(Y) \qquad \text{and} \qquad \Phi : D^b(Y) \xrightarrow{\sim} D^b(X).$$

(ii) The morphisms $\pi, \hat{\pi}$ are both flat.

Note that, by property (i) and our assumption that X, Y, S are all projective, the kernels of the relative integral functors Ψ and Φ both have finite homological dimensions, as complexes of $\mathscr{O}_X$-modules or $\mathscr{O}_Y$-modules, respectively [RMS, Proposition 2.10]. This ensures, that given any morphism of varieties $S' \to S$, the corresponding base changes $\Psi_{S'} : D(X_{S'}) \to D(Y_{S'})$ and $\Phi_{S'} : D(Y_{S'}) \to D(X_{S'})$ still take the bounded derived categories $D^b(X_{S'})$ and $D^b(Y_{S'})$ into each other [BBR, Section 6.1.1].

Let $\pi, \widehat{\pi}$ be as above. The following notations will be used throughout this article:

(1) For any variety W, we will write $D(W) = D^b(\text{Coh}(W))$ to denote the bounded derived category of coherent sheaves on W unless otherwise stated.
(2) For any closed point $s \in S$, we will write ι_s (resp. j_s) to denote the closed immersion of the fiber $X_s \to X$ (resp. $Y_s \to Y$) of π (resp. $\widehat{\pi}$) over s. When E is a coherent sheaf on X (resp. on Y), we will write $E|_s$ to denote the restriction $\iota_s^* E$ (resp. the restriction $j_s^* E$), and write $E|_s^L$ to denote the derived restriction $L\iota_s^* E$ (resp. the derived restriction $Lj_s^* E$).
(3) We will write $\mathcal{B}_X$ to denote the category of coherent sheaves E on X such that $E|_s$ is zero for a general closed point $s \in S$. We similarly define $\mathcal{B}_Y$.
(4) For any Abelian category $\mathcal{A}$ and any $E \in D(\mathcal{A})$, we will write $H^i(E)$ to denote the degree-i cohomology of E with respect to the standard t-structure on $D(\mathcal{A})$. When $\mathcal{B}$ is the heart of a t-structure on $D(\mathcal{A})$, for any $E \in D(\mathcal{A})$, we will write $H^i_{\mathcal{B}}(E)$ to denote the degree-i cohomology of E with respect to the t-structure with heart $\mathcal{B}$. We will also define, for any integers j, k,

$$D^{[j,k]}_{\mathcal{B}} := D^{[j,k]}_{\mathcal{B}}(\mathcal{A}) := \{E \in D(\mathcal{A}) : H^i_{\mathcal{B}}(E) = 0 \text{ for all } i \notin [j,k]\}.$$

(5) For each integer i, we will write $W_{i,X}$ to denote the category of coherent sheaves E on X such that

$$\Psi(E) \cong \widehat{E}[-i] \tag{2.1}$$

for some $\widehat{E} \in \text{Coh}(Y)$, and refer to objects in $W_{i,X}$ as Ψ-WIT$_i$ sheaves on X. For a Ψ-WIT$_i$ sheaf E on X, we will refer to $\widehat{E}$ in (2.1) as the transform of E. We similarly define $W_{i,Y}$ and Φ-WIT$_i$ sheaves for any integer i, with Ψ replaced by Φ and X replaced by Y in the definitions above.

(6) (WIT$_i$ sheaves) For any integer i, we will write $\Psi^i(-)$ to denote the composite functor $H^i(\Psi(-))$, where H^i is the cohomology functor with respect to the standard t-structure on $D(Y)$. Similarly, we write $\Phi^i(-) := H^i(\Phi(-))$.

(7) If a coherent sheaf E on X is supported on a finite number of fibers of π, then we will refer to it as a fiber sheaf. Similarly for coherent sheaves on Y.

(8) (Torsion pairs) Given an Abelian category $\mathcal{A}$, recall that a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$ is a pair of full subcategories of $\mathcal{A}$ satisfying the following two conditions:
   (a) Every $E \in \mathcal{A}$ fits in a short exact sequence in $\mathcal{A}$
   $$0 \to T \to E \to F \to 0$$
   where $T \in \mathcal{T}$ and $F \in \mathcal{F}$;
   (b) $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ for any $T \in \mathcal{T}, F \in \mathcal{F}$.

   Whenever we have a torsion pair $(\mathcal{T}, \mathcal{F})$ in an Abelian category $\mathcal{A}$, we will refer to $\mathcal{T}$ as the torsion class of the torsion pair, and $\mathcal{F}$ as the torsion-free class of the torsion pair. We will say that a subcategory $\mathcal{C}$ of an Abelian category $\mathcal{A}$ is a torsion class if it is the torsion class of a torsion pair in $\mathcal{A}$.

(9) Whenever we have a proper morphism $f : V \to W$ of noetherian schemes, we will write $\mathrm{Coh}(f)_b$ to denote the category of coherent sheaves E on V such that $\dim(f(\mathrm{supp}(E))) = b$ for any $b \geq 0$. We will also write $\mathrm{Coh}(f)_{\leq b}$ to denote the category of coherent sheaves E on X with $\dim(f(\mathrm{supp}(E))) \leq b$. For any $b \geq 0$, the category $\mathrm{Coh}(f)_{\leq b}$ is a Serre subcategory (i.e. closed under subobjects, extensions and quotients in $\mathrm{Coh}(V)$), and in particular, is the torsion class of a torsion pair in $\mathrm{Coh}(V)$. The category $\mathrm{Coh}(f)_0 = \mathrm{Coh}(f)_{\leq 0}$ is precisely the category of fiber sheaves on V. Also, when f is flat of relative dimension 1 and W is irreducible of dimension d, we have $\mathrm{Coh}(f)_{\leq d} = \mathcal{B}_V$.

(10) For any variety W, when $\mathcal{C}_1, \cdots, \mathcal{C}_n$ are subcategories of $D(W)$, we will write $\langle \mathcal{C}_1, \cdots, \mathcal{C}_n \rangle$ to denote the extension closure generated by the $\mathcal{C}_i$ in $D(W)$.

(11) For any noetherian scheme W and any integer $d \geq 0$, we will write $\mathrm{Coh}^{\leq d}(W)$ to denote the category of coherent sheaves on W supported in dimension at most d, write $\mathrm{Coh}^{=d}(W)$ to denote the category of pure d-dimensional coherent sheaves, and write $\mathrm{Coh}^{\geq d}(W)$ to denote the category of coherent sheaves with no nonzero subsheaves supported in dimension $d-1$ or lower.

(12) When W is a smooth projective variety, we sometimes write $\mathcal{T}_W$ to denote the category of coherent sheaves on W that are torsion, and write $\mathcal{F}_W$ to denote the category of coherent sheaves on W that are torsion-free.

(13) For a fixed variety W and a full subcategory $\mathcal{C}$ of $\mathrm{Coh}(W)$, we will define
$$\mathcal{C}^\circ := \{E \in \mathrm{Coh}(W) : \mathrm{Hom}_{\mathrm{Coh}(W)}(C, E) = 0 \text{ for all } C \in \mathcal{C}\}.$$

(14) In a noetherian Abelian category $\mathcal{A}$ (such as $\mathrm{Coh}(W)$ for a noetherian scheme W), whenever we have a full subcategory $\mathcal{C} \subseteq \mathcal{A}$ that is closed under extensions and quotients in $\mathcal{A}$, we have a torsion pair $(\mathcal{C}, \mathcal{C}^\circ)$ in $\mathcal{A}$ by [Pol, Lemma 1.1.3].

(15) Whenever we have a torsion pair $(\mathcal{T}, \mathcal{F})$ in an Abelian category $\mathcal{A}$, there is a corresponding t-structure $(D^{\leq 0}, D^{\geq 0})$ on the derived category $D(\mathcal{A})$

given by

$$D^{\leq 0} := \{E \in D(\mathcal{A}) : H^0(E) \in \mathcal{T}, H^i(E) = 0 \text{ for all } i > 0\},$$
$$D^{\geq 0} := \{E \in D(\mathcal{A}) : H^{-1}(E) \in \mathcal{F}, H^i(E) = 0 \text{ for all } i < -1 \}.$$

(16) (Elliptic fibrations) By an elliptic fibration, we will mean a proper flat morphism $\pi : X \to S$ such that the generic fiber of π is a smooth elliptic curve. (By the flatness of π, it follows that all fibers of π are 1-dimensional.) We will refer to π (or X) as an elliptic surface when $\dim X = 2$, and as an elliptic threefold when $\dim X = 3$.

In addition, we will say $\widehat{\pi}$ is a dual elliptic fibration, or say π and $\widehat{\pi}$ are a pair of dual elliptic fibrations, when $\pi, \widehat{\pi}$ are elliptic fibrations of the same dimension satisfying conditions (i) and (ii), where the kernels of Ψ and Φ in condition (i) are both coherent sheaves sitting at degree 0, flat over both X and Y, and we have $\Phi\Psi = \mathrm{id}_{D(X)}[-1], \Psi\Phi = \mathrm{id}_{D(Y)}[-1]$.

Note that, since Ψ and Φ are assumed to be relative integral functors, all 0-dimensional sheaves on X are Ψ-WIT_0, and are taken to pure 1-dimensional fiber sheaves on Y which are Φ-WIT_1.

Example 2.1. The prototypical examples of dual elliptic fibrations $\pi : X \to S$ and $\widehat{\pi} : Y \to S$ satisfying our definition above include:

- Elliptic surfaces $\pi : X \to S$ considered by Bridgeland in [Bri1], or elliptic threefolds $\pi : X \to S$ considered by Bridgeland-Maciocia in [BM]. In both cases, the fibration $\widehat{\pi} : Y \to S$ is constructed as a relative moduli space of stable sheaves on the fibers of π, and the singular fibers of π are not necessarily irreducible. If $\mathscr{P}$ denotes the universal sheaf on $Y \times X$ for the above moduli problem, then the relative integral functor $\Psi : D(X) \to D(Y)$ with kernel $\mathscr{P}$ is a Fourier-Mukai transform.
- Weierstrass fibrations $\pi : X \to S$ (which are elliptic fibrations) in the sense of [BBR, Section 6.2], where all the fibers are Gorenstein and geometrically integral. In this case, Y can be taken as the Altman-Kleiman compactification of the relative Jacobian of X, and $\Psi : D(X) \to D(Y)$ taken to be the relative Fourier-Mukai transform with the relative Poincaré sheaf as the kernel.

In both cases, a quasi-inverse $\Phi : D(Y) \to D(X)$ can always be constructed making $\pi, \widehat{\pi}$ a pair of dual fibrations in the sense above. In particular, the kernels of Ψ and Φ are both coherent sheaves sitting at degree 0, flat over both factors of $X \times Y$.

3. General constructions on fibrations

3.1. **Base change formulas.** Suppose $\pi : X \to S$ and $\widehat{\pi} : Y \to S$ are a pair of proper morphisms of varieties satisfying properties (i) and (ii) as in the beginning of Section 2. Then we have the following base change formula [BBR, (6.3)]:

$$j_{s*}\Psi_s(E) \cong \Psi(\iota_{s*}E) \quad \text{for all } E \in D(X_s). \tag{3.1}$$

Note that, this is where condition (ii) comes in, since (3.1) depends on the morphism $\hat{\pi}$ being flat.

Assuming additionally that the kernel for the relative integral functor Ψ (resp. Φ) has finite Tor-dimension as a complex of $\mathscr{O}_X$-modules (resp. $\mathscr{O}_Y$-modules), we have the following well-known observation as a consequence of the base change (3.1); we omit its proof:

Lemma 3.1. *For every closed point $s \in S$, the induced integral functors $\Psi_s : D(X_s) \to D(Y_s)$ and $\Phi_s : D(Y_s) \to D(X_s)$ are equivalences.*

The following is a second base change formula useful to us, which depends on π being flat [BBR, (6.2)]: with ι_s, j_s as above, for any $E \in D(X)$ we have

$$Lj_s^* \Psi(E) \cong \Psi_s(L\iota_s^* E) \tag{3.2}$$

This leads to the following observation that we will use frequently:

Lemma 3.2. *For any $E \in D(X)$, we have $\pi(\mathrm{supp}(E)) = \hat{\pi}(\mathrm{supp}(\Psi E))$.*

Proof. Take any $s \in S \setminus \pi(\mathrm{supp}(E))$. Then $0 = E|_s^L$, and we have $0 = \Psi_s(E|_s^L) \cong (\Psi E)|_s^L$ by the base change (3.2), i.e. $s \in S \setminus \hat{\pi}(\mathrm{supp}(\Psi E))$. In other words, we have $\hat{\pi}(\mathrm{supp}(\Psi E)) \subseteq \pi(\mathrm{supp}(E))$. By symmetry, we have equality. □

An immediate consequence of the base change formula (3.1) is the following:

Lemma 3.3. *If l, m are integers such that $\Psi^i(E) = 0$ for all $i \notin [l, m]$ and for all $E \in \mathrm{Coh}(X)$, then we have $\Psi_s^i(F) = 0$ for any closed point $s \in S$, $i \notin [l, m]$ and any $F \in \mathrm{Coh}(X_s)$.*

As a result, if Ψ is a relative integral functor that takes coherent sheaves on X to n-term complexes, then for any closed point $s \in S$, the base change Ψ_s also takes coherent sheaves on the fiber X_s to n-term complexes.

