The R_∞ and S_∞ properties for linear algebraic groups

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Abstract

In this paper we study twisted conjugacy classes and isogredience classes for automorphisms of reductive linear algebraic groups. We show that reductive linear algebraic groups over some fields of zero characteristic possess the R_∞ and S_∞ properties.

Keywords: R_∞-property, S_∞-property, linear algebraic groups, Reidemeister number.

1 Introduction

Let $\varphi : G \to G$ be an endomorphism of a group G. Then two elements x, y of G are said to be twisted φ-conjugate, if there exists a third element $z \in G$ such that $x = zy\varphi(z)^{-1}$. The equivalence classes are called the twisted conjugacy classes or the Reidemeister classes of φ. The Reidemeister number of φ denoted by $R(\varphi)$, is the number of those twisted conjugacy classes of φ. This number is either a positive integer or ∞ and we do not distinguish different infinite cardinal numbers. An infinite group G has the R_∞-property if for every automorphism φ of G the Reidemeister number of φ is infinite.

The interest in twisted conjugacy relations has its origins, in particular, in the Nielsen-Reidemeister fixed point theory (see, e.g. [7, 33]), in Arthur- Selberg theory (see, eg. [1, 49]), in algebraic geometry (see, e.g. [28]), in Galois cohomology [48] and in the theory of linear algebraic groups (see, e.g. [51]). In representation theory twisted conjugacy probably occurs first in Gantmacher's paper [20] (see, e.g [50, 44])

The problem of determining which classes of discrete infinite groups have the- R_∞ property is an area of active research initiated by Fel'shtyn and Hill in 1994

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[9]. Later, it was shown by various authors that the following groups have the R_∞-property: non-elementary Gromov hyperbolic groups [8, 38]; relatively hyperbolic groups [12]; Baumslag-Solitar groups $BS(m,n)$ except for $BS(1,1)$ [13], generalized Baumslag-Solitar groups, that is, finitely generated groups which act on a tree with all edge and vertex stabilizers infinite cyclic [37]; the solvable generalization Γ of $BS(1,n)$ given by the short exact sequence $1 \to \mathbb{Z}[\frac{1}{n}] \to \Gamma \to \mathbb{Z}^k \to 1$, as well as any group quasi-isometric to Γ [52]; a wide class of saturated weakly branch groups (including the Grigorchuk group [27] and the Gupta-Sidki group [29]) [11], Thompson's groups F [2] and T [3, 21]; generalized Thompson's groups $F_{n,0}$ and their finite direct products [23]; Houghton's groups [22, 34]; symplectic groups $\mathrm{Sp}(2n,\mathbb{Z})$, the mapping class groups Mod_S of a compact oriented surface S with genus g and p boundary components, $3g+p-4>0$, and the full braid groups $B_n(S)$ on $n>3$ strands of a compact surface S in the cases where S is either the compact disk D, or the sphere S^2 [14]; some classes of Artin groups of infinite type [35]; extensions of $\mathrm{SL}(n,\mathbb{Z})$, $\mathrm{PSL}(n,\mathbb{Z})$, $\mathrm{GL}(n,\mathbb{Z})$, $\mathrm{PGL}(n,\mathbb{Z})$, $\mathrm{Sp}(2n,\mathbb{Z})$, $\mathrm{PSp}(2n,\mathbb{Z})$, $n>1$, by a countable abelian group, and normal subgroups of $\mathrm{SL}(n,\mathbb{Z})$, $n>2$, not contained in the center [40]; $\mathrm{GL}(n,K)$ and $\mathrm{SL}(n,K)$ if $n>2$ and K is an infinite integral domain with trivial group of automorphisms, or K is an integral domain, which has a zero characteristic and for which $\mathrm{Aut}(K)$ is periodic [42]; irreducible lattices in a connected semisimple Lie group G with finite center and real rank at least 2 [41]; non-amenable, finitely generated residually finite groups [17] (this class gives a lot of new examples of groups with the R_∞-property); some metabelian groups of the form $\mathbb{Q}^n \rtimes \mathbb{Z}$ and $\mathbb{Z}[1/p]^n \rtimes \mathbb{Z}$ [15]; lamplighter groups $\mathbb{Z}_n \wr \mathbb{Z}$ if and only if $2|n$ or $3|n$ [25]; free nilpotent groups N_{rc} of rank $r=2$ and class $c \geqslant 9$ [26], N_{rc} of rank $r=2$ or $r=3$ and class $c \geq 4r$, or rank $r \geq 4$ and class $c \geq 2r$, any group N_{2c} for $c \geq 4$, every free solvable group S_{2t} of rank 2 and class $t \geq 2$ (in particular the free metabelian group $M_2 = S_{22}$ of rank 2), any free solvable group S_{rt} of rank $r \geq 2$ and class t big enough [46]; some crystallographic groups [6, 39]. Recently, in [5] it was proven that N_{rc}, $r>1$ has the R_∞-property if and only if $c \geq 2r$.

