On semiconjugate rational functions

F. Pakovich

Abstract

We investigate semiconjugate rational functions, that is rational functions A, B related by the functional equation $A \circ X = X \circ B$, where X is a rational function. We show that if A and B is a pair of such functions, then either A can be obtained from B by a certain iterative process, or A and B can be described in terms of orbifolds of non-negative Euler characteristic on the Riemann sphere.

1 Introduction

Let A, B be two rational functions of degree at least two on the Riemann sphere. The function B is said to be semiconjugate to the function A if there exists a non-constant rational function X such that the equality

$$A \circ X = X \circ B \tag{1}$$

holds. If X is invertible the functions A and B are called conjugate. The semiconjugacy relation plays an important role in the study of complex dynamical systems, and in this article we study triples A, B, X which satisfy relation (1).

Since condition (1) is not symmetric with respect to A and B, the semiconjugacy is not an equivalency relation. However, it is easy to see that if B is semiconjugate to A, and A is semiconjugate to C, then B is semiconjugate to C. Therefore, the semiconjugacy is a preorder on the set of rational functions. Having this in mind, we will use the notation $A \leq B$ for rational functions A and B satisfying equality (1) for some rational function X.

The problem of describing of semiconjugate rational functions can be considered as a generalization of the classical problem of describing of commuting rational functions, that is of rational functions A and X satisfying the functional equation

$$A \circ X = X \circ A. \tag{2}$$

The last problem was considered in the early twenties of the past century in the papers of Fatou, Julia, and Ritt [4], [5], [17]. In all these papers it was assumed that the considered commuting functions A and X have no iterate in common, that is

$$A^{\circ n} \neq X^{\circ m} \tag{3}$$

for all $n, m \in \mathbb{N}$. In particular, this assumption rules out "trivial" solutions of the form $A = R^{\circ m}$, $X = R^{\circ n}$, where R is an arbitrary rational function. Fatou and Julia used dynamical methods requiring an additional assumption that the Julia set of A or X does not coincide with the whole complex plane, while Ritt used a method of algebraic-topological character free of any assumptions about the Julia set. Briefly, the Ritt theorem states that if rational functions A and X commute and no iterate of A is equal to an iterate of X, then, up to a conjugacy, A and X are either powers, or Chebyshev polynomials, or Lattès functions.

Notice that both equations (1) and (2) are particular cases of the more general functional equation

$$A \circ X = Y \circ B \tag{4}$$

investigated for the first time by Ritt in the paper [18]. In this paper Ritt laid the foundation of the decomposition theory of rational functions, and constructed a comprehensive decomposition theory of polynomials. In particular, Ritt described solutions of (4) in the case where A, B, X, Y are polynomials, and the results of [18] can be applied to equation (1) in the polynomial case (see [7]). The Ritt theory may be extended to a decomposition theory of Laurent polynomials ([13]). However, more general results in this direction are not known.

A proof of the Ritt theorem, based on modern dynamical methods, was given by Eremenko [2] who pointed out that the Ritt result can be formulated in a natural way using the concept of orbifold introduced by Thurston [20]. Recall that a Riemann surface orbifold is a pair $\mathcal{O} = (R, \nu)$ consisting of a Riemann surface R and a ramification function $\nu : R \to \mathbb{N}$ which takes the value $\nu(z) = 1$ except at isolated set of points. The Euler characteristic of an orbifold $\mathcal{O} = (R, \nu)$ is defined by the formula

$$\chi(\mathcal{O}) = \chi(R) + \sum_{z \in R} \left(\frac{1}{\nu(z)} - 1 \right), \tag{5}$$

where $\chi(R)$ is the Euler characteristic of R. If R_1, R_2 are Riemann surfaces provided with ramification functions ν_1, ν_2, and $f : R_1 \to R_2$ is a holomorphic branched covering map, then f is said to be a covering map $f : \mathcal{O}_1 \to \mathcal{O}_2$ between orbifolds $\mathcal{O}_1 = (\mathcal{R}_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ if for any $z \in R_1$ the equality

$$\nu_2(f(z)) = \nu_1(z) \deg_z f \tag{6}$$

holds, where $\deg_z f$ is the local degree of f at the point z. If such a map has a finite degree d, then the Riemann-Hurwitz formula implies that

$$\chi(\mathcal{O}_1) = d\chi(\mathcal{O}_2). \tag{7}$$

In the above terms the Ritt theorem may be formulated as follows ([2]): if rational functions A and X commute and no iterate of A is equal to an iterate of X, then there exists an orbifold $\mathcal{O} = (R, \nu)$ of *zero* Euler characteristic with

R equal to $\mathbb{C} \setminus \{0\}$, $\mathbb{C}$, or $\mathbb{CP}^1$ such that the commutative diagram

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{A} & \mathcal{O} \\ \big\downarrow X & & \big\downarrow X \\ \mathcal{O} & \xrightarrow{A} & \mathcal{O} \end{array}$$

consists of covering maps between orbifolds. In this description power functions $z^{\pm n}$ correspond to covering maps preserving the orbifold $\mathcal{O} = (R, \nu)$, where $R = \mathbb{C} \setminus \{0\}$ and $\nu \equiv 1$. The Chebyshev polynomials $\pm T_n$, defined by the formula $T_n(\cos \varphi) = \cos(n\varphi)$, correspond to covering maps preserving the orbifold $\mathcal{O} = (R, \nu)$, where $R = \mathbb{C}$ and $\nu(-1) = \nu(1) = 2$. Finally, Lattès functions correspond to covering maps preserving an orbifold of zero characteristic $\mathcal{O}$ with $R = \mathbb{CP}^1$. In the last case formula (5) implies that the collection of ramification indices of $\mathcal{O}$ is either $(2,2,2,2)$, or one of the following triples $(3,3,3)$, $(2,4,4)$, $(2,3,6)$. Notice that the Ritt theorem provides no information about functions (commuting or not) that do share an iterate, and a description of such functions is known only in the polynomial case ([19], [17]). Thus, in a certain sense the classification of commuting rational functions is not yet completed.

In comparison with equation (2) equation (1) has many more solutions. Indeed, take arbitrary rational functions U_1, V_1 and set

$$B = V_1 \circ U_1, \quad A = U_1 \circ V_1.$$

Then the equality

$$(U_1 \circ V_1) \circ U_1 = U_1 \circ (V_1 \circ U_1) \tag{8}$$

implies that $A \leq B$. Similarly, $B \leq A$. Moreover, if now U_2, V_2 are rational functions such that the equality

$$U_1 \circ V_1 = V_2 \circ U_2 \tag{9}$$

holds, then the function $H = U_2 \circ V_2$ satisfies $A \leq H$ and $H \leq A$, implying that $B \leq H$ and $H \leq B$.

This motivates the following definition of an equivalence relation on the set of rational functions: $B \sim A$ if there exist rational functions U_i, V_i, $1 \leq i \leq n$, such that $B = V_1 \circ U_1$,

$$U_i \circ V_i = V_{i+1} \circ U_{i+1}, \quad 1 \leq i \leq n-1, \tag{10}$$

and $A = U_n \circ V_n$. Notice that since for any rational function W of degree one the equality

$$B = (B \circ W) \circ W^{-1}$$

holds, each equivalence class of $\sim$ is a union of conjugacy classes. Thus, since $B \sim A$ implies that $A \circ X = X \circ B$ and $B \circ Y = Y \circ A$ for

$$X = U_n \circ U_{n-1} \circ \cdots \circ U_1 \tag{11}$$

and

$$Y = V_1 \circ V_2 \circ \cdots \circ V_n,$$

the equivalence relation $\sim$ can be considered as a weaker form of the classical conjugacy relation whose classes consist of functions having "similar" although not "identical" dynamics.

Roughly speaking, our main result states that unless $B \sim A$ the relation $A \leq B$ implies very strong restrictions on A and B, which can be described in terms of orbifolds of *non-negative* Euler characteristic on the Riemann sphere. Namely, similarly to Lattès functions, such A and B can be characterized as maps "preserving" some orbifold on the Riemann sphere. However, since (7) implies that $\chi(\mathcal{O}) = 0$ for any self-covering map $f : \mathcal{O} \to \mathcal{O}$, we take as a basis the following weakened modification of the notion of covering.

A rational function f is called *a holomorphic map* $f : \mathcal{O}_1 \to \mathcal{O}_2$ between orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$ defined on $\mathbb{CP}^1$ if for any $z \in \mathbb{CP}^1$ instead of equality (6) a weaker condition

$$\nu_2(f(z)) \mid \nu_1(z)\deg_z f \tag{12}$$

holds. For fixed $\nu_2(z)$ a minimal possible value for $\nu_1(z)$ such that (12) holds is defined by the equality

$$\nu_2(f(z)) = \nu_1(z)\mathrm{GCD}(\deg_z f, \nu_2(f(z))). \tag{13}$$

If (13) is satisfied for all $z \in \mathbb{CP}^1$ we say that f is *a minimal holomorphic map* between orbifolds. Notice that any covering map between orbifolds is necessarily a minimal holomorphic map. The importance of the notion of minimal holomorphic map is explained by the fact that for any f and $\mathcal{O}$ there exists a uniquely defined orbifold $f^*\mathcal{O}$ such that $f : f^*\mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map, and the equality $(g \circ f)^*\mathcal{O} = f^*(g^*\mathcal{O})$ holds for any f, g, and $\mathcal{O}$.

If $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a holomorphic map between orbifolds, then instead of equality (7) the inequality

$$\chi(\mathcal{O}_1) \leq \chi(\mathcal{O}_2) \deg f \tag{14}$$

holds, and the equality is attained if and only if $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map. In particular, this implies that if $f : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic *self-map* and $\deg f > 1$, then $\chi(\mathcal{O}) \geq 0$. Moreover, if $\chi(\mathcal{O}) = 0$, then f in fact is a covering self-map and hence a Lattès function. Thus, the class of *minimal holomorphic self-maps* between orbifolds is a natural extension of the class of Lattès functions. Notice that although we assume that all considered orbifolds are defined on $\mathbb{CP}^1$, the functions $\pm z^n$ and $\pm T_n$ still belong to this new class. Indeed, it is easy to see that $z^{\pm n} : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map between orbifolds for any $\mathcal{O}$ defined by the conditions

$$\nu(0) = \nu(\infty) = m, \quad \mathrm{GCD}(n, m) = 1,$$

while $\pm T_n : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map between orbifolds for any $\mathcal{O}$ defined by the conditions

$$\nu(-1) = \nu(1) = 2, \quad \nu(\infty) = m, \quad \mathrm{GCD}(n, m) = 1.$$

In the above notation our main result may be formulated as follows.