Remark 3.4. From [BBR, Corollary 6.3], we know that, if $n := p + m_0$ where p is the dimension of the fibers of π, and m_0 is the largest index m such that $H^m(K) = 0$, where $K \in D(X \times_S Y)$ is the kernel of the relative integral functor Ψ, then we have, for any $E \in \mathrm{Coh}(X)$, the base change

$$\Psi^n(E)|_s \cong \Psi_s^n(E|_s) \quad \text{for any closed point } s \in S. \tag{3.3}$$

Given Lemma 3.3, we can think of the base change (3.3) as saying: for a coherent sheaf E on X, 'the right-most cohomology of $\Psi(E)$ vanishes if and only if the same holds on each fiber'.

Borrowing notation from [Lo2], we define, for any base change morphism $S' \to S$, the subcategory of $\mathrm{Coh}(X_{S'})$

$$\mathcal{B}_{i,X_{S'}} := \{E \in \mathrm{Coh}(X_{S'}) : \Psi_{S'}^i(E) = 0\}$$

for any integer i. We similarly define $\mathcal{B}_{i,Y_{S'}}$ (using the vanishing of $\Phi_{S'}^i$) for any morphism $S' \to S$. The interpretation of (3.3) at the end of Remark 3.4 can now be stated precisely as follows:

Lemma 3.5. *Let n be as in Remark 3.4. Then for any $E \in \mathrm{Coh}(X)$, we have $E \in \mathcal{B}_{n,X}$ if and only if $E|_s \in \mathcal{B}_{n,X_s}$ for every closed point $s \in S$.*

Proof. From Lemma 3.1, we know Ψ_s is an equivalence. The claim then follows from (3.3). □

When the morphism π has relative dimension 1 and the kernel of Ψ is a sheaf sitting at degree 0, we have $n = 1$ where n is as in Remark 3.4. It then follows that $\mathcal{B}_{1,X} = W_{0,X}$ (and similarly for Y). In other words, we have the following interpretation of WIT_0 sheaves on fibrations of relative dimension 1:

Lemma 3.6. *Suppose π has relative dimension 1, and the kernel of the integral functor Ψ is a sheaf (sitting at degree 0). Then for any $E \in \mathrm{Coh}(X)$, we have that E is Ψ-WIT_0 if and only if $E|_s$ is Ψ_s-WIT_0 for every closed point $s \in S$.*

Proof. This follows from Lemma 3.3, together with Lemma 3.5 with $n = 1$. □

Remark 3.7. Though innocuous-looking, Lemma 3.6 is a key lemma in this article. On an elliptic fibration $\pi : X \to S$, the stability of a sheaf F on X is related to the stability of the restriction of F to the generic fiber of π - but this relation often depends on the Chern classes of F (see [Bri1, Section 7.1] or [BM, Lemma 2.1]). By replacing stability with WIT_i properties, we obtain a framework that is more compatible with base change. For a fiber sheaf on an elliptic fibration that possesses a dual fibration, being $\mathrm{WIT}_i$is inherently related to the structure of its HN filtration with respect to slope stability on the fibers (see [BBR, Corollary 3.29]).

3.2. **Torsion pairs induced from fibers.** Given any morphism of algebraic varieties $\pi : X \to S$, we describe here two recipes for constructing torsion pairs: one restricts a torsion pair on X to torsion pairs on the fibers of π, while the other gives a torsion pair on X induced from torsion pairs on the fibers of π.

Take any subcategory $\mathcal{T}$ of $\mathrm{Coh}(X)$, and fix any closed point $s \in S$. Let ι_s denote the inclusion of the fiber $X_s \hookrightarrow X$. Consider the following two subcategories of $\mathrm{Coh}(X_s)$:

$$\mathcal{T}|_s := \{F \in \mathrm{Coh}(X_s) : F \cong E|_s \text{ for some } E \in \mathcal{T}\};$$
$$\mathcal{T}' := \{F \in \mathrm{Coh}(X_s) : \text{ there exists } E \twoheadrightarrow \iota_{s*}F \text{ in } \mathrm{Coh}(X) \text{ for some } E \in \mathcal{T}\}.$$

The inclusion $\mathcal{T}|_s \subseteq \mathcal{T}'$ is clear. When $\mathcal{T}$ is closed under taking quotients in $\mathrm{Coh}(X)$, we also have the inclusion $\mathcal{T}' \subseteq \mathcal{T}|_s$: for any $F \in \mathcal{T}'$, suppose E is an object in $\mathrm{Coh}(X)$ such that we have a surjection $E \twoheadrightarrow \iota_{s*}F$ in $\mathrm{Coh}(X)$. Then $\iota_{s*}F$ lies in $\mathcal{T}$, and we have $F \cong (\iota_{s*}F)|_s$.

In particular, when $\mathcal{T}$ is the torsion class of a torsion pair in $\mathrm{Coh}(X)$, the two subcategories $\mathcal{T}|_s$ and $\mathcal{T}'$ coincide.

Lemma 3.8. *Let $\mathcal{T}$ be a torsion class in* $\mathrm{Coh}(X)$*. Then, for each closed point $s \in S$, the category $\mathcal{T}|_s$ is a torsion class in* $\mathrm{Coh}(X_s)$*.*

Proof. To show that $\mathcal{T}|_s$ is a torsion class, it suffices to check that it is closed under quotients and extensions in $\mathrm{Coh}(X_s)$. That $\mathcal{T}|_s$ is closed under quotients is clear from the description $\mathcal{T}|_s = \mathcal{T}'$ above. Now, suppose we have $F_1, F_2 \in \mathcal{T}|_s$ and $F_i \cong E_i|_s$ for some $E_i \in \mathcal{T}$ for $i = 1, 2$. Consider any extension in $\mathrm{Coh}(X_s)$

$$0 \to F_1 \to F \to F_2 \to 0,$$

which pushforwards to a short exact sequence in $\mathrm{Coh}(X)$

$$0 \to \iota_{s*}F_1 \to \iota_{s*}F \to \iota_{s*}F_2 \to 0.$$

For each i, we have $\iota_{s*}F_i \in \mathcal{T}$, and so $\iota_{s*}F$ also lies in $\mathcal{T}$. Then $F \in \mathcal{T}|_s$ since $F \cong (\iota_{s*}F)|_s$. □

Definition 3.9. Suppose that, for each closed point $s \in S$, we are given a subcategory $\mathcal{T}_s$ of $\mathrm{Coh}(X_s)$. Then we set

$$\{\mathcal{T}_s\}^{\uparrow} := \{E \in \mathrm{Coh}(X) : E|_s \in \mathcal{T}_s \text{ for all closed points } s \in S\}. \tag{3.4}$$

Lemma 3.10. *Suppose that, for each closed point $s \in S$, we have a torsion class $\mathcal{T}_s$ in $\mathrm{Coh}(X_s)$. Then the category $\{\mathcal{T}_s\}^\uparrow$ is the torsion class of a torsion pair in $\mathrm{Coh}(X)$.*

Proof. Let us write $\mathcal{T}$ to denote $\{\mathcal{T}_s\}^\uparrow$. It suffices to check that $\mathcal{T}$ is closed under quotients and extensions in $\mathrm{Coh}(X)$. That $\mathcal{T}$ is closed under quotients is clear. Now, take any $E_1, E_2 \in \mathcal{T}$ and consider the extension

$$0 \to E_1 \to E \to E_2 \to 0 \quad \text{in } \mathrm{Coh}(X).$$

Fixing any closed point $s \in S$ and restricting the above short exact sequence to X_s, we get the exact sequence

$$E_1|_s \overset{\alpha}{\to} E|_s \to E_2|_s \to 0 \quad \text{in } \mathrm{Coh}(X_s).$$

Since $\mathcal{T}_s$ is closed under quotients, the image of α lies in $\mathcal{T}_s$. Then, because $\mathcal{T}_s$ is closed under extensions, we have $E|_s \in \mathcal{T}_s$. □

Remark 3.11. Given the constructions in Lemmas 3.8 and 3.10, it is natural to ask: are the two constructions described mutually inverse? In other words:

(a) Given a torsion class $\mathcal{T}$ on X, do we have $\{\mathcal{T}|_s\}^\uparrow = \mathcal{T}$?
(b) Given a torsion class $\mathcal{T}_s$ in $\mathrm{Coh}(X_s)$ for each closed point $s \in S$, do we have $(\{\mathcal{T}_s\}^\uparrow)|_s = \mathcal{T}_s$ for each closed point $s \in S$?

In Lemma 3.12 below, we show that the answer to the question (b) is 'yes'. On the other hand, even though we always have the inclusion $\mathcal{T} \subseteq \{\mathcal{T}|_s\}^\uparrow$, without further assumptions on the varieties X, S or the morphism π, the answer to the question (a) is *a priori* a 'no'.

Lemma 3.12. *Suppose that, for each closed point $s \in S$, we have a torsion class $\mathcal{T}_s$ in $\mathrm{Coh}(X_s)$. Then*

$$(\{\mathcal{T}_s\}^\uparrow)|_s = \mathcal{T}_s$$

for each closed point $s \in S$.

Proof. For any closed point $s \in S$ and $F \in \mathcal{T}_s$, we have $\iota_{s*}F \in \{\mathcal{T}_s\}^\uparrow$. Hence $F \cong \iota_s^*(\iota_{s*}F)$ lies in $(\{\mathcal{T}_s\}^\uparrow)|_s$. The other inclusion follows directly from the definitions. □

Example 3.13. Let $\pi : X \to S$ and $\hat{\pi} : Y \to S$ be a pair of proper morphisms satisfying conditions (i) and (ii) laid out in the beginning of Section 2, and let n be as in Remark 3.4. Then we have:

(a) $\mathcal{B}_{n,X}|_s = \mathcal{B}_{n,X_s}$ for any closed point $s \in S$, and
(b) $\{\mathcal{B}_{n,X}|_s\}^\uparrow = \{\mathcal{B}_{n,X_s}\}^\uparrow = \mathcal{B}_{n,X}$.

To see why (a) holds, fix any closed point $s \in S$. That $\mathcal{B}_{n,X}|_s \subseteq \mathcal{B}_{n,X_s}$ follows from (3.3). To show the other inclusion, take any $F \in \mathcal{B}_{n,X_s}$. By (3.1), we have $\iota_{s*}F \in \mathcal{B}_{n,X}$, and so $F \cong \iota_s^*\iota_{s*}F \in \mathcal{B}_{n,X}|_s$, giving us (a). In part (b), the first equality follows from part (a), while the second equality follows from Lemma 3.5. Therefore, $\mathcal{T} = \mathcal{B}_{n,X}$ is an example of a torsion class in $\mathrm{Coh}(X)$ for which the answer to question (a) in Remark 3.11 is 'yes'.

Remark 3.14. Suppose $\pi, \widehat{\pi}$ satisfy conditions (i) and (ii) in the beginning of Section 2, that they both have relative dimension 1, and that the kernels of Ψ and Φ are both sheaves sitting at degree 0. Then by Lemma 3.10, the category $\{\mathrm{Coh}^{\leq 0}(Y_s)\}^\uparrow$

is a torsion class in $\mathrm{Coh}(Y)$, and by Lemma 3.6, every $E \in \{\mathrm{Coh}^{\leq 0}(Y_s)\}^{\uparrow}$ is Φ-WIT$_0$. As a result, the category $\Phi(\{\mathrm{Coh}^{\leq 0}(Y_s)\}^{\uparrow})$ (which will be used frequently later on) is contained in $W_{1,X}$.

We briefly return to the question (a) in Remark 3.11. Let us write H^i to denote the i-th cohomology functor with respect to the standard t-structure on either $D(X)$ or $D(X_s)$, for any closed point $s \in S$. Given a torsion class $\mathcal{T}$ in $\mathrm{Coh}(X)$, let us also write $\mathcal{H}^i$ to denote the i-th cohomology functor with respect to the t-structure on $D(X)$ with heart $\langle \mathcal{T}^\circ[1], \mathcal{T}\rangle$, or the i-th cohomology functor with respect to the t-structure on $D(X_s)$ with heart $\langle (\mathcal{T}|_s)^\circ[1], \mathcal{T}|_s\rangle$, for any closed point $s \in S$. Then, for any coherent sheaf E on X, we have

$$E \in \mathcal{T} \text{ if and only if } \mathcal{H}^i(E) = 0 \text{ for all } i \neq 0,$$

and for any closed point $s \in S$

$$E \in \{\mathcal{T}|_s\}^{\uparrow} \text{ if and only if } \mathcal{H}^i(E|_s) = 0 \text{ for all } i \neq 0.$$

The condition that $\{\mathcal{T}|_s\}^{\uparrow} \subseteq \mathcal{T}$, which is equivalent to having a 'yes' to question (a) in Remark 3.11, is now equivalent to the condition

> For any coherent sheaf $E \in \mathrm{Coh}(X)$, if $\mathcal{H}^i(E|_s) = 0$ for all $i \neq 0$ and all closed points $s \in S$, then $\mathcal{H}^i(E) = 0$,

which can be thought of as a 'Nakayama's Lemma-type' statement.

