On tangent cones to Schubert varieties in type D_n

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1. Introduction and the main results

1.1. Let G be a complex reductive algebraic group, T a maximal torus in G, B a Borel subgroup in G containing T, and U the unipotent radical of B. Let Φ be the root system of G with respect to T, Φ^+ the set of positive roots with respect to B, Δ the set of simple roots, and W the Weyl group of Φ (see [Bo], [Hu1] and [Hu2] for basic facts about algebraic groups and root systems).

Denote by $\mathcal{F} = G/B$ the flag variety and by $X_w \subseteq \mathcal{F}$ the Schubert subvariety corresponding to an element w of the Weyl group W. Denote by $\mathcal{O} = \mathcal{O}_{p,X_w}$ the local ring at the point $p = eB \in X_w$. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$. The sequence of ideals

$$\mathcal{O} \supseteq \mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \ldots$$

is a filtration on $\mathcal{O}$. We define R to be the graded algebra

$$R = \operatorname{gr} \mathcal{O} = \bigoplus_{i \geq 0} \mathfrak{m}^i / \mathfrak{m}^{i+1}.$$

By definition, the *tangent cone* C_w to the Schubert variety X_w at the point p is the spectrum of R: $C_w = \operatorname{Spec} R$. Obviously, C_w is a subscheme of the tangent space $T_pX_w \subseteq T_p\mathcal{F}$. A hard problem in studying geometry of X_w is to describe C_w [BL, Chapter 7].

In 2011, D.Yu. Eliseev and A.N. Panov computed tangent cones C_w for all $w \in W$ in the case $G = \mathrm{SL}_n(\mathbb{C})$, $n \leq 5$ [EP]. Using their computations, A.N. Panov formulated the following Conjecture.

Conjecture 1.1. (A.N. Panov, 2011) *Let w_1, w_2 be involutions, i.e., $w_1^2 = w_2^2 = \mathrm{id}$. If $w_1 \neq w_2$, then $C_{w_1} \neq C_{w_2}$ as subschemes of $T_p\mathcal{F}$.*

One can easily check that it is enough to prove the Conjecture for irreducible root systems (see Remark 1.6 below). In 2013, D.Yu. Eliseev and the first author proved this Conjecture in types A_n, F_4 and G_2 [EI]. In [BIS], M.A. Bochkarev and the authors proved the Conjecture in types B_n and C_n. In this paper, we prove that the Conjecture is true if Φ is of type D_n and w_1, w_2 are basic involutions (see Definition 2.3). Precisely, our first main result is as follows.

Theorem 1.2. *Assume that every irreducible component of Φ is of type D_n , $n \geq 4$. Let w_1, w_2 be basic involutions in the Weyl group of Φ and $w_1 \neq w_2$. Then the tangent cones C_{w_1} and C_{w_2} do not coincide as subschemes of $T_p\mathcal{F}$.*

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Note that the similar question for other involutions in D_n and for the root systems E_6, E_7, E_8 remains open.

Now, let $\mathcal{A}$ be the symmetric algebra of the vector space $\mathfrak{m}/\mathfrak{m}^2$, or, equivalently, the algebra of regular functions on the tangent space T_pX_w. Since R is generated as $\mathbb{C}$-algebra by $\mathfrak{m}/\mathfrak{m}^2$, it is a quotient ring $R = \mathcal{A}/I$. By definition, the *reduced tangent cone* C_w^{red} to X_w at the point p is the common zero locus in T_pX_w of the polynomials $f \in I \subseteq \mathcal{A}$. Clearly, if $C_{w_1}^{\mathrm{red}} \neq C_{w_2}^{\mathrm{red}}$, then $C_{w_1} \neq C_{w_2}$. Our second main result is as follows.

Theorem 1.3. *Assume that every irreducible component of Φ is of type D_n, $n \geq 4$. Let w_1, w_2 be basic involutions in the Weyl group of Φ and $w_1 \neq w_2$. Then the reduced tangent cones $C_{w_1}^{\mathrm{red}}$ and $C_{w_2}^{\mathrm{red}}$ do not coincide as subvarieties of $T_p\mathcal{F}$.*

In [BIS], the similar result was obtained by M.A. Bochkarev for root systems of types A_n and C_n. Our proof for D_n is based on the similar idea. Note that the similar question for other involutions in D_n and for other root systems remains open.

The paper is organized as follows. In the next Subsection, we introduce the main technical tool used in the proof of Theorem 1.2. Namely, to each element $w \in W$ one can assign a polynomial d_w in the algebra of regular functions on the Lie algebra of the maximal torus T. These polynomials are called Kostant–Kumar polynomials [KK1], [KK2], [Ku], [Bi]. In [Ku] S. Kumar showed that if w_1 and w_2 are arbitrary elements of W and $d_{w_1} \neq d_{w_2}$, then $C_{w_1} \neq C_{w_2}$. We give three equivalent definitions of Kostant–Kumar polynomials and formulate their properties needed for the sequel.

In Section 2 we prove that if all irreducible components of Φ are of type D_n and w_1, w_2 are distinct basic involutions in W, then $d_{w_1} \neq d_{w_2}$, see Proposition 2.8. This implies that $C_{w_1} \neq C_{w_2}$ and proves Theorem 1.2. The proofs of Conjecture 1.1 for A_n, F_4, G_2, B_n and C_n presented in [EI] and [BIS] are based on the similar argument.

Section 3 contains the proof of Theorem 1.3. Namely, in Subsection 3.1 we describe connections of the geometry of tangent cones with the geometry of coadjoint B-orbits. Using these connections, in Subsection 3.2 we proof the result.

Of course, Theorem 1.2 is a corollary of Theorem 1.3. Nevertheless, we give in Section 2 an independent proof of the first Theorem based on computation of Kostant–Kumar polynomials. The reason is that we hope to prove Theorem 1.2 for all involutions in D_n using the same technique. At the contrary, there is no chance to prove Theorem 1.3 for non-basic involutions in D_n using arguments similar to presented in Section 3, see Remark 3.3 (ii) for the details.

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1.2. Let w be an element of the Weyl group W. Here we give precise definition of the Kostant–Kumar polynomial d_w, explain how to compute it in combinatorial terms, and show that it depends only on the scheme structure of C_w.

The torus T acts on the Schubert variety X_w by left multiplications (or, equivalently, by conjugations). The point p is invariant under this action, hence there is the structure of a T-module on the local ring $\mathcal{O}$. The action of T on $\mathcal{O}$ preserves the filtration by powers of the ideal $\mathfrak{m}$, so we obtain the structure of a T-module on the algebra $R = \operatorname{gr} \mathcal{O}$. By [Ku, Theorem 2.2], R can be decomposed into a direct sum of its finite-dimensional weight subspaces:

$$R = \bigoplus_{\lambda \in \mathfrak{X}(T)} R_\lambda.$$

Here $\mathfrak{h}$ is the Lie algebra of the torus T, $\mathfrak{X}(T) \subseteq \mathfrak{h}^*$ is the character lattice of T and $R_\lambda = \{f \in R \mid t.f = \lambda(t)f\}$ is the weight subspace of weight λ. Let Λ be the $\mathbb{Z}$-module consisting of all (possibly infinite) $\mathbb{Z}$-linear combinations of linearly independent elements e^λ, $\lambda \in \mathfrak{X}(T)$. The *formal character* of R is an element of Λ of the form

$$\operatorname{ch} R = \sum_{\lambda \in \mathfrak{X}(T)} m_\lambda e^\lambda,$$

where $m_\lambda = \dim R_\lambda$.

Now, pick an element $a = \sum_{\lambda \in \mathfrak{X}(T)} n_\lambda e^\lambda \in \Lambda$. Assume that there are finitely many $\lambda \in \mathfrak{X}(T)$ such that $n_\lambda \neq 0$. Given $k \geq 0$, one can define the polynomial

$$[a]_k = \sum_{\lambda \in \mathfrak{X}(T)} n_\lambda \cdot \frac{\lambda^k}{k!} \in S = \mathbb{C}[\mathfrak{h}].$$

Denote $[a] = [a]_{k_0}$, where k_0 is minimal among all non-negative numbers k such that $[a]_k \neq 0$. For instance, if $a = 1 - e^\lambda$, then $[a]_0 = 0$ and $[a] = [a]_1 = -\lambda$ (here we denote $1 = e^0$).

