JORDAN PROPERTY FOR CREMONA GROUPS

YURI PROKHOROV AND CONSTANTIN SHRAMOV

Abstract. Assuming Borisov–Alexeev–Borisov conjecture, we prove that there is a constant $J = J(n)$ such that for any rationally connected variety X of dimension n and any finite subgroup $G \subset \mathrm{Bir}(X)$ there exists a normal abelian subgroup $A \subset G$ of index at most J. In particular, we obtain that the Cremona group $\mathrm{Cr}_3 = \mathrm{Bir}(\mathbb{P}^3)$ enjoys the Jordan property.

1. Introduction

Unless explicitly stated otherwise, all varieties below are assumed to be defined over an algebraically closed field $\Bbbk$ of characteristic 0.

The *Cremona group* $\mathrm{Cr}_n(\Bbbk)$ is the group of birational transformations of the projective space $\mathbb{P}^n$. The group $\mathrm{Cr}_2(\Bbbk)$ and its subgroups have been a subject of research for many years (see [8], [9], [29] and references therein). The main philosophical observation is that this group is very large and it is "very far" from being a linear group. However, the system of its finite subgroups seems more accessible, and in particular happens to enjoy many features of finite subgroups in $\mathrm{GL}_n(\Bbbk)$ (which are actually not obvious even for the subgroups of $\mathrm{GL}_n(\Bbbk)$).

Theorem 1.1 (C. Jordan, see e. g. [6, Theorem 36.13]). *There is a constant $I = I(n)$ such that for any finite subgroup $G \subset \mathrm{GL}_n(\mathbb{C})$ there exists a normal abelian subgroup $A \subset G$ of index at most I.*

This leads to the following definition (cf. [26, Definition 2.1]).

Definition 1.2. A group Γ is called *Jordan* (alternatively, we say that Γ *has Jordan property*) if there is a constant J such that for any finite subgroup $G \subset \Gamma$ there exists a normal abelian subgroup $A \subset G$ of index at most J.

Theorem 1.1 implies that all linear algebraic groups over an arbitrary field $\Bbbk$ with $\mathrm{char}(\Bbbk) = 0$ are Jordan. The same question is of interest for other "large" groups, especially those that are more accessible for study on the level of finite subgroups than on the global level, in particular, for the groups of birational selfmaps of algebraic varieties. A complete answer is known in dimension at most 2. Moreover, already in dimension 2 it appears to be non-trivial, i. e. there are surfaces providing a positive answer to the question, as well as surfaces providing a negative answer.

First of all, the automorphism group of any curve is Jordan. The Cremona group of rank 2 is Jordan too.

Theorem 1.3 (J.-P. Serre [30, Theorem 5.3], [29, Théorème 3.1]). *The Cremona group $\mathrm{Cr}_2(\Bbbk)$ is Jordan.*

On the other hand, starting from dimension 2 one can construct varieties with non-Jordan groups of birational selfmaps.

Theorem 1.4 (Yu. Zarhin [32])**.** *Suppose that* $X \cong E \times \mathbb{P}^1$*, where* E *is an abelian variety of dimension* $\dim(E) > 0$*. Then the group* $\mathrm{Bir}(X)$ *is not Jordan.*

In any case, in dimension 2 it is possible to give a complete classification of surfaces with Jordan groups of birational automorphisms.

Theorem 1.5 (V. Popov [26, Theorem 2.32])**.** *Let* S *be a surface. Then the group* $\mathrm{Bir}(S)$ *is Jordan if and only if* S *is not birational to* $E \times \mathbb{P}^1$*, where* E *is an elliptic curve.*

Somehow, in higher dimensions the answer remained unknown even for a more particular question.

Question 1.6 (J.-P. Serre [30, 6.1])**.** *Is the group* $\mathrm{Cr}_n(\Bbbk)$ *Jordan?*

Question 1.6 asks about some kind of boundedness related to the geometry of rational varieties. It is not a big surprise that it appears to be related to another "boundedness conjecture", that is a particular case of the well-known Borisov–Alexeev–Borisov conjecture (see [3]).

Conjecture 1.7. *For a given positive integer* n*, Fano varieties of dimension* n *with terminal singularities are bounded, i. e. are contained in a finite number of algebraic families.*

Note that if Conjecture 1.7 holds in dimension n, then it also holds in all dimensions $k \leqslant n$.

The main purpose of this paper is to show that modulo Conjecture 1.7 the answer to Question 1.6 is positive even in the more general setting of rationally connected varieties (see Definition 3.2), and moreover the corresponding constant may be chosen in some uniform way. Namely, we prove the following.

Theorem 1.8. *Assume that Conjecture 1.7 holds in dimension* n*. Then there is a constant* $J = J(n)$ *such that for any rationally connected variety* X *of dimension* n *defined over an arbitrary (not necessarily algebraically closed) field* $\Bbbk$ *of characteristic* 0 *and for any finite subgroup* $G \subset \mathrm{Bir}(X)$ *there exists a normal abelian subgroup* $A \subset G$ *of index at most* J*.*

Note that Conjecture 1.7 is settled in dimension 3 (see [23]), so we have the following

Corollary 1.9. *The group* $\mathrm{Cr}_3(\Bbbk)$ *is Jordan.*

As an application of the method we use to prove Theorem 1.8, we can also derive some information about p-subgroups of Cremona groups.