The following observation on 1-dimensional closed subschemes of the total space of a fibration will be used from time to time:

Lemma 3.15. *Let $\pi : X \to S$ be a proper morphism of varieties of relative dimension 1. Let Z be any irreducible, 1-dimensional closed subscheme of X. Then Z is either of the following two types:*

(i) *Z is contained in a fiber of π;*
(ii) *for any $s \in S$, the intersection $Z \cap \pi^{-1}(s)$ is 0-dimensional if nonempty.*

Proof. Consider the locus

$$D := \{s \in S : Z \cap \pi^{-1}(s) \text{ is 1-dimensional}\}.$$

If D is empty, then Z is of type (ii). Therefore, let us suppose D is nonempty. Then $\pi^{-1}(D)$ is a closed subset of X by semicontinuity. Note that, the dimension of D must be exactly zero, or else Z would have dimension at least 2, a contradiction. That is, D is a finite number of closed points. Now, the intersection $\pi^{-1}(D) \cap Z$ is a 1-dimensional closed subset of Z. By the irreducibility of Z, we have $\pi^{-1}(D) \cap Z = Z$, and D consists of a single point, i.e. Z is of type (i). ☐

3.3. **More torsion classes.** In this section, we introduce a few more torsion classes in $\mathrm{Coh}(X)$ that depend on the geometry of the fibration. Suppose $\pi : X \to S$ and $\hat{\pi} : Y \to S$ are a pair of dual elliptic fibrations. We define

$$W'_{0,X} := \{E \in \mathrm{Coh}(X) : E|_s \text{ is } \Psi_s\text{-WIT}_0 \text{ for a general closed point } s \in S\},$$
$$W'_{1,X} := \{E \in \mathrm{Coh}(X) : E|_s \text{ is } \Psi_s\text{-WIT}_1 \text{ for a general closed point } s \in S\},$$

$$\mathfrak{T}_X := \langle W'_{0,X}, (\Phi(\{\mathrm{Coh}^{\leq 0}(Y_s)\}^{\uparrow}))\rangle. \tag{3.5}$$

Note that, by Lemma 3.3, for any closed point $s \in S$ and any coherent sheaf F on X_s, the functor Ψ_s takes F to a complex of length at most 2, sitting at degrees 0 and 1. Also, we have defined $W'_{0,X}, W'_{1,X}$ so that they contain sheaves that restrict to zero on a general fiber of π. That is, we have $\mathcal{B}_X \subseteq W'_{i,X}$ for $i = 0, 1$. In addition, we have $\mathcal{T}_X \subseteq W'_{0,X}$, i.e. $W'_{0,X}$ contains all the torsion sheaves on X - this is because for a torsion sheaf T on X and a general closed point $s \in S$, the restriction $T|_s$ must be 0-dimensional, which is Ψ_s-WIT$_0$.

We can similarly define $W'_{0,Y}, W'_{1,Y}$ and $\mathfrak{T}_Y$, with X replaced with Y and Ψ replaced with Φ.

Lemma 3.16. *We have*
- (i) $W_{0,X} \subseteq W'_{0,X} \subseteq \mathfrak{T}_X$*;*
- (ii) $\mathfrak{T}_X^\circ \subseteq (W_{0,X})^\circ = W_{1,X} \subseteq W'_{1,X}$.

Proof. Part (i) follows immediately from Lemma 3.6 and the definition of $\mathfrak{T}_X$.

In part (ii), the first inclusion follows from part (i), while the second inclusion follows from $(W_{0,X}, W_{1,X})$ being a torsion pair in $\mathrm{Coh}(X)$ (see, for instance, [BM, Lemma 9.2]). To show the last inclusion of part (ii), take any $E \in W_{1,X}$. By generic flatness, there exists a dense open subscheme $U \subseteq S$ such that both $E|_U$ and $(\widehat{E})|_U$ are flat. By base change [BBR, Proposition 6.1], the restriction $E|_U$ is Ψ_U-WIT$_1$. Then by [BBR, Corollary 6.3(iii)], we have that $E|_s$ is Ψ_s-WIT$_1$ for every closed point $s \in U$, i.e. $E \in W'_{1,X}$. □

Remark 3.17. Since $(W_{0,X}, W_{1,X})$ is a torsion pair in $\mathrm{Coh}(X)$, every coherent sheaf E on X fits in a short exact sequence in $\mathrm{Coh}(X)$

$$0 \to E_0 \to E \to E_1 \to 0 \tag{3.6}$$

where $E_0 \in W_{0,X}$ and $E_1 \in W_{1,X}$. By Lemma 3.16, we can also regard E_0, E_1 as objects in $W'_{0,X}, W'_{1,X}$, respectively.

Lemma 3.18. *For any $E \in \mathrm{Coh}(X)$, let E_0, E_1 be as in* (3.6)*. Then:*
- (i) *If $E \in W'_{i,X}$ (where $i = 0$ or 1), then $E_{1-i} \in \mathcal{B}_X$.*
- (ii) $W'_{1,X} \cap \mathcal{B}_X^\circ \subseteq W_{1,X}$.

Proof. For part (i), consider the short exact sequence (3.6):

$$0 \to E_0 \overset{\alpha}{\to} E \to E_1 \to 0. \tag{3.7}$$

For any closed point $s \in S$, we have the exact sequence

$$E_0|_s \overset{\alpha_s}{\to} E|_s \to E_1|_s \to 0. \tag{3.8}$$

To begin with, suppose $E \in W'_{0,X}$. Then for a general s, the restriction $E_1|_s$ is Ψ_s-WIT$_0$. On the other hand, the base change (3.3) gives $(\widehat{E_1})|_s \cong \Psi_s^1(E_1|_s)$. Thus $(\widehat{E_1})|_s = 0$ for a general $s \in S$, i.e. $\widehat{E_1} \in \mathcal{B}_X$.

Next, suppose $E \in W'_{1,X}$. By Lemma 3.6, the restriction $E_0|_s$ is Ψ_s-WIT$_0$ for every closed point $s \in S$. However, $E|_s$ is Ψ_s-WIT$_1$ for a general closed point $s \in S$ by assumption. Therefore, the map $\alpha|_s$ in (3.8) must be zero for a general closed point $s \in S$. In other words, the injection α in $\mathrm{Coh}(X)$ vanishes when we restrict to a general fiber over S. Hence $E_0 \in \mathcal{B}_X$, and the lemma is proved.

For part (ii), take any $E \in W'_{1,X} \cap \mathcal{B}_X^\circ$. Let E_0, E_1 be as in (3.6). From part (i), we know that $E_0 \in \mathcal{B}_X$, and hence must vanish, implying $E \in W_{1,X}$. □

Lemma 3.19. *We have*
- (i) *The category $W'_{0,X}$ is closed under quotients and extensions.*
- (ii) $W'_{1,X} \cap \mathcal{F}_X = (W'_{0,X})^\circ$.

(iii) *The category $\mathfrak{T}_X$ is closed under quotients and extensions in* $\mathrm{Coh}(X)$.

Proof. Part (i) is clear. For part (ii), let us first show that $W'_{1,X} \cap \mathcal{F}_X \subseteq (W'_{0,X})^\circ$. Take any morphism $\alpha : F \to E$ in $\mathrm{Coh}(X)$ where $E \in W'_{1,X} \cap \mathcal{F}_X, F \in W'_{0,X}$. We want to show that α is the zero map. Since $W'_{0,X}$ is closed under quotients by part (i), we can assume that α is an injection in $\mathrm{Coh}(X)$. Since E is torsion-free, we can also assume that F is torsion-free and nonzero. Then, for a general closed point $s \in S$, $F|_s$ is Ψ_s-WIT$_0$ while $E|_s$ is Ψ_s-WIT$_1$, implying $\alpha|_s = 0$ for a general s. Thus F must be torsion, a contradiction. Hence α must be zero.

For the other inclusion in (ii), take any $E \in (W'_{0,X})^\circ$ that is nonzero. By Remark 3.17, we have $E \in W'_{1,X}$. Since $W'_{0,X}$ contains all the torsion sheaves on X, we also know E is torsion-free. Thus (ii) is proved.

Given part (i), in order to show part (iii), it is enough to show that any quotient of an object F in $\Phi(\{\mathrm{Coh}^{\leq 0}(Y_s)\}^\uparrow)$ lies in $\mathfrak{T}_X$. Consider any surjection $F \overset{\alpha}{\twoheadrightarrow} F'$ in $\mathrm{Coh}(X)$. We can assume that F' lies in $(W'_{0,X})^\circ$. Therefore, from part (ii), we know F' is torsion-free and lies in $W'_{1,X}$. Now, consider the short exact sequence in $\mathrm{Coh}(X)$

$$0 \to K \to F \overset{\alpha}{\to} F' \to 0$$

where $K = \ker(\alpha)$. From Lemma 3.18(ii), we also know $F' \in W_{1,X}$. On the other hand, that $F \in W_{1,X}$ implies $K \in W_{1,X}$. The short exact sequence above is therefore taken by Ψ to the short exact sequence in $\mathrm{Coh}(Y)$

$$0 \to \widehat{K} \to \widehat{F} \to \widehat{F'} \to 0.$$

Since $\widehat{F} \in \{\mathrm{Coh}^{\leq 0}(Y_s)\}^\uparrow$, the same holds for $\widehat{F'}$, and so $F' \in \Phi(\{\mathrm{Coh}^{\leq 0}(Y_s)\}^\uparrow) \subseteq \mathfrak{T}_X$. Thus (iii) holds. □

By Lemma 3.19(iii), we now have a torsion pair $(\mathfrak{T}_X, \mathfrak{T}_X^\circ)$ in $\mathrm{Coh}(X)$. We set

$$\mathfrak{C}_X := \langle \mathfrak{T}_X^\circ[1], \mathfrak{T}_X \rangle.$$

For any subcategory $\mathcal{C}$ of $\mathrm{Coh}(X)$, where X is a smooth projective variety, we will define $\mathcal{C}^D$ to be the subcategory of $\mathrm{Coh}(X)$ consistant of all sheaves of the form $\mathscr{E}xt^c_X(F, \mathscr{O}_X)$, where $F \in \mathcal{C}$ and c is the codimension of the support of F.

Now we define, for an elliptic fibration $\widehat{\pi}$ of any dimension n and when Y is smooth,

$$\mathcal{B}_{Y,*} := \langle \mathrm{Coh}^{\leq 0}(Y), (W_{1,Y} \cap \mathrm{Coh}^{=1}(Y))^D \rangle. \tag{3.9}$$

Remark 3.20. In [HL, Definition 1.1.7], for a coherent sheaf E of codimension c on a smooth projective variety X, the notation E^D denotes the sheaf $\mathscr{E}xt^c_X(E, \omega_X)$ where ω_X is the canonical sheaf on X.

Lemma 3.21. *The category $\mathcal{B}_{Y,*}$ is closed under quotients and extensions in* $\mathrm{Coh}(Y)$.

Proof. Since $\mathcal{B}_{Y,*}$ is defined to be the category generated by $\mathrm{Coh}^{\leq 0}(Y)$ and $(W_{1,Y} \cap \mathrm{Coh}^{=1}(Y))^D$ via extensions, we only need to verify that it is closed under quotients.

Take any nonzero surjection $\alpha : E \twoheadrightarrow T$ in $\mathrm{Coh}(Y)$, where $E \in (W_{1,Y} \cap \mathrm{Coh}^{=1}(Y))^D$. We want to show that $T \in \mathcal{B}_{Y,*}$ as well. Note that, it suffices to assume T is pure 1-dimensional. By definition, $E \cong \mathscr{E}xt^{n-1}(F, \mathscr{O}_Y)$ for some

$F \in W_{1,Y} \cap \mathrm{Coh}^{=1}(Y)$. That F is pure 1-dimensional implies $E \cong F^\vee[n-1]$. Hence we have a short exact sequence in $\mathrm{Coh}(Y)$

$$0 \to K \to F^\vee[n-1] \xrightarrow{\alpha} T \to 0 \tag{3.10}$$

where $K := \ker(\alpha)$. Assuming that α is not an isomorphism, we have that K is a pure 1-dimensional sheaf. That is, all the terms in (3.10) are pure 1-dimensional sheaves. Considering (3.10) as an exact triangle in $D(Y)$, then dualising and shifting, we obtain the exact triangle in $D(Y)$

$$T^\vee[n-1] \xrightarrow{\alpha^\vee[n-1]} F \to K^\vee[n-1] \to T^\vee[n] \tag{3.11}$$

where all the terms $T^\vee[n-1], F$ and $K^\vee[n-1]$ are again pure 1-dimensional sheaves. As a result, we have a short exact sequence in $\mathrm{Coh}(Y)$

$$0 \to T^\vee[n-1] \xrightarrow{\alpha^\vee[n-1]} F \to K^\vee[n-1] \to 0.$$

Thus $T^\vee[n-1] \in W_{1,Y} \cap \mathrm{Coh}^{=1}(Y)$, and $T \cong \mathscr{E}xt_Y^{n-1}(T', \mathscr{O}_Y)$ where $T' := T^\vee[n-1]$. Hence $\mathcal{B}_{Y,*}$ is closed under quotients, and we are done. ☐

Remark 3.22. Since $\mathrm{Coh}(Y)$ is a Noetherian abelian category, we have another torsion pair $(\mathcal{B}_{Y,*}, (\mathcal{B}_{Y,*})^\circ)$ in $\mathrm{Coh}(Y)$ by [Pol, Lemma 1.1.3].

Let us define, for an elliptic fibration $\widehat{\pi} : Y \to S$ of any dimension,

$$\mathfrak{D}_Y := \langle \mathcal{B}_{Y,*}^\circ[1], \mathcal{B}_{Y,*} \rangle.$$

Remark 3.23. When Y is an elliptic surface, the objects of $\mathcal{B}_Y$ are exactly the fiber sheaves. Also, since 0-dimensional sheaves on Y are always Φ-WIT_0, the objects of $W_{1,Y} \cap \mathcal{B}_Y$ are exactly the pure 1-dimensional Φ-WIT_1 fiber sheaves in this case. Therefore, we have the following equivalent description of $\mathcal{B}_{Y,*}$ when Y is an elliptic surface:

$$\mathcal{B}_{Y,*} = \langle \mathrm{Coh}^{\leq 0}(Y), (W_{1,Y} \cap \mathcal{B}_Y)^D \rangle. \tag{3.12}$$

Remark 3.24. On the other hand, on an elliptic fibration of any dimension, if E is a Φ-WIT_1 pure 1-dimensional sheaf, then E cannot have any subsheaf lying in $\{\mathrm{Coh}^{\leq 0}(Y_s)\}^\uparrow$, which is contained in $W_{0,Y}$. That is, a Φ-WIT_1 pure 1-dimensional sheaf on an elliptic fibration of any dimension is a fiber sheaf.