Let A be the submodule of Λ consisting of all finite linear combinations. It is a commutative ring with respect to the multiplication $e^\lambda \cdot e^\mu = e^{\lambda+\mu}$. In fact, it is just the group ring of $\mathfrak{X}(T)$. Denote the field of fractions of the ring A by $Q \subseteq \Lambda$. To each element of Q of the form $q = a/b$, a, $b \in A$, one can assign the element

$$[q] = \frac{[a]}{[b]} \in \mathbb{C}(\mathfrak{h})$$

of the field of rational functions on $\mathfrak{h}$. Note that this element is well-defined [Ku].

There exists an involution $q \mapsto q^*$ on Q defined by

$$e^\lambda \mapsto (e^\lambda)^* = e^{-\lambda}.$$

It turns out [Ku, Theorem 2.2] that the character $\operatorname{ch} R$ belongs to Q, hence $(\operatorname{ch} R)^* \in Q$, too. Finally, we put

$$c_w = [(\operatorname{ch} R)^*], \; d_w = (-1)^{l(w)} \cdot c_w \cdot \prod_{\alpha \in \Phi^+} \alpha.$$

Here $l(w)$ is the length of w in the Weyl group W with respect to the set of simple roots Δ. Evidently, c_w and d_w belong to $\mathbb{C}(\mathfrak{h})$; in fact, d_w is a polynomial, i.e., it belongs to the algebra $S = \mathbb{C}[\mathfrak{h}]$ of regular functions on $\mathfrak{h}$, see [KK2] and [BL, Theorem 7.2.6].

Definition 1.4. Let w be an element of the Weyl group W. The polynomial $d_w \in S$ is called the *Kostant–Kumar polynomial* associated with w.

It follows from the definition that c_w and d_w depend only on the canonical structure of a T-module on the algebra R of regular functions on the tangent cone C_w. Thus, to prove that the tangent cones corresponding to elements w_1, w_2 of the Weyl group are distinct, it is enough to check that $c_{w_1} \neq c_{w_2}$, or, equivalently, $d_{w_1} \neq d_{w_2}$.

On the other hand, there is a purely combinatorial description of Kostant–Kumar polynomials. To give this description, we need some more notation. Let w, v be elements of W. Fix a reduced decomposition of the element $w = s_{i_1} \dots s_{i_l}$. (Here $\alpha_1, \dots, \alpha_n \in \Delta$ are simple roots and s_i is the simple reflection corresponding to α_i.) Put

$$c_{w,v} = (-1)^{l(w)} \cdot \sum \frac{1}{s_{i_1}^{\epsilon_1}\alpha_{i_1}} \cdot \frac{1}{s_{i_1}^{\epsilon_1} s_{i_2}^{\epsilon_2}\alpha_{i_2}} \cdot \ldots \cdot \frac{1}{s_{i_1}^{\epsilon_1} \dots s_{i_l}^{\epsilon_l}\alpha_{i_l}},$$

where the sum is taken over all sequences $(\epsilon_1, \ldots, \epsilon_l)$ of zeroes and units such that $s_{i_1}^{\epsilon_1} \ldots s_{i_l}^{\epsilon_l} = v$. Actually, the element $c_{w,v} \in \mathbb{C}(\mathfrak{h})$ depends only on w and v, not on the choice of a reduced decomposition of w [Ku, Section 3].

Example 1.5. Let $\Phi = A_n$. Put $w = s_1 s_2 s_1$. To compute $c_{w,\mathrm{id}}$, we should take the sum over two sequences, $(0, 0, 0)$ and $(1, 0, 1)$. Hence

$$c_{w,\mathrm{id}} = (-1)^3 \cdot \left(\frac{1}{\alpha_1 \alpha_2 \alpha_1} + \frac{1}{-\alpha_1(\alpha_1 + \alpha_2)\alpha_1} \right) = \frac{1}{\alpha_1 \alpha_2 (\alpha_1 + \alpha_2)}.$$

A remarkable fact is that $c_{w,\mathrm{id}} = c_w$, hence to prove that the tangent cones to Schubert varieties do not coincide as subschemes, we need only combinatorics of the Weyl group. Note also that for classical Weyl groups, elements $c_{w,v}$ are closely related to Schubert polynomials [Bi].

Finally, we will present an original definition of elements $c_{w,v}$ using so-called nil-Hecke ring (see [Ku] and [BL, Section 7.1]). The group W naturally acts on $\mathbb{C}(\mathfrak{h})$ by automorphisms. Denote by Q_W the vector space over $\mathbb{C}(\mathfrak{h})$ with basis $\{\delta_w,\ w \in W\}$. It is a ring with respect to the multiplication

$$f\delta_v \cdot g\delta_w = fv(g)\delta_{vw}.$$

This ring is called the *nil-Hecke ring*. To each i from 1 to n put

$$x_i = \alpha_i^{-1}(\delta_{s_i} - \delta_{\mathrm{id}}).$$

Let $w \in W$ and $w = s_{i_1} \ldots s_{i_l}$ be a reduced decomposition of w. Then the element

$$x_w = x_{i_1} \ldots x_{i_l}$$

does not depend on the choice of a reduced decomposition of w [KK1, Proposition 2.1].

Moreover, it turns out that $\{x_w,\ w \in W\}$ is a $\mathbb{C}(\mathfrak{h})$-basis of Q_W [KK1, Proposition 2.2], and

$$x_w = \sum\nolimits_{v \in W} c_{w,v}\delta_v.$$

Actually, if $w, v \in W$, then

$$\begin{aligned} &\text{a) } x_v \cdot x_w = \begin{cases} x_{vw}, & \text{if } l(vw) = l(v) + l(w), \\ 0, & \text{otherwise}, \end{cases} \\ &\text{b) } c_{w,v} = -v(\alpha_i)^{-1}(c_{ws_i,v} + c_{ws_i,vs_i}),\ \text{if } l(ws_i) = l(w) - 1, \\ &\text{c) } c_{w,v} = \alpha_i^{-1}(s_i(c_{s_iw,s_iv}) - c_{s_iw,v}),\ \text{if } l(s_iw) = l(w) - 1. \end{aligned} \tag{1}$$

The first property is proved in [KK1, Proposition 2.2]. The second and the third properties follow immediately from the first one and the definitions (see also the proof of [Ku, Corollary 3.2]).

Remark 1.6. i) Suppose Φ is a union of its subsystems Φ_1 and Φ_2 contained in mutually orthogonal subspaces. Let W_1, W_2 be the Weyl groups of Φ_1, Φ_2 respectively, so $W = W_1 \times W_2$. Denote $\Delta_1 = \Delta \cap \Phi_1 = \{\alpha_1, \ldots, \alpha_r\}$ and $\Delta_2 = \Delta \cap \Phi_2 = \{\beta_1, \ldots, \beta_s\}$, then

$$\mathbb{C}[\mathfrak{h}] \cong \mathbb{C}[\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s].$$

Given $v \in W_1$, denote by d_v^1 its Kostant–Kumar polynomial. We can consider d_v^1 as an element of $\mathbb{C}(\mathfrak{h})$ depending only on $\alpha_1, \ldots, \alpha_r$. We define $c_v^1 \in \mathbb{C}(\mathfrak{h})$ by the similar way. Given $v \in W_2$, we define $d_v^2 \in \mathbb{C}[\mathfrak{h}]$ and $c_v^2 \in \mathbb{C}(\mathfrak{h})$; they depend only on $\beta_1, \ldots, \beta_s$. Let $w \in W$, $w_1 \in W_1$, $w_2 \in W_2$ and $w = w_1 w_2$. Repeating literally the proof of [EI, Proposition 1.6], we obtain the following:

$$d_w = d_{w_1}^1 d_{w_2}^2,\ c_w = c_{w_1}^1 c_{w_2}^2.$$

Thus, to prove Theorem 1.2, it suffice to check it for irreducible root systems of type D_n, because $\mathbb{C}[\mathfrak{h}]$ is a unique factorization domain.