Theorem 1.10. *Assume that Conjecture 1.7 holds in dimension* n*. Then there is a constant* $L = L(n)$ *such that for any rationally connected variety* X *of dimension* n *defined over an arbitrary (not necessarily algebraically closed) field* $\Bbbk$ *of characteristic* 0 *and for any prime* $p > L$*, every finite* p*-subgroup of* $\mathrm{Bir}(X)$ *is an abelian group generated by at most* n *elements.*

Remark 1.11. An easy consequence of Theorem 1.10 is that if $\Bbbk$ is an algebraically closed fields of characteristic 0, and $m > n$ are positive integers, then there does not exist embedding of groups $\mathrm{Cr}_m(\Bbbk) \subset \mathrm{Cr}_n(\Bbbk)$. Indeed, for any p it is easy to construct an abelian p-group $A \subset \mathrm{GL}_m(\Bbbk) \subset \mathrm{Cr}_m(\Bbbk)$ that is not generated by less than m elements. Note that the same result is already known by [7, §1.6] or [4, Theorem B].

The plan of the proof of Theorem 1.8 (that is carried out in Section 4) is as follows. Given a rationally connected variety X and a finite group $G \subset \mathrm{Bir}(X)$, take a smooth regularization $\tilde{X}$ of G (see [31, Theorem 3]). We are going to show that $\tilde{X}$ has a point P fixed by a subgroup $H \subset G$ of bounded index and then apply Theorem 1.1 to H acting in the tangent space $T_P(\tilde{X})$. If $\tilde{X}$ is a G-Mori fiber space (see Section 2 for a definition), then, modulo Conjecture 1.7, we may assume that there is a *non-trivial* G-Mori fiber space structure $\tilde{X} \to S$, i. e. S is not a point. By induction one may suppose that there is a subgroup H of bounded index that fixes a point in S. Using the results of Section 3 (that are based on the auxiliary results of Section 2), we show that $\tilde{X}$ contains a G-invariant rationally connected subvariety. Furthermore, the same assertion holds for an arbitrary smooth $\tilde{X}$; this follows from the corresponding assertion for a G-Mori fiber space obtained by running a G-Minimal Model Program on $\tilde{X}$ by the results of Section 3. Using induction in dimension once again we conclude that there is actually a point in $\tilde{X}$ fixed by H.

The main technical result that allows us to prove Theorem 1.8 is Corollary 3.7 that lets us lift G-invariant rationally connected subvarieties along G-contractions. Actually, it has been essentially proved in [13, Corollary 1.7(1)]. The only new feature that we really need is the action of a finite group. Since this forces us to rewrite the statements and the proofs in any case, we use the chance to write down the details of the proof that were only sketched by the authors of [13]. We also refer a reader to [22] and [15] for ideas of similar flavour.

Acknowledgements. We would like to thank J.-P. Serre who attracted our attention to the questions discussed in this paper. We are also grateful to I. Cheltsov, O. Fujino, S. Gorchinskiy and A. Kuznetsov for useful discussions, and to a referee for interesting comments. A part of this work was written during the first author's stay at the Max-Planck-Institut für Mathematik in Bonn. He would like to thank MPIM for hospitality and support. Both authors were partially supported by RFBR grants No. 11-01-00336-a, 11-01-92613-KO_a, the grant of Leading Scientific Schools No. 5139.2012.1, and AG Laboratory SU-HSE, RF government grant ag. 11.G34.31.0023. The first author was partially supported by Simons-IUM fellowship. The second author was partially supported by the grants MK-6612.2012.1 and RFBR-12-01-33024.

2. Preliminaries

The purpose of this section is to establish several auxiliary results that will be used in Section 3. It seems that most of them are well known to experts, but we decided to include them for completeness since we did not manage to find proper references.

Throughout the rest of the paper we use the standard language of the singularities of pairs (see [21]). By *strictly log canonical* singularities we mean log canonical singularities that are not Kawamata log terminal. By a general point of a (possibly reducible) variety Z we will always mean a point in a Zariski open dense subset of Z. Whenever we speak about the canonical class, or the singularities of pairs related to a (normal) reducible variety, we define everything componentwise (note that connected components of a normal variety are irreducible).

Let X be a normal variety, let B be an effective $\mathbb{Q}$-divisor on X such that the $\mathbb{Q}$-divisor K_X+B is $\mathbb{Q}$-Cartier. A subvariety $Z \subset X$ is called *a center of non Kawamata log terminal singularities* (or *a center of non-klt singularities*) of the log pair (X, B) if $Z = \pi(E)$ for some divisor E on some log resolution $\pi : \hat{X} \to X$ with discrepancy $a(X, B; E) \leqslant -1$. A subvariety $Z \subset X$ is called *a center of non log canonical singularities* of the log pair (X, B) if Z is an image of some divisor with discrepancy strictly less than -1 on some log resolution. A center of non-klt singularities Z of the log pair (X, B) is called *minimal* if no other center of non-klt singularities of (X, B) is contained in Z.

Remark 2.1. In general it is not enough to consider one log resolution to detect all centers of non-klt singularities of a log pair, but the *union* of these centers can be figured out using one log resolution. Note that this does not mean that there is only a finite number of centers of non-klt singularities of a given log pair! Actually, the latter happens if and only if the log pair is log canonical.

Suppose that there is an action of some finite group G on X such that B is G-invariant. Let Z_1 be a center of non-klt singularities of the pair (X, B), let $Z_1, \ldots, Z_r$ be the G-orbit of the subvariety Z_1, and put $Z = \bigcup Z_i$. We say that Z is *a G-center of non-klt singularities of the pair* (X, B), and call Z a *minimal G-center of non-klt singularities* if no other G-center of non-klt singularities of the pair (X, B) is contained in Z. Note that one has $Z_i \cap Z_j = \varnothing$ for $i \neq j$ and each Z_i is normal (see [16, 1.5–1.6]).