4. Elliptic fibrations

In this section, we will consider a pair of dual elliptic fibrations $\pi : X \to S$ and $\widehat{\pi} : Y \to S$.

We first prove for elliptic surfaces, that the t-structure on $D(X)$ with heart $\mathfrak{T}_X$ is equivalent to the t-structure on $D(Y)$ with heart $\mathfrak{D}_Y$ (up to a shift) via a derived equivalence from $D(X)$ to $D(Y)$ - see Theorem 4.12. This result is a special case of Yoshioka's result [Yos3, Proposition 3.3.5]. We then extend the above result to the case of elliptic threefolds - see Theorem 4.24 and Theorem 4.26 for the precise statements; in this case, the two t-structures differ by a tilt (in the sense of [HRS, Chap. I, Sec. 2]). Below, we choose to discuss elliptic surfaces and elliptic threefolds separately because the two cases are interesting in their own rights.

Our central idea is to filter coherent sheaves on X or Y using the following torsion classes in $\mathrm{Coh}(X)$ (some of which are Serre subcategories) and their counterparts in $\mathrm{Coh}(Y)$, to the point that we understand the image under the Fourier-Mukai transform Ψ of any subfactor in the filtration:

$$W_{0,X}, W'_{0,X}, \{\mathrm{Coh}^{\leq 0}(X_s)\}^{\uparrow}, \mathfrak{T}_X, \mathcal{B}_{X,*}, \mathrm{Coh}^{\leq d}(X), \mathrm{Coh}(\pi)_{\leq d} \tag{4.1}$$

where $d \geq 0$.

In Yoshioka's work [Yos1], he considered an elliptic surface $\pi : X \to S$ with a zero section σ, where all the fibers of π are integral. After identifying a compactification $\widehat{\pi} : Y \to S$ of the relative Jacobian with π itself and using the Poincaré sheaf as the kernel, he proceeded to consider a Fourier-Mukai transform $\Psi : D(X) \to D(X)$. In [Yos1, Theorem 3.15] and [Yos1, Remark 3.5], Yoshioka proved an isomorphism between two moduli spaces of semistable sheaves on X where:

- one of the two moduli spaces parametrises pure 1-dimensional sheaves (so they have rank zero);
- the semistability is with respect to $\sigma + kf, k \gg 0$ where f is a fiber class for the fibration π;
- the isomorphism is induced by the composite functor $(\Psi(-))^\vee$, i.e. the Fourier-Mukai transform Ψ followed by the dualising functor on $D(X)$.

Later, in [Yos3, Proposition 3.4.5], he generalised the above results to the case of twisted semistable perverse coherent sheaves on dual elliptic surfaces that arise as resolutions of singularities.

Following Yoshioka's idea, we will study the functor that is the composition of the Fourier-Mukai transform from an elliptic fibration to its dual, followed by the dualising functor. From now on, we will write $\Lambda(-)$ to denote the composite functor $(\Psi(-))^\vee$, i.e. the Fourier-Mukai functor $\Psi(-) : D(X) \to D(Y)$ followed by the derived dual $-^\vee : D(Y) \to D(Y)$, irrespective of the dimensions of X and Y. We will also write $\Lambda^i(-)$ to denote $H^i(\Lambda(-))$, where $H^i(-)$ is the degree-i cohomology functor with respect to the standard t-structure on $D(Y)$.

4.1. t-structures on elliptic surfaces.

In this section, we assume that $\pi : X \to S$ and $\widehat{\pi} : Y \to S$ are a pair of dual elliptic surfaces.

Lemma 4.1. *Suppose X is an elliptic surface, and $E \in \mathcal{B}_X$. Let E_0, E_1 be as in (3.6). Then $\Lambda E \in D^{[0,1]}_{\mathrm{Coh}(Y)}$, and*

(i) $\Lambda^0 E \cong \mathscr{E}xt^1(\widehat{E_1}, \mathscr{O}_Y)$;

(ii) *there is a short exact sequence in* $\mathrm{Coh}(Y)$

$$0 \to \mathscr{E}xt^2(\widehat{E_1}, \mathscr{O}_Y) \to \Lambda^1 E \to \mathscr{E}xt^1(\widehat{E_0}, \mathscr{O}_Y) \to 0. \tag{4.2}$$

Proof. Take any $E \in \mathcal{B}_X$. From (3.6), we obtain the exact triangle in $D(Y)$

$$\widehat{E_0} \to \Psi(E) \to \widehat{E_1}[-1] \to \widehat{E_0}[1]. \tag{4.3}$$

Taking derived duals, we obtain the exact triangle

$$(\widehat{E_1})^\vee[1] \to \Lambda E \to (\widehat{E_0})^\vee \to (\widehat{E_1})^\vee[2]. \tag{4.4}$$

Since $\widehat{E_0}$ is Φ-WIT$_1$, it has no 0-dimensional subsheaves. Besides, since E_0 is a fiber sheaf, its transform $\widehat{E_0}$ remains a fiber sheaf. Hence $\widehat{E_0}$ is pure 1-dimensional, and thus $\mathscr{E}xt^i(\widehat{E_0}, \mathscr{O}_Y) = 0$ for all $i \neq 1$, meaning $(\widehat{E_0})^\vee \cong \mathscr{E}xt^1(\widehat{E_0}, \mathscr{O}_Y)[-1]$ is a 1-dimensional sheaf sitting at degree 1.

On the other hand, since E_1 is a fiber sheaf, the same holds for $\widehat{E_1}$, and so $\mathscr{E}xt^0(\widehat{E_1}, \mathscr{O}_Y) = 0$. The lemma then follows by taking the long exact sequence of cohomology of (4.4). $\square$

Lemma 4.2. *Suppose X is an elliptic surface, and $E \in \mathcal{T}_X \cap \mathcal{B}_X^\circ$. Then $\Lambda E \in D^{[0,1]}_{\mathrm{Coh}(Y)}$. Furthermore:*

(i) *If $E \in W_{0,X}$, then $\widehat{E}$ is a locally free sheaf.*
(ii) *$\Lambda^1 E$ is a 0-dimensional sheaf that is a quotient of $\mathscr{E}xt^2(\widehat{E_1}, \mathscr{O}_Y)$, where E_1 is as in* (3.6).

Proof. Take any $E \in \mathcal{T}_X \cap \mathcal{B}_X^\circ$, then E has no fiber subsheaves, and in particular, is pure 1-dimensional. Let E_i be as in (3.6). By [Lo1, Corollary 5.4], we know $\widehat{E_0}$ is a locally free sheaf. Thus part (i) holds.

By [Lo1, Lemma 2.6], we know E_1 is a fiber sheaf, and so $(\widehat{E_1})^\vee$ is a 2-term complex sitting at degrees 1 and 2, where the degree-2 cohomology is $\mathscr{E}xt^2(\widehat{E_1}, \mathscr{O}_Y)$, which is 0-dimensional. Since E_0 is a subsheaf of E, we have $E_0 \in \mathcal{T}_X \cap \mathcal{B}_X^\circ$. And by part (i), we know $\widehat{E_0}$ is locally free, and so $\mathscr{E}xt^1(\widehat{E_0}, \mathscr{O}_Y) = 0$. Taking the long exact sequence of cohomology of (4.4) then gives us part (ii) of the lemma. That $\Lambda E \in D^{[0,1]}_{\mathrm{Coh}(Y)}$ also follows from the long exact sequence. $\square$

Before we consider the images of sheaves supported in dimension 2 under the functor Λ, we prove:

Lemma 4.3. *Suppose $\pi : X \to S$ is an elliptic surface or an elliptic threefold. Suppose E is a pure d-dimensional, Ψ-WIT$_0$ sheaf on X.*

(i) *If $E \in \mathrm{Coh}(\pi)_{d-1}$, then $\widehat{E}$ is a pure sheaf of dimension d.*
(ii) *If $E \in \mathrm{Coh}(\pi)_d$ and E has no subsheaves E' in $\mathrm{Coh}(\pi)_{d-1}$, then $\widehat{E}$ is a pure sheaf of dimension $d+1$.*

Proof. (i): suppose E is a pure d-dimensional Ψ-WIT$_0$ sheaf lying in $\mathrm{Coh}(\pi)_{d-1}$. When $d = 3$, E is torsion-free, and the result is just [BM, Lemma 9.4]. When $d = 1$, E is a fiber sheaf, and so $\widehat{E}$ is a Φ-WIT$_1$ fiber sheaf, which is necessarily pure. Now, suppose $d = 2$. Then $\widehat{E} \in \mathrm{Coh}(\hat{\pi})_1$ by Lemma 3.2, and so $\dim \widehat{E} \leq 2$. Suppose there is a nonzero subsheaf T of $\widehat{E}$ where $T \in \mathrm{Coh}_{\leq 1}(Y)$. Since $\widehat{E} \in W_{1,Y}$, we have $T \in W_{1,Y}$ as well. In particular, T cannot have any subsheaf in $\{\mathrm{Coh}^{\leq 0}(Y_s)\}^\uparrow$ by Lemma 3.6. Hence T is forced to be a fiber sheaf. The injection $T \hookrightarrow \widehat{E}$ is then transformed under Φ to a nonzero map $\widehat{T} \to E$, implying E has a fiber subsheaf, contradicting its purity. Hence $\widehat{E}$ must be pure when $d = 2$. This finishes the proof of part (i).

(ii): suppose E is a pure d-dimensional Ψ-WIT$_0$ sheaf, except that now we suppose $E \in \mathrm{Coh}(\pi)_d$. Let us write $Z := \hat{\pi}(\operatorname{supp} \widehat{E}) = \pi(\operatorname{supp} E)$. In this case, the fiber $E|_s$ is 0-dimensional for a general closed point $s \in Z$. We will now show that, for a general closed point $s \in Z$, we have $\dim(\widehat{E}|_s) = 1$. If we write $\widehat{E}_{Z_{red}}$ for the pullback of $\widehat{E}$ along the base change $Z_{red} \hookrightarrow Z \hookrightarrow S$, then it is enough to show $\dim((\widehat{E}|_{Z_{red}})|_s) = 1$ for a general $s \in Z$. That is, we can assume that Z is reduced. Applying generic flatness to E and $\widehat{E}$ with respect to the morphism $X \times_S Z \to Z$ together with [BBR, Corollary 6.3(3)], we obtain that, for a general

$s \in Z$, the restriction $\widehat{E}|_s \cong \widehat{E|_s}$ is 1-dimensional, as wanted. Therefore, we have $\dim \widehat{E} = d+1$.

Now, suppose we have an injection $T \hookrightarrow \widehat{E}$ where $0 \neq T \in \mathrm{Coh}_{\leq d}(Y)$. Then T is Φ-WIT$_1$. We consider the different cases:

- When $d=2$, and X is of dimension 3: if $\hat{\pi}(\mathrm{supp}(T))$ is 2-dimensional (and hence equal to S), then T itself is 2-dimensional. This implies that $T|_s$ is 0-dimensional for a general closed point $s \in S$, and so $T \in W'_{0,Y}$. However, we also have $T \in W'_{1,Y}$ by Lemma 3.16(ii). Thus T lies in $\in W'_{0,Y} \cap W'_{1,Y}$, which is contained in $\mathcal{B}_Y$ by Lemma 3.18, i.e. $\dim(\widehat{\pi}(\mathrm{supp}(T))) \leq 1$. The injection $T \hookrightarrow \widehat{E}$ is then taken by Φ to a nonzero morphism $\widehat{T} \to E$, contradicting the assumption that E has no subsheaves in $\mathrm{Coh}(\pi)_{d-1}$.
- When $d=1$, and X is of dimension 2 or 3: by Lemma 3.15 and Remark 3.14, that $T \in W_{1,Y}$ implies T cannot have any subsheaf in $\{\mathrm{Coh}^{\leq 0}(Y_s)\}^{\uparrow}$ and must be a fiber 1-dimensional sheaf. Then the injection $T \hookrightarrow \widehat{E}$ is taken by Φ to a nonzero morphism $\widehat{T} \to E$, contradicting the assumption that E has no subsheaves lying in $\mathrm{Coh}(\pi)_{d-1}$.

