The cancelation norm and the geometry of biinvariant word metrics

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Abstract. We study biinvariant word metrics on groups. We provide an efficient algorithm for computing the biinvariant word norm on a finitely generated free group and we construct an isometric embedding of a locally compact tree into the biinvariant Cayley graph of a nonabelian free group. We investigate the geometry of cyclic subgroups. We observe that in many classes of groups cyclic subgroups are either bounded or detected by homogeneous quasimorphisms. We call this property the bq-dichotomy and we prove it for many classes of groups of geometric origin.

1. Introduction

The main object of study in the present paper are biinvariant word metrics on normally finitely generated groups. Let us recall definitions. Let G be a group generated by a symmetric set $S \subset G$. Let $\overline{S}$ denote the the smallest conjugation invariant subset of G containing the set S. The word norm of an element $g \in G$ associated with the sets S and $\overline{S}$ is denoted by $|g|$ and $\|g\|$ respectively:

$$|g| := \min\{k \in \mathbf{N} \mid g = s_1 \cdots s_k, \text{ where } s_i \in S\},$$
$$\|g\| := \min\{k \in \mathbf{N} \mid g = s_1 \cdots s_k, \text{ where } s_i \in \overline{S}\}.$$

The latter norm is conjugation invariant and defined if G is generated by $\overline{S}$ but not necessarily by S. If S is finite and G is generated by $\overline{S}$ then we say that G is *normally finitely generated*. This holds, for example, when G is a simple group and $S = \{g^{\pm 1}\}$ and $g \neq 1_G$.

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Another example is the infinite braid group B_∞ which is normally generated by one element twisting the two first strands.

Remark 1.A. The metric associated with the conjugation invariant norm is defined by $\mathbf{d}(g,h) := \|gh^{-1}\|$. It is biinvariant in the sense that both left and right actions of G on itself are by isometries. We focus in the paper exclusively on both conjugation invariant word norms and associated with them biinvariant metrics. Most of the arguments and computations are done for norms.

Since invariant sets are in general infinite, the corresponding word norms are not considered by the classical geometric group theory. The motivation for studying such norms comes from geometry and topology because transformation groups of manifolds often carry naturally defined conjugation invariant norms. The examples include the Hofer norm and the autonomous norm in symplectic geometry, fragmentation norms and the volume of the support norm in differential geometry and others, see for example [8, 10, 13, 15, 23, 25, 27].

Biinvariant word metrics are at present not well understood. It is known that for some nonuniform lattices in semisimple Lie groups (e.g. $\mathrm{SL}(n,\mathbf{Z})$, $n \geq 3$) biinvariant metrics are bounded [12, 21]. In general, the problem of understanding the biinvariant geometry of lattices in higher rank semisimple Lie groups is widely open.

The main tool for proving unboundedness of biinvariant word metrics are homogeneous quasimorphisms. Thus if a group admits a homogeneous quasimorphism that is bounded on a conjugation invariant generating set then the group is automatically unbounded with respect to the biinvariant word metric associated with this set. Examples include hyperbolic groups and groups of Hamiltonian diffeomorphisms of surfaces equipped with autonomous or fragmentation metrics [9, 10, 20]. If a group G is biinvariantly unbounded it is interesting to understand what metric spaces can be quasiisometrically embedded into G.

Before we discuss the content of the paper in greater detail let us recall a basic property of biinvariant word metrics on normally finitely generated groups.

Lipschitz properties of conjugation invariant norms on normally finitely generated groups. If a group Γ is normally finitely generated then every homomorphism $\Psi\colon \Gamma \to G$ is Lipschitz with respect to the norm $\|\circ\|$ on Γ and any conjugation invariant norm on G. In particular, two choices of such a finite set S produce Lipschitz equivalent metrics, so in this case we will refer to *the* word metric on a normally finitely generated group. Also, such a metric is maximal among biinvariant metrics.

The cancelation norm. Let G be a group generated by a symmetric set S and let w be a word in the alphabet S. The cancelation length $|w|_\times$ is defined to be the least number of letters to be deleted from w in order to obtain a word trivial in G. The cancelation norm of an element $g \in G$ is defined to be the minimal cancelation length of a representing word. We prove (Proposition 2.A) that the cancelation norm is equal to the conjugation invariant word norm associated with the generating set S.

In some cases the cancelation norm does not depend on the representing word. In particular, the following result is a consequence of a more general statement, see Proposition 2.E.

Theorem 1.B. *If G is either a right angled Artin group or a Coxeter group then the cancelation norm of an element does not depend on the representing word.*

Section 2.I provides an efficient algorithm for computing the cancelation length for nonabelian free groups. More precisely, we prove the following result.

Theorem 1.C. *Let $w \in \mathbf{F}_n$ be a word of standard length n. There exists an algorithm which computes the conjugation invariant word length of w. Its complexity is $O(n^3)$ in time and $O(n^2)$ in memory.*

A simple software for computing the biinvariant word norm on the free group on two generators can be downloaded from the website of MM, see [30].

Quasiisometric embeddings. One way of studying the geometry of a metric space X is to construct quasiisometric embeddings of understood metric spaces into X. In Section 3.A, we prove that the free abelian group $\mathbf{Z}^n$ with its standard word metric can be quasiisometrically embedded into a group G equipped with the biinvariant word metric provided G admits at least n linearly independent homogeneous quasimorphisms.

We then proceed to embedding of trees. We prove that there exists an isometric embedding of a locally compact tree in the biinvariant Cayley graph of a nonabelian free group. We first construct an isometric embedding of the one skeleton of the infinite unit cube

$$\square^\infty := \bigcup [0,1]^n$$

equipped with the ℓ^1-metric (Theorem 3.F). It is an easy observation that any locally compact tree with edges of unit lengths admits an isometric embedding into such a cube.

Theorem 1.D. *Let T be a locally compact tree with edges if unit lengths. There is an isometric embedding* $\mathrm{T} \to \mathbf{F}_2$ *into the Cayley graph of the free group on two generators with the biinvariant word metric associated with the standard generators.*

The geometry of cyclic subgroups. Let us recall that a function $q\colon G \to \mathbf{R}$ is called a *quasimorphism* if there exists a real number $A \geq 0$ such that

$$|q(gh) - q(g) - q(h)| \leq A$$

for all $g, h \in G$. A quasimorphism q is called *homogeneous* if in addition

$$q(g^n) = nq(g)$$

for all $n \in \mathbf{Z}$. The vector space of homogeneous quasimorphisms on G is denoted by $Q(G)$. It is straighforward to prove that a quasimorphism $q\colon G \to \mathbf{R}$ defined on a normally finitely generated group is Lipschitz with respect to the biinvariant word metric on G and the standard metric on the reals [21, Lemma 3.6]. For more details about quasimorphisms and their connections to different branches of mathematics see [14].

The geometry of a cyclic subgroup $\langle g \rangle \subset G$ is described by the growth rate of the function $n \mapsto \|g^n\|$. A priori this function can be anything from bounded to linear. If it is linear then the cyclic subgroup is called *undistorted* and *distorted* otherwise. It is an easy observation that if $\psi\colon G \to \mathbf{R}$ is a homogeneous quasimorphism and $\psi(g) \neq 0$ then g is undistorted. One of the main observations of this paper is that for many classes of groups of geometric origin a cyclic subgroup is either bounded or detected by a homogeneous quasimorphism.

Definition 1.E. A normally finitely generated group G satisfies the **bq-dichotomy** if every cyclic subgroup of G is either **b**ounded (with respect to the biinvariant word metric) or detected by a homogeneous **q**uasimorphism.

Remark 1.F. One can consider a weaker version of the above dichotomy when a cyclic subgroup is either bounded or undistorted. Since undistortedness is proved usually with the use of quasimorphism most of the proofs yield the stronger statement. There is one exception in this paper, Theorem 5.B, where we prove the weaker dichotomy for Coxeter groups and the stronger under an additional assumption. This is because we don't know how to extend quasimorphisms from a parabolic subgroup of a Coxeter group. More precisely, the following problem seems to be open:

Let $g \in W_T$, where W_T is a standard parabolic subgroup of a Coxeter group W. Does $\mathrm{scl}_{W_T}(g) > 0$ imply $\mathrm{scl}_W(g) > 0$?

Here, scl_G denotes the stable commutator length in G (see Calegari's book [14] for details.)