Suppose that X is a variety with only Kawamata log terminal singularities (in particular, this includes the assumptions that X is normal and the Weil divisor K_X is $\mathbb{Q}$-Cartier). A *G-contraction* is a G-equivariant proper morphism $f : X \to Y$ onto a normal variety Y such that f has connected fibers and $-K_X$ is f-ample (thus f is not only proper but projective). The variety X is called *a G-Mori fiber space* if X is projective and there exists a G-contraction $f : X \to Y$ with $\dim(Y) < \dim(X)$ and the relative G-equivariant Picard number $\rho^G(X/Y) = 1$. Furthermore, if Y is a point, then X is called a *G-Fano variety.*

Suppose that X is projective and $G\mathbb{Q}$-factorial, i. e. any G-invariant $\mathbb{Q}$-divisor on X is $\mathbb{Q}$-Cartier. If X is rationally connected (see Definition 3.2), then one can run a G-Minimal Model Program on X, as well as its relative versions, and end up with a G-Mori fibre space. This is possible due to [2, Corollary 1.3.3] and [25, Theorem 1], since rational connectedness implies uniruledness. Actually, [2] treats the case when G is trivial, but adding a finite group action does not make a big difference.

We start with proving some auxiliary statements that will be used in course of the proof of Theorem 1.8.

Suppose that V is a normal (irreducible) variety, and $f : V \to W$ is a proper morphism. Then for any curve $C \subset V$ contracted by f and any Cartier divisor D on V one has a well-defined intersection index $D \cdot C$, and one can consider a (finite dimensional) $\mathbb{R}$-vector space $\mathrm{N}_1(V/W)$ generated by the classes of curves in the fibers of f modulo numerical equivalence (see e. g. [18, §0-1]).

The following observation (see e. g. the proofs of [16, Theorem 1.10] and [17, Theorem 1]) is sometimes called *the perturbation trick.*

Lemma 2.2. *Let V be an irreducible normal quasi-projective variety, and $f : V \to W$ be a proper morphism to a variety W. Let D be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on V such that the log pair (V, D) is strictly log canonical. Suppose that a finite group G acts on V so that D is G-invariant. Let $Z \subset X$ be a minimal G-center of non-klt singularities of the log pair (V, D).*

Choose $\varepsilon > 0$ and a compact subset $\mathcal{K} \subset \mathrm{N}_1(V/W)$. Then there exists a G-invariant $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D' such that

- *the only centers of non-klt singularities of the log pair (V, D') are the irreducible components of Z;*
- *for any $\kappa \in \mathcal{K}$ one has $|(D - D') \cdot \kappa| < \varepsilon$.*

Proof. Let $Z_1, \ldots, Z_r$ be irreducible components of Z. Note that Z_i's are disjoint by [16, Proposition 1.5]. Let $\mathscr{M}$ be a linear system of very ample divisors such that $\mathrm{Bs}\,\mathscr{M} = Z_1$ and let $M_1 \in \mathscr{M}$ be a general element. Let $M_1, \ldots, M_l$ be the G-orbit of M_1, and let $M = \sum M_i$. For $0 < \theta \ll 1$ the subvariety Z_1 is the only center of non log canonical singularities for the log pair $(V, D + \theta M_1)$. Hence the only centers of non log canonical singularities for $(V, D + \theta M)$ are the subvarieties Z_i. Now take $\delta \in \mathbb{Q}_{>0}$ so that the log pair (V, D') is strictly log canonical, where $D' = (1 - \delta)D + \theta M$. By the above the only centers of non-klt singularities of (V, D') are Z_i's. Since $\theta \ll 1$, one has $\delta \ll 1$, which guarantees the existence of an appropriate ε. □

Remark 2.3. One can generalize Lemma 2.2 assuming that we start from a log pair that includes any formal linear combination of linear systems on the variety V with rational coefficients instead of a divisor D, and produce an effective $\mathbb{Q}$-divisor D'. Another version of the same assertion produces a movable linear system $\mathcal{D}'$ instead of a divisor D'. Note that neither Lemma 2.2 nor these generalizations require the morphism f to be equivariant with respect to the group G.

We will need the following Bertini-type statement.

Lemma 2.4 (cf. [28, Theorem 1.13]). *Let Z be a normal variety and D be an effective $\mathbb{Q}$-divisor on Z such that the log pair (Z, D) is Kawamata log terminal. Let $\mathscr{M}$ be a base point free linear system and let $M \in \mathscr{M}$ be a general member. Then*

- *the variety M is normal;*
- *the log pair $(M, D|_M)$ is Kawamata log terminal.*

Proof. Doing everything componentwise, we may assume that Z is connected. Since Z is normal, it is irreducible. The pair $(Z, D+\mathscr{M})$ is purely log terminal (see Definition 4.6 and Lemma 4.7.1 in [21]). Hence $(Z, D+M)$ is also purely log terminal (see [21, Theorem 4.8]). Thus by the inversion of adjunction (see e.g. [24, 5.50–5.51]) the variety M is normal and the pair $(M, D|_M)$ is Kawamata log terminal. □

The following is a relative version of the usual Kawamata subadjunction theorem.

Lemma 2.5. *Let V be an irreducible normal quasi-projective variety, and D be an effective $\mathbb{Q}$-divisor on V such that the log pair (V, D) is strictly log canonical. Let W be a normal quasi-projective variety, and $f : V \to W$ be a proper morphism with connected fibers such that $-(K_V + D)$ is f-ample.*

Suppose that G is a finite group acting on V. Let $Z \subset V$ be a minimal G-center of non-klt singularities of the log pair (V, D), and $T = f(Z) \subset W$. Let $Z_t = Z \cap f^{-1}(t)$ be a fiber of $f|_Z$ over a general point $t \in T$. Then