Theorem 1.1. *Let A, B and X be rational functions of degree at least two such that $A \circ X = X \circ B$. Then either $B \sim A$ and X satisfies* (11) *for some chain of transformations* (10)*, or there exist orbifolds $\mathcal{O}_1$, $\mathcal{O}_2$ of non-negative Euler characteristic on the Riemann sphere such that the commutative diagram*

$$\begin{array}{ccc} \mathcal{O}_1 & \xrightarrow{B} & \mathcal{O}_1 \\ \Big\downarrow X & & \Big\downarrow X \\ \mathcal{O}_2 & \xrightarrow{A} & \mathcal{O}_2 \end{array}$$

consists of minimal holomorphic maps between orbifolds. Furthermore, either $\chi(\mathcal{O}_1) = \chi(\mathcal{O}_2) = 0$ and A, B are Lattès functions, or $0 < \chi(\mathcal{O}_2) < \chi(\mathcal{O}_1)$. In the last case, the possible collections of ramification indices of $\mathcal{O}_1$ and $\mathcal{O}_2$ are the following: (n, n) or $(2, 2, n)$ for some $n \geq 2$, or one of the triples $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$. In addition, $\mathcal{O}_1$ may be a non-ramified sphere.

For example, for the solution

$$T_n \circ \frac{1}{2}\left(z^m + \frac{1}{z^m}\right) = \frac{1}{2}\left(z^m + \frac{1}{z^m}\right) \circ z^n$$

of (1) the orbifold $\mathcal{O}_2$ is defined by the conditions

$$\nu(-1) = \nu(1) = 2, \quad \nu(\infty) = m/d, \tag{15}$$

where $d = \mathrm{GCD}(n, m)$, while $\mathcal{O}_1$ is non-ramified. On the other hand, for the solution

$$T_n \circ T_m = T_m \circ T_n$$

the orbifold $\mathcal{O}_2$ is still defined by (15) but $\mathcal{O}_1$ is defined by the condition

$$\nu(-1) = \nu(1) = 2.$$

Notice that in the last example $A = B = T_n$ so $A \sim B$. However, the corresponding $X = T_m$ in general cannot be obtained by formula (11) so the second part of the theorem is applied.

If $B \not\sim A$, then the conditions imposed by Theorem 1.1 on A, B, and X are quite strong and provide a reasonably good description of solutions of (1). Different implications of these conditions are discussed in the body of the paper. In contrast, the paper contains essentially no information about the relation $A \sim B$. Notice however that this relation also is rather restrictive. Indeed, it follows easily from the definition, that any two equivalent rational functions are *isospectral* in the following sense: for each $n \geq 1$ the unordered lists of multipliers at fixed points of the iterates $A^{\circ n}$ and $B^{\circ n}$ are the same. By the result of McMullen ([8]), this implies in particular that unless A is a flexible Lattès function its equivalence class contains at most a *finite* number of conjugacy classes.

The paper is organized as follows. In the second section we review general properties of the functional equation

$$f \circ p = g \circ q \tag{16}$$

on compact Riemann surfaces, basing on the fiber product approach. In the third section we provide main definitions and results concerning orbifolds on Riemann surfaces and holomorphic maps between orbifolds.

In the fourth section we introduce the concept of a minimal holomorphic map between orbifolds, and study properties of such maps with respect to functional decompositions. In particular, we relate minimal holomorphic maps with functional equation (16). In the fifth section we describe an approach to the classification of minimal holomorphic self-maps between orbifolds of positive Euler characteristic on the Riemann sphere. In particular, we relate such maps with rational functions equivariant with respect to finite subgroups of $Aut(\mathbb{CP}^1)$. Finally, in the sixth section we prove Theorem 1.1 and provide a number of examples.

2 Equation $f \circ p = g \circ q$ and fiber products

In this section we recall, mostly without proofs, some general results related to the functional equation

$$h = f \circ p = g \circ q, \tag{17}$$

where $h : R \to \mathbb{CP}^1$, $p : R \to C_1$, $f : C_1 \to \mathbb{CP}^1$, $q : R \to C_2$, $g : C_2 \to \mathbb{CP}^1$ are holomorphic functions on compact Riemann surfaces. For more details we refer the reader to [13], Section 2 and 3.

Let $h : R \to \mathbb{CP}^1$ be a holomorphic function on a compact Riemann surface R, and $\mathcal{C}(h) = \{z_1, z_2, \dots, z_r\}$ the set of critical values of h. Fix a point $z_0 \in \mathbb{CP}^1 \setminus \mathcal{C}(h)$ and some loops γ_i around z_i, $1 \leq i \leq r$, such that $\gamma_1\gamma_2...\gamma_r = 1$ in $\pi_1(\mathbb{CP}^1 \setminus \mathcal{C}(h), z_0)$. Denote by δ_i, $1 \leq i \leq r$, a permutation of points of $h^{-1}\{z_0\}$ induced by the lifting of γ_i, $1 \leq i \leq r$, by h. The permutation group G_h generated by δ_i, $1 \leq i \leq r$, is called the monodromy group of h. Clearly,

$$\delta_1\delta_2...\delta_r = 1 \tag{18}$$

in G_h.

Let G be a group which acts transitively on a finite set X. Recall that a subset B of X is called a block of G, if for each $g \in G$ either $g(B) = B$ or $g(B) \cap B = \emptyset$. Clearly, if B is a block, then $\mathcal{B} = \{\sigma(B), \sigma \in G\}$ is a partition of X, which is called an imprimitvity system of G.

The monodromy group G_h is related to compositional properties of the function h as follows. If h can be decomposed into a composition $h = f \circ p$ of holomorphic functions $p : R \to C_1$ and $f : C_1 \to \mathbb{CP}^1$, where $\deg f = d$, then G_h has an imprimitivity system consisting of d blocks $\mathcal{A}_i = p^{-1}\{t_i\}$, $1 \leq i \leq d$, where $\{t_1, t_2, \dots, t_d\} = f^{-1}\{z_0\}$, and the permutation group induced by the action of G_h on these blocks is permutation isomorphic to the group G_f. Furthermore, any imprimitivity system of G_h arises from a decomposition of h, and decompositions $h = f \circ p$ and $h = g \circ q$, where $q : R \to C_2$, $g : C_2 \to \mathbb{CP}^1$, lead to the same imprimitivity system if and only there exists an isomorphism $\mu : C_2 \to C_1$ such that

$$f = g \circ \mu^{-1}, \quad p = \mu \circ q.$$

We will say that two holomorphic functions $p : R \to C_1$ and $q : R \to C_2$ have no non-trivial compositional common right factor, if the equalities

$$p = \widetilde{p} \circ w, \quad q = \widetilde{q} \circ w, \tag{19}$$

where $w : R \to \widetilde{R}$, $\widetilde{p} : \widetilde{R} \to C_1$, $\widetilde{q} : \widetilde{R} \to C_2$ are holomorphic functions, imply that $\deg w = 1$.

Theorem 2.1. *For any two fixed holomorphic functions $f : C_1 \to \mathbb{CP}^1$ and $g : C_2 \to \mathbb{CP}^1$ there exist holomorphic functions $h_j : R_j \to \mathbb{CP}^1$ (components of the fiber product of f and g) and $p_j : R_j \to C_1$, $q_j : R_j \to C_2$ such that the following conditions are satisfied:*

- $h_j = f \circ p_j = g \circ q_j$,
- $\sum_j \deg h_j = \deg f \deg g$,
- *for any solution h, p, q of* (17) *there exist an index j and a holomorphic function $w : R \to R_j$ such that $h = h_j \circ w$, $p = p_j \circ w$, $q = q_j \circ w$.* $\square$

Recall briefly the construction of the functions h_j. Let $S = \{z_1, z_2, \ldots, z_r\}$ be the union of $\mathcal{C}(f)$ and $\mathcal{C}(g)$. As above, fix a point z_0 from $\mathbb{CP}^1 \setminus S$ and small loops γ_i around z_i, $1 \le i \le r$, such that $\gamma_1\gamma_2...\gamma_r = 1$ in $\pi_1(\mathbb{CP}^1 \setminus S, z_0)$. Set $n = \deg f$, $m = \deg g$, and denote by $\alpha_i \in S_n$ (resp. $\beta_i \in S_m$) a permutation of points of $f^{-1}\{z_0\}$ (resp. $g^{-1}\{z_0\}$) induced by the lifting of γ_i, $1 \le i \le r$, by f (resp. g). Clearly, the permutations α_i (resp. β_i), $1 \le i \le r$, generate the monodromy group of f (resp. g) and

$$\alpha_1\alpha_2...\alpha_r = 1, \qquad \beta_1\beta_2...\beta_r = 1. \tag{20}$$

Define now permutations $\delta_1, \delta_2, \ldots, \delta_r \in S_{nm}$ as follows: consider the set of mn elements c_{j_1,j_2}, $1 \le j_1 \le n$, $1 \le j_2 \le m$, and set $(c_{j_1,j_2})^{\delta_i} = c_{j_1',j_2'}$, where

$$j_1' = j_1^{\alpha_i}, \qquad j_2' = j_2^{\beta_i}, \qquad 1 \le i \le r.$$

It is convenient to consider c_{j_1,j_2}, $1 \le j_1 \le n$, $1 \le j_2 \le m$, as elements of a $n \times m$ matrix M. Then the action of the permutation δ_i, $1 \le i \le r$, reduces to the permutation of rows of M in accordance with the permutation α_i and the permutation of columns of M in accordance with the permutation β_i.