Hence $\widehat{E}$ must be pure of dimension $d+1$, finishing the proof of (ii). □

Lemma 4.3 also yields the following results on reflexivity of sheaves under Fourier-Mukai transforms:

Lemma 4.4. *Let $\pi : X \to S$ be an elliptic surface or an elliptic threefold. Suppose E is a Ψ-WIT$_0$ torsion-free sheaf. Then $\widehat{E}$ is torsion-free, and is reflexive whenever*

$$\mathrm{Ext}^1(W_{0,X} \cap \mathrm{Coh}(\pi)_{\leq 0}, E) = 0.$$

Proof. By Lemma 4.3(i), we know $\widehat{E}$ is pure of codimension 0, so we have a short exact sequence

$$0 \to \widehat{E} \to (\widehat{E})^{DD} \to T \to 0 \tag{4.5}$$

where $(\widehat{E})^{DD}$, being the double dual of $\widehat{E}$ (where the 'dual' of a sheaf is in the sense of [HL, Definition 1.1.7]), is torsion-free and reflexive, while T is a coherent sheaf of codimension at least 2 (which implies $T \in \mathcal{B}_Y$). Note that $\mathrm{Ext}^1(W_{0,Y} \cap \mathcal{B}_Y, \widehat{E}) \cong \mathrm{Hom}(W_{1,X} \cap \mathcal{B}_X, E) = 0$ since E is torsion-free. Now, take any $A \in W_{0,Y} \cap \mathcal{B}_Y$. Applying the functor $\mathrm{Hom}(A, -)$ to (4.5), we get $\mathrm{Hom}(A,T) = 0$. In other words, we have $\mathrm{Hom}(W_{0,Y} \cap \mathcal{B}_Y, T) = 0$. This implies that $T \in W_{1,Y}$, and so $(\widehat{E})^{DD}$ is also Φ-WIT$_1$. On the other hand, since $\dim T \leq 1$, Lemma 3.15 and Remark 3.14 together imply that T must be a fiber sheaf. That is, we have $T \in W_{1,Y} \cap \mathrm{Coh}(\widehat{\pi})_{\leq 0}$. The lemma then follows from

$$\mathrm{Ext}^1(W_{1,Y} \cap \mathrm{Coh}(\widehat{\pi})_{\leq 0}, \widehat{E}) \cong \mathrm{Ext}^1(W_{0,X} \cap \mathrm{Coh}(\pi)_{\leq 0}, E).$$

□

Corollary 4.5. *Suppose X is an elliptic threefold where all the fibers are Cohen-Macaulay with trivial dualising sheaves. If E is a Ψ-WIT$_0$ reflexive torsion-free sheaf, then $\widehat{E}$ is also a reflexive torsion-free sheaf.*

Proof. Since E is reflexive and torsion-free, we have $\mathrm{Ext}^1(\mathrm{Coh}^{\leq 1}(X), E) = 0$ by [CL, Lemma 4.21]. The corollary then follows from Lemma 4.4. □

Lemma 4.6. *Let X be an elliptic threefold. Suppose that:*

- *$E \in W_{0,X}$;*
- *$\mathrm{Hom}(\Phi(\mathrm{Coh}^{\leq 0}(Y)), E) = 0$; and*
- *$\widehat{E}$ is a pure sheaf of dimension at least 2.*

Then $\mathscr{E}xt^2(\widehat{E}, \mathscr{O}_Y) = 0$, and $\Lambda(E) \in D^{[0,1]}_{\mathrm{Coh}(Y)}$. In particular, if $\widehat{E}$ is pure of dimension 2, then $\widehat{E}$ is reflexive.

Proof. Consider the canonical short exact sequence

$$0 \to \widehat{E} \to (\widehat{E})^{DD} \to T \to 0 \tag{4.6}$$

in $\mathrm{Coh}(Y)$ where $T \in \mathrm{Coh}^{\leq 1}(Y)$. Since $(\widehat{E})^{DD}$ is a reflexive sheaf on a threefold, we have $\mathscr{E}xt^i((\widehat{E})^{DD}, \mathscr{O}_Y) = 0$ for $i \geq 2$ regardless of whether $\widehat{E}$ is of dimension 2 or 3 (see [HL, Proposition 1.1.6(ii), Proposition 1.1.10(4')]). On the other hand, from (4.6) we have the exact sequence

$$\mathscr{E}xt^2((\widehat{E})^{DD}, \mathscr{O}_Y) \to \mathscr{E}xt^2(\widehat{E}, \mathscr{O}_Y) \to \mathscr{E}xt^3(T, \mathscr{O}_Y) \to 0,$$

which gives $\mathscr{E}xt^2(\widehat{E}, \mathscr{O}_Y) \cong \mathscr{E}xt^3(T, \mathscr{O}_Y)$. We will now show that $\mathscr{E}xt^3(T, \mathscr{O}_Y)$ vanishes by showing that T is pure 1-dimensional.

Let Q be the maximal 0-dimensional subsheaf of T; we can pull back the short exact sequence (4.6) along the inclusion $Q \hookrightarrow T$, to obtain the following commutative diagram of short exact sequences in $\mathrm{Coh}(Y)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{E} & \longrightarrow & F & \longrightarrow & Q & \longrightarrow & 0\ . \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widehat{E} & \longrightarrow & (\widehat{E})^{DD} & \longrightarrow & T & \longrightarrow & 0 \end{array}$$

Then F is necessarily pure of dimension at least 2, since it is a subsheaf of $(\widehat{E})^{DD}$. However, we have $\mathrm{Ext}^1(Q, \widehat{E}) \cong \mathrm{Hom}(Q, \widehat{E}[1]) \cong \mathrm{Hom}(\Phi(Q), E)$, which vanishes since $\mathrm{Hom}(\Phi(\mathrm{Coh}^{\leq 0}(Y)), E) = 0$ by assumption, implying $F \cong \widehat{E} \oplus Q$, contradicting the purity of F unless $Q = 0$. Hence T is pure 1-dimensional, and we obtain $\Lambda(E) \in D^{[0,1]}_{\mathrm{Coh}(Y)}$. The last assertion of the lemma follows from [HL, Proposition 1.1.10]. $\square$

Corollary 4.7. *Suppose $\pi : X \to S$ is an elliptic threefold. Suppose E lies in $\mathrm{Coh}^{\leq 1}(X) \cap \mathrm{Coh}(\pi)_1$ and has no fiber subsheaves. Then E is Ψ-WIT_0, and its transform $\widehat{E}$ is a 2-dimensional reflexive sheaf.*

Proof. By Lemma 3.15, we have $E \in \Phi(\{\mathrm{Coh}^{\leq 0}(Y_s)\}^{\uparrow})$, which implies E is Ψ-WIT_0 by Remark 3.14. That E has no fiber subsheaves implies $\widehat{E}$ is pure of dimension 2 by Lemma 4.3(ii). Lemma 4.6 gives us the reflexivity of $\widehat{E}$. $\square$

Lemma 4.8. *Suppose X is an elliptic surface, and $E \in \mathrm{Coh}^{=2}(X) \cap W'_{0,X}$. Then $\Lambda E \in D^{[0,1]}_{\mathrm{Coh}(Y)}$, and $\Lambda^1 E$ is a 0-dimensional sheaf.*

Proof. By Lemma 3.18, we have $E_1 \in \mathcal{B}_X$. On the other hand, $\widehat{E_0}$ is pure 2-dimensional by Lemma 4.3. Hence in the exact triangle (4.4), the complex $(\widehat{E_0})^\vee$

lies in $D^{[0,1]}_{\mathrm{Coh}(Y)}$, while $\Lambda E_1 = (\widehat{E_1})^\vee[1]$ also lies in $D^{[0,1]}_{\mathrm{Coh}(Y)}$ by Lemma 4.1. As a result, we have $\Lambda E \in D^{[0,1]}_{\mathrm{Coh}(Y)}$. In particular, we have the exact sequence

$$\mathscr{E}xt^2(\widehat{E_1}, \mathscr{O}_Y) \to \Lambda^1 E \to \mathscr{E}xt^1(\widehat{E_0}, \mathscr{O}_Y) \to 0. \tag{4.7}$$

Since $\widehat{E_0}$ is pure 2-dimensional, the sheaf $\mathscr{E}xt^1(\widehat{E_0}, \mathscr{O}_Y)$ is 0-dimensional, as is $\mathscr{E}xt^2(\widehat{E_1}, \mathscr{O}_Y)$. Hence $\Lambda^1 E$ itself is 0-dimensional. □

Lemma 4.9. *Let X be an elliptic surface, and suppose $E \in \Phi(\{\mathrm{Coh}^{\leq 0}(Y_s)\}^\uparrow)$. Then*

- (i) *$\Lambda E \in D^{[0,1]}_{\mathrm{Coh}(Y)}$ and $\Lambda^1 E$ is a 0-dimensional sheaf;*
- (ii) *E is torsion-free if and only if $\Lambda E \cong \mathscr{E}xt^1(\widehat{E}, \mathscr{O}_Y)$ is a pure 1-dimensional sheaf (lying in $\{\mathrm{Coh}^{\leq 0}(Y_s)\}^\uparrow$) sitting at degree 0.*

Proof. By Lemma 3.6, the category $\Phi(\{\mathrm{Coh}^{\leq 0}(Y_s)\}^\uparrow)$ is contained in $W_{1,X}$. Take any $E \in \Phi(\{\mathrm{Coh}^{\leq 0}(Y_s)\}^\uparrow)$. Then $\widehat{E}$ is Φ-WIT$_0$. Consider the short exact sequence

$$0 \to F_0 \to \widehat{E} \to F_1 \to 0 \tag{4.8}$$

in $\mathrm{Coh}(Y)$ where F_0 is the maximal 0-dimensional subsheaf of $\widehat{E}$. Then both F_0, F_1 are Φ-WIT$_0$, and the dimension of F_1 is 1 if $F_1 \neq 0$; we also have the exact triangle in $D(Y)$

$$F_1^\vee \to (\widehat{E})^\vee \to F_0^\vee \to F_1^\vee[1].$$

Here, $F_0^\vee$ is a 0-dimensional sheaf sitting at degree 2, while $F_1^\vee \cong \mathscr{E}xt^1(F_1, \mathscr{O}_Y)[-1]$. On the other hand, we have $\Psi(E) \cong \widehat{E}[-1]$, and so $\Lambda E \cong (\widehat{E})^\vee[1] \in D^{[0,1]}_{\mathrm{Coh}(Y)}$. Besides, the exact triangle above gives $\Lambda^1 E \cong H^2((\widehat{E})^\vee) \cong H^2(F_0^\vee)$, which is a 0-dimensional sheaf. Thus part (i) holds.

Now, the transform of (4.8) is a short exact sequence in $\mathrm{Coh}(X)$

$$0 \to \widehat{F_0} \to E \to \widehat{F_1} \to 0. \tag{4.9}$$

Since $\widehat{F_0}$ is a fiber sheaf, it must be zero when E is torsion-free, in which case the argument in part (i) shows that $\Lambda E \cong \mathscr{E}xt^1(\widehat{E}, \mathscr{O}_Y)$ is a pure 1-dimensional sheaf sitting at degree 0.

For the 'if' part of part (ii), suppose that $\Lambda E \cong \mathscr{E}xt^1(\widehat{E}, \mathscr{O}_Y)$ is a pure 1-dimensional sheaf sitting at degree 0. Then from the computation of part (i), we see that F_0 vanishes and $E \cong \widehat{F_1}$. However, if $\widehat{F_1}$ had a torsion subsheaf F', then it would be a Ψ-WIT$_1$ fiber sheaf by [Lo1, Lemma 2.6]. This means that F_1 itself would also have a fiber subsheaf, contradicting that F_1 is a pure 1-dimensional sheaf lying in $\{\mathrm{Coh}^{\leq 0}(Y_s)\}^\uparrow$. Thus part (ii) is proved. □

Lemma 4.10. *Suppose X is an elliptic surface and $E \in \mathrm{Coh}(X)$ is a pure 2-dimensional sheaf such that:*

- *$E \in W'_{1,X}$;*
- *$\mathrm{Hom}(\Phi(\{\mathrm{Coh}^{\leq 0}(Y_s)\}^\uparrow), E) = 0$.*

Then $\Lambda(E[1]) \in D^{[0,1]}_{\mathrm{Coh}(Y)}$ and $\Lambda^1(E[1])$ sits in an exact sequence in $\mathrm{Coh}(Y)$ of the form

$$\mathscr{E}xt^1(A, \mathscr{O}_Y) \to \Lambda^1(E[1]) \to \mathscr{E}xt^1(B, \mathscr{O}_Y) \to 0$$

where A is pure 2-dimensional (hence $\mathscr{E}xt^1(A, \mathscr{O}_Y)$ is 0-dimensional), and $B \in W_{1,Y} \cap \mathcal{B}_Y$ (in particular, B is a pure 1-dimensional fiber sheaf).

Proof. Let E_0, E_1 be as in (3.6). By Lemma 3.18, we know $E_0 \in \mathcal{B}_X$. However, we are assuming E to be pure, and so $E = E_1$, and $\Lambda(E[1]) \cong (\widehat{E_1})^\vee$.

Suppose T is the maximal torsion subsheaf of $\widehat{E_1}$. Note that, $\widehat{E_1}$ has no 0-dimensional subsheaves, or else there would be a nonzero map from a fiber sheaf to E_1, contradicting the purity of E. Therefore, T must be a pure 1-dimensional sheaf if it is nonzero, and we have the short exact sequence in $\mathrm{Coh}(Y)$

$$0 \to T \to \widehat{E_1} \to \widehat{E_1}/T \to 0$$

where $\widehat{E_1}/T$ is a pure 2-dimensional sheaf. We thus obtain an exact triangle in $D(Y)$

$$(\widehat{E_1}/T)^\vee \to (\widehat{E_1})^\vee \to T^\vee \to (\widehat{E_1}/T)^\vee[1].$$

Since $\widehat{E_1}/T$ is pure 2-dimensional, the sheaf $\mathscr{E}xt^2(\widehat{E_1}/T, \mathscr{O}_Y)$ vanishes, and so $(\widehat{E_1}/T)^\vee$ has cohomology only in degrees 0 and 1. Also, since T is pure 1-dimensional, we have $\mathscr{E}xt^0(T, \mathscr{O}_Y) = 0 = \mathscr{E}xt^2(T, \mathscr{O}_Y)$. Hence $T^\vee$ is a pure 1-dimensional sheaf sitting at degree 1. Thus $\Lambda(E[1]) \cong (\widehat{E_1})^\vee$ is a 2-term complex sitting in degrees 0 and 1, and we have the exact sequence in $\mathrm{Coh}(Y)$

$$\mathscr{E}xt^1(\widehat{E_1}/T, \mathscr{O}_Y) \to H^1((\widehat{E_1})^\vee) \to \mathscr{E}xt^1(T, \mathscr{O}_Y) \to 0. \tag{4.10}$$

where $\mathscr{E}xt^1(\widehat{E_1}/T, \mathscr{O}_Y)$ is a 0-dimensional sheaf.