The only example known to the authors of a group which does not satisfy bq-dichotomy is provided by Muranov in [32]. He constructs a group G with unbounded (but distorted) elements not detectable by a homogeneous quasimorphism. His group G is finitely generated but not finitely presented. We know no finitely presented example. Also, we know no example of an undistorted subgroup not detected by a homogeneous quasimorphism.

Remark 1.G. Observe that if G satisfies the bq-dichotomy then if $\mathrm{scl}_G(g) = 0$ then the cyclic subgroup $\langle g \rangle$ is bounded, due to a theorem of Bavard [2].

It is interesting to understand to what extent the bq-dichotomy is true. To sum up let us make a list of groups that satisfy the bq-dichotomy:

- Coxeter groups with even exponents - Theorem 5.B,
- finite index subgroups of mapping class groups of closed oriented surfaces (possibly with punctures) - Theorem 5.D,
- Artin braid groups (both pure and full) on a finite number of strings - Theorem 5.E,
- spherical braid groups (both pure and full) on a finite number of strings - Theorem 5.F,

- finitely generated nilpotent groups - Theorem 5.H. We actually prove that the commutator subgroup $[G, G]$ is bounded in G,
- finitely generated solvable groups whose commutator subgroups are finitely generated and nilpotent, e.g. lattices in simply connected solvable Lie groups - Theorem 5.K,
- $\mathrm{SL}(n, \mathbf{Z})$ - for $n = 2$ it is proved by Polterovich and Rudnick [33]; for $n > 2$ the groups are bounded,
- lattices in certain Chevalley groups [21] (the groups are bounded in this case),
- hyperbolic groups - due to Calegari and Fujiwara (Theorem 3.56 in [14]). They prove there that if g is a nontorsion element such that no positive power of g is conjugate to its inverse then it is detected by a homogeneous quasimorphisms. On the other hand it follows from Lemma 4.B that if a positive power of g is conjugate to its inverse then g generates a bounded cyclic subgroup.),
- right angled Artin groups - Theorem 4.F,
- Baumslag-Solitar groups and fundamental groups of some graph of groups - Theorem 5.O.

1.H. **Bounded elements.** Let $[x, y] = xyx^{-1}y^{-1}$ and ${}^t x = txt^{-1}$. In many cases we prove that an element $g \in G$ generates a bounded cyclic subgroup by making the observation that the element $[x, t]$ in the group

$$\Gamma := \left\langle x, t \,|\, \left[x, {}^t x\right] = 1 \right\rangle$$

generates a bounded subgroup of Γ. Then we construct nontrivial homomorphism $\Psi\colon \Gamma \to G$ such that $\Psi[x, t] = g$. The examples include Baumslag-Solitar groups, nonabelian braid groups B_n, $\mathrm{SL}(2, \mathbf{Z}[1/2])$, and HNN extensions of abelian groups, e.g. $\mathrm{Sol}(3, \mathbf{Z})$, Heisenberg groups and lamplighter groups (see Section 4.A).

1.I. **Elements detected by a quasimorphism.** In some cases it is easy to provide examples of elements detected by a nontrivial homogeneous quasimorphism, for example any nontrivial element in a free group has this property. Generalizing this observation yields the following result (Section 4.E).

Theorem 1.J. *Let G be one of the following groups:*

(1) a right angled Artin group,

(2) the commutator subgroup in a right angled Coxeter group,

(3) a pure braid group.

Then for every nontrivial element $g \in G$ there exists a homogeneous quasi-morphism ψ such that $\psi(g) \neq 0$. In particular, every nontrivial cyclic subgroup in G is biinvariantly undistorted.

We say that a group is *quasi-residually real* if it satisfies the property from the statement of the above theorem. Of course, a quasi-residually real group satisfies the bq-dichotomy.

2. The cancelation norm

Let $G = \langle S \mid R \rangle$ be a presentation of G, where S is a finite symmetric set of generators. Let $w = s_1 \ldots s_n$ be a word in the alphabet S. The number

$$|w|_\times := \min\{k \in \mathbf{N} \mid s_1 \ldots \widehat{s_{i_1}} \ldots \widehat{s_{i_k}} \ldots s_n = 1 \text{ in } G\}$$

is called the *cancelation length* of the word w. In other words, the cancelation length is the smallest number of letters we need to cross out from w in order to obtain a word representing the neutral element. The number

$$|g|_\times := \min\{|w|_\times \in \mathbf{N} \mid w \text{ represents } g \text{ in } G\}$$

is called the *cancelation norm* of $g \in G$.

The sequence of indices $i_1, \ldots, i_k$ so that deleting the letters $s_{i_1}, \ldots, s_{i_k}$ makes the word $w = s_1 \ldots s_n$ trivial is called the trivializing sequence of w. We will sometimes abuse the terminology and we will call the sequence of letters $s_{i_1}, \ldots, s_{i_n}$ trivializing. In this terminology the cancelation length is the minimal length of a trivializing sequence.

Proposition 2.A. *Let G be finitely normally generated by a symmetric set $S \subset G$. The cancelation norm is equal to the biinvariant word norm associated with S.*

Proof. Let $g = \prod_{i=1}^k w_i^{-1} s_i w_i$ then $(s_1, \ldots, s_k)$ is a trivializing sequence for g, and hence $|g|_\times \leq \|g\|$.

Let $g = u_0 s_1 u_1 \cdots s_k u_k$ with $(s_1, \ldots, s_k)$ being a trivializing sequence. Then $g = \prod_{i=1}^k w_i^{-1} s_i w_i$ with $w_i = \prod_{j=i}^k u_j$. Thus $\|g\| \leq |g|_\times$. □

Let $G = \langle S \mid R \rangle$. A relation $v = w$ in R is called *balanced* if it has the following property: if $\bar{v}$ is the word obtained from v by deleting k letters then there exist k letters in w such that deleting them produces a word $\bar{w}$ such that $\bar{v} =_G \bar{w}$ in G. The following lemma is straightforward to prove and is left to the reader.

Lemma 2.B. *If $G = \langle S \mid R \rangle$ and $v = w$ is a balanced relation in R then*

$$|xvy|_\times = |xwy|_\times$$

for any words x, y.

Example 2.C. Coxeter groups and right angled Artin groups admit presentations whose all relations are balanced. Indeed, observe that there exists a presentation of a Coxeter group with relations of the form $s = s^{-1}$ and $st \ldots s = ts \ldots t$ or $(st)^n = (ts)^n$. The presentation with balanced relations of a right angled Artin group has relations of the form $st = ts$.

The proof of the following observation is straightforward and is left to the reader.

Proposition 2.D.

(1) *Let $G_i = \langle S_i | R_i \rangle$, for $i \in \{1, 2\}$, be two presentations whose all relations are balanced and with disjoint S_1 and S_2. Let $R_0 = \{s_1 s_2 = s_2 s_1 | s_i \in S_i\}$. Then $\langle S_1 \cup S_2 | R_0 \cup R_1 \cup R_2 \rangle$ is a presentation of $G_1 \times G_2$ with all relations balanced.*

(2) *Let $G_i = \langle S_i | R_i \rangle$, for $i \in \{1, 2\}$, be two presentations whose all relations are balanced. Assume that the subgroups of G_1 and G_2 generated by $T = S_1 \cap S_2$ are isomorphic (by the isomorphism which is the identity on T). Then $G_1 *_{\langle T \rangle} G_2 = \langle S_1 \cup S_2 | R_1 \cup R_2 \rangle$ has all relations balanced.*

Proposition 2.E. *Let $G = \langle S | R \rangle$ be a presentation whose all relations are balanced. Let u and v be two words in alphabet S representing the same element $g \in G$. Then $|v|_\times = |w|_\times$. In particular, the cancelation norm of g is equal to the cancelation length of any word representing g.*

Remark 2.F. The last statement for Coxeter groups was obtained by Dyer in [18].

Proof. Suppose that $v = xy$ and $w = xr^{-1}ty$, where $r = t$ is a relation from R and x, y are any words. Then we have that

$$\begin{aligned} |w|_\times &= |xr^{-1}ty|_\times \\ &= |xr^{-1}ry|_\times \\ &= |xy|_\times = |v|_\times, \end{aligned}$$

where the second equality follows from Lemma 2.B. If the words v and w represent the same element in G then w can be obtained from v be performing a sequence of the operations above. This implies the statement. □

Example 2.G. Let $G = \langle x, t \,|\, x^5 = tx^2t^{-1} \rangle$ be a Baumslag-Solitar group. In this case the cancelation length is not well defined since, for example, the cancelation lengths of x^5 and of tx^2t^{-1} are distinct but these words represent the same element.