- *Z_t is normal;*
- *Z_t is irreducible;*

- *there exists an effective $\mathbb{Q}$-divisor D_Z on Z such that $K_Z + D_Z$ is $\mathbb{Q}$-Cartier, the log pair $(Z_t, D_Z|_{Z_t})$ is Kawamata log terminal and*

$$K_{Z_t} + D_{Z|_{Z_t}} \sim_{\mathbb{Q}} (K_V + D)|_{Z_t}.$$

Proof. By Lemma 2.2 we may assume that Z is the only G-center of non-klt singularities of the log pair (V, D). Furthermore, since an intersection of centers of non-klt singularities is again a center of non-klt singularities (see [16, Proposition 1.5]), we conclude that each of the connected components of Z is irreducible, because otherwise the pairwise intersections of irreducible components of Z would be a (non-empty) union of G-centers of non-klt singularities of the pair (V, D). Applying [16, Theorem 1.6] to connected components of Z, one obtains that Z is normal (note that connected components of Z are minimal centers of non-klt singularities of (V, D)). Moreover, a general fiber Z_t is connected by the Nadel–Shokurov connectedness theorem (see e. g. [5, Theorem 3.2]). Hence Z_t is irreducible.

To proceed we may drop the action of the group G and assume that T is a point. Indeed, let $W' \subset W$ be a general hyperplane section, and $t \in W'$ be a general point (which is the same as to choose t to be a general point of W, and then to choose a general hyperplane section $W' \ni t$). Put $V' = f^{-1}(W')$. By Lemma 2.4 the variety V' is normal. Let $\varphi : \tilde{V} \to V$ be a log resolution of (V, D), and let $\tilde{V}'$ be the proper transform of V'. Since V' is a general member of a base point free linear system, φ is also a log resolution of the log pair $(V, D + V')$. Therefore, φ induces a log resolution of $(V', D|_{V'})$. This implies that the log pair $(V', D|_{V'})$ is log canonical and the irreducible components of $Z' = Z|_{V'}$ are its minimal centers of non-klt singularities. Replacing $f : (V, D) \to W$ by

$$f|_{V'} : (V', D|_{V'}) \to W'$$

and repeating this process $\mathrm{codim}_W(T)$ times, we get the situation where T is a point, and $Z = Z_t$ (in particular, Z is projective, normal, and irreducible).

With these reductions done, we apply Kawamata's subadjunction theorem (see e. g. [17, Theorem 1] or [11, Theorem 1.2]) to conclude that there exists an effective $\mathbb{Q}$-divisor D_Z on Z such that $K_Z + D_Z$ is $\mathbb{Q}$-Cartier, the log pair (Z, D_Z) is Kawamata log terminal and

$$K_Z + D_Z \sim_{\mathbb{Q}} (K_V + D)|_Z.$$

□

Remark 2.6. A usual form of the Kawamata's subadjunction theorem (as in [17] and [11]) requires the ambient variety to be projective. Therefore, if one wants to be as accurate as possible, the end of the proof of Lemma 2.5 should be read as follows. Assuming that T is a point, we know that Z is projective; as above, we can also suppose that Z is the only center of non-klt singularities of (V, D). Taking a log canonical closure $(\bar{V}, \bar{D})$ of the log pair (V, D) as in [14, Corollary 1.2], we see that Z is still a minimal center of non-klt singularities of the new pair $(\bar{V}, \bar{D})$, and all other centers of non-klt singularities of $(\bar{V}, \bar{D})$ are disjoint from Z. Now [11, Theorem 1.2] implies the assertion of Lemma 2.5. Since this step is more or less obvious, we decided not to include it in the proof to save space (and readers attention) for more essential points.

Another interesting moment in the proof of Lemma 2.5 that we want to emphasize is that we do not care about the action of the group G anywhere apart from the equivariant perturbation trick at the very beginning (in particular, the morphism f is not required to be G-equivariant, cf. Remark 2.3). On the other hand, it seems that one cannot replace this G-perturbation by a non-equivariant perturbation performed at some later step, since otherwise we would not know that the fiber Z_t is connected, and thus it would remain undecided if we have occasionally got rid of some components of Z or not. This is crucial for us, since we are going to obtain a G-invariant subvariety Z with controllable fibers.

3. Rationally connected subvarieties

In this section we develop techniques to "pull-back" invariant rationally connected subvarieties under contractions appearing in the Minimal Model Program. Basically we follow the ideas of [13].

Recall the following standard definitions.

Definition 3.1 (see e. g. [27, Lemma-Definition 2.6]). A (normal irreducible) variety X is called a *variety of Fano type* if there exists an effective $\mathbb{Q}$-divisor Δ on X such that the pair (X, Δ) is Kawamata log terminal and $-(K_X + \Delta)$ is nef and big.

Definition 3.2 (see e. g. [20, §IV.3]). An irreducible variety X is called *rationally connected* if for two general points $x_1, x_2 \in X$ there exists a rational map $t : C \dashrightarrow X$, where C is a rational curve, such that the image $t(C)$ contains x_1 and x_2.

In particular, a point is a rationally connected variety. Furthermore, rational connectedness is birationally invariant, and an image of a rationally connected variety under any rational map is again rationally connected.

The following is an easy consequence of Lemma 2.5.

Lemma 3.3. *Let $f : V \to W$ be a G-contraction from a quasi-projective variety V with Kawamata log terminal singularities. Choose an effective G-invariant $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D_W on W, and put $D = f^*D_W$.*

Let $Z \subset V$ be a minimal G-center of non-klt singularities of the log pair (V, D), and $T = f(Z) \subset W$. Let $Z_t = Z \cap f^{-1}(t)$ be a fiber of $f|_Z$ over a general point $t \in T$. Then Z_t is a variety of Fano type. In particular, Z_t is rationally connected.