To finish off the proof, observe that T has no nonzero subsheaves lying in the category $\{\mathrm{Coh}^{\leq 0}(Y_s)\}^\uparrow$. For, if T had such a subsheaf T', then the image of the composition $T' \hookrightarrow T \hookrightarrow \widehat{E_1} = \widehat{E}$ under Φ would give us a nonzero element in $\mathrm{Hom}(\Phi(\{\mathrm{Coh}^{\leq 0}(Y_s)\}^\uparrow), E)$, contradicting our assumption. Thus T must be a fiber sheaf by Lemma 3.15. Since $E = E_1$ is pure 2-dimensional, we have $\mathrm{Hom}(\mathcal{B}_Y \cap W_{0,Y}, \widehat{E_1}) = 0$, meaning the Φ-WIT$_0$ part of T vanishes, i.e. T is Φ-WIT$_1$ as claimed. □

Pulling the above results together, we can now characterise the image of the heart $\mathfrak{C}_X$ under the functor Λ:

Proposition 4.11. *Let X be an elliptic surface. Then for any $E \in \mathfrak{C}_X$, we have:*

(i) $\Lambda E \in D^{[0,1]}_{\mathrm{Coh}(Y)}$;
(ii) $\Lambda^1 E \in \mathcal{B}_{Y,*}$.

Proof. We have the following inclusions of torsion classes in $\mathrm{Coh}(X)$:

$$\mathcal{B}_X \subseteq \mathcal{T}_X = \mathrm{Coh}^{\leq 1}(X) \subseteq W'_{0,X}.$$

Given any $E \in W'_{0,X}$, we can first find a short exact sequence in $\mathrm{Coh}(X)$

$$0 \to E^0 \to E \to E^1 \to 0$$

where $E^0 \in \mathrm{Coh}^{\leq 1}(X)$ and $E^1 \in \mathrm{Coh}^{=2}(X) \cap W'_{0,X}$, and then another short exact sequence in $\mathrm{Coh}(X)$

$$0 \to E^{0,0} \to E^0 \to E^{0,1} \to 0$$

where $E^{0,0} \in B_X$ and $E^{0,1} \in \mathrm{Coh}^{\leq 1}(X) \cap \mathcal{B}_X^\circ$. Setting $E'' := E^{0,0}$ and $E' := E^0$, we obtain a filtration in $\mathrm{Coh}(X)$

$$E'' \subseteq E' \subseteq E$$

where $E'' \in \mathcal{B}_X$, $E'/E'' \in \mathcal{T}_X \cap \mathcal{B}_X^\circ$ and $E/E' \in W'_{0,X} \cap \mathrm{Coh}^{=2}(X)$. Since $\mathfrak{T}_X$ is defined as the extension closure $\langle W'_{0,X}, \Phi(\{\mathrm{Coh}^{\leq 0}(Y_s)\}^\uparrow)\rangle$, Lemmas 4.1, 4.2, 4.8 and 4.9 together imply $\Lambda(\mathfrak{T}_X) \subset D^{[0,1]}_{\mathrm{Coh}(Y)}$.

Now, by the definition of $\mathfrak{T}_X$ and Lemma 3.16(ii), we see that any object in $(\mathfrak{T}_X)^\circ$ satisfies the hypotheses of Lemma 4.10. This gives us $\Lambda(\mathfrak{T}_X^\circ[1]) \subset D^{[0,1]}_{\mathrm{Coh}(Y)}$. Together with the last paragraph, we now have $\Lambda(\mathfrak{C}_X) \subset D^{[0,1]}_{\mathrm{Coh}(Y)}$, i.e. we have part (i) of the proposition. Part (ii) of the proposition follows from the computations in Lemmas 4.1, 4.2, 4.8, 4.9 and 4.10. □

We now have the following theorem:

Theorem 4.12. *When X is a smooth elliptic surface, the functor $\Lambda(-) := (\Psi(-))^\vee$ induces an equivalence between the t-structure $(D^{\leq 0}_{\mathfrak{C}}, D^{\geq 0}_{\mathfrak{C}})$ on $D(X)$, and the t-structure $(D^{\leq 0}_{\mathfrak{D}}, D^{\geq 0}_{\mathfrak{D}})$ on $D(Y)$. Equivalently, Λ induces an equivalence of hearts*

$$\mathfrak{C}_X \overset{\sim}{\to} \mathfrak{D}_Y[-1]. \tag{4.11}$$

Proof. By Remark 3.22 and Lemma 3.19, the two categories $\mathfrak{C}_X, \mathfrak{D}_Y$ are both hearts of t-structures. Proposition 4.11(i) shows that the functor Λ takes $\mathfrak{C}_X$ to a heart of the form $\langle \mathcal{F}[1], \mathcal{T}\rangle[-1]$, for some torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathrm{Coh}(Y)$. Moreover, Proposition 4.11(ii) shows that $\mathcal{T} \subseteq \mathcal{B}_{Y,*}$. Therefore, to prove the theorem, it remains to show that $\mathcal{B}_{Y,*} \subseteq \mathcal{T}$. That is, it remains to show that every object $E \in \mathcal{B}_{Y,*}$ appears as the degree-1 cohomology of $\Lambda E'$ for some $E' \in \mathfrak{C}_X$. Furthermore, from the construction of $\mathcal{B}_{Y,*}$ and Remark 3.23, it is enough to consider the following two cases:

(1) when $E \cong \mathscr{O}_y$ is a skyscraper sheaf of length 1 supported at a closed point $y \in Y$;
(2) when $E \in (W_{1,Y} \cap \mathcal{B}_Y)^D$.

In case (1), observe that $\widehat{\mathscr{O}_y} \in \mathcal{B}_X \subset \mathfrak{C}_X$, and $\Lambda(\widehat{\mathscr{O}_y}) \cong (\mathscr{O}_y[-1])^\vee \cong \mathscr{O}_y^\vee[1] \cong \mathscr{O}_y[-1]$. This shows that $\mathscr{O}_y \in \mathcal{T}$.

In case (2), suppose $E \in (W_{1,Y} \cap \mathcal{B}_Y)^D$. Then $E \cong \mathscr{E}xt^1_Y(F, \mathscr{O}_Y)$ for some $F \in W_{1,Y} \cap \mathcal{B}_Y$. Then $\widehat{F} \in \mathcal{B}_X \subset \mathfrak{C}_X$, and $\Lambda\widehat{F} \cong F^\vee \cong \mathscr{E}xt^1(F, \mathscr{O}_Y)[-1] = E[-1]$, showing $E \in \mathcal{T}$. This completes the proof of the theorem. □

In [Yos3, Section 3.3], Yoshioka considered a torsion pair $(\overline{\mathfrak{T}}, \overline{\mathfrak{F}})$ in a category of perverse sheaves. When the category of perverse sheaves coincides with $\mathrm{Coh}(X)$, the torsion class $\overline{\mathfrak{T}}$ in $\mathrm{Coh}(X)$ is the category of objects $E \in \mathrm{Coh}(X)$ such that in its relative Harder-Narasimhan filtration with respect to π (see, for instance, [Yos3, (3.5)]), all the subfactors satisfy the inequality $\mu_f \geq 0$. Here, μ_f is a slope function defined as follows: writing f to denote the fiber class of the morphism π, and for any $F \in \mathrm{Coh}(X)$, we set

$$\mu_f(F) := \begin{cases} \frac{c_1(F)\cdot f}{\mathrm{rk}\, F} & \text{if } \mathrm{rk}\,(F) > 0, \\ +\infty & \text{if } \mathrm{rk}\,(F) = 0. \end{cases}$$

When $\pi : X \to S$ is a smooth elliptic surface with a section and integral fibers, it has a relative compactified Jacobian $\widehat{\pi} : Y \to S$ that is a fine moduli space and a universal 'Poincaré' sheaf. For instance, $\widehat{\pi}$ can be taken to be the Altman-Kleiman compactification [BBR, Remark 6.33]. Under this setting, Theorem 4.12

can be considered as a special case (the 'untwisted' case, and where there are no singularities to resolve) of [Yos3, Proposition 3.3.6], in the following precise sense:

Lemma 4.13. *Suppose $\pi : X \to S$ is a smooth elliptic surface with a section and integral fibers, and $\widehat{\pi} : Y \to S$ is a compactification of the relative Jacobian of X, and $\Psi : D(X) \to D(Y)$ is the relative Fourier-Mukai transform with the Poincaré sheaf as the kernel. Then the torsion classes $\overline{\mathfrak{T}}$ and $\mathfrak{T}_X$ in $\mathrm{Coh}(X)$ coincide.*

Proof. From our definition (3.5) of $\mathfrak{T}_X$, it is clear that $\mu_f(F) \geq 0$ for any $F \in \mathfrak{T}_X$. Since $\mathfrak{T}_X$ is closed under quotients, the right-most subfactor in the relative HN filtration of F must have $\mu_f \geq 0$. Hence $\mu_f(F') \geq 0$ for all the subfactors F' in the relative HN filtration of F, and so $\mathfrak{T}_X \subseteq \overline{\mathfrak{T}}$.

For the other inclusion, take any $E \in \overline{\mathfrak{T}}$. Consider the short exact sequence in $\mathrm{Coh}(X)$

$$0 \to E' \to E \to E'' \to 0$$

where $E' \in W'_{0,X}$ and $E'' \in (W'_{0,X})^\circ$. Then E'' is torsion-free and Φ-WIT$_1$. Note that the left-most subfactor in the relative HN filtration of E'' must have $\mu_f \leq 0$, for otherwise that subfactor would lie in $W'_{0,X}$ by [BBR, Corollary 3.29], a contradiction. Since $\overline{\mathfrak{T}}$ is a torsion class in $\mathrm{Coh}(X)$, we have $E'' \in \overline{\mathfrak{T}}$. Therefore, all the subfactors in the relative HN filtration of E'' have $\mu_f = 0$. As a result, $\widehat{E''}|_s$ is a 0-dimensional sheaf for a general closed point $s \in S$ (again by [BBR, Corollary 3.29]). In particular, we have $\widehat{E''} \in \mathrm{Coh}^{\leq 1}(Y)$. By Lemma 3.15, we can then fit $\widehat{E''}$ in a short exact sequence in $\mathrm{Coh}(Y)$

$$0 \to (\widehat{E''})_0 \to \widehat{E''} \to (\widehat{E''})_1 \to 0$$

where $(\widehat{E''})_0 \in \{\mathrm{Coh}^{\leq 0}(Y_s)\}^\uparrow$ and $(\widehat{E''})_1 \in \mathcal{B}_Y$. Note that, all the terms in the above short exact sequence are Φ-WIT$_0$, and so we obtain $E'' \in \langle \mathcal{B}_X, \Phi(\{\mathrm{Coh}^{\leq 0}(Y_s)\}^\uparrow)\rangle \subseteq \mathfrak{T}_X$. Thus $E \in \mathfrak{T}_X$. □

4.2. **t-structures on elliptic threefolds.** In this section, we prove an analogue of Theorem 4.12 on dual elliptic threefolds $\pi : X \to S$ and $\widehat{\pi} : Y \to S$. The strategy is the same as that on elliptic surfaces - we analyse the images of various categories of coherent sheaves under the functor $\Lambda(-) := (\Psi(-))^\vee$.

Note that, since we defined a fiber sheaf on X to be a sheaf supported on a finite number of fibers of π, now that $\pi : X \to S$ is an elliptic threefold, the category $\mathcal{B}_X$ coincides with $\mathrm{Coh}(\pi)_{\leq 1}$, and is strictly larger than the category of fiber sheaves on X, which is precisely $\mathrm{Coh}(\pi)_{\leq 0}$.

Lemma 4.14. *Suppose X is an elliptic threefold. Let E be any fiber sheaf on X. Then:*

(i) *If E is 0-dimensional, then $\Lambda(E) \cong \mathscr{E}xt^2(\widehat{E}, \mathscr{O}_Y)[-2]$.*
(ii) *If E is 1-dimensional and Ψ-WIT$_0$, then $\Lambda(E) \cong (\widehat{E})^\vee \cong \mathscr{E}xt^2(\widehat{E}, \mathscr{O}_Y)[-2]$.*
(iii) *If E is 1-dimensional and Ψ-WIT$_1$, then $\Lambda(E) \cong (\widehat{E})^\vee[1]$ lies in $D^{[1,2]}_{\mathrm{Coh}(Y)}$, with $\Lambda^1 E \cong \mathscr{E}xt^2(\widehat{E}, \mathscr{O}_Y)$ being a pure 1-dimensional fiber sheaf (if nonzero), and $\Lambda^2 E \cong \mathscr{E}xt^3(\widehat{E}, \mathscr{O}_Y)$ being a 0-dimensional sheaf.*

Overall, if E is an arbitrary fiber sheaf on X, then $\Lambda(E)$ only has cohomology at degrees 1 and 2, and $\Lambda^2 E \in \mathcal{B}_{Y,*}$.