Let Ψ belongs to $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$. We consider an outer automorphism $\Psi \in \mathrm{Out}(G)$ as a collection of ordinary automorphisms $a \in \mathrm{Aut}(G)$. We say that two automorphisms $a, b \in \Psi$ are *similar* (or *isogredient*) if $b = \varphi_h a \varphi_h^{-1}$ for some $h \in G$, where $\varphi_h(g) = hgh^{-1}$ an inner automorphism induced by the element h (see [38]). Let $\mathfrak{S}(\Psi)$ be the set of isogredience classes of automorphisms representing Ψ. Denote by $S(\Psi)$ the cardinality of the set $\mathfrak{S}(\Psi)$. A group G is called an S_∞-*group* if for any Ψ the set $\mathfrak{S}(\Psi)$ is infinite, i. e. $S(\Psi)=\infty$ (see [18]).

In this paper we study the R_∞ and S_∞ properties for linear algebraic groups. First results in this direction were obtained for some classes of Chevalley groups by Nasybullov in [43].

In Section 3 we extend the previous result from [43] and prove

Theorem 2. *Let G be a Chevalley group of the type Φ over the field F of zero* *characteristic. If the transcendence degree of F over $\mathbb{Q}$ is finite, then G possesses the R_∞-property.*

The following main theorem is proved in Section 4.

Theorem 3. *Let F be such an algebraically closed field of zero characteristic that the transcendence degree of F over $\mathbb{Q}$ is finite. If the reductive linear algebraic group G over the field F has a nontrivial quotient group $G/R(G)$, where $R(G)$ is the radical of G, then G possesses the R_∞-property.*

These theorems can not be generalized to groups over a field of non-zero characteristic. It follows from the following theorem of Steinberg [51, Theorem 10.1].

Theorem. *Let G be a connected linear algebraic group and φ be an endomorphism of G onto G. If φ has a finite set of fixed points, then $G = \{x\varphi(x^{-1}) \mid x \in G\}$.*

We would like to point out that R. Steinberg [51] calls by an automorphism of a linear algebraic group a bijective endomorphism which is a morphism and its inverse is a morphism too. However, throughout the paper we understand an automorphism of a linear algebraic group as an automorphism of an abstract group, i. e. a bijective endomorphism of a group.

Any semisimple linear algebraic group over an algebraically closed field of positive characteristic possesses an automorphism φ with finitely many fixed points (Frobenius morphism, see [47, §3.2]), therefore, this group coincides with the set $\{x\varphi(x^{-1}) \mid x \in G\} = [e]_\varphi$, hence $R(\varphi) = 1$ and such a group can not have the R_∞-property.

If $T_1, T_2, \ldots$ are algebraically independent over $\mathbb{Q}$ elements, then the fields $\overline{\mathbb{Q}}$, $\overline{\mathbb{Q}(T_1, \ldots, T_k)}$ ($k \geq 1$) are algebraically closed fields of zero characteristic with finite transcendence degree over $\mathbb{Q}$. Then the reductive linear algebraic groups over these fields possess the R_∞-property.

In the Section 5 we prove that an infinite reductive linear algebraic group G over the field F of zero characteristic and finite transcendence degree over $\mathbb{Q}$ which possesses an automorphism φ with a finite Reidemeister number is a torus.

In the Section 6 we prove the following

Theorem 5. *Let F be such an algebraically closed field of zero characteristic that the transcendence degree of F over $\mathbb{Q}$ is finite. If the reductive linear algebraic group G over the field F has a nontrivial quotient group $G/R(G)$, then G possesses the S_∞-property.*

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2 Preliminaries

In this paragraph we recall some preliminary statements which are used in the paper. A lot of used results are thoroughly written in [43], the reader can use it as a background material.

Symbols I_n and $O_{n\times m}$ mean the identity $n\times n$ matrix and the $n\times m$ matrix with zero entries, respectively. If A an $n\times n$ matrix and B an $m\times m$ matrix, then the symbol $A\oplus B$ denotes the direct sum of the matrices A and B, i. e. the block-diagonal $(m+n)\times(m+n)$ matrix

$$\left(\begin{array}{c|c} A & O_{n\times m} \\ \hline O_{m\times n} & B \end{array}\right).$$

It is obvious that for a pair of $n\times n$ matrices A_1, A_2 and for a pair of $m\times m$ matrices B_1, B_2 we have $(A_1\oplus B_1)(A_2\oplus B_2)=A_1A_2\oplus B_1B_2$, $(A_1\oplus B_1)^{-1}=A_1^{-1}\oplus B_1^{-1}$.