In general, the permutation group generated by δ_i, $1 \le i \le r$, is not transitive on the set c_{j_1,j_2}, $1 \le j_1 \le n$, $1 \le j_2 \le m$. However, on each transitivity set U_j the induced permutations $\delta_i(j)$, $1 \le i \le r$, satisfy the equality

$$\delta_1(j)\delta_2(j)\ldots\delta_r(j) = 1.$$

By the Riemann existence theorem, this implies that there exist compact Riemann surfaces R_j and holomorphic functions $h_j: R_j \to \mathbb{CP}^1$ non-ramified outside S such that the permutations $\delta_i(j)$, $1 \le i \le r$, are induced by the lifting of γ_i by h_j. Moreover, it is easy to see by construction that the intersections of the transitivity set U_j with the rows of M form an imprimitivity system $\Omega_f(j)$ for the group generated by $\delta_i(j)$, $1 \le i \le r$, such that the permutations of blocks of $\Omega_f(j)$ induced by $\delta_i(j)$, $1 \le i \le r$, coincide with α_i. Similarly, the intersections of U_j with the columns of M form an imprimitivity system $\Omega_g(j)$ such that the permutations of blocks of $\Omega_g(j)$ induced by $\delta_i(j)$, $1 \le i \le r$, coincide with β_i. These imprimitivity systems correspond to decompositions $h_j = f \circ p_j = g \circ q_j$ for some functions p_j and q_j.

Corollary 2.1. *Let h, f, p, g, q be rational functions satisfying* (17)*. Then p and q have no non-trivial common compositional right factor if and only if for any $z \in \mathbb{CP}^1$ the local degrees $\deg_z p$ and $\deg_z q$ are coprime.*

Proof. If p and q have no non-trivial common compositional right factor, then without loss of generality we can assume that $h = h_j$, $p = p_j$, $q = q_j$. Clearly,

$$\deg_z h_j = \deg_z p_j \deg_{p_j(z)} f = \deg_z q_j \deg_{q_j(z)} g. \tag{21}$$

On the other hand, the definition of the permutations δ_i, $1 \le i \le r$, yields that for any $z \in \mathbb{CP}^1$ the equality

$$\deg_z h_j = \mathrm{LCM}(\deg_{p_j(z)} f, \deg_{q_j(z)} g) \tag{22}$$

holds. It follows now from equalities (21) and (22) that $\deg_z p_j$ and $\deg_z q_j$ are coprime.

In the other direction, if w is a common compositional right factor of p and q, and z_0 is any critical point of w, then the chain rule implies that $\deg_{z_0} p$ and $\deg_{z_0} q$ have a non-trivial common divisor. Therefore, since any rational function of degree greater than one has critical points, if $\deg_z p$ and $\deg_z q$ are coprime for any $z \in \mathbb{CP}^1$, p and q may not have a non-trivial common compositional right factor. $\square$

Let h, f, p, g, q be a solution of (17), and $\mathcal{A}$ and $\mathcal{B}$ imprimitivity systems of the group G_h corresponding to the decompositions $h = f \circ p$ and $h = g \circ q$ respectively. We say that the solution h, f, p, g, q is *good* if any block of $\mathcal{A}$ intersects with any block of $\mathcal{B}$ and this intersection consists of a unique element. This is equivalent to the requirement that the fiber product of f and g has a unique component and p and q have no non-trivial common compositional right factor.

Notice that a solution of (17) in rational functions h, f, p, g, q is good if and only if the algebraic curve

$$\mathcal{E}(f, g): \; f(x) - g(y) = 0$$

is irreducible and $\mathbb{C}(p, q) = \mathbb{C}(z)$. Indeed, irreducible components of $\mathcal{E}(f, g)$ correspond to irreducible components of the fiber product of f and g (see [13], Proposition 2.4). On the other hand, by the Lüroth theorem, any subfield $K \subset \mathbb{C}(z)$, $K \neq \mathbb{C}$, has the form $K = \mathbb{C}(w)$ for some $w \in \mathbb{C}(z)$, implying that (19) holds for some rational functions $\widetilde{p}, \widetilde{q}, w$ with $\deg w > 1$ if and only if $\mathbb{C}(p,q) \neq \mathbb{C}(z)$.

The construction of the fiber product implies easily the following statement.

Lemma 2.1. *A solution h, f, p, g, q of* (17) *is good whenever any two of the following three conditions are satisfied:*

- *the fiber product of f and g has a unique component,*
- *p and q have no non-trivial common compositional right factor,*
- $\deg f = \deg q, \quad \deg g = \deg p.$ □

Finally, let us mention the following property of good solutions of (17).

Lemma 2.2. *Let h, f, p, g, q be a good solution of* (17)*, z_1 a point from the set $g^{-1}\{z_0\}$, and $\sigma \in G_h$ a permutation which maps the set $q^{-1}\{z_1\}$ to itself. Then the permutation induced by σ on $q^{-1}\{z_1\}$ has the same cyclic structure as the permutation induced by σ on blocks $\mathcal{A}_i = p^{-1}\{t_i\}$, $1 \leq i \leq d$, where $\{t_1, t_2, \dots, t_d\} = f^{-1}\{z_0\}$.*

Proof. Since the set $q^{-1}\{z_1\}$ is a block, it follows from the definition of a good solution that there is a natural one-to-one correspondence between the elements of $q^{-1}\{z_1\}$ and the blocks $\mathcal{A}_i$, $1 \leq i \leq d$. Furthermore, the action of any $\sigma \in G_h$ which maps $q^{-1}\{z_1\}$ to itself obviously respects this correspondence. □

3 Orbifolds on Riemann surfaces

In this section we recall main definitions and results related to Riemann surface orbifolds (see [10], Appendix E).

A pair $\mathcal{O} = (R, \nu)$ consisting of a Riemann surface R and a ramification function $\nu : R \to \mathbb{N}$ which takes the value $\nu(z) = 1$ except at isolated points is called an orbifold. The Euler characteristic of an orbifold $\mathcal{O} = (R, \nu)$ is defined by the formula

$$\chi(\mathcal{O}) = \chi(R) + \sum_{z \in R} \left(\frac{1}{\nu(z)} - 1 \right),$$

where $\chi(R)$ is the Euler characteristic of R.

If R_1, R_2 are Riemann surfaces provided with ramification functions ν_1, ν_2, and $f : R_1 \to R_2$ is a holomorphic branched covering map, then f is called a *covering map* $f : \mathcal{O}_1 \to \mathcal{O}_2$ *between orbifolds* $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ if for any $z \in R_1$ the equality

$$\nu_2(f(z)) = \nu_1(z) \deg_z f \tag{23}$$

holds, where $\deg_z f$ is the local degree of f at the point z. If for any $z \in R_1$ instead of equality (23) a weaker condition

$$\nu_2(f(z)) \mid \nu_1(z)\deg_z f \tag{24}$$

holds, then f is called a *holomorphic map* $f : \mathcal{O}_1 \to \mathcal{O}_2$ *between orbifolds* $\mathcal{O}_1$ and $\mathcal{O}_2$.

A universal cover of an orbifold $\mathcal{O}$ is a covering map between orbifolds $\theta_{\mathcal{O}} : \widetilde{\mathcal{O}} \to \mathcal{O}$ such that $\widetilde{R}$ is simply connected and $\widetilde{\nu}(z) \equiv 1$. If $\theta_{\mathcal{O}}$ is such a map, then there exists a group $\Gamma_{\mathcal{O}}$ of conformal automorphisms of $\widetilde{R}$ such that the equality $\theta_{\mathcal{O}}(z_1) = \theta_{\mathcal{O}}(z_2)$ holds for $z_1, z_2 \in \widetilde{R}$ if and only if $z_1 = \sigma(z_2)$ for some $\sigma \in \Gamma_{\mathcal{O}}$. A universal cover exists and is unique up to a conformal isomorphism of $\widetilde{R}$, unless $\mathcal{O}$ is the Riemann sphere with one ramified point, or $\mathcal{O}$ is the Riemann sphere with two ramified points for which $\nu(z_1) \neq \nu(z_2)$. Unless stated otherwise, below we will assume that considered orbifolds have a universal cover. Abusing notation we will use the symbol $\widetilde{\mathcal{O}}$ both for the orbifold and for the Riemann surface $\widetilde{R}$.

If $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds, then for any choice of $\theta_{\mathcal{O}_1}$ and $\theta_{\mathcal{O}_2}$ there exists a conformal isomorphism $\tau : \widetilde{\mathcal{O}_1} \to \widetilde{\mathcal{O}_2}$ such that the diagram

$$\begin{array}{ccc} \widetilde{\mathcal{O}_1} & \xrightarrow{\tau} & \widetilde{\mathcal{O}_2} \\ \downarrow{\scriptstyle \theta_{\mathcal{O}_1}} & & \downarrow{\scriptstyle \theta_{\mathcal{O}_2}} \\ \mathcal{O}_1 & \xrightarrow{f} & \mathcal{O}_2 \end{array} \tag{25}$$

is commutative. More generally, the following proposition holds.

Proposition 3.1. *Let $f : \mathcal{O}_1 \to \mathcal{O}_2$ be a holomorphic map between orbifolds. Then for any choice of $\theta_{\mathcal{O}_1}$ and $\theta_{\mathcal{O}_2}$ there exist a holomorphic map $F : \widetilde{\mathcal{O}_1} \to \widetilde{\mathcal{O}_2}$ and a homomorphism $\varphi : \Gamma_{\mathcal{O}_1} \to \Gamma_{\mathcal{O}_2}$ such that diagram*

$$\begin{array}{ccc} \widetilde{\mathcal{O}_1} & \xrightarrow{F} & \widetilde{\mathcal{O}_2} \\ \downarrow{\scriptstyle \theta_{\mathcal{O}_1}} & & \downarrow{\scriptstyle \theta_{\mathcal{O}_2}} \\ \mathcal{O}_1 & \xrightarrow{f} & \mathcal{O}_2 \end{array} \tag{26}$$

is commutative and for any $\sigma \in \Gamma_{\mathcal{O}_1}$ the equality

$$F \circ \sigma = \varphi(\sigma) \circ F \tag{27}$$

holds. The map F is defined by $\theta_{\mathcal{O}_1}$, $\theta_{\mathcal{O}_2}$, and f uniquely up to the transformation $F \to g \circ F$, where $g \in \Gamma_{\mathcal{O}_2}$. In the other direction, for any holomorphic map $F : \widetilde{\mathcal{O}_1} \to \widetilde{\mathcal{O}_2}$ which satisfies (27) *for some homomorphism $\varphi : \Gamma_{\mathcal{O}_1} \to \Gamma_{\mathcal{O}_2}$ there exists a uniquely defined holomorphic map between orbifolds $f : \mathcal{O}_1 \to \mathcal{O}_2$ such that diagram* (26) *is commutative. The holomorphic map F is an isomorphism if and only if f is a covering map between orbifolds.*

Proof. Assume that (24) holds. Let F be the complete analytic continuation of the germ $\theta_{\mathcal{O}_2}^{-1}\circ f\circ\theta_{\mathcal{O}_1}$, where $\theta_{\mathcal{O}_2}^{-1}$ is a germ of a branch of the function inverse to $\theta_{\mathcal{O}_2}$. Clearly, the local multiplicity of the map $f\circ\theta_{\mathcal{O}_1}$ at a point $z\in\widetilde{\mathcal{O}_1}$ equals

$$\deg_z\theta_{\mathcal{O}_1}\deg_{\theta_{\mathcal{O}_1}(z)}f=\nu(\theta_{\mathcal{O}_1}(z))\deg_{\theta_{\mathcal{O}_1}(z)}f.$$

On the other hand, the order of the permutation of branches of the function inverse to $\theta_{\mathcal{O}_2}$ induced by the analytic continuation along a small loop around the point $(f\circ\theta_{\mathcal{O}_1})(z)$ is equal to $\nu_2((f\circ\theta_{\mathcal{O}_1})(z))$. Therefore, condition (24) implies that the function F has no ramification points. Since $\widetilde{\mathcal{O}_1}$ is simply connected, we conclude that F is single valued.