ii) Now, let $G \cong G_1 \times G_2$, where G_1, G_2 are reductive subgroups of G, $T_i = T \cap G_i$ is a maximal torus in G_i, $i = 1, 2$, and the root system of G_i with respect to T_i is isomorphic to Φ_i. Then $B_i = B \cap G_i$ is a Borel subgroup in G_i containing T_i. Denote by $\mathcal{F}_i = G_i/B_i$ the corresponding flag variety. Then $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ and $T_p\mathcal{F} = T_p\mathcal{F}_1 \times T_p\mathcal{F}_2$ as algebraic varieties. If $w \in W$ and $w = w_1w_2$, $w_i \in W_i$, $i = 1, 2$, then $C_w^{\mathrm{red}} \cong C_{w_1,G_1}^{\mathrm{red}} \times C_{w_2,G_2}^{\mathrm{red}}$ as affine varieties. Here $C_{w_i,G_i}^{\mathrm{red}}$, $i = 1, 2$, denotes the tangent cone to the Schubert subvariety X_{w_i} of the flag variety $\mathcal{F}_i$. Furthermore, note that w is an involution if and only if w_1 and w_2 are involutions, too. This means that it suffice to prove that Theorem 1.3 holds for all irreducible root system of type D_n.

2. Non-reduced tangent cones

2.1. Throughout this Section, Φ denotes an irreducible root system of type D_n, $n \geq 4$. In this Subsection, we briefly recall some facts about Φ. Let $\epsilon_1, \ldots, \epsilon_n$ be the standard basis of the Euclidean space $\mathbb{R}^n$. As usual, we identify the set Φ^+ of positive roots with the following subset of $\mathbb{R}^n$:

$$D_n^+ = \{\epsilon_i - \epsilon_j,\ \epsilon_i + \epsilon_j,\ 1 \leq i < j \leq n\},$$

so W can be considered as a subgroup of the orthogonal group $O(\mathbb{R}^n)$.

Let $S_{\pm n}$ denote the symmetric group on $2n$ letters $1, \ldots, n, -n, \ldots, -1$. The Weyl group W is isomorphic to the *even-signed hyperoctahedral group*, that is, the subgroup of $S_{\pm n}$ consisting of permutations $w \in S_{\pm n}$ such that $w(-i) = -w(i)$ for all $1 \leq i \leq n$, and $\#\{i > 0 \mid w(i) < 0\}$ is even. The isomorphism is given by

$$\begin{aligned} s_{\epsilon_i - \epsilon_j} &\mapsto (i, j)(-i, -j), \\ s_{\epsilon_i + \epsilon_j} &\mapsto (i, -j)(-i, j). \end{aligned}$$

Here s_α is the reflection in the hyperplane orthogonal to a root α. In the sequel, we will identify W with the even-signed hyperoctahedral group.

Remark 2.1. i) Note that every $w \in W$ is completely determined by its restriction to the subset $\{1, \ldots, n\}$. This allows us to use the usual two-line notation: if $w(i) = w_i$ for $1 \leq i \leq n$, then we will write $w = \begin{pmatrix} 1 & 2 & \ldots & n \\ w_1 & w_2 & \ldots & w_n \end{pmatrix}$. For instance, if $\Phi = D_5$, then

$$s_{\epsilon_1+\epsilon_5} s_{\epsilon_2+\epsilon_4} s_{\epsilon_2-\epsilon_4} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -5 & -2 & 3 & -4 & -1 \end{pmatrix}.$$

ii) Note also that the set of simple roots has the following form: $\Delta = \{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_1 = \epsilon_1 - \epsilon_2, \ldots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n$, and $\alpha_n = \epsilon_{n-1} + \epsilon_n$.

We say that v is less or equal to w with respect to the *Bruhat order*, written $v \leq w$, if some reduced decomposition for v is a subword of some reduced decomposition for w. It is well-known that this order plays the crucial role in many geometric aspects of theory of algebraic groups. For instance, the Bruhat order encodes the incidences among Schubert varieties, i.e., X_v is contained in X_w if and only if $v \leq w$. It turns out that $c_{w,v}$ is non-zero if and only if $v \leq w$ [Ku, Corollary 3.2]. For example, $c_w = c_{w,\mathrm{id}}$ is non-zero for *any* w, because id is the smallest element of W with respect to the Bruhat order. Note that given $v, w \in W$, there exists $g_{w,v} \in S = \mathbb{C}[\mathfrak{h}]$ such that

$$c_{w,v} = g_{w,v} \cdot \prod_{\alpha > 0,\ s_\alpha v \leq w} \alpha^{-1}, \tag{2}$$

see [Dy] and [BL, Theorem 7.1.11]

There exists a nice combinatorial description of the Bruhat order on the even-signed hyperoctahedral group. Namely, given $w \in W$, denote by X_w the $2n \times 2n$ matrix of the form

$$(X_w)_{i,j} = \begin{cases} 1, & \text{if } w(j) = i, \\ 0 & \text{otherwise.} \end{cases}$$

The rows and the columns of this matrix are indicated by the numbers $1, \ldots, n, -n, \ldots, 1$. It is called the 0–1 matrix, permutation matrix or rook placement for w. Define the matrix R_w by putting its (i,j)th element to be equal to the rank of the lower left $(n-i+1) \times j$ submatrix of X_w. In other words, $(R_w)_{i,j}$ is just the number or rooks located non-strictly to the South-West from (i,j).

Example 2.2. Let $n = 4$, $w = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -2 & 4 & 1 & -3 \end{pmatrix}$. Here we draw the matrices X_w and R_w (rooks are marked by $\otimes$):

$X_w =$

	1	2	3	4	−4	−3	−2	−1
1			$\otimes$					
2								$\otimes$
3					$\otimes$			
4		$\otimes$						
−4							$\otimes$	
−3				$\otimes$				
−2	$\otimes$							
−1						$\otimes$		

, $R_w =$

	1	2	3	4	−4	−3	−2	−1
1	1	2	3	4	5	6	7	8
2	1	2	2	3	4	5	6	7
3	1	2	2	3	4	5	6	6
4	1	2	2	3	3	4	5	5
−4	1	1	1	2	2	3	4	4
−3	1	1	1	2	2	3	3	3
−2	1	1	1	1	1	2	2	2
−1	0	0	0	0	0	1	1	1

.

Let $w \in W$. Given $a, b \in \{1, 2, \ldots, n\}$, we say that $[-a, a] \times [-b, b]$ is an *empty rectangle* for w, if

$$\{i \in [\pm n] \mid |i| \geq b \text{ and } |w(i)| \geq a\} = \varnothing.$$

Here $[\pm n] = \{1, \ldots, n, -n, \ldots, -1\}$. For instance, in the previous example $[-4,4] \times [-3,3]$ and $[-4,4] \times [-4,4]$ are empty rectangles for w. Let X and Y be matrices with integer entries. We say that $X \leq Y$ if $X_{i,j} \leq Y_{i,j}$ for all i, j. It turns out that given v, $w \in W$, $v \leq w$ if and only if

i) $R_v \leq R_w$;

ii) for all $a, b \in \{1, \ldots, n\}$, if $[-a, a] \times [-b, b]$ is an empty rectangle for both v and w and $(R_v)_{-(a-1),b-1} = (R_w)_{-(a-1),b-1}$, then $(R_v)_{-(a-1),n} \equiv (R_w)_{-(a-1),n} \pmod 2$. (3)

(See, e.g., [BB, Theorem 8.2.8].)