Proof. By Lemma 2.5 a general fiber Z_t is a normal irreducible variety (so that we may assume $\dim(Z_t) > 0$), and there exists an effective $\mathbb{Q}$-divisor D_Z on Z such that $K_Z + D_Z$ is $\mathbb{Q}$-Cartier, the log pair $(Z_t, D_Z|_{Z_t})$ is Kawamata log terminal and

$$K_{Z_t} + D_Z|_{Z_t} \sim_{\mathbb{Q}} (K_V + D)|_{Z_t}.$$

Since Z_t is an irreducible variety such that the restriction of D to Z_t is trivial, and the restriction of $-K_V$ to Z_t is ample, we see that Z_t is a variety of Fano type. The last assertion of the lemma follows from [33, Theorem 1] or [13, Corollary 1.13]. □

Now we are ready to prove the main technical result of this section.

Lemma 3.4 (cf. [13, Corollary 1.7(1)]). *Let $f : V \to W$ be a G-contraction from a quasi-projective variety with Kawamata log terminal singularities onto a quasi-projective variety W. Let $T \subsetneq W$ be a G-invariant irreducible subvariety. Then there exists a G-invariant (irreducible) subvariety $Z \subsetneq V$ such that $f|_Z : Z \to T$ is dominant and a general fiber of $f|_Z$ is rationally connected.*

Proof. Take $k \gg 0$, and choose $H_1, \ldots, H_k$ to be general divisors from some (very ample) linear system $\mathcal{H}$ with $\operatorname{Bs} \mathcal{H} = T$. Adding the images of the divisors H_i under the action of G to the set $\{H_1, \ldots, H_k\}$ we may assume that this set is G-invariant. Put $D_W = \sum H_i$, and $D = f^*D_W$. Let c be the log canonical threshold of the log pair (V, D) over a general point of T (this makes sense since the log canonical threshold in a neighborhood of a point $P \in V$ is an upper semi-continuous function of P, and T is irreducible). Note that we can assume that for any center L of non-klt singularities of (V, cD) one has $f(L) \subset T$ by the construction of D.

Let S be a union of all centers of non-klt singularities of the log pair (V, cD) that *do not* dominate T. Then T is not contained in the set $f(S)$. Indeed, the union $\mathcal{Z}$ of centers of non-klt singularities of (V, cD) is a union of a finite number of centers of non-klt singularities of (V, cD) by Remark 2.1. By definition of c we conclude that the log pair (V, cD) has a center of non-klt singularities Z_1 that dominates T.

Let $Z_1, \ldots, Z_r$ be the G-orbit of the subvariety Z_1, and put $Z = \bigcup Z_i$. Put $W^o = W \setminus f(S)$ and $V^o = f^{-1}(W^o)$, and note that $Z \cap V^o$ is a minimal G-center of non-klt singularities of the log pair $(V^o, cD|_{V^o})$. Lemma 3.3 implies that the fiber Z_t of the morphism $f|_Z$ over a general point $t \in T \cap W^o$ is rationally connected. □

Remark 3.5. In the case when f is an isomorphism over a general point of T the proof of Lemma 3.4 produces the strict transform of T on V as a resulting subvariety Z.

Rationally connected varieties enjoy the following important property (see [12, Corollary 1.3] for the proof over $\mathbb{C}$; the case of an arbitrary field of characteristic 0 follows by the usual Lefschetz principle).

Theorem 3.6. *Let $f : X \to Y$ be a dominant morphism of proper varieties over $\Bbbk$. Assume that both Y and a general fiber of f are rationally connected. Then X is also rationally connected.*

Together with the previous considerations this enables us to lift G-invariant rationally connected varieties via G-contractions. Namely, the following immediate consequences of Lemma 3.4 and Theorem 3.6 will be used in the proof of Theorem 1.8.

Corollary 3.7. *Let $f : V \to W$ be a G-contraction from a quasi-projective variety with Kawamata log terminal singularities onto a quasi-projective variety W. Let $T \subsetneq W$ be a G-invariant rationally connected subvariety. Then there exists a G-invariant rationally connected subvariety $Z \subsetneq V$ that dominates T.*

Proof. Apply Lemma 3.4 to obtain a subvariety $Z \subsetneq V$ that maps to a rationally connected variety T with a rationally connected general fiber. Theorem 3.6 applied to (a desingularization of a compactification of) Z completes the proof. □

The following is just a small modification of Corollary 3.7, but we find it useful to state it to have a result allowing us to lift rationally connected subvarieties via (equivariant) flips.

Corollary 3.8. *Let $f : V \to W$ be a G-contraction from a quasi-projective variety with Kawamata log terminal singularities onto a quasi-projective variety W. Consider a diagram of G-equivariant morphisms*

$$\begin{array}{ccc} V & & V' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & W & \end{array}$$

Suppose that there exists a G-invariant rationally connected subvariety $Z' \subset V'$ such that $f'(Z') \neq W$. Then there exists a G-invariant rationally connected subvariety $Z \subsetneq V$.

Proof. Apply Corollary 3.7 to the rationally connected variety $T = f'(Z') \subsetneq W$. □

Corollaries 3.7 and 3.8 imply the following assertion.

Lemma 3.9. *Let V be a projective variety with an action of a finite group G. Suppose that V has Kawamata log terminal $G\mathbb{Q}$-factorial singularities. Let $f : V \dashrightarrow W$ be a birational map that is a result of a G-Minimal Model Program ran on V. Let $F \subset G$ be a subgroup. Suppose that there exists an F-invariant rationally connected subvariety $T \subsetneq W$. Then there exists an F-invariant rationally connected subvariety $Z \subsetneq V$.*

Proof. Induction in the number of steps of the G-Minimal Model Program using Corollaries 3.7 and 3.8 (note that any G-contraction is also an F-contraction). □

In particular, Lemma 3.9 implies the following assertion (it will not be used directly in the proof of our main theorems, but still we suggest that it deserves being mentioned).