Furthermore, if $\widehat{F}$ is another function which makes diagram (26) commutative, then $\theta_{\mathcal{O}_2}\circ F=\theta_{\mathcal{O}_2}\circ\widehat{F}$. Thus, for any $z\in\widetilde{R}_1$ we have $\widehat{F}(z)=g_z\circ F(z)$ for some $g_z\in\Gamma_{\mathcal{O}_2}$. Since $\Gamma_{\mathcal{O}_2}$ is countable this implies easily that $\widehat{F}\equiv g\circ F$ for some $g\in\Gamma_{\mathcal{O}_2}$. Finally, since (26) implies that for any $\sigma\in\Gamma_{\mathcal{O}_1}$ the equality

$$\theta_{\mathcal{O}_2}\circ F=\theta_{\mathcal{O}_2}\circ(F\circ\sigma)$$

holds, we have:

$$F\circ\sigma=\varphi(\sigma)\circ F$$

for some $\varphi(\sigma)\in\Gamma_{\mathcal{O}_2}$, and it is easy to see that the correspondence $\sigma\to\varphi(\sigma)$ is a homomorphism.

In the other direction, if (27) holds, then F maps any orbit of $\Gamma_{\mathcal{O}_1}$ to an orbit of $\Gamma_{\mathcal{O}_2}$, implying that the function $f=\theta_{\mathcal{O}_2}\circ F\circ\theta_{\mathcal{O}_1}^{-1}$ is well defined and holomorphic. Further, (26) implies that

$$\deg_z(f\circ\theta_{\mathcal{O}_1})=\deg_z(\theta_{\mathcal{O}_2}\circ F).$$

Therefore, since

$$\deg_z(f\circ\theta_{\mathcal{O}_1})=\deg_z\theta_{\mathcal{O}_1}\deg_{\theta_{\mathcal{O}_1}(z)}f=\nu_1(\theta_{\mathcal{O}_1}(z))\deg_{\theta_{\mathcal{O}_1}(z)}f$$

and

$$\begin{aligned}\deg_z(\theta_{\mathcal{O}_2}\circ F)=\deg_zF\deg_{F(z)}\theta_{\mathcal{O}_2}=&\\ =\deg_zF\,\nu_2((\theta_{\mathcal{O}_2}\circ F)(z))=\deg_zF\,\nu_2((f\circ\theta_{\mathcal{O}_1})(z)),&\end{aligned}$$

we conclude that

$$\nu_1(\theta_{\mathcal{O}_1}(z))\deg_{\theta_{\mathcal{O}_1}(z)}f=\deg_zF\,\nu_2((f\circ\theta_{\mathcal{O}_1})(z)),\tag{28}$$

implying (24). Moreover, it follows from (28) that F is locally and therefore globally invertible if and only if (23) holds. □

If $f:\mathcal{O}_1\to\mathcal{O}_2$ is a covering map between orbifolds with compact support, then the Riemann-Hurwitz formula implies that

$$\chi(\mathcal{O}_1)=d\chi(\mathcal{O}_2),\tag{29}$$

where $d=\deg f$. More generally, the following statement is true.

Proposition 3.2. *Let $f : \mathcal{O}_1 \to \mathcal{O}_2$ be a holomorphic map between orbifolds with compact support. Then*

$$\chi(\mathcal{O}_1) \leq \chi(\mathcal{O}_2) \deg f, \tag{30}$$

and the equality holds if and only if $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds.

Proof. Denote by S_1 (resp. S_2) the set of ramified points of $\mathcal{O}_1$ (resp. $\mathcal{O}_2$) and by $\mathcal{C}(f)$ the set of critical values of f. Set

$$S = f(S_1) \cup \mathcal{C}(f), \quad \widehat{R}_2 = R_2 \setminus S, \quad \widehat{R}_1 = f^{-1}\{\widehat{R}_2\}.$$

Observe that $S_2 \subseteq S$ since (24) implies that whenever $\nu_2(f(z))$ is greater than one at least one of the numbers $\deg_z f$ and $\nu_1(z)$ also is greater than one. Since $f : \widehat{R}_1 \to \widehat{R}_2$ is a covering map between surfaces, we have:

$$\chi(\widehat{R}_1) = d\chi(\widehat{R}_2),$$

where $d = \deg f$. Furthermore, it follows from (24) that

$$\frac{1}{\nu_1(z)} \leq \frac{\deg_z f}{\nu_2(f(z))}$$

implying that

$$\sum_{\substack{x \in R_1 \\ f(x) = f(z)}} \frac{1}{\nu_1(x)} \leq \frac{d}{\nu_2(f(z))}, \tag{31}$$

where the equality holds if and only if (23) holds for any $x \in f^{-1}\{z\}$.

Since removing a point from a surface reduces the Euler characteristic by one, we have:

$$\chi(\mathcal{O}_1) = \chi(R_1) + \sum_{x \in R_1} \left(\frac{1}{\nu_1(x)} - 1 \right) = \chi(R_1) + \sum_{\substack{x \in R_1 \\ f(x) \in S}} \left(\frac{1}{\nu_1(x)} - 1 \right) =$$

$$= \chi(\widehat{R}_1) + \sum_{\substack{x \in R_1 \\ f(x) \in S}} \frac{1}{\nu_1(x)} = d\chi(\widehat{R}_2) + \sum_{\substack{x \in R_1 \\ f(x) \in S}} \frac{1}{\nu_1(x)}.$$

It follows now from (31) that

$$\chi(\mathcal{O}_1) \leq d\chi(\widehat{R}_2) + \sum_{z \in S} \frac{d}{\nu_2(z)} = d\chi(\mathcal{O}_2),$$

where the equality holds if and only if (23) holds for any $z \in f^{-1}\{S\}$. Since the definition of S implies that (23) is satisfied for $z \notin f^{-1}\{S\}$, we conclude that the equality in (30) holds if and only if $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds. □

Corollary 3.1. *Let $f : \mathcal{O}_1 \to \mathcal{O}_2$ be a holomorphic map between orbifolds with compact supports. Assume that* $\deg f > 1$ *and $\chi(\mathcal{O}_1) = \chi(\mathcal{O}_2) = l$. Then $l \geq 0$. Furthermore, $l = 0$ if and only if f is a covering map.* □

4 Minimal maps and decompositions

In this section we introduce the concept of a minimal holomorphic map between orbifolds, and establish some properties of such maps related to functional decompositions.

Let R_1, R_2 be Riemann surfaces, and $f: R_1 \to R_2$ a holomorphic branched covering map. Assume that R_2 is provided with a ramification function ν_2. In order to define a ramification function ν_1 on R_1 so that f would be a holomorphic map between orbifolds $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ we must satisfy condition (24), and it is easy to see that for any $z \in R_1$ a minimal possible value for $\nu_1(z)$ is defined by the equality

$$\nu_2(f(z)) = \nu_1(z)\mathrm{GCD}(\deg_z f, \nu_2(f(z))). \tag{32}$$

In case if (32) is satisfied for any $z \in R_1$ we say that f is a *minimal holomorphic map between orbifolds* $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$.

It follows from the definition that for any orbifold $\mathcal{O} = (R, \nu)$ and holomorphic branched covering map $f: R' \to R$ there exists a *unique* orbifold structure ν' on R' such that f is a minimal holomorphic map between corresponding orbifolds. We will denote the corresponding orbifold by $f^*\mathcal{O}$.

Lemma 4.1. *Any covering map between orbifolds $f: \mathcal{O}_1 \to \mathcal{O}_2$ is a minimal holomorphic map. In particular, the equality $\mathcal{O}_1 = f^*\mathcal{O}_2$ holds. A minimal holomorphic map $f: \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map if and only if $\deg_z f \mid \nu_2(f(z))$ for any $z \in R_1$.*

Proof. Follows from the corresponding definitions. $\square$

Minimal holomorphic maps possess the following fundamental property with respect to compositions.