2.2. In this Subsection, we introduce some more notation and prove technical, but crucial Lemma 2.7. We define the maps $\mathrm{row}\colon \Phi^+ \to \mathbb{Z}$ and $\mathrm{col}\colon \Phi^+ \to \mathbb{Z}$ by

$$\mathrm{row}(\epsilon_i - \epsilon_j) = j, \ \mathrm{row}(\epsilon_i + \epsilon_j) = -j,$$
$$\mathrm{col}(\epsilon_i - \epsilon_j) = \mathrm{col}(\epsilon_i + \epsilon_j) = i.$$

For any $k \in [\pm n]$, put

$$\mathcal{R}_k = \{\alpha \in \Phi^+ \mid \mathrm{row}(\alpha) = k\},$$
$$\mathcal{C}_k = \{\alpha \in \Phi^+ \mid \mathrm{col}(\alpha) = k\}.$$

The set $\mathcal{R}_k$ (resp. $\mathcal{C}_k$) is called the kth *row* (resp. the kth *column*) of Φ^+.

Definition 2.3. An involution $w \in W$ is called *basic*, if

$$\{i \in \{1, \ldots, n\} \mid w(i) = -i\} = \varnothing.$$

Definition 2.4. Let $\sigma \in W$ be a basic involution. We define the *support* $\mathrm{Supp}(\sigma)$ of the involution σ by the following rule:

$$\begin{aligned} &\text{if } 1 \leq i < j \leq n \text{ and } \sigma(i) = j, \text{ then } \epsilon_i - \epsilon_j \in \mathrm{Supp}(\sigma),\\ &\text{if } 1 \leq i < j \leq n \text{ and } \sigma(i) = -j, \text{ then } \epsilon_i + \epsilon_j \in \mathrm{Supp}(\sigma). \end{aligned}$$

By definition, $\mathrm{Supp}(\sigma)$ is an orthogonal subset of Φ^+. Note that

$$\sigma = \prod_{\beta \in \mathrm{Supp}(\sigma)} s_\beta,$$

where the product is taken in any fixed order. Note that for any k one has

$$|\mathrm{Supp}(\sigma) \cap \mathcal{C}_k| \leq 1, \ |\mathrm{Supp}(\sigma) \cap \mathcal{R}_k| \leq 1.$$

Note also that if w is not basic, then, in general, there are several different ways to define $\mathrm{Supp}(w)$, see Remark 3.3 (ii) below.

Example 2.5. Let $\Phi = D_6$ and $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ -6 & 2 & 5 & 4 & 3 & -1 \end{pmatrix}$. Then

$$\mathrm{Supp}(\sigma) = \{\epsilon_1 + \epsilon_6, \epsilon_3 - \epsilon_5\}.$$

Remark 2.6. i) Denote the set of involutions (resp. of basic involutions) by $I(W)$ (resp. by $B(W)$). By [Ig2, Proposition 2.3], if $\sigma, \tau \in I(W)$, then

$$R_\sigma \leq R_\tau \text{ if and only if } R^*_\sigma \leq R^*_\tau, \tag{4}$$

where R^*_w is the strictly lower-triangular part of R_w, i.e.,

$$(R^*_w)_{i,j} = \begin{cases} (R_w)_{i,j} & \text{if } i > j, \\ 0, & \text{if } i \leq j. \end{cases}$$

ii) Using Formulas (3) or (4), one can easily check that if $\alpha \in \mathcal{C}_1$ and $\beta \notin \mathcal{C}_1$, then $s_\alpha \nleq s_\beta$. One can also check that

$$s_{\epsilon_1 - \epsilon_2} < \ldots < s_{\epsilon_1 - \epsilon_n}, \ s_{\epsilon_1 + \epsilon_n} < \ldots < s_{\epsilon_1 + \epsilon_2}.$$

Further, $s_{\epsilon_1 - \epsilon_i} < s_{\epsilon_1 + \epsilon_j}$ for all $i, j \in \{1, \ldots, n\}$ such that $i < n$ or $j < n$, but $s_{\epsilon_1 - \epsilon_n} \not< s_{\epsilon_1 + \epsilon_n}$ and $s_{\epsilon_1 + \epsilon_n} \not< s_{\epsilon_1 - \epsilon_n}$.

The following Lemma plays the crucial role in the proof of Theorem 1.2 (cf. [EI, Lemmas 2.4, 2.5] and [BIS, Lemma 2.6]).

Lemma 2.7. *Let $w \in W$ be a basic involution. If $\mathrm{Supp}(w) \cap \mathcal{C}_1 = \varnothing$, then α divides d_w in the polynomial ring $\mathbb{C}[\mathfrak{h}]$ for all $\alpha \in \mathcal{C}_1$. If $\mathrm{Supp}(w) \cap \mathcal{C}_1 = \{\beta\}$, then β does not divide d_w in $\mathbb{C}[\mathfrak{h}]$.*

Proof. Denote by $\widetilde{W}$ the subgroup of W generated by $s_2, \ldots, s_n$. Suppose $\mathrm{Supp}(w) \cap \mathcal{C}_1 = \varnothing$, then $w \in \widetilde{W}$. Denote by $\widetilde{\Phi}$ the root system corresponding to $\widetilde{W}$; in fact, $\widetilde{\Phi}^+ = \Phi^+ \setminus \mathcal{C}_1$.

Let $\widetilde{d}_w \in \widetilde{S} = \mathbb{C}[\alpha_2, \ldots, \alpha_n]$ be the Kostant–Kumar polynomial of w considered as an element of $\widetilde{W}$; define $\widetilde{c}_w \in \mathbb{C}(\alpha_2, \ldots, \alpha_n)$ by the similar way. Since $\widetilde{W}$ is a parabolic subgroup of W, the length of w as an element of $\widetilde{W}$ equals the length of w as an element of W. Further, any reduced decomposition for w in $\widetilde{W}$ is a reduced decomposition for w in W. This means that $\widetilde{c}_w = c_w$, so

$$d_w = (-1)^{l(w)} \cdot \prod_{\alpha\in\Phi^+} \alpha \cdot c_w = (-1)^{l(w)} \cdot \prod_{\alpha\in\mathcal{C}_1} \alpha \cdot \prod_{\alpha\in\widetilde{\Phi}^+} \alpha \cdot \widetilde{c}_w = \widetilde{d}_w \cdot \prod_{\alpha\in\mathcal{C}_1} \alpha.$$

In particular, α divides d_w for all $\alpha \in \mathcal{C}_1$.

Now, suppose $\mathrm{Supp}(w) \cap \mathcal{C}_1 = \{\beta\}$. By [Hu2, Proposition 1.10], there exists a unique $v \in \widetilde{W}$ such that $w = uv$ and $l(us_i) = l(u) + 1$ for all $2 \le i \le n$ (or, equivalently, $u(\alpha_i) > 0$ for all $2 \le i \le n$). Furthermore, $l(w) = l(u) + l(v)$. One can easily check that

$$\begin{aligned}
&\text{if } \beta = \epsilon_1 - \epsilon_j \text{ (i.e., } w(1) = j\text{), then}\\
&u = s_{j-1} \ldots s_2 s_1\\
&= \begin{cases} \begin{pmatrix} 1 & 2 & 3 & \ldots & j-1 & j & j+1 & \ldots & n-1 & n \\ j & 1 & 2 & \ldots & j-2 & j-1 & j+1 & \ldots & n-1 & n \end{pmatrix}, & \text{if } j < n,\\ \begin{pmatrix} 1 & 2 & 3 & \ldots & n-1 & n \\ n & 1 & 2 & \ldots & n-2 & n-1 \end{pmatrix}, & \text{if } j = n, \end{cases}\\
&\text{if } \beta = \epsilon_1 + \epsilon_j \text{ (i.e., } w(1) = -j\text{), then}\\
&u = s_j s_{j+1} \ldots s_{n-1} s_n s_{n-2} s_{n-3} \ldots s_2 s_1\\
&= \begin{cases} \begin{pmatrix} 1 & 2 & 3 & \ldots & j-1 & j & j+1 & \ldots & n-1 & n \\ -j & 1 & 2 & \ldots & j-2 & j-1 & j+1 & \ldots & n-1 & -n \end{pmatrix}, & \text{if } j < n,\\ \begin{pmatrix} 1 & 2 & 3 & \ldots & n-1 & n \\ -n & 1 & 2 & \ldots & n-2 & -(n-1) \end{pmatrix}, & \text{if } j = n. \end{cases}
\end{aligned}$$