Proof. The proofs of statements (i), (ii) and (iii) are all straightforward, and we omit them here. Given any fiber sheaf E on X, i.e. $E \in \mathrm{Coh}(\pi)_0$, we can find a short exact sequence in $\mathrm{Coh}(X)$

$$0 \to E^0 \to E \to E^1 \to 0$$

where $E^0 \in \mathrm{Coh}(\pi)_0 \cap W_{0,X}$ and $E^1 \in \mathrm{Coh}(\pi)_0 \cap W_{1,X}$. We can then find another short exact sequence in $\mathrm{Coh}(X)$

$$0 \to E^{0,0} \to E^0 \to E^{0,1} \to 0$$

where $E^{0,0} \in \mathrm{Coh}^{\leq 0}(X)$ and $E^{0,1} \in \mathrm{Coh}(\pi)_0 \cap W_{0,X} \cap \mathrm{Coh}^{=1}(X)$. As a result, we obtain a filtration in $\mathrm{Coh}(X)$

$$E^{0,0} \subseteq E^0 \subseteq E$$

where $E^{0,0} \in \mathrm{Coh}^{\leq 0}(X)$, $E^0/E^{0,0} \in \mathrm{Coh}^{=1}(X) \cap W_{0,X}$ and $E/E^0 \in \mathrm{Coh}^{=1}(X) \cap W_{1,X}$. The final claim of the lemma then follows from statements (i) through (iii). □

Lemma 4.15. *Suppose X is an elliptic threefold. Let $E \in \mathrm{Coh}^{\leq 1}(X)$ be such that E has no fiber subsheaves. Then:*

(i) *$E \in \mathrm{Coh}(\pi)_1$.*
(ii) *$E \in W_{0,X}$, and $\widehat{E}$ is a 2-dimensional reflexive sheaf.*
(iii) *$\Lambda(E) \cong (\widehat{E})^\vee \cong \mathscr{E}xt^1(\widehat{E}, \mathscr{O}_Y)[-1] \in \left(\mathrm{Coh}^{=2}(Y) \cap \Psi(\{\mathrm{Coh}^{\leq 0}(X_s)\}^\uparrow)\right)[-1]$.*

Proof. Part (i) follows from Lemma 3.15. Part (ii) is just Corollary 4.7, and part (iii) follows easily from part (ii) and [HL, Proposition 1.1.10]. □

Lemma 4.16. *Suppose X is an elliptic threefold. Let $E \in \mathrm{Coh}^{=2}(X) \cap \mathrm{Coh}(\pi)_1$. Then, with E_0, E_1 as in (3.6), we have:*

(i) *If $E_0 \neq 0$, then $\widehat{E_0}$ is pure 2-dimensional and reflexive, and $\Lambda(E_0) \cong \mathscr{E}xt^1(\widehat{E_0}, \mathscr{O}_Y)[-1]$;*
(ii) *$\widehat{E_1} \in \mathrm{Coh}^{\leq 2}(Y)$, and $\Lambda(E_1) \in D^{[0,2]}_{\mathrm{Coh}(Y)}$ with $H^0(\Lambda(E_1)) \cong \mathscr{E}xt^1(\widehat{E_1}, \mathscr{O}_Y)$ (which is pure of dimension 2 if nonzero) and $H^2(\Lambda(E_1)) \cong \mathscr{E}xt^3(\widehat{E_1}, \mathscr{O}_Y)$.*

Overall, $\Lambda(E) \in D^{[0,2]}_{\mathrm{Coh}(Y)}$ with $\Lambda^0 E \cong \mathscr{E}xt^1(\widehat{E_1}, \mathscr{O}_Y)$, $\Lambda^1 E \in \mathrm{Coh}^{\leq 2}(Y)$ and $\Lambda^2 E \in \mathrm{Coh}^{\leq 0}(Y)$.

Proof. If $E_0 \neq 0$, then E_0 is also pure of dimension 2. Then by Lemma 4.3(i), $\widehat{E_0}$ is pure 2-dimensional. That E_0 is pure 2-dimensional also implies the vanishing $\mathrm{Hom}(\Phi(\mathrm{Coh}^{\leq 0}(Y)), E_0) = 0$. Hence $\widehat{E_0}$ is reflexive by Lemma 4.6, and part (i) holds.

For part (ii), note that Lemma 3.2 gives

$$\dim(\hat{\pi}(\mathrm{supp}(\widehat{E_1}))) = \dim(\pi(\mathrm{supp}(E_1))) = 1,$$

and so $\widehat{E_1} \in \mathrm{Coh}^{\leq 2}(Y)$ and $\Lambda(E_1) \in D^{[0,2]}_{\mathrm{Coh}(Y)}$. If $\widehat{E_1}$ is 2-dimensional, then since it is of codimension 1, the sheaf $\mathscr{E}xt^1(\widehat{E_1}, \mathscr{O}_Y)$ is nonzero and is also pure 2-dimensional. The rest of the lemma is clear. □

Lemma 4.17. *Suppose X is an elliptic threefold. Let $E \in \mathrm{Coh}^{=2}(X) \cap \mathrm{Coh}(\pi)_2$, and suppose E has no subsheaves lying in $\mathrm{Coh}(\pi)_1$. Let E_0, E_1 be as in (3.6). Then $\widehat{E_0} \in \mathrm{Coh}^{=3}(Y)$, and*

$$\Lambda E \in \langle \mathrm{Coh}^{\geq 2}(Y), \mathrm{Coh}^{\leq 2}(Y)[-1], \mathcal{B}_{Y,*}[-2]\rangle.$$

Proof. Since $\dim E = \dim(\pi(\mathrm{supp}(E))) = \dim S$, we know $E|_s$ is 0-dimensional for a general closed point $s \in S$. As a result, $E \in W'_{0,X}$ and if we let E_0, E_1 be as in (3.6), then Lemma 3.18 says that $E_1 \in \mathcal{B}_X$.

That E has no subsheaves in $\mathrm{Coh}(\pi)_1$ implies E_0 also has no subsheaves in $\mathrm{Coh}(\pi)_1$. Therefore, by Lemma 4.3(ii), we obtain that $\widehat{E_0}$ is pure of dimension 3. On the other hand, since E_0 is pure 2-dimensional, Lemma 4.6 gives us $\mathscr{E}xt^2(\widehat{E_0}, \mathscr{O}_Y) = 0$. Thus $\Lambda E_0 \cong (\widehat{E_0})^\vee \in D^{[0,1]}_{\mathrm{Coh}(Y)}$ with $\Lambda^0(E_0) \in \mathrm{Coh}^{=3}(Y)$ and $\Lambda^1(E_0) \in \mathrm{Coh}^{\leq 1}(Y)$.

Now, since $E_1 \in \mathcal{B}_X = \mathrm{Coh}(\pi)_{\leq 1}$, we can fit E_1 in a short exact sequence in $\mathrm{Coh}(X)$

$$0 \to T_1 \to E_1 \to E_1/T_1 \to 0 \tag{4.12}$$

where $T_1 \in \mathrm{Coh}^{\leq 1}(X)$ and $E_1/T_1 \in \mathrm{Coh}^{=2}(X) \cap \mathrm{Coh}(\pi)_1$. We can further fit T_1 in a short exact sequence in $\mathrm{Coh}(X)$

$$0 \to T_{1,f} \to T_1 \to T_1/T_{1,f} \to 0 \tag{4.13}$$

where $T_{1,f} \in \mathrm{Coh}(\pi)_0$, and $T_1/T_{1,f} \in \mathrm{Coh}^{=1}(X) \cap (\mathrm{Coh}(\pi)_0)^\circ$. Now, by Lemmas 4.14, 4.15 and 4.16, as well as the filtrations (4.12) and (4.13), we see that

$$\Lambda E_1 \in \langle \mathrm{Coh}^{=2}(Y), \mathrm{Coh}^{\leq 2}(Y)[-1], \mathcal{B}_{Y,*}[-2]\rangle.$$

This, together with the previous paragraph, gives us

$$\Lambda E \in \langle \mathrm{Coh}^{\geq 2}(Y), \mathrm{Coh}^{\leq 2}(Y)[-1], \mathcal{B}_{Y,*}[-2]\rangle$$

as claimed. □

Lemma 4.18. *Suppose X is an elliptic threefold. Let $E \in W'_{0,X} \cap \mathrm{Coh}^{=3}(X)$, and let E_0, E_1 be as in (3.6). Then:*

- (i) *$E_1 \in \mathcal{B}_X$;*
- (ii) *$E_0 \neq 0$ and $\widehat{E_0} \in \mathrm{Coh}^{=3}(Y)$;*
- (iii) *$\Lambda(E_0) \in D^{[0,2]}_{\mathrm{Coh}(Y)}$ with $\Lambda^0(E_0) \cong \mathscr{E}xt^0(\widehat{E_0}, \mathscr{O}_Y)$, $\Lambda^1(E_0) \in \mathrm{Coh}^{\leq 1}(Y)$ and $\Lambda^2(E_0) \in \mathrm{Coh}^{\leq 0}(Y)$.*
- (iv) *$\Lambda^0(E)$ is nonzero, supported in dimension 3 and lies in $\mathrm{Coh}^{\geq 2}(Y)$.*

Overall, we have $\Lambda E \in D^{[0,2]}_{\mathrm{Coh}(Y)}$ with $\Lambda^1 \in \mathrm{Coh}^{\leq 2}(Y)$ and $\Lambda^2 E \in \mathcal{B}_{Y,*}$.

Proof. Part (i) follows from Lemma 3.18. If $E_0 = 0$, then $E = E_1$ would be torsion by part (i), a contradiction; thus $E_0 \neq 0$. That $\widehat{E_0}$ is pure 3-dimensional follows from Lemma 4.3(i). Thus part (ii) holds, and part (iii) follows easily.

Now, by part (i), we know $E_1 \in \mathcal{B}_X$. From the second half of the proof of Lemma 4.17, we also know that

$$\Lambda E_1 \in \langle \mathrm{Coh}^{=2}(Y), \mathrm{Coh}^{\leq 2}(Y)[-1], \mathcal{B}_{Y,*}[-2]\rangle,$$

and so $\Lambda^0(E_1)$ is a pure 2-dimensional sheaf if nonzero. From (3.6), we have the exact triangle in $D^b(Y)$

$$\Lambda(E_1) \to \Lambda(E) \to \Lambda(E_0) \to \Lambda(E_1)[1]$$

and the associated long exact sequence of cohomology:

$$0 \to \Lambda^0(E_1) \to \Lambda^0(E) \overset{\alpha}{\to} \Lambda^0(E_0) \to \Lambda^1(E_1) \to \cdots .$$

From parts (ii) and (iii), we know that $\Lambda^0(E_0) \neq 0$ is nonzero and pure 3-dimensional. Since $\Lambda^1(E_1) \in \mathrm{Coh}^{\leq 2}(Y)$ from above, we see that α is nonzero. Thus $\mathrm{im}\,(\alpha)$ is nonzero and is pure 3-dimensional. Now, we also know $\Lambda^0(E_1) \in \mathrm{Coh}^{=2}(Y)$ from above, and so $\Lambda^0(E)$ must be nonzero, supported in dimension 3, and lies in $\mathrm{Coh}^{\geq 2}(Y)$. □

Lemma 4.19. *Suppose X is an elliptic threefold, and $E \in \Phi(\{\mathrm{Coh}^{\leq 0}(Y_s)\}^{\uparrow})$. Then $\Lambda(E) \in D^{[0,2]}_{\mathrm{Coh}(Y)}$ where $\Lambda^0(E) \cong \mathscr{E}xt^1(T_2, \mathscr{O}_Y)$ is a pure 2-dimensional sheaf if nonzero (here, T_2 denotes the pure 2-dimensional component of $\widehat{E}$), $\Lambda^1(E) \in \mathrm{Coh}^{\leq 1}(Y)$ and $\Lambda^2(E) \in \mathrm{Coh}^{\leq 0}(Y)$.*

Proof. By Lemma 3.6, the sheaf E is Ψ-WIT$_1$. Thus $\Lambda(E) \cong (\widehat{E}[-1])^\vee \cong (\widehat{E})^\vee[1]$. Since $\widehat{E} \in \{\mathrm{Coh}^{\leq 0}(Y_s)\}^{\uparrow} \subseteq \mathrm{Coh}^{\leq 2}(Y)$, we know $(\widehat{E})^\vee \in D^{[1,3]}_{\mathrm{Coh}(Y)}$, and so $\Lambda(E) \in D^{[0,2]}_{\mathrm{Coh}(Y)}$.

Let T_1 be the maximal 1-dimensional subsheaf of $\widehat{E}$, and let $T_2 := \widehat{E}/T_1$. From the short exact sequence $0 \to T_1 \to \widehat{E} \to T_2 \to 0$ in $\mathrm{Coh}(Y)$, we obtain an exact triangle in $D(Y)$

$$T_1 \to \widehat{E} \to T_2 \to T_1[1].$$

Dualising this exact triangle and taking the long exact sequence of cohomology, we obtain $\Lambda^0(E) \cong H^1((\widehat{E})^\vee) \cong H^1(T_2^\vee) \cong \mathscr{E}xt^1(T_2, \mathscr{O}_Y)$. The rest of the lemma follows easily from the long exact sequence. □

Lemma 4.20. *Suppose X is an elliptic threefold. Let $E \in \mathrm{Coh}^{=3}(X) \cap W'_{1,X}$ and suppose $\mathrm{Hom}(\Phi(\{\mathrm{Coh}^{\leq 0}(Y_s)\}^{\uparrow}), E) = 0$. Then $E \in W_{1,X}$, and $\Lambda(E[1]) \in D^{[0,2]}_{\mathrm{Coh}(Y)}$ with $\Lambda^0(E[1]) \cong \mathscr{E}xt^0(E', \mathscr{O}_Y)$, where E' is the torsion-free part of $\widehat{E}$ and is Φ-WIT$_0$, and $\Lambda^1(E[1]) \in \mathrm{Coh}^{\leq 2}(Y)$ while $\Lambda^2(E[1]) \in \mathcal{B}_{Y,*}$.*

Proof. With E as given, let E_0, E_1 be as in (3.6). Then $E_0 \in \mathcal{B}_X$ by Lemma 3.18. Since E is pure 3-dimensional, we have $E_0 = 0$ and so $E \in W_{1,X}$. Hence $\Lambda(E[1]) \cong (\widehat{E})^\vee$.