Theorem 4.1. *Let $f: R'' \to R'$ and $g: R' \to R$ be holomorphic branched covering maps, and $\mathcal{O} = (R, \nu)$ an orbifold. Then*

$$(g \circ f)^*\mathcal{O} = f^*(g^*\mathcal{O}).$$

Proof. Let $f^*(g^*\mathcal{O}) = (R'', \nu_1)$ and $g^*\mathcal{O} = (R', \nu_2)$. Since

$$f: f^*(g^*\mathcal{O}) \to g^*\mathcal{O}$$

and

$$g: g^*\mathcal{O} \to \mathcal{O}$$

are minimal holomorphic maps, for any $z \in R''$ we have:

$$\nu_2(f(z)) = \nu_1(z)\mathrm{GCD}\Big(\deg_z f, \nu_2(f(z))\Big) \tag{33}$$

and

$$\nu\Big((g \circ f)(z)\Big) = \nu_2(f(z))\mathrm{GCD}\Big(\deg_{f(z)} g, \nu\Big((g \circ f)(z)\Big)\Big). \tag{34}$$

In order to prove the theorem, we only must show that

$$\nu\Big((g\circ f)(z)\Big) = \nu_1(z)\mathrm{GCD}\Big(\deg{}_z(g\circ f), \nu\Big((g\circ f)(z)\Big)\Big). \tag{35}$$

Observe first that for any positive integers a, b, c the equality

$$\mathrm{GCD}(ab, c) = \mathrm{GCD}(a, c)\mathrm{GCD}\left(b, \frac{c}{\mathrm{GCD}(a, c)}\right) \tag{36}$$

holds. Indeed, the last statement is equivalent to the statement that for any non-negative integers α, β, γ the equality

$$\min\{\alpha+\beta, \gamma\} = \min\{\alpha, \gamma\} + \min\{\beta, \gamma - \min\{\alpha, \gamma\}\} \tag{37}$$

holds. If $\min\{\alpha, \gamma\} = \gamma$, then clearly $\gamma \le \alpha + \beta$ and inequality (37) is true. On the other hand, if $\min\{\alpha, \gamma\} = \alpha$, then (37) reduces to to the obvious equality

$$\min\{\alpha+\beta, \gamma\} = \alpha + \min\{\beta, \gamma - \alpha\}.$$

Setting in (36)

$$a = \deg{}_{f(z)} g, \quad b = \deg{}_z f, \quad c = \nu\Big((g\circ f)(z)\Big).$$

and using the formulas

$$\deg{}_z(g\circ f) = \deg{}_z f \deg{}_{f(z)} g \tag{38}$$

and (34), we have:

$$\begin{aligned}\mathrm{GCD}\Big(\deg{}_z(g\circ f), \nu\Big((g\circ f)(z)\Big)\Big) &= \\ = \mathrm{GCD}\Big(\deg{}_{f(z)} g, \nu\Big((g\circ f)(z)\Big)\Big)&\mathrm{GCD}\Big(\deg{}_z f, \nu_2(f(z))\Big).\end{aligned} \tag{39}$$

Since equalities (33) and (34) imply the equality

$$\nu\Big((g\circ f)(z)\Big) = \nu_1(z)\mathrm{GCD}\Big(\deg{}_z f, \nu_2(f(z))\Big)\mathrm{GCD}\Big(\deg{}_{f(z)} g, \nu\Big((g\circ f)(z)\Big)\Big)$$

equality (35) follows now from equality (39). □

Corollary 4.1. *Let $f : \mathcal{O}_1 \to \mathcal{O}'$ and $g : \mathcal{O}' \to \mathcal{O}_2$ be minimal holomorphic maps (resp. covering maps) between orbifolds. Then $g\circ f : \mathcal{O}_1 \to \mathcal{O}_2$ is a minimal holomorphic map (resp. a covering map).*

Proof. It follows from the conditions that

$$\mathcal{O}' = g^*\mathcal{O}_2, \quad \mathcal{O}_1 = f^*(\mathcal{O}') = f^*(g^*\mathcal{O}_2).$$

Therefore, by Theorem 4.1,

$$\mathcal{O}_1 = (g \circ f)^* \mathcal{O}_2,$$

implying that $g \circ f : \mathcal{O}_1 \to \mathcal{O}_2$ is a minimal holomorphic map. Furthermore, the equalities

$$\nu'(f(z)) = \nu_1(z) \deg_z f, \tag{40}$$

and

$$\nu_2\Big((g \circ f)(z)\Big) = \nu'(f(z)) \deg_{f(z)} g \tag{41}$$

imply the equality

$$\nu_2\Big((g \circ f)(z)\Big) = \nu_1(z) \deg_z (g \circ f). \quad \square \tag{42}$$

Corollary 4.2. *Let $f : R_1 \to R'$ and $g : R' \to R_2$ be holomorphic branched covering maps, and $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ orbifolds. Assume that $g \circ f : \mathcal{O}_1 \to \mathcal{O}_2$ is a minimal holomorphic map (resp. a covering map). Then $f : \mathcal{O}_1 \to g^* \mathcal{O}_2$ and $g : g^* \mathcal{O}_2 \to \mathcal{O}_2$ are minimal holomorphic maps (resp. covering maps).*

Proof. Set $\mathcal{O}' = g^* \mathcal{O}_2$. By condition, $\mathcal{O}_1 = (g \circ f)^* \mathcal{O}_2$. Therefore, by Theorem 4.1,

$$\mathcal{O}_1 = f^*(\mathcal{O}'),$$

implying that $f : \mathcal{O}_1 \to \mathcal{O}'$ is a minimal holomorphic map.

Further, if $g \circ f : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map, then it follows from (42) that

$$\deg_{f(z)} g \mid \nu_2\Big((g \circ f)(z)\Big),$$

implying that $g : \mathcal{O}' \to \mathcal{O}_2$ is a covering map. Now, equalities (42) and (41) imply equality (40). $\square$

With each holomorphic function $f : R_1 \to R_2$ between compact Riemann surfaces one can associate in a natural way two orbifolds $\mathcal{O}_1^f = (R_1, \nu_1^f)$ and $\mathcal{O}_2^f = (R_2, \nu_2^f)$, setting $\nu_2^f(z)$ equal to the least common multiple of local degrees of f at the points of the preimage $f^{-1}\{z\}$, and

$$\nu_1^1(z) = \frac{\nu_2^f(f(z))}{\deg_z f}.$$

By construction, f is a covering map between orbifolds $f : \mathcal{O}_1^f \to \mathcal{O}_2^f$. Notice that by Lemma 4.1 the equality

$$\mathcal{O}_1^f = f^* \mathcal{O}_2^f \tag{43}$$

holds.

Lemma 4.2. *Orbifolds $\mathcal{O}_1^f$ and $\mathcal{O}_2^f$ have a universal cover.*

Proof. Equality (18) implies that f may not have only one critical value. Moreover, if f has two critical values, then, by transitivity of G_h, the corresponding permutations have the same order equal to $\deg f$. Therefore, $\mathcal{O}_2^f$ has a universal cover.

Show now that $\mathcal{O}_1^f$ also has a universal cover. Let $\theta : \widetilde{\mathcal{O}} \to \mathcal{O}_2^f$ be a universal cover of $\mathcal{O}_2^f$, and $\widehat{\theta}$ the complete analytic continuation of a germ $f^{-1} \circ \theta$, where f^{-1} is a germ of a branch of the algebraic function inverse to f. It follows from the equality

$$\deg_z \theta = \nu_2^f(\theta(z)), \quad z \in \widetilde{\mathcal{O}}, \tag{44}$$

and the definition of $\mathcal{O}_2^f$ that $\widehat{\theta}$ has no local branching. Therefore, since $\widetilde{\mathcal{O}}$ is simply connected, $\widehat{\theta}$ is single valued. Moreover, $f \circ \widehat{\theta} = \theta$. It follows now from equality (43) and Corollary 4.2 that $\widehat{\theta} : \widetilde{\mathcal{O}} \to \mathcal{O}_1^f$ is a covering map between orbifolds. Since $\widetilde{\mathcal{O}}$ is non-ramified, this implies that $\widehat{\theta}$ is a universal cover of $\mathcal{O}_1^f$. □

Theorem 4.2. *Let h, f, p, g, q be a good solution of the equation $h = f \circ p = g \circ q$. Then the commutative diagram*

$$\begin{array}{ccc} \mathcal{O}_1^q & \xrightarrow{p} & \mathcal{O}_1^f \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle f} \\ \mathcal{O}_2^q & \xrightarrow{g} & \mathcal{O}_2^f \end{array} \tag{45}$$

consists of minimal holomorphic maps between orbifolds

Proof. Denote by C the Riemann surface on which g is defined. Let $z \in C$ be a point, $\rho \subset C$ a small free loop around z, and $z_1 \in \rho$ a point such that $g(z_1) = z_0$ is a regular value of h. Then a permutation of points of $h^{-1}\{z_0\}$ corresponding to the analytic continuation of h^{-1} along the curve $g(\rho) \subset \mathbb{CP}^1$ induces a permutation σ_1 of points of $q^{-1}\{z_1\}$ as well as a permutation σ_2 of points of $f^{-1}\{z_0\}$. Furthermore, by Lemma 2.2 the permutations σ_1 and σ_2 have the same cyclic structure. In particular, since the order of σ_1 is equal to $\nu_2^q(z)$ by construction of $\mathcal{O}_2^q$, the order of σ_2 also is equal to $\nu_2^q(z)$.

On the other hand,

$$\sigma_2 = \sigma_3^{\deg_z g},$$

where σ_3 is a permutation of points $f^{-1}\{z_0\}$ induced by the analytic continuation of h^{-1} along a small free loop $\widetilde{\rho}$ around $g(z)$. Since the order of σ_3 is equal to $\nu_2^f(g(z))$, this yields that the order of σ_2 is equal to

$$\frac{\nu_2^f(g(z))}{\mathrm{GCD}(\deg_z g, \nu_2^f(g(z))},$$

implying the equality

$$\nu_2^f(g(z)) = \nu_2^q(z)\mathrm{GCD}(\deg_z g, \nu_2^f(g(z)). \tag{46}$$

Thus, $g : \mathcal{O}_2^q \to \mathcal{O}_2^f$ is a minimal holomorphic map between orbifolds.

Further, since $g : \mathcal{O}_2^q \to \mathcal{O}_2^f$ is a minimal holomorphic map, it follows from Corollary 4.1 that $g \circ q : \mathcal{O}_1^q \to \mathcal{O}_2^f$ also is a minimal holomorphic map. Finally, since $f \circ p = g \circ q$, it follows now from Corollary 4.2 taking into account equality (43) that $p : \mathcal{O}_1^q \to \mathcal{O}_1^f$ also is a minimal holomorphic map. $\square$

5 Minimal self-maps and equivariant functions

Let f be a rational function of degree at least two such that $f : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic self-map between orbifolds for some $\mathcal{O}$ defined on $\mathbb{CP}^1$. Then by Corollary 3.1 the inequality $\chi(\mathcal{O}) \geq 0$ holds and the equality attains if and only if f is a covering map. In turn, the condition that $f : \mathcal{O} \to \mathcal{O}$ is a covering map for some $\mathcal{O}$ with $\chi(\mathcal{O}) = 0$ is equivalent to the condition that f is a Lattès function (see [11], Theorem 4.1). Thus, minimal holomorphic self-maps $f : \mathcal{O} \to \mathcal{O}$ for $\mathcal{O}$ with $\chi(\mathcal{O}) = 0$ are exactly Lattès functions whose properties are well-established (see [11] for a comprehensive survey of these properties). In contrast, rational functions which are minimal holomorphic self-maps $f : \mathcal{O} \to \mathcal{O}$ for $\mathcal{O}$ with $\chi(\mathcal{O}) > 0$ seem to be a completely new object. In this section we provide a characterization of such functions, and outline an approach to their classification.