Proposition 3.10. *Let W be a quasi-projective variety with terminal singularities acted on by a finite group G so that W is $G\mathbb{Q}$-factorial. Let $f : V \to W$ be a G-equivariant resolution of singularities of W. Suppose that there exists a G-invariant rationally connected subvariety $T \subsetneq W$. Then there exists a G-invariant rationally connected subvariety $Z \subsetneq V$.*

Proof. Run a relative G-Minimal Model Program on V over W (this is possible due to an equivariant version of [2, Corollary 1.4.2]) to obtain a variety V_n that is a relatively minimal model over W together with a series of birational modifications

$$V = V_0 \overset{f_1}{\dashrightarrow} \ldots \overset{f_n}{\dashrightarrow} V_n \xrightarrow{f_{n+1}} W.$$

Then f_{n+1} is small by the Negativity Lemma (see e. g. [19, 2.19]). Thus $G\mathbb{Q}$-factoriality of W implies that f_{n+1} is actually an isomorphism. Now the assertion follows from Lemma 3.9. □

4. Jordan property

In this section we will prove Theorem 1.8. Before we proceed let us introduce the following notion.

Definition 4.1. Let $\mathcal{C}$ be some set of varieties. We say that $\mathcal{C}$ *has almost fixed points* if there is a constant $J = J(\mathcal{C})$ such that for any variety $X \in \mathcal{C}$ and for any finite subgroup $G \subset \mathrm{Aut}(X)$ there exists a subgroup $F \subset G$ of index at most J acting on X with a fixed point.

Theorem 1.8 will be implied by the following auxiliary result.

Theorem 4.2. *Let $\mathcal{R}(n)$ be the set of all rationally connected varieties of dimension n. Assume that Conjecture 1.7 holds. Then $\mathcal{R}(n)$ has almost fixed points.*

Remark 4.3. In the proof of Theorem 1.8 we will only use the particular case of Theorem 4.2 for smooth rationally connected varieties. However, it is more convenient to prove it without any assumptions on singularities. In any case, it does not make a big difference (see Corollary 4.5 below).

Sometimes it would be convenient to restrict ourselves to non-singular varieties when proving assertions like Theorem 4.2. It is possible by the following (nearly trivial) observation.

Lemma 4.4. *Let $\mathcal{C}$ be some set of varieties, and let $\mathcal{C}' \subset \mathcal{C}$. Suppose that for any $X \in \mathcal{C}$ and for any finite group $G \subset \mathrm{Aut}(X)$ there is a variety $X' \in \mathcal{C}'$ with $G \subset \mathrm{Aut}(X')$ and a G-equivariant surjective morphism $X' \to X$. Then $\mathcal{C}$ has almost fixed points if and only if $\mathcal{C}'$ does.*

Proof. An image of a fixed point under an equivariant morphism is again a fixed point. □

Corollary 4.5. *The set $\mathcal{R}(n)$ of rationally connected varieties of dimension n has almost fixed points if and only if the set $\mathcal{R}'(n)$ of non-singular rationally connected varieties does.*

To prove Theorem 4.2 we will need its particular case concerning Fano varieties.

Lemma 4.6. *Let $\mathcal{F}(n)$ be the set of all Fano varieties of dimension n with terminal singularities, and assume that Conjecture 1.7 holds in dimension n. Then $\mathcal{F}(n)$ has almost fixed points.*

Proof. Using Noetherian induction, one can show that there exists a positive integer m such that for any $X \in \mathcal{F}(n)$ the divisor $-mK_X$ is very ample and gives an embedding

$$X \hookrightarrow \mathbb{P}^{\dim|-mK_X|}.$$

So we may assume that any $X \in \mathcal{F}(n)$ admits an embedding $X \hookrightarrow \mathbb{P}^N$ for some $N = N(n)$ (that does not depend on X) as a subvariety of degree at most $d = d(n)$. Moreover, the action of $G \subset \mathrm{Aut}(X)$ is induced by an action of some linear group $\Gamma \subset \mathrm{GL}_{N+1}(\mathbb{C})$. By Theorem 1.1 there exists an abelian subgroup $\Gamma_0 \subset \Gamma$ of index at most $I = I(N+1)$. Let $G_0 \subset G$ be the image of Γ_0 under the natural projection from Γ to G. Take linear independent Γ_0-semi-invariant sections

$$s_1, \ldots, s_{N+1} \in H^0(X, -mK_X).$$

They define G_0-invariant hyperplanes $H_1, \ldots, H_{N+1} \subset \mathbb{P}^N$. Let k be the minimal positive integer such that

$$X \cap H_1 \cap \ldots \cap H_k = \{P_1, \ldots, P_r\}$$

is a finite (G_0-invariant) set. Then $r \leqslant d$. Since the stabilizer $G_1 \subset G_0$ of P_1 is a subgroup of index at most $r \leqslant d$, the assertion of the lemma follows. □

Lemma 4.6 allows us to derive a slightly wider particular case of Theorem 4.2 involving G-Mori fiber spaces from the assertion of Theorem 4.2 for lower dimensions.

Lemma 4.7. *Suppose that the sets $\mathcal{R}(k)$ of rationally connected varieties of dimension k have almost fixed points for $k \leqslant n-1$, and assume that Conjecture 1.7 holds in dimension n. Then there is a constant $J = J(n)$ such that for any finite group G and for any rationally connected G-Mori fiber space $\phi : M \to S$ with $\dim(M) = n$ there is a finite subgroup of index at most J in G acting on M with a fixed point.*

Proof. Let $\phi : M \to S$ be a rationally connected G-Mori fiber space of dimension n. We are going to show that there is a constant J that does not depend on M and G such that there exists a subgroup $H \subset G$ of index at most J acting on M with a fixed point. By Lemma 4.6 we may suppose that $1 \leqslant \dim(S) \leqslant n-1$.