Corollary 2.H. *Let G be either a Coxeter group or a right angled Artin group generated by a set S. The inclusion $P_T \subset G$ of the standard parabolic subgroup associated with a subset $T \subset S$ is an isometry with respect to biinvariant word metrics associated with the sets T and S.* □

2.I. An algorithm for computing the cancelation norm on a free group.

Lemma 2.J. *If x is a generator of a free group $\mathbf{F}_n$ and $w \in \mathbf{F}_n$ then*

$$\|xw\| = \min\left\{1 + \|w\|, \min\{\|u\| + \|v\|, \text{ where } w = ux^{-1}v\}\right\}.$$

Proof. The sequence $x, x_1, \ldots, x_n$ is minimal trivializing for the word xw if and only if the sequence $x_1, \ldots, x_n$ is minimal trivializing for the word w. This implies that if x is contained in a minimal trivializing sequence then $\|xw\| = 1 + \|w\|$.

Suppose that x is not contained in a minimal sequence trivializing xw. Then the word w must contain a letter equal to x^{-1} that is not contained in a minimal trivializing sequence $x_1, \ldots, x_n$ for w and with which x may be canceled out. This implies that $w = ux^{-1}v$ and there exists k such that the sequence $x_1, \ldots, x_k$ minimally trivializes u and $x_{k+1}, \ldots, x_n$ minimally trivializes v. This implies that

$$\|w\| = \|u\| + \|v\|.$$

□

Proof of Theorem 1.C. Assume that we have a reduced word v of standard length k and we know biinvariant lengths of all its proper connected subwords. We can compute $\|v\|$ in time k by processing the word from the beginning to the end in order to find patterns as in Lemma 2.J and computing the minimum.

Let $w = w_1 w_2 \dots w_n$ be a reduced word written in the standard generators. In order to compute $\|w\|$ we need to compute biinvariant lengths of all its connected subwords $w_i w_{i+1} \dots w_j$. Thus we proceed as follows: first we compute biinvariant lengths of all words of standard length 3 (words of length 1 and 2 always have biinvariant lengths 1 and 2, respectively), then biinvariant lengths of all words of standard length 4 and so on.

In order to find computational complexity of this problem assume that we have computed biinvariant lengths of all connected subwords of standard length less then k. There are no more then n subwords of standard length k. Thus to compute biinvariant length of all subwords of standard length k we perform no more then Cnk operations for some constant C.

Thus the complexity of our algorithm is

$$\Sigma_{k=1}^{n} Cnk = O(n^3)$$

During computations we need to remember only lengths of subwords. Since there are $O(n^2)$ subwords, we used $O(n^2)$ memory. □

Remark 2.K. There is no obvious algorithm computing the conjugation invariant norm even for groups where the word problem is solvable. However, it follows from Proposition 2.E that we can find an algorithm for computing the conjugation invariant word norm for groups admitting a presentation whose all relations are balanced and with solvable word problem. But even then, we need to check all possible subsequences of the chosen word which makes the algorithm exponential in time.

3. Quasiisometric embeddings

3.A. **Quasiisometric embeddings of $\mathbf{Z}^n$.** We say that a map

$$f\colon (X, d_X) \to (Y, d_Y)$$

is a quasiisometric embedding if f is a quasiisometry on its image.

Lemma 3.B ([10]). *Suppose that* $\dim Q(G) \geq n$. *Then there exist* n *quasimorphisms* $q_1, \ldots, q_n \in Q(G)$ *and* $g_1, \ldots, g_n \in G$ *such that* $q_i(g_j) = \delta_{ij}$.

Theorem 3.C. *Suppose that* $\dim Q(G) \geq n$. *Then there exists a quasi-isometric embedding* $\mathbf{Z}^n \to G$, *where* $\mathbf{Z}^n$ *is equipped with the standard word metric and* G *is equipped with the biinvariant word metric.*

Proof. Let $q_1, \ldots, q_n \colon G \to \mathbf{R}$ be linearly independent homogeneous quasimorphisms and let $g_1, \ldots, g_n \in G$ be such that $q_i(g_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

We define $\Psi \colon \mathbf{Z}^n \to G$ by $\Psi(k_1, \ldots, k_n) = g_1^{k_1} \cdots g_n^{k_n}$ and observe that

$$\left\| \prod_i g_i^{k_i} \right\| \leq c \sum_i |k_i|,$$

where $c = \max_i \|g_i\|$. On the other hand, for every $j \in \{1, ..., n\}$ we have

$$c_j \left\| \prod_i g_i^{k_i} \right\| \geq \left| q_j \left(\prod_i g_i^{k_i} \right) \right| \geq |k_j| - nd_j,$$

where d_j is the defect of the quasimorphism q_j and c_j is its Lipschitz constant. Taking $C := \max\{c, nc_1, ..., nc_n\}$ and $D := C \sum_i nd_i$ and combining the two inequalities we obtain

$$\frac{1}{C} \sum_i |k_i| - D \leq \left\| \prod_i g_i^{k_i} \right\| \leq C \sum_i |k_i|.$$

□

It follows from the above theorem that if the space of homogeneous quasimorphisms of a group G is infinite dimensional then there exists a quasiisometric embedding $\mathbf{Z}^n \to G$ for every natural number $n \in \mathbf{N}$.

Examples 3.D. Groups for which the space of homogeneous quasimorphisms is infinite dimensional include:

(1) a nonabelian free group $\mathbf{F}_m$ [11],

(2) Artin braid groups on 3 and more strings, and braid groups of a hyperbolic surface [5],

(3) a non-elementary hyperbolic group [20],

(4) a finitely generated group which satisfies the small cancelation condition $C'(1/12)$ [1],

(5) mapping class group of a surface of positive genus [5],

(6) a nonabelian right angled Artin group [3],

(7) groups of Hamiltonian diffeomorphisms of compact orientable surfaces [19, 22].

3.E. **Embeddings of trees.**

Theorem 3.F. *There is an isometric embedding $\square^\infty \to \mathbf{F}_2$ of the vertex set of the infinite dimensional unit cube with the ℓ^1-metric into the free group on two generators with the biinvariant word metric coming from the standard generators.*

Proof. Let $\mathbf{F}_2$ be the free group generated by elements a and b and let $\square^n = \{0,1\}^n$ denote the n-dimensional cube. Let $\square^n$ be embedded into $\square^{n+1}$ as $\square^n \times \{0\}$. For an arbitrary isometric embedding

$$\psi_n \colon \square^n \to \mathbf{F}_2$$

we construct an extension to

$$\psi_{n+1} \colon \square^{n+1} \to \mathbf{F}_2$$

as follows. Take an element $g = b^{4k}ab^{-4k}$, where $k > |\psi(v)|$ for every $v \in \square^n$. Define $\psi_{n+1}(v,0) = \psi_n(v)$ and $\psi_{n+1}(v,1) = g\psi_n(v)$. Since the multiplication from the left is an isometry of the biinvariant metric, ψ_{n+1} is an isometry on both $\square^n \times \{0\}$ and $\square^n \times \{1\}$. Hence what we need to show is that

$$d((v,0),(w,1)) = \|\psi_{n+1}(v,0)\psi_{n+1}(w,1)^{-1}\|$$

for every $v, w \in \square^n$. From the definition of ψ_{n+1} we have that

$$\|\psi_{n+1}(v,0)\psi_{n+1}(w,1)^{-1}\| = \|\psi_n(v)\psi_n(w)^{-1}b^{4k}a^{-1}b^{-4k}\|$$

We shall show that every minimal sequence trivializing

$$\psi_n(v)\psi_n(w)^{-1}b^{4k}a^{-1}b^{-4k}$$

contains the last letter a^{-1}, thus has the length

$$\|\psi_n(v)\psi_n(w)^{-1}\| + 1 = d((v,0),(w,1)).$$

To see that assume on the contrary, that a^{-1} is not in a minimal trivializing sequence. Then it has to cancel out with some letter a in $\psi_n(v)\psi_n(w)^{-1}$. But $|\psi_n(v)\psi_n(w)^{-1}| < 2k$, so in order to make the cancelation possible, one has to cross out at least $2k+1$ letters b between $\psi_n(v)\psi_n(w)^{-1}$ and a^{-1}. Since

$$2k+1 > |\psi_n(v)\psi_n(w)^{-1}| + 1 \geq \|\psi_n(v)\psi_n(w)^{-1}\| + 1,$$

such trivializing sequence cannot be minimal.