For instance, consider the case $\beta = \epsilon_1 + \epsilon_j$ (the case $\beta = \epsilon_1 - \epsilon_j$ can be considered similarly). Recall that W acts on $\mathbb{C}(\mathfrak{h})$ by automorphisms. Using (1) and arguing as in the proof of [EI, Lemma 2.5], one can easily show that

$$c_w = -\frac{c_{us_1,g_0} g_0(c_{v,g_0^{-1}})}{\beta} - \sum_{g\le u,\ g^{-1}\le v,\ g\ne g_0} \frac{c_{us_1,g} g(c_{v,g^{-1}})}{g(\alpha_1)} = \beta^{-1} \cdot g_0(c_{v,g_0^{-1}}) \cdot \frac{K}{L} + \frac{M}{N} \tag{5}$$

(cf. Formula (7) from [EI]). Here

$$\begin{aligned}
&g_0 = us_1 = s_j s_{j+1} \ldots s_{n-1} s_n s_{n-2} s_{n-3} \ldots s_2\\
&= \begin{cases} \begin{pmatrix} 1 & 2 & 3 & \ldots & j-1 & j & j+1 & \ldots & n-1 & n \\ 1 & -j & 2 & \ldots & j-2 & j-1 & j+1 & \ldots & n-1 & -n \end{pmatrix}, & \text{if } j < n,\\ \begin{pmatrix} 1 & 2 & 3 & \ldots & n-1 & n \\ 1 & -n & 2 & \ldots & n-2 & -(n-1) \end{pmatrix}, & \text{if } j = n, \end{cases}
\end{aligned}$$

and K, L and $M, N \in \mathbb{C}[\mathfrak{h}]$ are pairs of coprime polynomials such that β divides neither K nor N.

To prove that β does not divide d_w, it is enough to show that $c_{v,g_0^{-1}} \ne 0$, i.e., $v \ge g_0^{-1}$ (or, equivalently, $v^{-1} \ge g_0$). Arguing as in the proof of [BIS, Lemma 2.6], we obtain $R_{v^{-1}} \ge R_{g_0}$. Thus, it remains to check that the second condition in the definition of the Bruhat order is satisfied. Suppose that $[-a, a] \times [-b, b]$ is an empty rectangle for both v^{-1} and g_0 and $(R_{g_0})_{-(a-1),b-1} = (R_{v^{-1}})_{-(a-1),b-1}$. We must prove that $(R_{g_0})_{-(a-1),n} \equiv (R_{v^{-1}})_{-(a-1),n} \pmod 2$.

If $j < n$, then $g_0(n) = -n$, so there are no empty rectangles for g_0, hence $j = n$. In this case, $a = n$ and $b \geq 3$. For example, on the picture below we draw X_{g_0} for $n = 5$, $b = 4$. Entries from the empty rectangle $[-5, 5] \times [-4, 4]$ are grey.

	1	2	3	4	5	−5	−4	−3	−2	−1
1	⊗									
2			⊗							
3				⊗						
4						⊗				
5									⊗	
−5		⊗								
−4					⊗					
−3							⊗			
−2								⊗		
−1										⊗

Clearly, $(R_{g_0})_{-(n-1),b-1} = 0$, hence $(R_{v^{-1}})_{-(n-1),b-1} = 0$. At the same time, $(R_{g_0})_{-(n-1),n} = 1$, so we must check that $(R_{v^{-1}})_{-(n-1),n}$ is odd.

By definition,

$$(R_{v^{-1}})_{-(n-1),n} = \#\{i \in \{1, \ldots, n\} \mid v^{-1}(i) \in \{-1, \ldots, -(n-1)\}\}.$$

On the other hand, $v^{-1}(2) = wu(2) = w(1) = -n$, hence $\#\{i \in \{1, \ldots, n\} \mid v^{-1}(i) = -n\} = \#\{n\} = 1$. Since the number

$$(R_{v^{-1}})_{-(n-1),n} + 1 = \#\{i \in \{1, \ldots, n\} \mid v^{-1}(i) < 0\}$$

is even (by definition of W), we conclude that $(R_{v^{-1}})_{-(n-1),n}$ is odd, as required. □

2.3. Things now are ready for the proof of our first main result, Theorem 1.2. The proof immediately follows from the Proposition 2.8 below (cf. [EI, Propositions 2.6, 2.7, 2.8] and [BIS, Propositions 2.7, 2.8]). Our goal is to check that if σ, τ are distinct basic involutions in W, then their Kostant–Kumar polynomials do not coincide, and, consequently, the tangent cones C_σ and C_τ do not coincide as subschemes of $T_p\mathcal{F}$. We will proceed by induction on n (the base is trivial).

Proposition 2.8. *Let $\sigma, \tau \in W$ be distinct basic involutions. Then $d_\sigma \neq d_\tau$.*

Proof. If $\mathrm{Supp}(\sigma) \cap \mathcal{C}_1 \neq \mathrm{Supp}(\tau) \cap \mathcal{C}_1$, then one can repeat literally the proof of [BIS, Proposition 2.7] to obtain the result. Namely, if $\mathrm{Supp}(\sigma) \cap \mathcal{C}_1 = \{\beta\}$ and $\mathrm{Supp}(\tau) \cap \mathcal{C}_1 = \varnothing$, then β does not divide d_σ by the previous Lemma. But, thanks to formula (2), β divides d_τ, so $d_\sigma \neq d_\tau$. On the other hand, suppose that $\mathrm{Supp}(\sigma) \cap \mathcal{C}_1 = \beta$, $\mathrm{Supp}(\tau) \cap \mathcal{C}_1 = \beta'$, $\beta \not< \beta'$, then β divides d_τ (by formula (2)), but β does not divide d_σ (by the previous Lemma), so $d_\sigma \neq d_\tau$.

From now on, we may assume that $\mathrm{Supp}(\sigma) \cap \mathcal{C}_1 = \mathrm{Supp}(\tau) \cap \mathcal{C}_1$. If $\mathrm{Supp}(\sigma) \cap \mathcal{C}_1 = \mathrm{Supp}(\tau) \cap \mathcal{C}_1 = \varnothing$, then the inductive assumption completes the proof. Suppose $\mathrm{Supp}(\sigma) \cap \mathcal{C}_1 = \mathrm{Supp}(\tau) \cap \mathcal{C}_1 = \{\beta\}$. Let u be as in the proof of Lemma 2.7. There are two cases:

$$\text{i) } \beta = \epsilon_1 - \epsilon_j, \text{ i.e., } w(1) = j,$$
$$\text{ii) } \beta = \epsilon_1 + \epsilon_j, \text{ i.e., } w(1) = -j.$$

If $\beta = \epsilon_1 - \epsilon_j$, then one can repeat literally the proof of Case (i) of [BIS, Proposition 2.8], so we may assume that $\beta = \epsilon_1 + \epsilon_j$. Arguing as in the proof of Case (ii) of [BIS, Proposition 2.8], we obtain that

$$c_{v,g_0^{-1}} = c_{v_2,\mathrm{id}} \cdot \prod_{i=3}^{n-1} (\epsilon_2 - \epsilon_i)^{-1} \cdot \prod_{i=j+1}^{n-1} (\epsilon_2 + \epsilon_i)^{-1} \cdot (\epsilon_2 + \epsilon_n)^{-2}.$$

Here $w = aw_2a^{-1}$, $a = s_2s_3 \ldots s_{n-2}s_ns_{n-1} \ldots s_{j+1}s_j$, $w_2 = u_2v_2$, $\mathrm{Supp}(w_2) \cap \mathcal{C}_1 = \{\alpha_1\}$, $u_2 = s_1$, and $v_2 \in \widetilde{W}$ is an involution.