Consider an exact sequence of groups

$$1 \to G_\phi \longrightarrow G \xrightarrow{\theta} G_S \to 1,$$

where the action of G_ϕ is fiberwise with respect to ϕ and G_S is the image of G in $\mathrm{Aut}(S)$. Note that S is rationally connected since so is M. By assumption there is a constant J_1 that does not depend on S and G such that there exists a subgroup $F_S \subset G_S$ of index at most J_1 acting on S with a fixed point. Let $P \in S$ be one of the points fixed by F_S.

Define a subgroup $F \subset G$ to be the preimage of the subgroup $F_S \subset G_S$ under the homomorphism θ. Then $\phi : M \to S$ is an F-contraction. By Corollary 3.7 applied to the group F and the contraction ϕ there exists an F-invariant rationally connected subvariety $Z \subset M$ such that $\phi(Z) = P$. In particular, $\dim(Z) < n$. By assumption there is a constant J_2 that does not depend on Z and F such that there is a subgroup $H \subset F$ of index at most J_2 acting on Z (and thus on X) with a fixed point. The assertion follows since $[G : F] = [G_S : F_S] \leqslant J_1$. □

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. Let X be a non-singular (or terminal) rationally connected variety of dimension n, and $G \subset \mathrm{Aut}(X)$ be a finite subgroup. By Corollary 4.5 it is enough to show that there is a constant J that does not depend on X and G such that there exists a subgroup $H \subset G$ of index at most J acting on X with a fixed point.

Run a G-Minimal Model Program on X, resulting in a G-Mori fiber space X' and a rational map $f : X \dashrightarrow X'$ that factors into a sequence of G-contractions and G-flips. By Lemma 4.7 there is a constant J_1 that does not depend on X' (and thus on X) and G such that there exists a subgroup $F \subset G$ of index at most J_1 acting on X' with a fixed point. Using Lemma 3.9 applied to the group F, we obtain an F-invariant rationally connected subvariety $Z \subsetneq X$.

The rest of the argument is similar to that in the proof of Lemma 4.7. Using induction in n, we see that there is a constant J_2 that does not depend on Z and F such that there is a subgroup $H \subset F$ of index at most J_2 having a fixed point on Z (and thus on X), and the assertion of the theorem follows. □

Remark 4.8. To prove Theorem 4.2 one could actually use a weaker version of Lemma 4.7. For this purpose it is sufficient to know that the G-Mori fiber space contains an F-invariant rationally connected subvariety $Z' \subsetneq X'$ for some subgroup $F \subset G$ of bounded index, without assuming that Z' is a point.

Now we are going to derive Theorem 1.8 from Theorem 4.2. We will need the following easy observation.

Lemma 4.9. *Let G be a group and $H \subset G$ be a subgroup of finite index $[G : H] = j$. Suppose that H has some property $\mathcal{P}$ that is preserved under intersections and under conjugation in G. Then there exists a normal subgroup $H' \subset G$ of finite index $[G : H'] \leqslant j^j$ such that H' also enjoys the property $\mathcal{P}$.*

Proof. Let $H_1 = H, \ldots, H_r \subset G$ be the subgroups that are conjugate to H. Then $r \leqslant j$, and $H' = \bigcap H_i$ is normal and has the property $\mathcal{P}$. It remains to notice that $[G : H'] \leqslant j^r$. □

Proof of Theorem 1.8. We may assume that the field $\Bbbk$ is algebraically closed. Let X be a rationally connected variety of dimension n, and $G \subset \mathrm{Bir}(X)$ be a finite group. Let $\tilde{X}$ be a regularization of G, i. e. $\tilde{X}$ is a projective variety with an action of G and a G-equivariant birational map $\xi : \tilde{X} \dashrightarrow X$ (see [31, Theorem 3]). Taking a G-equivariant resolution of singularities (see [1]), one can assume that $\tilde{X}$ is smooth. Note that $\tilde{X}$ is rationally connected since so is X. By Theorem 4.2 there is a constant J_1 that does not depend on $\tilde{X}$ and G (and thus on X) such that there exists a subgroup $F \subset G$ of index at most J_1 and a point $P \in X$ fixed by A. The action of F on the Zariski tangent space $T_P(\tilde{X}) \cong \Bbbk^n$ is faithful (see e. g. [10, Lemma 2.7(b)]). By Theorem 1.1 applied to $\mathrm{GL}_n(\Bbbk)$ there is a constant J_2 (again independent of anything except for n) such that F has an abelian subgroup A of index at most J_2. The assertion follows by Lemma 4.9. □

Finally, we prove Theorem 1.10.

Proof of Theorem 1.10. We may assume that the field $\Bbbk$ is algebraically closed. Let X be a rationally connected variety of dimension n, and let $G \subset \mathrm{Bir}(X)$ be a finite p-group. Arguing as in the proof of Theorem 1.8, we obtain an abelian subgroup $F \subset G$ of index $[G : F]$ bounded by some constant L (that does not depend on X and G) with an embedding $F \subset \mathrm{GL}_n(\Bbbk)$. The latter implies that the abelian p-group F is generated by at most n elements. On the other hand, if $p > L$, then the index of any subgroup of G is at least p, so that the subgroup F coincides with G. □

References

[1] D. Abramovich and J. Wang. Equivariant resolution of singularities in characteristic 0. *Math. Res. Lett.*, 4(2-3):427–433, 1997.

[2] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, 23(2):405–468, 2010.

[3] A. Borisov. Boundedness theorem for Fano log-threefolds. *J. Algebraic Geom.*, 5(1):119–133, 1996.

[4] S. Cantat. Morphisms between Cremona groups, and characterization of rational varieties. preprint `http://perso.univ-rennes1.fr/serge.cantat/Articles/autbir.pdf`, to appear in Compositio Math.