Now take an arbitrary ψ_0 and construct a sequence of isometries ψ_n. Then $\psi_\infty = \bigcup_{n=0}^{\infty} \psi_n$ is an isometric embedding of $\square^\infty$. $\square$

Proof of Theorem 1.D. Let T be a locally compact tree with edges of unit length. Then T isometrically embeds into the cube $\square^\infty$ as follows. Let v be a vertex of T and w be a vertex of $\square^\infty$. We map a star of v isometrically into a star of w. We then continue the procedure inductively. It is possible because the star of any vertex of the cube has countably infinitely many edges. $\square$

4. Biinvariant geometry of cyclic subgroups

4.A. **Bounded cyclic subgroups.**

Lemma 4.B. *Let $\Gamma := \langle x, t \mid [x, {}^t x] = 1 \rangle$. The following identity holds in Γ:*

$$[x,t]^n = [x^n, t]\,.$$

In particular, the cyclic subgroup generated by $[x,t]$ is bounded by two (with respect to the generating set $\{x^{\pm 1}, t^{\pm 1}\}$).

Proof. The identity is true for $n=1$. Let us assume that it is true for some n. We then obtain that

$$\begin{aligned}
[x,t]^{n+1} &= x^n t x^{-n} \left(t^{-1} x t\right) x^{-1} t^{-1} \\
&= x^n t \left(t^{-1} x t\right) x^{-n} x^{-1} t^{-1} \\
&= x^{n+1} t x^{-(n+1)} t^{-1}.
\end{aligned}$$

The statement follows by induction. $\square$

Examples 4.C. In the following examples we prove boundedness of a cyclic subgroup of a group G by constructing a relevant homomorphism $\Psi\colon \Gamma \to G$.

(1) Let

$$\mathrm{BS}(p,q) = \langle a, t \,|\, t a^p t^{-1} = a^q \rangle$$

be the Baumslag-Solitar group, where $q > p$ are integers. Let $\Psi\colon \Gamma \to \mathrm{BS}(p,q)$ be defined by $\Psi(x) = a^p$ and $\Psi(t) = t$. It follows that the cyclic subgroup generated by $[\Psi(t), \Psi(x)]$ is bounded. Since $[t, a^p] = a^{p-q}$ we obtain that the cyclic subgroup generated by a is bounded.

(2) Let $A \in \mathrm{SL}(2, \mathbf{Z})$ and let $G = \mathbf{Z} \ltimes_A \mathbf{Z}^2$ be the associated semidirect product. If $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ then G has the following presentation

$$G = \left\langle x, y, t \,\middle|\, [x,y] = 1, {}^t x = x^a y^c, {}^t y = x^b y^d \right\rangle.$$

Note that $\Psi\colon \Gamma \to G$ given by $\Psi(t) = t$ and $\Psi(x) \in \mathbf{Z}^2 \subset G$ is a well defined homomorphism.

If A has two distinct real eigenvalues, for example if A is the Arnold cat matrix, then every element in the kernel generates a bounded cyclic subgroup. If $A \neq \mathrm{Id}$ has eigenvalues equal to one then the center of G is bounded (cf. Theorem 5.H and 5.K).

(3) Consider the integer lamplighter group

$$\mathbf{Z} \wr \mathbf{Z} = \mathbf{Z} \ltimes \mathbf{Z}^\infty.$$

where $\mathbf{Z}^\infty$ denotes the group of all integer valued sequences $\{a_i\}_{i \in \mathbf{Z}}$. The generator t of $\mathbf{Z}$ acts by the shift and hence the conjugation of $\{a_i\}$ by t has the following form

$$t\{a_i\}t^{-1} = \{a_{i+1}\}.$$

Since for every sequence $\{a_i\}$ there exists a sequence $\{b_i\}$ such that $a_i = b_{i+1} - b_i$ we get that $\{a_i\} = t\{b_i\}t^{-1}\{b_i\}^{-1}$. Let $\Psi\colon \Gamma \to \mathbf{Z} \wr \mathbf{Z}$ be defined by $\Psi(x) = \{b_i\}$ and $\Psi(t) = t$. This shows that every element in the commutator subgroup of the lamplighter group generates a bounded cyclic subgroup.

(4) Let $G = \mathrm{SL}(2, \mathbf{Z}[1/2])$. Define

$$\Psi(x) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$$
$$\Psi(t) = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2^{-1} \end{smallmatrix}\right)$$

It well defines a homomorphism since $\Psi({}^t x) = \left(\begin{smallmatrix} 1 & 4 \\ 0 & 1 \end{smallmatrix}\right)$. Consequently we get that $\left(\begin{smallmatrix} 1 & -3 \\ 0 & 1 \end{smallmatrix}\right) = \Psi([x,t])$ generates a bounded cyclic subgroup. More generally, it implies that the subgroups of elementary matrices are bounded. It is known that every element of G can be written as a product of up to five elementary matrices [28] (see also [14, Example 5.38]). Hence we obtain that the whole group G is bounded.

(5) Let B_k be the braid group on $k \geq 2$ strings and let $i\colon B_n \to B_{2n}$ be a natural inclusion on the first n strings. Assume, that g is in the image of i. Let $\Delta = (\sigma_1 \ldots \sigma_{n-1}) \ldots (\sigma_1\sigma_2)(\sigma_1)$ (Δ is a half-twist Garside fundamental braid) where σ_i's are the standard Artin generators of the braid group B_n. The conjugation $\Delta g \Delta^{-1}$ flips g, thus $[\Delta g \Delta^{-1}, g] = e$. For example, if $g = \sigma_1 \in B_4$, then $\Delta g \Delta^{-1} = \sigma_3$ and $\sigma_1\sigma_3^{-1}$ is bounded in B_4.

(6) Let $\Delta \in B_n$ be as above and let $g = \sigma_{i_1} \ldots \sigma_{i_k} \in B_n$ be any element. The conjugation by Δ acts on g as follows
$$\Delta \sigma_{i_1}\sigma_{i_2} \ldots \sigma_{i_k} \Delta^{-1} = \sigma_{n-i_1}\sigma_{n-i_2} \ldots \sigma_{n-i_k}.$$
This implies that every braid of the form
$$g = \sigma_{i_1}\sigma_{i_2} \ldots \sigma_{n-i_2}^{-1}\sigma_{n-i_1}^{-1}$$
is conjugate via Δ to its inverse. Consequently, $[g^n, \Delta] = g^{2n}$ which implies that the cyclic subgroup generated by g is bounded by $2\|\Delta\| + \|g\|$. For example, $\sigma_1\sigma_2^{-1} \in B_3$ generates a bounded cyclic subgroup.

(7) It is a well-known fact that the center of B_3 is a cyclic group generated by Δ^2 (for definition of Δ see item (5) above). We have a central extension
$$1 \to \langle \Delta^2 \rangle \to B_3 \xrightarrow{\Psi} \mathrm{PSL}(2, \mathbf{Z}) \to 1$$
where $\Psi(\sigma_1) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\Psi(\sigma_2) = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$. Denote
$$J = \Psi(\Delta) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right).$$
Let $M \in \mathrm{PSL}(2, \mathbf{Z})$ be a symmetric matrix. It has two orthogonal eigenspaces (over $\mathbf{R}$) with reciprocal eigenvalues. The rotation J swaps the eigenspaces which implies $M^J = M^{-1}$. Moreover, there exists a braid g in B_3 such that g is conjugate to g^{-1} and $\Psi(g) = M$. Indeed, any symmetric matrix is of the form $[J, N]$ for some $N \in \mathrm{PSL}(2, \mathbf{Z})$. Let h be a lift of N to B_3 and take $g = [\Delta, h]$. Then
$$\Delta^{-1} g \Delta = h\Delta^{-1}h^{-1}\Delta = h\Delta^{-1}\Delta^2 h^{-1}\Delta^{-2}\Delta = [h, \Delta] = g^{-1}.$$
By the same argument as in item (6) above, g generates a bounded subgroup. For example the image of an element $\sigma_1\sigma_2^{-1}$ is Arnold's cat matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Since there are infinitely many conjugacy classes

of symmetric matrices in $\mathrm{PSL}(2, \mathbf{Z})$, there are infinitely many conjugacy classes of bounded cyclic subgroups in B_3. It should be compared to the group of pure braids P_3, which is a finite index subgroup of B_3, but due to Theorem 4.H every nontrivial element in P_3 is undistorted.

(8) Let $f, h\colon M \to M$ be homeomorphisms of a manifold such that $h(\mathrm{supp}(f)) \cap \mathrm{supp}(f) = \emptyset$. Then the commutator $[f, h]$ is bounded with respect to any biinvariant metric on a group of homeomorphisms containing f and h.