Now, consider the involutions σ and τ. Put $\sigma = uv_\sigma$, $\tau = uv_\tau$, where u is as above. Put also $\sigma = a\sigma_2a^{-1}$, $\tau = a\tau_2a^{-1}$, $\sigma_2 = u_2v_\sigma^2$, $\tau_2 = u_2v_\tau^2$, where $u_2 = s_1$. By the inductive assumption, $c_{v_\sigma^2,\mathrm{id}} \neq c_{v_\tau^2,\mathrm{id}}$, hence $c_{v_\sigma,g_0^{-1}} \neq c_{v_\tau,g_0^{-1}}$. Arguing as in the last two paragraphs of the proof of [EI, Proposition 2.8], one can conclude the proof.

Namely, one can easily deduce from formula (5) that if $c_\sigma = c_\tau$, then β divides $P_\sigma Q_\tau - P_\tau Q_\sigma$, where P_σ and Q_σ (resp. P_τ and Q_τ) are coprime polynomials such that $g_0(c_{v_\sigma,g_0^{-1}}) = P_\sigma/Q_\sigma$ (resp. $g_0(c_{v_\tau,g_0^{-1}}) = P_\tau/Q_\tau$). But these polynomials belong to the subalgebra of $\mathbb{C}[\mathfrak{h}]$ generated by $\alpha_2, \ldots, \alpha_n$, so $c_{v_\sigma,g_0^{-1}} = c_{v_\tau,g_0^{-1}}$, a contradiction. □

3. Reduced tangent cones

3.1. In this Section we will prove our second main result, Theorem 1.3. Throughout the Section, we will assume that every Φ is of type D_n, $n \geq 4$. In this Subsection, we briefly describe connections between tangent cones and coadjoint orbits of U, the unipotent radical of the Borel subgroup B.

Denote by $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{n}$ the Lie algebras of G, B, U respectively, then $T_p\mathcal{F}$ is naturally isomorphic to the quotient space $\mathfrak{g}/\mathfrak{b}$. Using the Killing form on $\mathfrak{g}$, one can identify the latter space with the dual space $\mathfrak{n}^*$. The group B acts on $\mathcal{F}$ by conjugation. Since p is B-stable, B acts on the tangent space $T_p\mathcal{F} \cong \mathfrak{n}^*$. This action is called *coadjoint*. We denote the result of coadjoint action by $b.\lambda$, $b \in B$, $\lambda \in \mathfrak{n}^*$. In 1962, A.A. Kirillov discovered that orbits of this action play an important role in representation theory of B and U, see, e.g., [Ki1], [Ki2]. We fix a basis $\{e_\alpha,\ \alpha \in \Phi^+\}$ of $\mathfrak{n}$ consisting of root vectors. Let $\{e_\alpha^*,\ \alpha \in \Phi^+\}$ be the dual basis of $\mathfrak{n}^*$. Let $w \in W$ be a basic involution. Put

$$f_w = \sum_{\beta \in \mathrm{Supp}(w)} e_\beta^* \in \mathfrak{n}^*.$$

Definition 3.1. We say that the B-orbit Ω_w and the U-orbit Θ_w of f_w are *associated* with the involution w.

One can easily check that $\Theta_w \subset \Omega_w \subseteq C_w^{\mathrm{red}}$. Further, C_w^{red} is B-stable (in fact, the tangent cone to an arbitrary Schubert variety is B-stable). Orbits associated with involutions were studied by A.N. Panov [Pa] and the second author [Ig1], [Ig2], [Ig3], [Ig4] (see also the Kostant's papers [Ko1], [Ko2], [Ko2] for the connections with the center of enveloping algebra of $\mathfrak{n}$). In particular, it was shown in [Ig3, Theorem 1.2] that

$$\dim \Theta_w = l(w) - |\mathrm{Supp}(w)|. \tag{6}$$

We need the following corollary of this fact (cf. [Ig1, Proposition 4.1] and [Ig2, Theorem 3.1]).

Lemma 3.2. *If $w \in W$ is a basic involution, then*

$$\dim \Omega_w = l(w). \tag{7}$$

Proof. Denote $D = \mathrm{Supp}(w)$. Let $\xi \colon D \to \mathbb{C}^\times$ be a map. Denote by $\Theta_{w,\xi}$ the U-orbit of the linear form

$$f_{w,\xi} = \sum_{\beta \in D} \xi(\beta) e_\beta^*.$$

In particular, $f_w = f_{w,\xi_0}$, where $\xi_0(\beta) = 1$ for all $\beta \in D$.

Without loss of generality, we can identify G with the group $\mathrm{SO}_{2n}(\mathbb{C})$ of all invertible $2n \times 2n$ matrices g of determinant 1 such that $g^tJg = J$, where J is the symmetric $2n \times 2n$ with 1's on the antidiagonal and 0's elsewhere. Then T (resp. B and U) is the group of all diagonal (resp. upper-triangular and upper-triangular with 1's on the diagonal) matrices from G. Moreover, $\mathfrak{g}$ is the algebra of $2n \times 2n$ matrices x of zero trace satisfying $x^tJ + Jx = 0$, and $\mathfrak{h}$ (resp. $\mathfrak{b}$ and $\mathfrak{n}$) is the algebra of all diagonal (resp. upper-triangular and upper-triangular with 0's on the diagonal) matrices from $\mathfrak{g}$. Using Killing form of $\mathfrak{g}$, one can identify $\mathfrak{n}^*$ with the space $\mathfrak{n}^t$ of all lower-triangular matrices from $\mathfrak{g}$ with 0's on the diagonal. Under this identification, the coadjoint action of B has a simple form

$$b.\lambda = (b\lambda b^{-1})_{\mathrm{low}},\ b \in B,\ \lambda \in \mathfrak{n}^*, \tag{8}$$

where A_{low} denotes the strictly lower-triangular part of a matrix A.

First, we claim that if $\xi_1 \neq \xi_2$, then $\Theta_{w,\xi_1} \neq \Theta_{w,\xi_2}$. Indeed, let $\widetilde{U}$ be the group of all $2n \times 2n$ upper-triangular matrices with 1's on the diagonal. This group acts on the space $\widetilde{\mathfrak{n}}$ of all upper-triangular $2n \times 2n$ matrices with 0's on the diagonal by the adjoint action, hence one can consider the dual (coadjoint) action of this group on the space $\widetilde{\mathfrak{n}}^*$. Using Killing form of $\mathfrak{gl}_{2n}(\mathbb{C})$, one can identify $\widetilde{\mathfrak{n}}^*$ with the space $\widetilde{\mathfrak{n}}^t$ of all lower-triangular $2n \times 2n$ matrices with 0's on the diagonal. Under this identification, the coadjoint action of $\widetilde{U}$ is given again by formula (8). Let $\widetilde{\Theta}_{w,\xi} \subset \widetilde{\mathfrak{n}}^*$ be the $\widetilde{U}$-orbit of $f_{w,\xi}$, then, clearly, $\Theta_{w,\xi} \subseteq \widetilde{\Theta}_{w,\xi}$ for any ξ. Since w is an involution in $S_{\pm n}$, it follows from [Pa, Theorem 1.4] that $\widetilde{\Theta}_{w,\xi_1} \neq \widetilde{\Theta}_{w,\xi_2}$. Thus, $\Theta_{w,\xi_1} \neq \Theta_{w,\xi_2}$, as required.

Second, we claim that $\Omega_w = \bigcup_\xi \Theta_{w,\xi}$, where the union is taken over all maps from D to $\mathbb{C}^\times$. Indeed, is is well-known that the *exponential map*

$$\exp\colon \mathfrak{n} \to U,\ x \mapsto \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

is an isomorphism of affine varieties. Given $\alpha \in \Phi^+$, $s \in \mathbb{C}^\times$, put

$$\begin{aligned} &x_\alpha(s) = \exp se_\alpha = 1 + se_\alpha,\ x_{-\alpha}(s) = x_\alpha(s)^t,\\ &w_\alpha(s) = x_\alpha(s)x_{-\alpha}(-s^{-1})x_\alpha(s),\ h_\alpha(s) = w_\alpha(s)w_\alpha(1)^{-1}. \end{aligned}$$

Note that $h_\alpha(s)$ belongs to T.