If $\chi(\mathcal{O}) > 0$ and $\mathcal{O}$ is neither non-ramified sphere nor one of two orbifolds without the universal cover, then the collection of ramification indices of $\mathcal{O}$ is either (n, n), or $(2, 2, n)$ for some $n \geq 2$, or one of the following triples $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$. Further, for all these collections the universal cover $\widetilde{\mathcal{O}}$ of $\mathcal{O}$ is $\mathbb{CP}^1$, and the corresponding groups $\Gamma_{\mathcal{O}}$ are finite subgroups of the automorphism group of $\mathbb{CP}^1$, namely, the cyclic, dihedral, tetrahedral, octahedral, and icosahedral. Accordingly, the functions $\theta_{\mathcal{O}}$ are rational functions of degree n, $2n$, 12, 24, and 60 which can be characterized as regular covers of the sphere by the sphere (see e.g. [6]).

Theorem 5.1. *Let A and F be rational functions of degree at least two and $\mathcal{O} = (\mathbb{CP}^1, \nu)$ an orbifold with $\chi(\mathcal{O}) > 0$ such that $A : \mathcal{O} \to \mathcal{O}$ is a holomorphic map between orbifolds and the diagram*

$$\begin{array}{ccc} \widetilde{\mathcal{O}} & \xrightarrow{F} & \widetilde{\mathcal{O}} \\ \downarrow \theta_{\mathcal{O}} & & \downarrow \theta_{\mathcal{O}} \\ \mathcal{O} & \xrightarrow{A} & \mathcal{O} \end{array} \tag{47}$$

is commutative. Then the following conditions are equivalent.

1. *The holomorphic map A is a minimal holomorphic map.*
2. *The homomorphism $\varphi : \Gamma_{\mathcal{O}} \to \Gamma_{\mathcal{O}}$ defined by the equality*

$$F \circ \sigma = \varphi(\sigma) \circ F, \quad \sigma \in \Gamma_{\mathcal{O}}, \tag{48}$$

   is an automorphism of $\Gamma_{\mathcal{O}}$.

3. The triple F, A, $\theta_{\mathcal{O}}$ is a good solution of the equation

$$A \circ \theta_{\mathcal{O}} = \theta_{\mathcal{O}} \circ F. \tag{49}$$

Proof. By Lemma 2.1, in order to prove $2 \Leftrightarrow 3$ it is enough to show that φ is an automorphism if and only if the functions F and $\theta_{\mathcal{O}}$ have no non-trivial common compositional right factor. Since $\theta_{\mathcal{O}}$ is a regular cover, any compositional right factor of $\theta_{\mathcal{O}}$ of degree greater than one has the form $\theta_{\mathcal{O}'}$, where $\Gamma_{\mathcal{O}'} \neq \{e\}$ is a subgroup of $\Gamma_{\mathcal{O}}$. Therefore, the functions F and $\theta_{\mathcal{O}}$ have a non-trivial common compositional right factor if and only if there exists $\Gamma_{\mathcal{O}'} \subseteq \Gamma_{\mathcal{O}}$ such that $F(z_1) = F(z_2)$ whenever $z_2 = \sigma(z_1)$ for some $\sigma \in \Gamma_{\mathcal{O}'}$. On the other hand, by (48), such a subgroup exists if and only of φ is not a monomorphism. This proves the equivalence $2 \Leftrightarrow 3$.

By Lemma 2.1 and Corollary 2.1, in order to prove $1 \Leftrightarrow 3$, it is enough to show that $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map between orbifolds if and only if $\deg_z \theta_{\mathcal{O}}$ and $\deg_z F$ are coprime for any $z \in \mathbb{CP}^1$. Since

$$\deg_z \theta_{\mathcal{O}} \deg_{\theta_{\mathcal{O}}(z)} A = \deg_z F \deg_{F(z)} \theta_{\mathcal{O}},$$

the last condition is equivalent to the equality

$$\deg_{F(z)} \theta_{\mathcal{O}} = \deg_z \theta_{\mathcal{O}} \mathrm{GCD}(\deg_{\theta_{\mathcal{O}}(z)} A, \deg_{F(z)} \theta_{\mathcal{O}}). \tag{50}$$

On the other hand, since $\deg_z \theta_{\mathcal{O}} = \nu(\theta_{\mathcal{O}}(z))$ and

$$\deg_{F(z)} \theta_{\mathcal{O}} = \nu(\theta_{\mathcal{O}}(F(z))) = \nu(A(\theta_{\mathcal{O}}(z))),$$

equality (50) is equivalent to the equality

$$\nu(A(\theta_{\mathcal{O}}(z))) = \nu(\theta_{\mathcal{O}}(z)) \mathrm{GCD}\Big(\deg_{\theta_{\mathcal{O}}(z)} A, \nu(A(\theta_{\mathcal{O}}(z)))\Big),$$

which in turn is equivalent to the requirement that $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic map. $\square$

Corollary 5.1. *Let $\mathcal{O} = (\mathbb{CP}^1, \nu)$ be an orbifold with $\chi(\mathcal{O}) > 0$, and $A : \mathcal{O} \to \mathcal{O}$ a minimal holomorphic map between orbifolds. Then for any decomposition $A = U \circ V$ the ramification collection of $U^*\mathcal{O}$ coincides with the one of $\mathcal{O}$.*

Proof. Denote $U^*\mathcal{O}$ by $\mathcal{O}'$. It follows from Corollary 4.2 and Proposition 3.1 that there exist holomorphic maps F_U and F_V which make the diagram

$$\begin{array}{ccccc}
\widetilde{\mathcal{O}} & \xrightarrow{F_V} & \widetilde{\mathcal{O}'} & \xrightarrow{F_U} & \widetilde{\mathcal{O}} \\
\downarrow \theta_{\mathcal{O}} & & \downarrow \theta_{\mathcal{O}'} & & \downarrow \theta_{\mathcal{O}} \\
\mathcal{O} & \xrightarrow{V} & \mathcal{O}' & \xrightarrow{U} & \mathcal{O}
\end{array}$$

commutative. Since

$$\chi(\mathcal{O}) \leq \chi(\mathcal{O}') \deg V$$

by Proposition 3.2, the inequality $\chi(\mathcal{O}') > 0$ holds, implying that $\widetilde{\mathcal{O}'} = \mathbb{CP}^1$. Furthermore, there exist homomorphisms

$$\varphi_V : \Gamma_{\mathcal{O}} \to \Gamma_{\mathcal{O}'}, \quad \varphi_U : \Gamma_{\mathcal{O}'} \to \Gamma_{\mathcal{O}}$$

such that

$$F_V \circ \sigma = \varphi_V(\sigma) \circ F_V, \quad \sigma \in \Gamma_{\mathcal{O}}, \qquad F_U \circ \sigma = \varphi_U(\sigma) \circ F_U, \quad \sigma \in \Gamma_{\mathcal{O}'}.$$

Since the function $F_U \circ F_V$ makes diagram (47) commutative, Theorem 5.1 implies that the composition of homomorphisms

$$\varphi_U \circ \varphi_V : \Gamma_{\mathcal{O}} \to \Gamma_{\mathcal{O}}$$

is an automorphism. Therefore, $\Gamma_{\mathcal{O}'} \cong \Gamma_{\mathcal{O}}$, and the ramification collection of $\mathcal{O}'$ coincides with the one of $\mathcal{O}$. $\square$

Theorem 5.1 reduces the study of minimal holomorphic maps $f : \mathcal{O} \to \mathcal{O}$ with $\chi(\mathcal{O}) > 0$ to the study of rational functions such that (48) holds for some finite subgroup $\Gamma_{\mathcal{O}}$ of $\mathbb{CP}^1$ and an automorphism $\varphi : \Gamma_{\mathcal{O}} \to \Gamma_{\mathcal{O}}$. For example, consider an orbifold $\mathcal{O}$ with ramification

$$\nu(0) = n, \quad \nu(\infty) = n.$$

In this case, $\Gamma_{\mathcal{O}} = \mathbb{Z}/n\mathbb{Z}$ is generated by the transformation

$$E_n : z \to e^{2\pi i/n} z,$$

and $\theta_{\mathcal{O}} = z^n$. Since a homomorphism $\varphi : \Gamma_{\mathcal{O}} \to \Gamma_{\mathcal{O}}$ is an automorphism if and only if

$$\varphi(E_n) : z \to e^{2\pi i r/n} z, \tag{51}$$

for some r, $1 \leq r \leq n-1$, with $\mathrm{GCD}(r, n) = 1$, a rational function F satisfies (48) for some automorphism φ of $\Gamma_{\mathcal{O}}$ if and only if F/z^r is $\Gamma_{\mathcal{O}}$-invariant for some r as above, that is if and only if

$$F = z^r R(z^n)$$

for some rational function R. Accordingly, since the function $A = z^r R^n(z)$ makes corresponding diagram (47) commutative, minimal holomorphic maps $f : \mathcal{O} \to \mathcal{O}$ have the form $z^r R^n(z)$, and the correspondence between minimal holomorphic maps and functions satisfying (48) is given by the commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{O}} & \xrightarrow{z^r R(z^n)} & \widetilde{\mathcal{O}} \\ \big\downarrow z^n & & \big\downarrow z^n \\ \mathcal{O} & \xrightarrow{z^r R^n(z)} & \mathcal{O}. \end{array} \tag{52}$$

More generally, it was shown in [1] that for finite subgroups Γ of $Aut(\mathbb{CP}^1)$ the problem of describing of Γ-equivariant functions, that is of functions F

satisfying (48) for the *identity* automorphism φ, may be reduced to the classical problem of describing of homogeneous Γ-invariant polynomials solved by Klein in [6]. For the group $\Gamma = S_4$, say, this solves the problem. Indeed, since F corresponding to A in (47) is defined up to the change $F \to g \circ F$, where $g \in \Gamma_{\mathcal{O}}$ (see Proposition 3.1), the automorphism φ is defined up to the change $\varphi \to g \circ \varphi \circ g^{-1}$. Therefore, since all automorphisms of S_4 are inner, without loss of generality we may assume that $\varphi(\sigma) = \sigma$. For Γ equal to A_4 or A_5 the group $Out(\Gamma)$ consists of two elements, implying that in these cases in addition to the identity automorphism we must consider one additional automorphism, and one can try to extend the approach of [1] so that to cover this case too. Notice also that although a function F satisfying (48) is not necessary $\Gamma_{\mathcal{O}}$-equivariant, it follows from the finiteness of $\Gamma_{\mathcal{O}}$ that some *iterate* of F is $\Gamma_{\mathcal{O}}$-equivariant.