4.D. **Unbounded cyclic subgroups not detected by quasimorphisms.** Let G be the simple finitely generated group constructed by Muranov in [32]. The following facts are proved in the Main Theorem of his paper:

- every cyclic subgroup of G is distorted with respect to the biinvariant word metric; in particular, G does not admit nontrivial homogeneous quasimorphisms (Main Theorem (3)).
- G contains cyclic subgroups unbounded with respect to the commutator length (Main Theorem (1)); in particular, they are unbounded with respect to the biinvariant word metric.

4.E. **Cyclic groups detected by homogeneous quasimorphisms.** A group G is called *quasiresidually real* if for every element $g \in G$ there exists a homogeneous quasimorphism $q\colon G \to \mathbf{R}$ such that $q(g) \neq 0$. It is equivalent to the existence of an unbounded quasimorphism on the cyclic subgroup generated by g.

Free groups are quasiresidually real as well as torsion free hyperbolic groups. It immediately follows that every element in such a group is undistorted. The purpose of this section is to prove the following results.

Theorem 4.F. *A right angled Artin group is quasiresidually real.*

Theorem 4.G. *A commutator subgroup of a right angled Coxeter group is quasiresidually real.*

Theorem 4.H. *A pure braid group on any number of strings is quasiresidually real.*

We need to introduce some terminology and state some lemmas before the proof. The definitions and basic properties of rank-one elements can be found in [6].

Lemma 4.I (Bestvina-Fujiwara)**.** *Assume that G acts on a proper* $\mathrm{CAT}(0)$ *or hyperbolic space X by isometries and $g \in G$ is a rank-one isometry. If no positive power of g is conjugate to a positive power of g^{-1} then there is a homogeneous quasimorphism $q\colon G \to \mathbf{R}$ which is nontrivial on the cyclic subgroup generated by g.*

Proof. Let $x_0 \in X$ be the basepoint and $\sigma = [x_0, gx_0]$ be a geodesic interval. If α is a piecewise geodesic path in X then let $|\alpha|_g$ be the maximal number of nonoverlaping translates of σ in α such that every subpath of α which connects two consecutive translates of σ is a geodesic segment. Let $c_g\colon G \times G \to \mathbf{R}$ be defined by

$$c_g(x,y) := \inf_\alpha(|\alpha| - |\alpha|_g),$$

where α ranges over all piecewise geodesic paths from x to y.

Let $\Psi_g\colon G \to \mathbf{R}$ be defined by

$$\Psi_g(h) = c_g(x_0, h(x_0)) - c_g(h(x_0), x_0)$$

and it follows from [6, the proof of Theorem 6.3] that there exists $k > 0$ such that Ψ_{g^k} is unbounded on the cyclic group generated by g. Homogenizing Ψ_{g^k} yields a required quasimorphism $q\colon G \to \mathbf{R}$. □

Lemma 4.J. *Let G be a group acting on a proper* $\mathrm{CAT}(0)$ *space X by isometries. Assume that $g \in G$ is a rank one isometry. Then*

$$xg^nx^{-1} \neq g^{-m}$$

for all $x \in G$ and $m, n > 0$ provided that $m \neq n$. If G is torsion free the above holds also if $m = n$.

Proof. Suppose otherwise that there exists $x \in G$ and m, n such that

$$xg^nx^{-1} = g^{-m}. \tag{1}$$

Assume that $m = n$. Then we have that

$$x^2g^nx^{-2} = xg^{-n}x^{-1} = g^n,$$

which means that g^n and x^2 commute. Moreover, a group generated by g^n and x^2 is of rank two. To prove it assume otherwise that there exist $r \in G$ and k, l such that $g^n = r^l$ and $x^2 = r^k$. Take the k-th power of (1)

$$xg^{kn}x^{-1} = g^{-kn}.$$

Together with $g^{kn} = r^{kl} = x^{2l}$ it gives that $x^{4l} = e$. Which is a contradiction.

Since g^n is an element of the free abelian subgroup of rank two, it follows from the flat torus theorem that its axis lies in some flat. Thus g^n, and consequently g, cannot be a rank one isometry.

Assume now that $m \neq n$ and take the k-th power of (1)

$$xg^{kn}x^{-1} = g^{km}.$$

Let P be a point on the axis $L \subset X$ on which g acts by a translation by d units. Since $xg^{km}x^{-1}(P) = (g^{-1})^{kn}(P)$, the image of a geodesic between $x^{-1}(P)$ and $g^{km}x^{-1}(P)$ with respect to x is contained in the axis L.

Let $l := \mathrm{d}(x^{-1}(P), P)$, where d is the distance function on X. Applying the triangle inequality we get that

$$\begin{aligned} kmd &= \mathrm{d}\left(P, g^{km}(P)\right) \\ &\leq \mathrm{d}\left(P, x^{-1}(P)\right) + \mathrm{d}\left(x^{-1}(P), g^{km}x^{-1}(P)\right) + \mathrm{d}\left(g^{km}x^{-1}(P), g^{km}(P)\right) \\ &= 2l + \mathrm{d}\left(x^{-1}(P), g^{km}x^{-1}(P)\right). \end{aligned}$$

This and a similar additional computation imply that

$$kmd - 2l \leq \mathrm{d}\left(P, xg^{km}x^{-1}(P)\right) \leq kmd + 2l.$$

On the other hand, $\mathrm{d}\left((g^{-1})^{kn}(P), P\right) = knd$ which implies that

$$(g^{-1})^{kn}(P) \neq xg^{km}x^{-1}(P)$$

for k large enough which contradicts (1). □

Let A_Δ be the right angled Artin group defined by the graph Δ. The presentation complex X_Δ of A_Δ is a two dimensional complex with one vertex and with edges corresponding to generators and two dimensional cells corresponding to relations. It is a union of two dimensional tori. Its universal covering $\widetilde{X}_\Delta$ is a CAT(0) square complex. Let $\Delta' \subset \Delta$ be a full subgraph. Then

(1) the homomorphism $\pi\colon A_\Delta \to A_{\Delta'}$ defined by

$$\pi(v) := \begin{cases} v & \text{if } v \in \Delta' \\ 1 & \text{if } v \notin \Delta' \end{cases}$$

is well defined and surjective;

(2) every quasimorphism $q\colon A_{\Delta'} \to \mathbf{R}$ extends to A_Δ.

If Δ' is a bipartite graph then the subgroup $A_{\Delta'} \subset A_\Delta$ is called a *join* subgroup.

Proof of Theorem 4.F. Let $g \in A_\Delta$ be a nontrivial element of a right angled Artin group. Suppose that no conjugate of g is contained in a join subgroup. Then, according to Berhstock-Charney [3, Theorem 5.2], g acts on the universal cover $\widetilde{X}_\Delta$ of the presentation complex as a rank one isometry.

Thus, since A_Δ is torsion-free, we can apply Lemma 4.J and consequently Lemma 4.I to g.

If g is an element of a join subgroup then we project it to one of the factors repeatedly until no conjugate of g is contained in a join subgroup and then we apply the above construction and extend the obtained quasimorphism to A_Δ. □

The right angled Coxeter group given by the graph Δ is a group defined by the following presentation

$$W_\Delta = \langle v \in \Delta |\, v^2 = 1, [v, v'] = 1 \text{ iff } (v, v') \text{ is an edge in } \Delta \rangle,$$

As in the case of right angled Artin groups, we have a well defined projection π for an arbitrary full subgraph Δ' and the notion of a join subgroup.

The natural CAT(0) complex on which W_Δ acts geometrically is the Davis cube complex Σ_Δ (see Davis [17] for more details).

Proof of Theorem 4.G. First we prove that the commutator subgroup W'_Δ of W_Δ is torsion-free. Let $g \in W'_\Delta$ be a torsion element. By the CAT(0) property it stabilizes a cube in Σ_Δ. It follows from the definition of the Davis complex that stabilizers of cubes are conjugate to spherical subgroups (i.e. subgroups generated by vertices of some clique). Note that an abelianization of W_Δ equals $\oplus_{v\in\Delta}\mathbf{Z}/2\mathbf{Z}$ and spherical subgroups, as well as its conjugates project injectively into the abelianization. Thus g is a trivial element.

Now the argument is analogous to the proof of Theorem 4.F. Suppose that $g \in W'_\Delta$ is an element such that no conjugate of g is contained in a join subgroup. According to [16, Proposition 4.5], g acts on Σ_Δ as a rank one isometry. Now we apply Lemma 4.J and 4.I to g and W'_Δ.