Let $\xi\colon D \to \mathbb{C}^\times$ be a map, $\alpha \in D$ be a root. To any number $s \in \mathbb{C}^\times$, denote by $\sqrt{s}$ a complex number such that $(\sqrt{s})^2 = s$. One can trivially check by direct matrix calculations that

$$h_\alpha(\sqrt{s}).f_{w,\xi} = s\xi(\alpha)e_{\alpha^*} + \sum_{\beta\in D,\ \beta\neq\alpha} \xi(\beta)e^*_\beta.$$

Thus,

$$\left(\prod_{\alpha\in D} h_\alpha\left(\sqrt{\xi(\alpha)}\right)\right).f_w = f_{w,\xi},$$

so $\Theta_{w,\xi} \subset \Omega_w$.

On the other hand, $B = U \rtimes T$ as algebraic groups. Since T is generated by $h_\alpha(s)$, $\alpha \in \Phi^+$, $s \in \mathbb{C}^\times$, we see that if $h \in T$, then $h.f_{w,\xi} = f_{w,\xi'}$ for some map $\xi'\colon D \to \mathbb{C}^\times$. Thus, if $g \in B$ and $g = uh$, $u \in U$, $h \in T$, then $g.f_w = u.f_{w,\xi}$ for some ξ, so $\Omega_w = \bigcup_\xi \Theta_{w,\xi}$, as required.

Third, let Z_B (resp. Z_U and Z_T) be the stabilizer of f_w under the coadjoint action of B (resp. of U and T). Then

$$\begin{aligned} \dim \Omega_w &= \dim B - \dim Z_B,\\ \dim \Theta_w &= \dim U - \dim Z_U. \end{aligned}$$

If $g = uh \in Z_B$, $u \in U$, $h \in T$, then

$$g.f_w = u.(h.f_w) = u.f_{w,\xi}$$

for some ξ. If $f_w \neq f_{w,\xi}$, then $\Theta_w \neq \Theta_{w,\xi}$. Hence $f_w = f_{w,\xi}$, so $h \in Z_T$ and $u \in Z_U$. It follows that the map

$$Z_U \times Z_T \to Z_B \colon (u, h) \mapsto uh$$

is an isomorphism of algebraic varieties, so

$$\dim Z_B = \dim Z_U + \dim Z_T.$$

Finally, it follows that $X = \bigcup_\xi \{f_{w,\xi}\}$ is the T-orbit of f_w (the union is taken over all maps from D to $\mathbb{C}^\times$). Thus, using (6), we conclude that

$$\begin{aligned}\dim \Omega_w &= \dim B - \dim Z_B \\ &= \dim U + \dim T - \dim Z_U - \dim Z_T \\ &= \dim \Theta_w + \dim X = l(w) - |D| + |D| = l(w).\end{aligned}$$

The proof is complete. □

Remark 3.3. i) Since $\dim C_w^{\text{red}} = \dim X_w = l(w)$, we conclude that $\overline{\Omega}_w$, the closure of Ω_w, is an irreducible component of C_w^{red} of maximal dimension. (In fact, C_w^{red} is equidimensional.)

ii) If w is not basic, then there are several different ways to define $\text{Supp}(w)$. For example, if $n = 4$ and $w = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & -3 & -4 \end{pmatrix}$, then there are three subsets $D \subset \Phi^+$ such that $w = \prod_{\beta \in D} s_\beta$:

$$\begin{aligned}&\{\epsilon_1 - \epsilon_2, \epsilon_1 + \epsilon_2, \epsilon_3 - \epsilon_4, \epsilon_3 + \epsilon_4\},\\ &\{\epsilon_1 - \epsilon_3, \epsilon_1 + \epsilon_3, \epsilon_2 - \epsilon_4, \epsilon_2 + \epsilon_4\},\\ &\{\epsilon_1 - \epsilon_4, \epsilon_1 + \epsilon_4, \epsilon_2 - \epsilon_3, \epsilon_2 + \epsilon_3\}.\end{aligned}$$

(For some reasons, the first candidate is "the best", see [De], [S].)

So, one can define Θ_w and Ω_w using one of the definitions of the support of w. But there is no chance that formula (7) holds for all non-basic involutions. Indeed, one can repeat literally the proof of the previous Lemma to obtain $\dim \Omega_w = \dim \Theta_w + |\text{Supp}(w)|$. But if w is not basic, then the dimension of $\dim \Theta_w$ can be strictly less than $l(w) - |\text{Supp}(w)|$, see [Ig3, Theorem 1.2]. That's why we restrict our attention to the case of basic involutions.

Now, assume that G' is a reductive subgroup of G'', T' (resp. T'') is a maximal torus of G' (resp. of G''), $T' = T'' \cap G'$, B' (resp. B'') is a Borel subgroup of G' (resp. of G'') containing T' (resp. T''), $B' = B'' \cap G'$, and Φ' (resp. Φ'') is the root system of G' (resp. of G'') with respect to T' (resp. to T''). We denote by W' (resp. by W'') the Weyl group of Φ' (resp. of Φ''). Denote by $\mathcal{F}' = G'/B'$, $\mathcal{F}'' = G''/B''$ the flag varieties. Put $p' = eB' \in \mathcal{F}'$, $p'' = eB'' \in \mathcal{F}''$. Let U' (resp. U'') be the unipotent radical of B' (resp. of B''), $U' = U'' \cap B'$. Denote also by $\mathfrak{g}'$, $\mathfrak{b}'$, $\mathfrak{n}'$ the Lie algebras of G', B', U' respectively. Define $\mathfrak{g}''$, $\mathfrak{b}''$, $\mathfrak{n}''$ by the similar way. One can consider the dual space $\mathfrak{n}'^* \cong \mathfrak{g}'/\mathfrak{b}'$ as a subspace of $\mathfrak{n}''^* \cong \mathfrak{g}''/\mathfrak{b}''$. Hence we can consider $T_{p'}\mathcal{F}'$ as a subspace of $T_{p''}\mathcal{F}''$.

Pick involutions $w_1, w_2 \in W'$. Let C_i' be the reduced tangent cone at the point p' to the Schubert subvariety X_{w_i}' of the flag variety $\mathcal{F}'$, $i = 1, 2$. Similarly, let C_i'' be the reduced tangent cone at p'' to the Schubert subvariety X_{w_i}'' of $\mathcal{F}''$, $i = 1, 2$. Denote by l' (resp. by l'') the length function on the Weyl group W' (resp. on W''). Assume $C_1' = C_2'$. This implies that

$$l'(w_1) = l'(w_2).$$

Note that $C'_i \subseteq C''_i$, hence $B''.C'_i \subseteq C''_i$, $i = 1, 2$. Denote by $\Omega'_{w_i} \subseteq \mathfrak{n}'^*$ the coadjoint B'-orbit associated with the involution w_i, $i = 1, 2$; define Ω''_{w_i} by the similar way. It follows from formula (7) that

$$\begin{aligned} l''(w_i) &= \dim C''_i \geq \dim B''.C'_i \geq \dim B''.\Omega'_{w_i} \\ &= \dim \Omega''_{w_i} = l''(w_i), \end{aligned}$$

because $\Omega''_{w_i} = B''.\Omega'_{w_i}$. This implies $l''(w_i) = \dim C''_i = \dim B''.C'_i$. But $C'_1 = C'_2$, thus $\dim C''_1 = \dim C''_2$. We obtain the following result:

$$\text{if } C'_1 = C'_2, \text{ then } l''(w_1) = l''(w_2). \tag{9}$$

3.2. In this Subsection, we prove Theorem 1.3: if w_1, w_2 are basic involutions in the Weyl group W of type D_n, $n \geq 4$, and $w_1 \neq w_2$, then $C^{\mathrm{red}}_{w_1} \neq C^{\mathrm{red}}_{w_2}$ as subvarieties of $T_p\mathcal{F}$. Let W'' be of type D_{n+2}. Let

$$D^+_{n+2} = \{\eta_i - \eta_j,\ \eta_i + \eta_j,\ 1 \leq i < j \leq n+2\},$$

where $\{\eta_i\}_{i=1}^{n+2}$ is the standard basis of $\mathbb{R}^{n+2}$. Pick numbers k_1, k_2 such that $1 \leq k_1 < k_2 \leq n+2$. Put $P = \{k_1, k_2\}$, $Q = \{1, \ldots, n+2\} \setminus P$, and

$$\begin{aligned} \widetilde{W} &= \{w \in W'' \mid w(i) = i \text{ for all } i \in P\}, \\ \widetilde{W}_2 &= \{w \in W'' \mid w(i) = i \text{ for all } i \in Q\}, \\ W' &= \{w \in W'' \mid w(P) = P,\ w(Q) = Q\} = \widetilde{W} \times \widetilde{W}_2. \end{aligned}$$