6 Proof of Theorem 1.1 and examples

We start from proving Theorem 1.1 in the case where $\mathbb{C}(B, X) = \mathbb{C}(z)$. We will call solutions of (1) satisfying this condition *primitive*. By Lemma 2.1, a solution A, X, B of (1) is primitive if and only if the corresponding solution of (17) given by

$$f = q = X, \quad p = B, \quad g = A,$$

is good. Clearly, any solution A, X, B of (1) with $\deg X = 1$ is primitive. The corresponding functions A and B are conjugated and therefore equivalent.

Theorem 6.1. *Let A, X, B be a primitive solution of* (1) *with* $\deg X > 1$. *Then* $\chi(\mathcal{O}_1^X) \geq 0$, $\chi(\mathcal{O}_2^X) \geq 0$, *and the commutative diagram*

$$\begin{tikzcd} \mathcal{O}_1^X \arrow[r, "B"] \arrow[d, "X"] & \mathcal{O}_1^X \arrow[d, "X"] \\ \mathcal{O}_2^X \arrow[r, "A"] & \mathcal{O}_2^X \end{tikzcd} \tag{53}$$

consists of minimal holomorphic maps between orbifolds. Furthermore, either $\chi(\mathcal{O}_1^X) = \chi(\mathcal{O}_2^X) = 0$ *and A, B are Lattès functions, or* $0 < \chi(\mathcal{O}_2^X) < \chi(\mathcal{O}_1^X)$. *In the last case, possible collections of ramification indices of $\mathcal{O}_1^X$ and $\mathcal{O}_2^X$ are following:* (n, n) *or* $(2, 2, n)$ *for some* $n \geq 2$, *or one of the triples* $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$. *In addition, $\mathcal{O}_1^X$ may be a non-ramified sphere.*

Proof. Since $A : \mathcal{O}_2^X \to \mathcal{O}_2^X$ and $B : \mathcal{O}_1^X \to \mathcal{O}_1^X$ are minimal holomorphic maps between orbifolds by Theorem 4.2, it follows from Corollary 3.1 that $\chi(\mathcal{O}_1^X) \geq 0$ and $\chi(\mathcal{O}_2^X) \geq 0$. Furthermore, since $X : \mathcal{O}_1^X \to \mathcal{O}_2^X$ is a covering map, formula (29) implies that either $\chi(\mathcal{O}_1^X) = \chi(\mathcal{O}_2^X) = 0$, or $0 < \chi(\mathcal{O}_2^X) < \chi(\mathcal{O}_1^X)$. In the first case, A and B are Lattès functions. On the other hand, if $\chi(\mathcal{O}_2^X) > 0$, then the collections of ramification indices of $\mathcal{O}_1^X$ and $\mathcal{O}_2^X$ have the required form since $\mathcal{O}_1^X$ and $\mathcal{O}_2^X$ have a universal cover by Lemma 4.2, and the condition $\deg X > 1$ implies that $\mathcal{O}_2^X$ is distinct from the non-ramified sphere. $\square$

The proof of Theorem 1.1 in the general case reduces to the primitive case as follows. Let A, X, B be a non-primitive solution of (1) and U_1 any rational function such that

$$\mathbb{C}(X, B) = \mathbb{C}(U_1) \tag{54}$$

so that

$$X = X_1 \circ U_1, \quad B = V_1 \circ U_1 \tag{55}$$

for some $X_1, V_1 \in \mathbb{C}(z)$ such that $\mathbb{C}(X_1, V_1) = \mathbb{C}(z)$. Clearly, (1) and (55) imply that

$$A \circ X_1 = X_1 \circ (U_1 \circ V_1).$$

Thus, $A, X_1, U_1 \circ V_1$ is another solution of (1). Continuing in this way, define X_{i+1}, V_{i+1}, and U_{i+1}, $i \geq 1$, as rational functions satisfying the equalities

$$\mathbb{C}(X_i, U_i \circ V_i) = \mathbb{C}(U_{i+1}), \tag{56}$$

$$X_i = X_{i+1} \circ U_{i+1}, \tag{57}$$

$$U_i \circ V_i = V_{i+1} \circ U_{i+1}, \tag{58}$$

and set

$$B_i = U_i \circ V_i.$$

Clearly,

$$A \circ X_i = X_i \circ B_i.$$

Furthermore, since $\deg X_{i+1} < \deg X_i$ whenever $\deg U_{i+1} > 1$, the equality $\mathbb{C}(X_l, B_l) = \mathbb{C}(z)$ holds for some $l \geq 1$, and hence A, X_l, B_l is a primitive solution of (1).

By construction, if $\deg X_l = 1$, then $B \sim A$. Otherwise, by Theorem 6.1, the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_1^{X_l} & \xrightarrow{B_l} & \mathcal{O}_1^{X_l} \\ \big\downarrow X_l & & \big\downarrow X_l \\ \mathcal{O}_2^{X_l} & \xrightarrow{A} & \mathcal{O}_2^{X_l} \end{array}$$

consists of minimal holomorphic maps between orbifolds. Set

$$\mathcal{O} = U_l^* \mathcal{O}_1^{X_l}.$$

Since $B_l = U_l \circ V_l$, it follows from Corollary 4.2 and Corollary 5.1 that

$$V_l : \mathcal{O}_1^{X_l} \to \mathcal{O}, \quad U_l : \mathcal{O} \to \mathcal{O}_1^{X_l}$$

are minimal holomorphic maps between orbifolds, and $\mathcal{O}$ has the same ramification collection as $\mathcal{O}_1^{X_l}$ (and hence the same Euler characteristic). It follows now from Corollary 4.1, taking into account the equality

$$B_{l-1} = U_{l-1} \circ V_{l-1} = V_l \circ U_l,$$

that

$$X_l \circ U_l : \mathcal{O} \to \mathcal{O}_2^{X_l}, \quad B_{l-1} : \mathcal{O} \to \mathcal{O}$$

are minimal holomorphic maps between orbifolds. Since

$$X = X_l \circ U_l \circ U_{l-1} \circ \cdots \circ U_1, \tag{59}$$

continuing in this way we see that the conclusion of Theorem 1.1 holds for $\mathcal{O}_2 = \mathcal{O}_2^{X_l}$ and some orbifold $\mathcal{O}_1$ whose ramification collection coincides with the one of $\mathcal{O}_1^{X_l}$. □

The simplest examples of solutions of (1) are obtained from diagram (47), where F, A, and $\mathcal{O}$ satisfy the conditions of Theorem 5.1. For example, diagram (52) provides a family of such examples. Moreover, since for any finite subgroup Γ of $Aut(\mathbb{CP}^1)$ there exist Γ-equivariant rational functions F (see [1]), similar examples can be given for any finite subgroup Γ of $Aut(\mathbb{CP}^1)$.

More generally, assume that the automorphism φ in (48) satisfies the equality

$$\varphi(\Gamma_{\mathcal{O}'}) = \Gamma_{\mathcal{O}'}$$

for some subgroup $\Gamma_{\mathcal{O}'}$ of $\Gamma_{\mathcal{O}}$ (notice that if F is $\Gamma_{\mathcal{O}}$-equivariant this is true for any subgroup $\Gamma_{\mathcal{O}'}$ of $\Gamma_{\mathcal{O}}$). Then by Proposition 3.1 there exists a rational function B such that

$$B \circ \theta_{\mathcal{O}'} = \theta_{\mathcal{O}'} \circ F. \tag{60}$$

On the other hand, it follows from $\Gamma_{\mathcal{O}'} \subset \Gamma_{\mathcal{O}}$ that

$$\theta_{\mathcal{O}} = X \circ \theta_{\mathcal{O}'} \tag{61}$$

for some rational function X. Thus, the diagram

$$\begin{array}{ccc}
\widetilde{\mathcal{O}'} & \xrightarrow{F} & \widetilde{\mathcal{O}'} \\
\Big\downarrow{\scriptstyle \theta_{\mathcal{O}'}} & & \Big\downarrow{\scriptstyle \theta_{\mathcal{O}'}} \\
\mathcal{O}' & \xrightarrow{B} & \mathcal{O}' \\
\Big\downarrow{\scriptstyle X} & & \Big\downarrow{\scriptstyle X} \\
\mathcal{O} & \xrightarrow{A} & \mathcal{O}
\end{array} \tag{62}$$

is commutative, implying that A, X, and B satisfy (1).

In order to illustrate the above construction, consider an orbifold $\mathcal{O}$ with

$$\nu(1) = \nu(-1) = 2, \quad \nu(\infty) = n.$$

In this case $\Gamma_{\mathcal{O}} = D_{2n}$ is generated by the transformations

$$\alpha : z \to e^{2\pi i/n} z, \quad \beta : z \to \frac{1}{z},$$

and

$$\theta_{\mathcal{O}} = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right). \tag{63}$$

Clearly, the function $F = z^m$, where $\mathrm{GCD}(n,m) = 1$, satisfies (48) for some automorphism φ, and corresponding diagram (47) has the form

$$T_m \circ \frac{1}{2}\left(z^n + \frac{1}{z^n}\right) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right) \circ z^m.$$

Further, F maps the cyclic subgroup $\Gamma_{\mathcal{O}'} \subset \Gamma_{\mathcal{O}}$, generated by β and corresponding to the orbifold $\mathcal{O}'$ defined by the condition $\nu(-1) = \nu(1) = 2$, to itself, and equality (61) has the form

$$\frac{1}{2}\left(z^n + \frac{1}{z^n}\right) = T_n \circ \frac{1}{2}\left(z + \frac{1}{z}\right).$$