If g is in a join subgroup, we project g together with W'_Δ on the infinite factor. The projection of a commutator subgroup is again a commutator subgroup, thus it is torsion-free. Hence the assumption of Lemmas 4.J and 4.I are satisfied. Thus we apply the same argument as in Theorem 4.F constructing a quasimorphism which can be extended to W'_Δ. □

Before the proof of Theorem 4.H let us recall basic properties and definitions of braid and pure braid groups. Denote by D_n an open two dimensional disc with n marked points. The braid group on n strings, denoted B_n, is a group of isotopy classes of orientation-preserving homeomorphisms of D_n which permute marked points (this is the mapping class group of a disc with n punctures). A class of a homeomorphism which fixes all marked points is called a pure braid. The group of all pure braids on n strings, denoted P_n, is a finite index normal subgroup of B_n.

Let $g > 1$. Denote by $\mathcal{MCG}_g^n$ the mapping class group of a closed hyperbolic surface Σ_g with n punctures. In [7] J. Birman showed that B_n naturally embeds into $\mathcal{MCG}_g^n$. More precisely, let D be an embedded disc in Σ_g which contains n punctures. Then a mapping class group of this n punctured disc D is a subgroup of $\mathcal{MCG}_g^n$. Let us identify B_n with this subgroup. In the same way we identify P_n with a subgroup of the pure mapping class group $\mathcal{PMCG}_g^n$. Note that $\mathcal{PMCG}_g^n$ is a finite index subgroup of $\mathcal{MCG}_g^n$.

It follows from the Nielsen-Thurston decomposition in $\mathcal{MCG}_g^n$ that for every $\gamma \in B_n < \mathcal{MCG}_g^n$ there exists N, pseudo-Anosov braids $\gamma_1, \gamma_2, \ldots, \gamma_m \in B_n$ and Dehn twists $\delta_1, \delta_2, \ldots, \delta_n \in B_n$ such that

$$\gamma^N = \gamma_1\gamma_2 \ldots \gamma_m \delta_1^{m_1}\delta_2^{m_2} \ldots \delta_n^{m_n},$$

where all elements in the above decomposition pairwise commute. Moreover, the support of each element is contained in a connected component of the disc D, is bounded by a simple curve and contains non empty subset of marked points.

Following [4, Section 4] we call an element γ **chiral** if it is not conjugate to its inverse. Note that if two elements in $B_n < \mathcal{MCG}_g^n$ are conjugate in $\mathcal{MCG}_g^n$, then they are conjugate in B_n. Similarly, if two elements in $P_n < \mathcal{PMCG}_g^n$ are conjugate in $\mathcal{PMCG}_g^n$, then they are conjugate in P_n. It follows that γ is chiral in B_n if and only if it is chiral in $\mathcal{MCG}_g^n$, and the same statement holds for groups P_n and

$\mathcal{PMCG}_g^n$. The following lemma is a straightforward consequence of Theorem 4.2 from [4].

Lemma 4.K (Bestvina-Bromberg-Fujiwara)**.** *Let Σ be a closed orientable surface, possibly with punctures. Let G be a finite index subgroup of the mapping class group of Σ. Consider a nontrivial element $\gamma \in G$ and Nielsen-Thurston decomposition of its appropriate power as above. Assume that every element from the decomposition is chiral and nontrivial powers of any two elements from the decomposition are not conjugate in G. Then there is a homogeneous quasimorphism on G which takes a non zero value on γ.*

A group G is said to be **biorderable** if there exists a linear order on G which is invariant under left and right translations. For example the pure braid group on any number of strings is biorderable [34].

Lemma 4.L. *Let G be a biorderable group. Then $xy^mx^{-1} \neq y^{-n}$ for every $y \neq e, x \in G$ and positive m, n.*

In particular, every nontrivial element in a biorderable group is chiral.

Proof. Let $<$ be a biinvariant order on G. Assume on the contrary that $xy^mx^{-1} = y^{-n}$. Without loss of generality we can assume that $y > e$. Then $y^m > e$, we can conjugate the inequality by x which gives that $y^{-n} = xy^mx^{-1} > e$. Thus $e > y^n$, that is $e > y$. We got a contradiction. □

Proof of Theorem 4.H. Let γ be a nontrivial pure braid on n strings. We will show that there is a homogeneous quasimorphism on P_n nontrivial on γ. After passing to a power of γ we can write that

$$\gamma = \gamma_1\gamma_2 \ldots \gamma_m\delta_1^{m_1}\delta_2^{m_2} \ldots \delta_n^{m_n}$$

where γ_i and δ_i are as in the discussion above. Since P_n is a finite index subgroup of B_n we can find M such that all γ_i^M and δ_i^M are in P_n. Thus passing to even bigger power of γ we can assume that the braids arising in the decomposition are pure.

Lemma 4.L implies that every element from the decomposition is chiral, and so it is chiral in $\mathcal{PMCG}_g^n$. Let x and y be two distinct elements among γ_i and $\delta_i^{m_i}$. From the definition of the decomposition, simple curves associated to x and y bound disjoint subsets of marked points. Since isotopy classes from P_n preserve marked points pointwise, powers of x and y cannot be conjugate by a pure braid and hence by no element of $\mathcal{PMCG}_g^n$.

The assumptions of Lemma 4.K are satisfied, hence γ is detectable by homogeneous quasimorphism. Note that this homogeneous quasimorphism is defined on the whole group $\mathcal{PMCG}_g^n$, and a quasimorphism on P_n, which detects γ, is a restriction of the above quasimorphism to the subgroup $P_n < \mathcal{PMCG}_g^n$. □

5. The bq-dichotomy

The purpose of this section is to prove the bq-dichotomy for various classes of groups. We introduce a family of auxiliary groups which detects bounded elements.

Lemma 5.A. *Let $\bar{m} = (m_0, m_1, \dots, m_k)$ be a sequence of integers such that $\frac{1}{m_0} + \frac{1}{m_1} + \dots + \frac{1}{m_k} = 0$. Define*

$$\Gamma(\bar{m}) = \langle x_0, \dots, x_k, t_1, \dots, t_k \mid ({}^{t_i}x_0)^{m_0} = x_i^{m_i}, [x_j, x_k] = e \rangle$$

Then $g = x_0 x_1 \dots x_k$ generates a bounded cyclic subgroup.

Proof. Let $N = m_0 m_1 \dots m_k$ and $a_i = \frac{N}{m_i}$. From the assumption on m_i we have that $a_0 + a_1 + \dots + a_k = 0$. For any n we obtain that

$$\begin{aligned}
g^{nN} &= x_0^{nN} x_1^{nN} \dots x_k^{nN} \\
&= x_0^{nm_0a_0} x_1^{nm_1a_1} \dots x_k^{nm_ka_k} \\
&= x_0^{nm_0a_0} ({}^{t_1}x_0)^{nm_0a_1} \dots ({}^{t_k}x_0)^{nm_0a_k} \\
&= x_0^{nm_0(-a_1-a_2-\dots-a_k)} ({}^{t_1}x_0)^{nm_0a_1} \dots ({}^{t_k}x_0)^{nm_0a_k} \\
&= ({}^{t_1}x_0)^{nm_0a_1} x_0^{-nm_0a_1} \dots ({}^{t_k}x_0)^{nm_0a_k} x_0^{-nm_0a_k} \\
&= [t_1, x_0^{nm_0a_1}] \dots [t_k, x_0^{nm_0a_k}].
\end{aligned}$$

It shows that g^{nN} is bounded by $2k$ for every n. Hence g generates a bounded subgroup. □

Let us remark that $\Gamma(1, -1)$ is isomorphic to the group Γ defined in Section 4.B. We start with a somewhat weaker statement for Coxeter groups.

Theorem 5.B. *Let W be a Coxeter group and let $g \in W$.*

- *The cyclic subgroup $\langle g \rangle$ is either bounded or undistorted.*

- *Let $W_T = W_{T_1} \times W_{T_2}$ be a standard parabolic subgroup such that both W_{T_1} and W_{T_2} are infinite and standard parabolic. If the standard projection $W \to W_T$ is well defined for all W_T of the above form then W satisfies the bq-dichotomy.*

Remark 5.C. Let S be the standard generating set for W. The property in the second item of the theorem holds if for every $s \in S \setminus T$ and $t \in T$ the exponent in the relation $(st)^m$ is even.