Let Φ' (resp. $\widetilde{\Phi}$) be the root system of W' (resp. of $\widetilde{W}$). Clearly, Φ' (resp. $\widetilde{\Phi}$) is of type $D_n \times A_1 \times A_1$ (resp. of type D_n). Put $G'' = \mathrm{SO}_{2n+4}(\mathbb{C})$ and denote by G' (resp. by $\widetilde{G}$) the subgroup of G corresponding to Φ' (resp. to $\widetilde{\Phi}$), then $G' \cong \mathrm{SO}_n(\mathbb{C}) \times \mathrm{SO}_2(\mathbb{C})$. Put also

$$\begin{aligned} A &= \{1, \ldots, k_1 - 1\}, \\ B &= \{k_1 + 1, \ldots, k_2 - 1\}, \\ C &= \{k_2 + 1, \ldots, n+2\}. \end{aligned}$$

Now, let $\Phi = D_n$. We can assume without loss of generality that $G = \mathrm{SO}_n(\mathbb{C})$. We identify Φ with $\widetilde{\Phi}$ by the map $\epsilon_k \mapsto \eta_{k'}$, where

$$k' = \begin{cases} k, & \text{if } k \leq k_1 - 1, \\ k+1, & \text{if } k_1 \leq k \leq k_2 - 2, \\ k+2, & \text{if } k_2 - 1 \leq k \leq n. \end{cases}$$

This identifies G (resp. W) with $\widetilde{G}$ (resp. with $\widetilde{W}$). We denote the image in $\widetilde{W}$ of an element $w \in W$ under this identification by $\widetilde{w}$. Let $w \in W$ be an involution. Arguing as in the proof of [BIS, Lemma 3.2], we obtain the following result.

Lemma 3.4. i) *If $w' = \widetilde{w} s_{\eta_{k_1} - \eta_{k_2}}$, then the length of w' in the Weyl group W'' equals*

$$l''(w') = 2(k_2 - k_1 - 1) + 4|\widetilde{w}(A) \cap B^-| + 4|\widetilde{w}(A) \cap A^-| + 4|\widetilde{w}(A) \cap C^{\pm}| + l(w) + 1.$$

ii) *If $w' = \widetilde{w} s_{\eta_{k_1} + \eta_{k_2}}$, then*

$$l''(w') = 2(k_2 - k_1 - 1) + 4|\widetilde{w}(A) \cap A^-| + 4|\widetilde{w}(A) \cap B^-| + 4|C| + l(w) + 1.$$

(By a slight abuse of notation, here we consider $\widetilde{w}$ as an element of $S_{\pm(n+2)}$ and, at the same time, as an element of $\widetilde{W}$, i.e., as an element of W'' such that $\widetilde{w}(k_1) = k_1$ and $\widetilde{w}(k_2) = k_2$.)

Proof of Theorem 1.3. Assume $C_{w_1}^{\mathrm{red}} = C_{w_2}^{\mathrm{red}}$. In particular,

$$l(w_1) = \dim C_{w_1}^{\mathrm{red}} = \dim C_{w_2}^{\mathrm{red}} = l(w_2).$$

Since $w_1 \neq w_2$, there exists $1 \leq k \leq n$ such that $w_1(\epsilon_i) = w_2(\epsilon_i)$ for $1 \leq i \leq k-1$, and $w_1(\epsilon_k) \neq w_2(\epsilon_k)$. Assume without loss of generality that $w_1(\epsilon_k) < w_2(\epsilon_k)$, i.e., $w_2(\epsilon_k) - w_1(\epsilon_k)$ is a sum of positive roots. Note that $w_1(\epsilon_k) \neq \pm\epsilon_k$, because $w_1(\epsilon_i) = w_2(\epsilon_i)$ for all i from 1 to $k-1$. Put $k_1 = k+1$, so $A = \{1, \ldots, k\}$ and $\widetilde{w}_1(a) = \widetilde{w}_2(a)$ for all $a \in A \setminus \{k\}$. We consider three different cases.

i) Suppose $w_1(\epsilon_k) < 0$, $w_2(\epsilon_k) > 0$. Here we put $k_2 = n+2$, so $C = \varnothing$ and

$$(\widetilde{w}_i(A) \cap A^-) \cup (\widetilde{w}_i(A) \cap B^-) = \widetilde{w}_i(A) \cap \{-1, \ldots, -(n+2)\},\ i = 1, 2.$$

Let $w_i' = \widetilde{w}_i s_{\eta_{k_1} - \eta_{k_2}}$, $i = 1, 2$. Since

$$\widetilde{w}_1(A) \cap \{-1, \ldots, -(n+2)\} = \widetilde{w}_2(A) \cap \{-1, \ldots, -(n+2)\} \cup \{k\},$$

Lemma 3.4 (i) shows that $l''(w_1') \neq l''(w_2')$. On the other hand, $C_{w_1}^{\mathrm{red}} = C_{w_2}^{\mathrm{red}}$ implies $C_1' = C_2'$, which contradicts (9).

ii) Next, suppose $w_1(\epsilon_k) = \epsilon_{m_1} > 0$, $w_2(\epsilon_k) = \epsilon_{m_2} > 0$. Note that $m_1 > m_2 \geq k$, because $w_1(\epsilon_k) < w_2(\epsilon_k)$ and $w_1(\epsilon_i) = w_2(\epsilon_i)$ for all i from 1 to $k-1$. Here we put $k_2 = m_1 + 1$, so $\widetilde{w}_1(k) \in C$ and $\widetilde{w}_2(k) \in B$. By Lemma 3.4 (i), $l''(w_1') \neq l''(w_2')$, where $w_i' = \widetilde{w}_i s_{\eta_{k_1} - \eta_{k_2}}$, $i = 1, 2$. But $C_1' = C_2'$, a contradiction.

iii) Finally, suppose $w_1(\epsilon_k) = -\epsilon_{m_1} < 0$, $w_2(\epsilon_k) = -\epsilon_{m_2} < 0$. Note that $m_2 > m_1 > k$, because $w_1(\epsilon_k) < w_2(\epsilon_k)$ and $w_1(\epsilon_i) = w_2(\epsilon_i)$ for all i from 1 to $k-1$. Here we put $k_2 = m_2 + 1$, so $\widetilde{w}_1(k) \in B^-$ and $\widetilde{w}_2(k) \in C^-$. By Lemma 3.4 (ii), $l''(w_1') \neq l''(w_2')$, where $w_i' = \widetilde{w}_i s_{\eta_{k_1} + \eta_{k_2}}$, $i = 1, 2$. On the other hand, $C_1' = C_2'$. This contradicts (9). The result follows. □

Remark 3.5. Actually, for $\Phi = B_n$, one can introduce the notion of basic involution literally as for D_n. It is easy to check that the previous proposition is true for basic involutions in type B_n.

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