That E is torsion-free implies

$$\mathrm{Hom}(\mathcal{B}_Y \cap W_{0,Y}, \widehat{E}) = 0. \tag{4.14}$$

Now, let T be the maximal torsion subsheaf of $\widehat{E}$. Note that T could well be the same as $\widehat{E_0}$. Also, let T_1 be the maximal 1-dimensional subsheaf of T and let $T_2 := T/T_1$, which is pure 2-dimensional if nonzero. By the vanishing (4.14), T_1 is pure 1-dimensional, and must be Φ-WIT$_1$ (by Remark 3.14 and Lemma 3.15). We can consider the images of T_1, T_2 and $\widehat{E}/T$ under derived dual $(-)^\vee$ separately:

- $T_1^\vee \cong \mathscr{E}xt^2(T_1, \mathscr{O}_Y)[-2] \in \mathcal{B}_{Y,*}[-2]$.

- $T_2^\vee \in D^{[1,2]}_{\mathrm{Coh}(Y)}$ where $H^1(T_2^\vee) \cong \mathscr{E}xt^1(T_2, \mathscr{O}_Y)$ is pure 2-dimensional, and $H^2(T_2^\vee) \cong \mathscr{E}xt^2(T_2, \mathscr{O}_Y)$ is 0-dimensional.
- $(\widehat{E}/T)^\vee \in D^{[0,2]}_{\mathrm{Coh}(Y)}$ where $H^0((\widehat{E}/T)^\vee) \cong \mathscr{E}xt^0(\widehat{E}/T, \mathscr{O}_Y)$ (and $\widehat{E}/T$, being the quotient of a Φ-WIT$_0$ sheaf, is again Φ-WIT$_0$). Also, $H^1((\widehat{E}/T)^\vee)) \cong \mathscr{E}xt^1(\widehat{E}/T, \mathscr{O}_Y)$ is 1-dimensional while $H^2((\widehat{E}/T)^\vee)) \cong \mathscr{E}xt^2(\widehat{E}/T, \mathscr{O}_Y)$ is 0-dimensional.

From above, we see that $H^0(T^\vee) = 0$. From the short exact sequence in $\mathrm{Coh}(Y)$

$$0 \to T \to \widehat{E} \to \widehat{E}/T \to 0,$$

we obtain the long exact sequence

$$0 \to H^0((\widehat{E}/T)^\vee) \to H^0((\widehat{E})^\vee) \to H^0(T^\vee) \to \cdots.$$

Thus $H^0((\widehat{E})^\vee) \cong H^0((\widehat{E}/T)^\vee) \cong \mathscr{E}xt^0(\widehat{E}/T, \mathscr{O}_Y)$ where $\widehat{E}/T$ is Φ-WIT$_0$ and pure 3-dimensional if nonzero. The rest of the lemma is clear. □

Lemma 4.21. *If $E \in \mathfrak{T}_X^\circ$, then E satisfies the hypotheses of Lemma 4.20.*

Proof. Let $E \in \mathfrak{T}_X^\circ$. Since $\mathfrak{T}_X$ contains all the torsion sheaves and contains $\Phi(\{\mathrm{Coh}^{\leq 0}(Y_s)\}^\uparrow)$, we see that E is pure 3-dimensional and satisfies the vanishing $\mathrm{Hom}(\Phi(\{\mathrm{Coh}^{\leq 0}(Y_s)\}^\uparrow), E) = 0$. Also, since $W'_{0,X} \subset \mathfrak{T}_X$ by definition, we have $E \in (W'_{0,X})^\circ$. From Remark 3.17, we get $E \in W'_{1,X}$. Thus E satisfies all the hypotheses of Lemma 4.20. □

Remark 4.22. For ease of reference, let us list here the conclusions of some of the lemmas above. Suppose that $\pi : X \to S$ and $\widehat{\pi} : Y \to S$ are a pair of dual elliptic threefolds. Then

- Lemma 4.14: for fiber sheaves E, we have $\Lambda E \in \langle \mathrm{Coh}(\widehat{\pi})_0[-1], \mathcal{B}_{Y,*}[-2]\rangle$.
- Lemma 4.15: for $E \in \mathrm{Coh}^{\leq 1}(X)$ without fiber subsheaves, we have $\Lambda E \in \Big(\mathrm{Coh}^{=2}(Y) \cap \Psi(\{\mathrm{Coh}^{\leq 0}(X_s)\}^\uparrow)\Big)[-1]$.
- Lemma 4.16: for $E \in \mathrm{Coh}^{=2}(X) \cap \mathrm{Coh}(\pi)_1$, we have
$$\Lambda E \in \langle \mathrm{Coh}^{=2}(Y), \mathrm{Coh}^{\leq 2}(Y)[-1], \mathrm{Coh}^{\leq 0}(Y)[-2]\rangle.$$
- Lemma 4.17: for $E \in \mathrm{Coh}^{=2}(Y) \cap \mathrm{Coh}(\pi)_2 \cap (\mathrm{Coh}(\pi)_{\leq 1})^\circ$, we have
$$\Lambda E \in \langle \mathrm{Coh}^{\geq 2}(Y), \mathrm{Coh}^{\leq 2}(Y)[-1], \mathcal{B}_{Y,*}[-2]\rangle.$$
- Lemma 4.18: for $E \in W'_{0,X} \cap \mathrm{Coh}^{=3}(X)$, we have
$$\Lambda E \in \langle \mathrm{Coh}^{\geq 2}(Y), \mathrm{Coh}^{\leq 2}(Y)[-1], \mathcal{B}_{Y,*}[-2]\rangle.$$
- Lemma 4.19: for $E \in \Phi(\{\mathrm{Coh}^{\leq 0}(Y_s)\}^\uparrow)$, we have
$$\Lambda E \in \langle \mathrm{Coh}^{=2}(Y) \cap \{\mathrm{Coh}^{\leq 0}(Y_s)\}^\uparrow, \mathrm{Coh}^{\leq 1}(Y)[-1], \mathrm{Coh}^{\leq 0}(Y)[-2]\rangle.$$
- Lemma 4.20: for $E \in \mathrm{Coh}^{=3}(X) \cap W'_{1,X}$ with $\mathrm{Hom}(\Phi(\{\mathrm{Coh}^{\leq 0}(Y_s)\}^\uparrow), E) = 0$, we have
$$\Lambda(E[1]) \in \langle \mathrm{Coh}^{=3}(Y), \mathrm{Coh}^{\leq 2}(Y)[-1], \mathcal{B}_{Y,*}[-2]\rangle.$$

Lemma 4.23. *Give an exact triangle $E' \to E \to E'' \to E'[1]$ in $D(X)$, if both $\Lambda E', \Lambda E''$ lie in the extension closure*

$$\langle \mathrm{Coh}^{\geq 2}(Y), \mathrm{Coh}^{\leq 2}(Y)[-1], \mathcal{B}_{Y,*}[-2] \rangle,$$

then so does ΛE.

Proof. From the long exact sequence of cohomology for the exact triangle $E' \to E \to E'' \to E'[1]$, we have the following exact sequences:

$$\begin{aligned} 0 \to \Lambda^0 E'' \to &\Lambda^0 E \to \Lambda^0 E', \\ \Lambda^1 E'' \to &\Lambda^1 E \to \Lambda^1 E', \text{ and} \\ \Lambda^2 E'' \to &\Lambda^2 E \to \Lambda^2 E' \to 0. \end{aligned}$$

Since $\Lambda^0 E'', \Lambda^0 E' \in \mathrm{Coh}^{\geq 2}(Y)$, we have $\Lambda^0 E$. Also, that $\Lambda^1 E'', \Lambda^1 E' \in \mathrm{Coh}^{\leq 2}(Y)$ implies $\Lambda^1 E$ since $\mathrm{Coh}^{\leq 2}(Y)$ is a Serre subcategory of $\mathrm{Coh}(Y)$. And finally, that $\Lambda^2 E'', \Lambda^2 E' \in \mathcal{B}_{Y,*}$ implies Λ^2 since $\mathcal{B}_{Y,*}$ is closed under quotients and extensions in $\mathrm{Coh}(Y)$. □

Combining the above lemmas together, we now have:

Theorem 4.24. *Suppose X is an elliptic threefold. We have*

$$\Lambda(\mathfrak{C}_X) \subseteq \langle \mathrm{Coh}^{\geq 2}(Y), \mathrm{Coh}^{\leq 2}(Y)[-1], \mathcal{B}_{Y,*}[-2] \rangle. \tag{4.15}$$

Proof. We begin by showing that any sheaf in $W'_{0,X}$ can be filtered by sheaves of the types from the various lemmas above. The idea is to use the fact that $W'_{0,X}$ is a torsion class (hence closed under quotients in $\mathrm{Coh}(X)$), and that we have the following nested sequence of Serre subcategories in $\mathrm{Coh}(X)$:

$$\mathrm{Coh}(\pi)_{\leq 0} \subset \mathrm{Coh}^{\leq 1}(X) \subset \mathrm{Coh}(\pi)_{\leq 1} \subset \mathrm{Coh}^{\leq 2}(X).$$

Fix any $E \in W'_{0,X}$. Then E fits in a short exact sequence in $\mathrm{Coh}(X)$

$$0 \to E^{0,0} \to E \to E^{0,1} \to 0$$

where $E^{0,0} \in \mathrm{Coh}^{\leq 2}(X)$ and $E^{0,1} \in \mathrm{Coh}^{=3}(X) \cap W'_{0,X}$. We can then fit $E^{0,0}$ in a short exact sequence in $\mathrm{Coh}(X)$

$$0 \to E^{1,0} \to E^{0,0} \to E^{1,1} \to 0$$

where $E^{1,0} \in \mathrm{Coh}^{\leq 2}(X) \cap \mathrm{Coh}(\pi)_{\leq 1} = \mathrm{Coh}(\pi)_{\leq 1}$ and $E^{1,1}$ lies in $\mathrm{Coh}^{\leq 2}(X) \cap (\mathrm{Coh}(\pi)_{\leq 1})^\circ$, which is the same as $\mathrm{Coh}^{=2}(X) \cap \mathrm{Coh}(\pi)_2 \cap (\mathrm{Coh}(\pi)_{\leq 1})^\circ$. In turn, we can fit $E^{1,0}$ in a short exact sequence in $\mathrm{Coh}(X)$

$$0 \to E^{2,0} \to E^{1,0} \to E^{2,1} \to 0$$

where $E^{2,0} \in \mathrm{Coh}(\pi)_{\leq 1} \cap \mathrm{Coh}^{\leq 1}(X) = \mathrm{Coh}^{\leq 1}(X)$ and $E^{2,1}$ lies in $\mathrm{Coh}(\pi)_{\leq 1} \cap (\mathrm{Coh}^{\leq 1}(X))^\circ$, which is the same as $\mathrm{Coh}^{=2}(X) \cap \mathrm{Coh}(\pi)_1$. Next, we can fit $E^{2,0}$ in a short exact sequence in $\mathrm{Coh}(X)$

$$0 \to E^{3,0} \to E^{2,0} \to E^{3,1} \to 0$$

where $E^{3,0} \in \mathrm{Coh}^{\leq 1}(X) \cap \mathrm{Coh}(\pi)_0$ and $E^{3,1} \in \mathrm{Coh}^{\leq 1}(X) \cap (\mathrm{Coh}(\pi)_0)^\circ$. Overall, we have constructed a filtration of E

$$E^{3,0} \subseteq E^{2,0} \subseteq E^{1,0} \subseteq E^{0,0} \subseteq E$$

where the subfactors are:

- $E^{3,0}$, which satisfies the hypotheses of Lemma 4.14;

- $E^{2,0}/E^{3,0} \cong E^{3,1}$, which satisfies the hypotheses of Lemma 4.15;
- $E^{1,0}/E^{2,0} \cong E^{2,1}$, which satisfies the hypotheses of Lemma 4.16;
- $E^{0,0}/E^{1,0} \cong E^{1,1}$, which satisfies the hypotheses of Lemma 4.17;
- $E/E^{0,0} \cong E^{0,1}$, which satisfies the hypotheses of Lemma 4.18.

As a result (see Remark 4.22, for instance), each of the subfactors of E listed above is contained in the category

$$\langle \mathrm{Coh}^{\geq 2}(Y), \mathrm{Coh}^{\leq 2}(Y)[-1], \mathcal{B}_{Y,*}[-2] \rangle. \tag{4.16}$$

Therefore, $\Lambda(W'_{0,X})$ is contained in the category (4.16). Besides, the two categories $\Lambda(\Phi(\{\mathrm{Coh}^{\leq 0}(Y_s)\}^{\uparrow}))$ and $\Lambda(\mathfrak{T}_X^{\circ}[1])$ are also contained in the category (4.16) by Lemmas 4.19, 4.20 and 4.21. The inclusion (4.15) thus follows from Lemma 4.23. □

Remark 4.25. Since $\mathcal{B}_{Y,*} \subseteq \mathfrak{T}_X$, Theorem 4.24 immediately gives

$$\Lambda(\mathfrak{C}_X) \subseteq \langle \mathfrak{C}_Y, \mathfrak{C}_Y[-1], \mathfrak{C}_Y[-2] \rangle. \tag{4.17}$$

Now we have the following theorem, which can be considered as the analogue of Theorem 4.12 on elliptic threefolds:

Theorem 4.26. *Suppose X is a smooth elliptic threefold. Then the heart $\Lambda(\mathfrak{C}_X)$ differs from the heart $\mathfrak{D}_Y[-2]$ by one tilt.*

Proof. Since $\mathcal{B}_{Y,*} \subseteq \mathrm{Coh}^{\leq 1}(Y)$, we have $\mathrm{Coh}^{\geq 2}(Y) \subseteq \mathcal{B}_{Y,*}^{\circ}$. The theorem then follows from Theorem 4.24 and [BMT, Proposition 2.3.2(b)]. □

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