Proof. We proceed by induction on a number of Coxeter generators. If there is only one generator the theorem is obvious. Let $n \in \mathbf{N}$ be a natural number and W be a Coxeter group generated by n Coxeter generators. Assume that the theorem is true for Coxeter groups generated by less than n Coxeter generators. Let $g \in W$ be a nontorsion element. There are two cases:

Case 1: The element g acts as a rank one isometry on the Davis complex. If no positive power of g is conjugate to a positive power of g^{-1} then we can apply Lemma 4.I to obtain a homogeneous quasimorphism nonvanishing on g. Otherwise we have that $xg^mx^{-1} = g^{-n}$ for some $x \in W$ and positive $m, n \in \mathbf{N}$. By Lemma 4.J it follows that $m = n$. There is a homomorphism

$$\Psi\colon \Gamma(m, -m) \to W$$

defined on generators as $\Psi(x_0) = \Psi(x_1) = g$, $\Psi(t_1) = x$. Thus the element $\Psi(x_0x_1) = g^2$ (as well as g) generates a bounded cyclic subgroup.

Case 2: g does not act as a rank one isometry on the Davis complex. Then, according to Caprace and Fujiwara [16, Proposition 4.5], g is contained in a parabolic subgroup P that is either

(1) equal to $P_1 \times P_2$, where P_1 is finite parabolic and P_2 is parabolic and affine of rank at least three or

(2) equal to $P_1 \times P_2$, where both P_1 and P_2 are infinite parabolic.

In the first case, both P_1 and P_2 are bounded [31] and so is their product and hence g generates a bounded subgroup.

In the second case we project g to the factors and the first statement follows by induction because the inclusion of a parabolic subgroup is an isometry due to Corollary 2.H.

If the projection of g to one of the factors is detectable by a homogeneous quasimorphism then this quasimorphisms extends to the product P. Thus if W satisfies the assumption of the second statement, we pull back the latter to W using the projection $W \to W_T$ and the conjugation $xW_Tx^{-1} = P$. Otherwise g generates a bounded subgroup in both P_1 and P_2. Indeed, we proceed by induction since the assumption of the second statement is inherited by parabolic subgroups. Thus g also generates a bounded subgroup in $P_1 \times P_2$ and hence in W. □

Theorem 5.D. *The bq-dichotomy holds for a finite index subgroup of the mapping class group of a closed surface possibly with punctures.*

Proof. Let us recall some notions from [4, Section 4]. We say that two chiral elements of a group G are equivalent if some of theirs non-trivial powers are conjugate. An equivalence class $\{\gamma_0, \gamma_1, \ldots, \gamma_n\}$ of this relation is called **inessential**, if there is a sequence of numbers $\bar{m} = (m_0, \ldots, m_n)$ such that elements $\gamma_i^{m_i}$ are pairwise conjugate and $\Sigma \frac{1}{m_i} = 0$. Let $h = \gamma_0 \ldots \gamma_n$, where all γ_i's commute. Note that there is a homomorphism

$$\Psi\colon \Gamma(\bar{m}) \to G$$

defined on the generators as $\Psi(x_i) = \gamma_i$. From the Lemma 5.A it follows that $\Psi(x_0 \ldots x_n) = h$ generates a bounded subgroup. When γ is not chiral, it generates a bounded subgroup due to a homomorphism from $\Gamma(1, -1)$ defined by $\Psi(x_0) = \Psi(x_1) = \gamma$.

Let $\gamma \in G$. By the same argument as in the proof of Theorem 4.H we can assume that γ has a Nielsen-Thurston decomposition within G (that is, elements of the decomposition are in G). Assume that there is no homogeneous quasimorphism which takes non zero value on γ. Then by [4, Theorem 4.2] in the decomposition of γ we have either not chiral elements, or chiral elements which can be divided into inessential equivalence classes. Hence we can write that

$$\gamma = c_1 \ldots c_m h_1 \ldots h_n,$$

where c_i are not chiral and h_i are products of elements from inessential class. In both cases they generate bounded subgroups. Since $c's$ and $h's$ commute, we have that

$$\gamma^k = c_1^k \ldots c_n^k h_1^k \ldots h_n^k.$$

Thus γ generates a bounded subgroup in G. □

Theorem 5.E. *The bq-dichotomy holds for Artin braid groups.*

Proof. Let $\gamma \in B_n < \mathcal{MCG}_g^n$, where $g > 1$. Recall that two braids in B_n are conjugate in B_n if and only if they are conjugate in $\mathcal{MCG}_g^n$. Hence an equivalence class $\{\gamma_0, \gamma_1, \ldots, \gamma_n\}$, where each $\gamma_i \in B_n$, is essential (respectively inessential) in B_n if and only if it is essential (respectively inessential) in $\mathcal{MCG}_g^n$. Similarly if γ is not chiral in B_n, then it is not chiral in $\mathcal{MCG}_g^n$.

Assume that there is no homogeneous quasimorphism on $\mathcal{MCG}_g^n$ which takes non zero value on γ. Then by [4, Theorem 4.2] in the Nielsen-Thurston decomposition of γ in $\mathcal{MCG}_g^n$ we have either not chiral elements, or chiral elements which can be divided into inessential equivalence classes. Since each element in the Nielsen-Thurston decomposition of γ lies in B_n, and the notion of equivalence class and chirality is the same in $B_n < \mathcal{MCG}_g^n$ and in $\mathcal{MCG}_g^n$, it follows that there is no homogeneous quasimorphism on B_n which takes non zero value on γ. Hence we can write that

$$\gamma = c_1 \ldots c_m h_1 \ldots h_n,$$

where c_i's are not chiral in B_n and h_i's are products of elements from inessential class in B_n. In both cases they generate bounded subgroups (see the discussion in the proof of the previous case). Since $c's$ and $h's$ commute, we have that

$$\gamma^k = c_1^k \ldots c_n^k h_1^k \ldots h_n^k.$$

Thus γ generates a bounded subgroup in B_n. □

Theorem 5.F. *The bq-dichotomy holds for spherical braid groups (both pure and full).*

Proof. **The case of spherical pure braid groups $P_n(S^2)$.** Recall that $P_n(S^2)$ is a fundamental group of an ordered configuration space of n different points in a two-sphere S^2. As before we denote by

$\mathcal{MCG}_0^n$ the mapping class group of the n punctured sphere, and by $\mathcal{PMCG}_0^n$ the pure mapping class group of the n punctured sphere. Since $P_n(S^2)$ are trivial for $n = 1, 2$, we assume that $n > 2$. It is well known fact that $P_n(S^2)$ is isomorphic to a direct product of $\mathbf{Z}/\mathbf{Z}_2$ and $\mathcal{PMCG}_0^n$, see e.g. [24]. Since we already proved the statement for finite index subgroups of mapping class groups, the proof of this case follows.

The case of spherical braid groups $B_n(S^2)$. The group $B_n(S^2)$ is a fundamental group of a configuration space of n different points in a two-sphere S^2. It is known that the group $\mathcal{MCG}_0^n$ is isomorphic to $B_n(S^2)/\langle\Delta^2\rangle$, where Δ is the Garside fundamental braid, see [29]. In particular, Δ^2 lies in the center of $B_n(S^2)$ and $\Delta^4 = 1_{B_n(S^2)}$. Let

$$\Pi\colon B_n(S^2) \to B_n(S^2)/\langle\Delta^2\rangle \cong \mathcal{MCG}_0^n$$

be the projection homomorphism. Since $\Delta^4 = 1_{B_n(S^2)}$ and Δ^2 is central, every homogeneous quasimorphism on $\mathcal{MCG}_0^n$ defines a homogeneous quasimorphism on $B_n(S^2)$ and vice versa. In addition, if two elements $\Pi(x), \Pi(y) \in \mathcal{MCG}_0^n$ commute or are conjugate in $\mathcal{MCG}_0^n$, then x and y commute or are conjugate up to the multiplication by a torsion element Δ^2 in $B_n(S^2)$.