Thus, we arrive to the diagram

$$\begin{array}{ccc}
\widetilde{\mathcal{O}'} & \xrightarrow{z^m} & \widetilde{\mathcal{O}'} \\
\Big\downarrow \frac{1}{2}\left(z+\frac{1}{z}\right) & & \Big\downarrow \frac{1}{2}\left(z+\frac{1}{z}\right) \\
\mathcal{O}' & \xrightarrow{T_m} & \mathcal{O}' \\
\Big\downarrow T_n & & \Big\downarrow T_n \\
\mathcal{O} & \xrightarrow{T_m} & \mathcal{O}
\end{array}$$

and to the series of semiconjugate functions

$$T_m \circ T_n = T_n \circ T_m. \tag{64}$$

Examples of solutions of (1) involving Lattès functions can be constructed similarly. We start from a covering map $A : \mathcal{O} \to \mathcal{O}$ for some orbifold $\mathcal{O}$ of zero Euler characteristic and a function F which makes the diagram

$$\begin{array}{ccc}
\widetilde{\mathcal{O}} & \xrightarrow{F} & \widetilde{\mathcal{O}} \\
\Big\downarrow \theta_{\mathcal{O}} & & \Big\downarrow \theta_{\mathcal{O}} \\
\mathcal{O} & \xrightarrow{A} & \mathcal{O}
\end{array} \tag{65}$$

commutative (notice that since A is a covering map, F is an isomorphism and hence has the form $F = az + b$, $a, b \in \mathbb{C}$). Since $\theta_{\mathcal{O}}$ is transcendental, diagram (65) by itself does not provide now any rational solution of (1). However, it is easy to see that if $\Gamma_{\mathcal{O}'} \subseteq \Gamma_{\mathcal{O}}$ is a subgroup corresponding to another orbifold $\mathcal{O}'$ with $\chi(\mathcal{O}') = 0$ such that φ in (27) satisfies the condition $\varphi(\Gamma_{\mathcal{O}'}) \subseteq \Gamma_{\mathcal{O}'}$, then there exists a rational function X such that diagram (62) is commutative and the corresponding Lattès functions B and A are semiconjugate.

For example, let $\mathcal{O}$ be an orbifold with ramification $(2,2,2,2)$. For such an orbifold the group $\Gamma_{\mathcal{O}}$ is generated by translations by elements of some lattice L of rank two in $\mathbb{C}$ and the transformation $z \to -z$. The universal cover $\theta_{\mathcal{O}}$ is the Weierstrass function $\wp_L$ corresponding to L. Clearly, the function $F = mz$, where $m \geq 2$ is an integer, satisfies (27). The corresponding Lattès function $A = R_{L,m}$ is a rational function satisfying the equality $\wp_L(mz) = R_{L,m} \circ \wp_L$. Let now L' be a sublattice of L. Then $\wp_L = X \circ \wp_{L'}$ for some rational function X, and $F(L') \subset L'$, implying that

$$R_{L,m} \circ X = X \circ R_{L',m}.$$

In conclusion, let us make several comments regarding the equivalence relation $\sim$. First, observe that for a rational function F its equivalence class may simply coincide with its conjugacy class. This is the case for examples for any rational function F which cannot be decomposed into a composition of rational functions of lesser degrees. Notice that in fact any generic rational function F is indecomposable since the condition $F = A \circ B$ imposes algebraic restrictions on the coefficients of F. Another example of a function whose equivalence class coincides with its conjugacy class is the function $F = z^n$. Although this function is decomposable, any decomposition $A \circ B$ of z^n has the form

$$A = z^d \circ \mu, \qquad B = \mu^{-1} \circ z^{n/d}$$

for some $d|n$ and a Möbius transformation μ, and therefore any transformation of the form $A \circ B \to B \circ A$ leads to a conjugated function.

Further, by the result of McMullen cited in the introduction, any equivalence class contains at most a finite number of conjugacy classes. However, there is no uniform bound on the number of such classes. Indeed, for any rational function R and a natural number d we have:

$$z^2 \circ zR(z^{2^d}) = zR^2(z^{2^{d-1}}) \circ z^2,$$

$$z^2 \circ zR^2(z^{2^{d-1}}) = zR^4(z^{2^{d-2}}) \circ z^2,$$

$$z^2 \circ zR^4(z^{2^{d-2}}) = zR^8(z^{2^{d-3}}) \circ z^2,$$

$$\dots$$

$$z^2 \circ zR^{2^{d-1}}(z^2) = zR^{2^d}(z) \circ z^2,$$

implying that

$$z^2 \circ zR(z^{2^d}) \sim z^2 \circ zR^2(z^{2^{d-1}}) \sim z^2 \circ zR^4(z^{2^{d-2}}) \sim \cdots \sim z^2 \circ zR^{2^{d-1}}(z^2). \quad (66)$$

Setting for example $R = z - 1$, it is easy to see that the corresponding functions

$$F_i = z^2(z^{2^{d-i+1}} - 1)^{2^i}, \quad i = 1, \dots d,$$

in (66) cannot be conjugated. Indeed, since $F_i^{-1}\{\infty\} = \{\infty\}$, $1 \leq i \leq d$, the equality

$$F_i = \mu \circ F_j \circ \mu^{-1}, \quad i \neq j, \tag{67}$$

where μ is a Möbius transformation, implies that $\mu = \alpha z + \beta$, $\alpha, \beta \in \mathbb{C}$. Furthermore, denoting by n the common degree of the functions F_i, $1 \leq i \leq d$, and comparing the coefficients of z^{n-1} in both parts of equality (67), we conclude that $\beta = 0$. Finally, (67) cannot be satisfied for $\mu = \alpha z$ since polynomials in the left and in the right parts of (67) have different collections of monomials with non-zero coefficients.

Acknowledgments. The author is grateful to A. Eremenko and C. McMullen for discussions, and to the Max-Planck-Institut fuer Mathematik for the hospitality and the support. Besides, the author would like to thank the referee for many valuable comments.

ADDENDUM

The very first version of this paper was published in the form of a preprint in 2011. Below we mention some publications related to semiconjugate rational functions that appeared afterward.

In the paper [3] functional equation (1) was related with the problem of describing of Jordan curves in $\mathbb{C}$ invariant under a rational function. On the other hand, it was shown in the paper [9] that equation (1) is closely related to the problem of describing of algebraic curves $\mathcal{C}$ in $\mathbb{C}^2$ invariant under maps of the form $F : (x, y) \to (f(x), g(y))$, where f, g are polynomials of degree at least two. In particular, the paper [9] contains a detailed analysis of equation (1) in the polynomial case, based on the Ritt theory of decompositions of polynomials.

Another approach to equation (1) in the polynomial case, using results of [12] about polynomials sharing preimages of compact sets, was proposed in [15]. In particular, it was shown in [15] that in the polynomial case the conditions $A \leq B$ and $B \leq A$ hold simultaneously if and only if $A \sim B$. Notice however that methods of both papers [9] and [15] seem to be restricted to the polynomial case.

It the paper [14] the methods of this paper were applied to the functional equation $A \circ C = D \circ B$, where A, B, C, D are rational functions. In particular, it was shown in [14] that if A, B, C, D satisfy this equation and the algebraic curve

$$A(x) - D(y) = 0$$

is irreducible, then whenever $\deg D \geq 84 \deg A$ the inequality $\chi(\mathcal{O}_2^A) \geq 0$ holds. Finally, further development of ideas and methods of this paper was given in the paper [16] devoted to quantitative aspects of the description of solutions of (1) for fixed B. In particular, it was shown in [16] that if B is neither a Lattès function nor conjugated to $z^{\pm d}$ or $\pm T_d$, then, up to some natural transformations, the number of A and X satisfying (1) is finite and can be effectively bounded in terms of $\deg B$ only.

References

[1] P. Doyle, C. McMullen, *Solving the quintic by iteration,* Acta Math. 163 (1989), no. 3-4, 151-180.

[2] A. Eremenko, *Some functional equations connected with the iteration of rational functions* (Russian), Algebra i Analiz 1 (1989), 102-116; translation in Leningrad Math. J. 1 (1990), 905-919.

[3] A. Eremenko, *Invariant curves and semiconjugacies of rational functions*, Fundamenta Math., 219, 3 (2012) 263-270.

[4] P. Fatou, *Sur l'iteration analytique et les substitutions permutables,* J. Math. Pures Appl. (9), 2, 1923, 343-384.

[5] G. Julia, *Mémoire sur la permutabilité des fractions rationelles*, Ann. Sci. École Norm. Sup. 39 (3) (1922), 131-215.

[6] F. Klein, *Lectures on the icosahedron and the solution of equations of the fifth degree,* Dover Publications, Inc., New York, N.Y., 1956. xvi+289 pp,

[7] H. Inou, *Extending local analytic conjugacies,* Trans. Amer. Math. Soc. 363 (2011), no. 1, 331-343,

[8] C. McMullen, *Families of rational maps and iterative root-finding algorithms,* Ann. of Math., 125, No. 3 (1987), 467-493.

[9] A. Medvedev, T. Scanlon, *Invariant varieties for polynomial dynamical systems,* Annals of Mathematics, 179 (2014), no. 1, 81 - 177.

[10] J. Milnor, *Dynamics in one complex variable*, Princeton Annals in Mathematics 160. Princeton, NJ: Princeton University Press (2006).

[11] J. Milnor, *On Lattès maps,* Dynamics on the Riemann Sphere. Eds. P. Hjorth and C. L. Petersen. A Bodil Branner Festschrift, European Mathematical Society, 2006, pp. 9-43.

[12] F. Pakovich, *On polynomials sharing preimages of compact sets, and related questions*, Geom. Funct. Anal, 18, No. 1, 163-183 (2008).

[13] F. Pakovich, *Prime and composite Laurent polynomials*, Bull. Sci. Math., 133 (2009), 693-732.

[14] F. Pakovich, *On the genus of an algebraic curve A(x)-B(y)=0*, preprint, arXiv:1505.01007.

[15] F. Pakovich, *Polynomial semiconjugacies, decompositions of iterations, and invariant curves*, Ann. Sc. Norm. Super. Pisa Cl. Sci., to appear.

[16] F. Pakovich, *Finiteness theorems for commuting and semiconjugate rational functions*, preprint, arxiv: 1604.04771.

[17] J. F. Ritt, *Permutable rational functions,* Trans. Amer. Math. Soc. 25 (1923), 399-448.

[18] J. F. Ritt, *Prime and composite polynomials,* Trans. Amer. Math. Soc., 23, 1922, 51-66.

[19] J. F. Ritt, *On the iteration of rational functions,* Trans. Amer. Math. Soc. 21 (1920), no. 3, 348-356.

[20] W. Thurston, *The geometry and topology of 3-manifolds*, mimographed notes, Princeton, 1979.