[5] A. Corti. Singularities of linear systems and 3-fold birational geometry. In *Explicit birational geometry of 3-folds*, *London Math. Soc. Lecture Note Ser.*, vol. **281**, 259–312. Cambridge Univ. Press, Cambridge, 2000.

[6] C. W. Curtis and I. Reiner. *Representation theory of finite groups and associative algebras*. Pure and Applied Mathematics, Vol. XI. Interscience Publishers, a division of John Wiley & Sons, New York-London, 1962.

[7] M. Demazure. Sous-groupes algébriques de rang maximum du groupe de Cremona. *Ann. Sci. École Norm. Sup. (4)*, 3:507–588, 1970.

[8] J. Déserti. *Some properties of the Cremona group*, *Ensaios Matemáticos [Mathematical Surveys]*, vol. **21**. Sociedade Brasileira de Matemática, Rio de Janeiro, 2012.

[9] I. V. Dolgachev and V. A. Iskovskikh. Finite subgroups of the plane Cremona group. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I*, *Progr. Math.*, vol. **269**, 443–548. Birkhäuser Boston Inc., Boston, MA, 2009.

[10] H. Flenner and M. Zaidenberg. Locally nilpotent derivations on affine surfaces with a $\mathbb{C}^*$-action. *Osaka J. Math.*, 42(4):931–974, 2005.

[11] O. Fujino and Y. Gongyo. On canonical bundle formulas and subadjunctions. *Michigan Math. J.*, 61(2):255–264, 2012.

[12] T. Graber, J. Harris, and J. Starr. Families of rationally connected varieties. *J. Am. Math. Soc.*, 16(1):57–67, 2003.

[13] C. D. Hacon and J. McKernan. On Shokurov's rational connectedness conjecture. *Duke Math. J.*, 138(1):119–136, 2007.

[14] C. D. Hacon and C. Xu. Existence of log canonical closures. *Invent. Math.*, 192(1):161–195, 2013.

[15] A. Hogadi and C. Xu. Degenerations of rationally connected varieties. *Trans. Amer. Math. Soc.*, 361(7):3931–3949, 2009.

[16] Y. Kawamata. On Fujita's freeness conjecture for 3-folds and 4-folds. *Math. Ann.*, 308(3):491–505, 1997.

[17] Y. Kawamata. Subadjunction of log canonical divisors. II. *Amer. J. Math.*, 120(5):893–899, 1998.

[18] Y. Kawamata, K. Matsuda, and K. Matsuki. Introduction to the minimal model problem. In *Algebraic geometry, Sendai, 1985*, *Adv. Stud. Pure Math.*, vol **10**, 283–360. North-Holland, Amsterdam, 1987.

[19] J. Kollár, editor. *Flips and abundance for algebraic threefolds*. Société Mathématique de France, Paris, 1992. Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992).

[20] J. Kollár. *Rational curves on algebraic varieties*, *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*, vol. **32**. Springer-Verlag, Berlin, 1996.

[21] J. Kollár. Singularities of pairs. In *Algebraic geometry—Santa Cruz 1995*, *Proc. Sympos. Pure Math.*, vol. **62**, 221–287. Amer. Math. Soc., Providence, RI, 1997.

[22] J. Kollár. A conjecture of Ax and degenerations of Fano varieties. *Israel J. Math.*, 162:235–251, 2007.

[23] J. Kollár, Y. Miyaoka, S. Mori, and H. Takagi. Boundedness of canonical **Q**-Fano 3-folds. *Proc. Japan Acad. Ser. A Math. Sci.*, 76(5):73–77, 2000.

[24] J. Kollár and S. Mori. *Birational geometry of algebraic varieties*, *Cambridge Tracts in Mathematics*, vol. **134**. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[25] Y. Miyaoka and S. Mori. A numerical criterion for uniruledness. *Ann. of Math. (2)*, 124(1):65–69, 1986.

[26] V. L. Popov. On the Makar-Limanov, Derksen invariants, and finite automorphism groups of algebraic varieties. In *Peter Russell's Festschrift, Proceedings of the conference on Affine Algebraic Geometry held in Professor Russell's honour, 1–5 June 2009, McGill Univ., Montreal.*, *Centre de Recherches Mathématiques CRM Proc. and Lect. Notes*, vol. **54**, 289–311, 2011.

[27] Y. Prokhorov and V. V. Shokurov. Towards the second main theorem on complements. *J. Algebraic Geom.*, 18(1):151–199, 2009.

[28] M. Reid. Canonical 3-folds. In *Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, 273–310. Sijthoff & Noordhoff, Alphen aan den Rijn, 1980.

[29] J.-P. Serre. Le groupe de Cremona et ses sous-groupes finis. *Séminaire Bourbaki, Nov. 2008, 61-'eme anné*, (1000), 2008-2009.

[30] J.-P. Serre. A Minkowski-style bound for the orders of the finite subgroups of the Cremona group of rank 2 over an arbitrary field. *Mosc. Math. J.*, 9(1):193–208, 2009.

[31] H. Sumihiro. Equivariant completion. *J. Math. Kyoto Univ.*, 14:1–28, 1974.

[32] Y. G. Zarhin. Theta groups and products of abelian and rational varieties. *Proc. Edinburgh Math. Soc.*, 57(1):299–304, 2014.

[33] Q. Zhang. Rational connectedness of log **Q**-Fano varieties. *J. Reine Angew. Math.*, 590:131–142, 2006.

Steklov Institute of Mathematics, 8 Gubkina street, Moscow 119991, Russia
and
Laboratory of Algebraic Geometry, GU-HSE, 7 Vavilova street, Moscow 117312, Russia

E-mail address: `prokhoro@gmail.com`

E-mail address: `costya.shramov@gmail.com`