Let $\gamma \in B_n(S^2)$. Assume that there is no homogeneous quasimorphism on $B_n(S^2)$ which takes non zero value on γ. Then there is no homogeneous quasimorphism on $\mathcal{MCG}_0^n$ which takes non zero value on $\Pi(\gamma)$. Then by [4, Theorem 4.2] in the Nielsen-Thurston decomposition of $\Pi(\gamma)$ in $\mathcal{MCG}_0^n$ we have either not chiral elements, or chiral elements which can be divided into inessential equivalence classes. Hence we can write that

$$\Pi(\gamma) = \Pi(c_1)\ldots\Pi(c_m)\Pi(h_1)\ldots\Pi(h_n),$$

where $\Pi(c_i)$ are not chiral in $\mathcal{MCG}_0^n$ and $\Pi(h_i)$ are products of elements from inessential class in $\mathcal{MCG}_0^n$. As before, $\Pi(\gamma)$ generates a bounded subgroup in $\mathcal{MCG}_0^n$ and since $\Delta^4 = 1_{B_n(S^2)}$ and Δ^2 is central, γ generates a bounded subgroup in $B_n(S^2)$. □

5.G. **The bq-dichotomy for nilpotent groups.** Let us recall that a group G is said to be boundedly generated if there are cyclic subgroups $C_1, \ldots, C_n$ of G such that $G = C_1 \ldots C_n$. It is known that a finitely generated nilpotent group has bounded generation [36]. In the proof below we shall use a trivial observation that if group is boundedly generated by bounded cyclic subgroups then it is bounded.

Theorem 5.H. *Let N be a finitely generated nilpotent group. Then the commutator subgroup $[N, N]$ is bounded in N. Consequently, N satisfies the bq-dichotomy.*

In the proof of the theorem we will use the following observation. Its straightforward proof is left to the reader.

Lemma 5.I. *Let $K \lhd H < G$ be a sequence of groups such that K is normal in G. If K is bounded in G and every cyclic subgroup of H/K is bounded in G/K then every cyclic subgroup of H is bounded in G.*

Proof of Theorem 5.H. Let $N_i \subset N$ be the lower central series. That is $N_0 = N$, $N_1 = [N, N]$ and $N_{i+1} = [N, N_i]$. Since N is nilpotent $N_i = 0$ for $i > k$ and the last nontrivial term N_k is central.

Observe first that N_k is bounded in N. Let $x \in N$ and let $y \in N_{k-1}$. Then $z = [x, y] \in N_k$ is central and a direct calculation shows that $z^n = [x^n, y]$. Since N is finitely generated, we know that all N_i are finitely generated as well, according to Baer [26, page 232]. Now we have that N_k is finitely generated by products of commutators of the above form and, since N_k is abelian, these elements generate bounded (in N) cyclic subgroups. This implies that N_k is bounded in N as claimed.

The quotient series N_i/N_k is the lower central series for N/N_k and by the same argument as above we obtain that N_{k-1}/N_k is bounded in N/N_k. Applying Lemma 5.I to the diagram

$$\begin{tikzcd} N_k \arrow[r] \arrow[d] & N_k \arrow[d] \\ N_{k-1} \arrow[r] \arrow[d] & N \arrow[d] \\ N_{k-1}/N_k \arrow[r] & N/N_k \end{tikzcd}$$

we get that every cyclic subgroup in N_{k-1} is bounded in N. Again, this implies that N_{k-1} is bounded in N. Repeating this argument for N/N_{k-1} we obtain that N_{k-2} is bounded in N. The statement follows by induction. $\square$

5.J. **The bq-dichotomy for solvable groups.**

Theorem 5.K. *Let G be a finitely generated solvable group such that its commutator subgroup is finitely generated and nilpotent. Then the commutator subgroup $[G, G]$ is bounded in G. Consequently, it satisfies the bq-dichotomy.*

Proof. Let us first proof the statement under an additional assumption that G is metabelian (hence $[G, G]$ is a finitely generated abelian group). Let $x, y, t \in G$ and consider the element $[t, [x, y]] \in [G, [G, G]]$.

Observe that it generates a bounded subgroup in G because $[x,y]$ commutes with ${}^t[x,y]$ and we can apply Lemma 4.B. Since the subgroup $[G,[G,G]] \subset [G,G]$ is finitely generated abelian, it is boundedly generated by cyclic subgroups bounded in G and hence $[G,[G,G]]$ is bounded in G.

Consider the following diagram.

$$\begin{tikzcd}
G_2 = [G,[G,G]] \arrow[r] \arrow[d] & {[G,[G,G]]} \arrow[d] \\
G_1 = [G,G] \arrow[r] \arrow[d] & G \arrow[d] \\
G_1/G_2 \arrow[r] & G/G_2
\end{tikzcd}$$

Since $[G/G_2,[G/G_2,G/G_2]]$ is trivial, G/G_2 is metabelian and nilpotent. Hence, due to Theorem 5.H, we get that $G_1/G_2 = [G/G_2, G/G_2]$ is bounded in G/G_2. It then follows from Lemma 5.I that $[G,G]$ is bounded in G.

Let us prove the statement for a general G. Since the commutator subgroup $G^1 = [G,G]$ is finitely generated and nilpotent we have, according to Theorem 5.H, that $G^2 = [G^1,G^1]$ is bounded in G_1 and hence in G.

$$\begin{tikzcd}
G^2 = [G^1,G^1] \arrow[r] \arrow[d] & {[G^1,G^1]} \arrow[d] \\
G^1 = [G,G] \arrow[r] \arrow[d] & G \arrow[d] \\
G^1/G^2 \arrow[r] & G/G^2
\end{tikzcd}$$

Since G/G^2 is metabelian and $G^1/G^2 = [G/G^2, G/G^2]$ is finitely generated (because G^1 is) we have that G^1/G^2 is bounded in G/G^2, due to the first part of the proof. Again, by Lemma 5.I, we get that $[G,G]$ is bounded in G as claimed. $\square$

Remark 5.L. The integer lamplighter group $G = \mathbf{Z} \wr \mathbf{Z}$ is solvable and finitely generated but its commutator subgroup is abelian of infinite rank. The proof still works in this case because we have that $[G,[G,G]] = [G,G]$. We also showed it directly in Example 4.C (3).

5.M. **The bq-dichotomy for graph of groups.** For an introduction to graph of groups see Serre [35].

Lemma 5.N. *Let $\mathbf{A}$ be a graph of groups and let $G_{\mathbf{A}}$ be its fundamental group. Assume that $g \in G_{\mathbf{A}}$ is not conjugate to an element of the vertex group. Then g is either detectable by a homogenous quasimorphism or $\langle g\rangle$ is bounded with respect to the conjugation invariant norm.*

Proof. Consider the action of $G_{\mathbf{A}}$ on the Bass-Serre tree $T_{\mathbf{A}}$. The action of g on $T_{\mathbf{A}}$ does not have a fixpoint, for $G_{\mathbf{A}}$ acts on $T_{\mathbf{A}}$ without edge inversions and the stabilizers of vertices are conjugate to the vertex groups. Thus g acts by a hyperbolic isometry and it is automatically of rank-one. By Lemma 4.I, g is either detectable by a homogeneous quasimorphism, or it is conjugate to g^{-1}, hence $\langle g\rangle$ is bounded. $\square$

Let $H \subset G$ be a subgroup. We say that H satisfies the *relative bq-dichotomy (with respect to G)* if every cyclic subgroup of H is either bounded in G or it is detected by a homogeneous quasimorphism $q\colon G \to \mathbf{R}$. The following result is a straightforward application of the above lemma.

Theorem 5.O. *Let $\mathbf{A}$ be a graph of groups and let $G_{\mathbf{A}}$ be its fundamental group. If each vertex subgroup of $G_{\mathbf{A}}$ satisfies the relative bq-dichotomy then $G_{\mathbf{A}}$ satisfies the bq-dichotomy.* $\square$

Example 5.P. Baumslag-Solitar groups satisfy the bq-dichotomy. Indeed, Baumslag-Solitar groups are HNN extensions of the infinite cyclic group $\mathbf{Z}$. The graph of groups in this case has one vertex and one edge. By virtue of Example 4.C (1) the vertex group is bounded and we can apply Theorem 5.O.

Example 5.Q. The groups $\Gamma(\overline{m})$ (defined in Lemma 5.A) satisfy the bq-dichotomy. We keep the notation from Lemma 5.A. The group $\Gamma(\bar{m})$ is the fundamental group of the graph of groups associated with a rose with k petels. The vertex group is $\mathbf{Z}^{k+1}$ generated by $x_0, \ldots, x_k$ and the edge groups are cyclic generated by $x_i^{m_i}$. The elements x_i are detected by a homomorphism $h\colon \Gamma(\bar{m}) \to \mathbf{Z}$ defined by $h(t_i) = 0$ and $h(x_i) = a_i$. The kernel of this homomorphism is bounded, according to Lemma 5.A. Consequently, the bq-dichotomy follows from Theorem 5.O.

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