Symbols $G\times H$ and $G\circ H$ mean the direct product and the central product of the groups G and H, respectively.

If g is an element of the group G, then φ_g denotes an inner automorphism induced by the element g. The following lemma can be found in [16, Corollary 2.5].

Lemma 1. *Let φ be an automorphism of the group G and φ_g be an inner automorphism of the group G. Then $R(\varphi\varphi_g)=R(\varphi)$.*

The next lemma is proved in [40, Lemma 2.1]

Lemma 2. *Let*

$$1\to N\to G\to A\to 1$$

be an exact sequence of groups. Suppose that N is a characteristic subgroup of G and that A possesses the R_∞-property, then G also possesses the R_∞-property.

Here we prove similar result for the S_∞-property.

Lemma 3. *Let*

$$1\to N\to G\to A\to 1$$

be an exact sequence of groups. Suppose that N is a characteristic subgroup of G and that A possesses the S_∞-property, then G also possesses the S_∞-property.

Proof. Let φ be an automorphism of the group G. Since N is a characteristic subgroup of G then φ induces an automorphism $\overline{\varphi}$ of the group A. Since the group A has the S_∞-property then there exists an infinite set of elements $\overline{g}_1,\overline{g}_2,\dots$ of the group A such that $\varphi_{\overline{g}_i}\overline{\varphi}$ and $\varphi_{\overline{g}_j}\overline{\varphi}$ are not isogredient for $i\neq j$.

Suppose that $S(\varphi \mathrm{Inn}(G)) < \infty$. Then there exists a pair of isogredient automorphisms in the set $\varphi_{g_1}\varphi, \varphi_{g_2}\varphi, \dots$. Suppose that $\varphi_{g_i}\varphi$ and $\varphi_{g_j}\varphi$ are isogredient for $i \neq j$. Then for some element $h \in G$ we have

$$\varphi_{g_i}\varphi = \varphi_h \varphi_{g_j} \varphi \varphi_h^{-1}.$$

From this equality we have the following equality in the group $\mathrm{Aut}(A)$

$$\varphi_{\overline{g}_i}\overline{\varphi} = \varphi_{\overline{h}} \varphi_{\overline{g}_j} \overline{\varphi} \varphi_{\overline{h}}^{-1},$$

but it contradicts to the choice of the elements $\overline{g}_1, \overline{g}_2, \dots$ □

Let ν be a map from the set of rational numbers $\mathbb{Q}$ to the set 2^π of all subsets of the set of prime numbers π, which acts on the irreducible fraction $x = a/b$ by the rule

$$\nu(x) = \{\textit{all the prime devisors of } a\} \cup \{\textit{all the prime devisors of } b\}.$$

The proof of the following lemma is presented in [43, Lemma 5].

Lemma 4. *Let F be a field of zero characteristic and $x_1, \dots, x_k$ be elements of F which are algebraically independent over the field $\mathbb{Q}$. Let x_{k+1} be such an element of F, that the elements $x_1, \dots, x_{k+1}$ are algebraically dependent over $\mathbb{Q}$. Let δ be an automorphism of the field F which acts on this elements by the rule*

$$\delta : x_i \mapsto t_0 t_i x_i, \quad i = 1, \dots, k+1,$$

where $t_0, \dots, t_{k+1} \in \mathbb{Q}$ and $t_1, \dots, t_{k+1}$ are not equal to 1. If $\nu(t_i) \cap \nu(t_j) = \varnothing$ for $i \neq j$, then $x_{k+1} = 0$.

Using this lemma we prove the following auxiliary statement.

Lemma 5. *Let F be such a field of zero characteristic, that the transcendence degree of F over $\mathbb{Q}$ is finite. If the automorphism δ of the field F acts on the elements z_1, $z_2, \dots$ of the field F by the rule*

$$\delta : z_i \mapsto \alpha a_i z_i,$$

where $\alpha \in F$, $1 \neq a_i \in \mathbb{Q} \subseteq F$ and $\nu(a_i) \cap \nu(a_j) = \varnothing$ for $i \neq j$, then there are only a finite number of non-zero elements in the set $z_1, z_2, \dots$.

Proof. If all the elements $z_1, z_2, \dots$ are equal to zero then there is nothing to prove. Hence we can consider that there exists a non-zero element in the set $z_1, z_2, \dots$ Without loosing of generality we can consider that $z_1 \neq 0$ (Otherwise we can reenumerate the elements $z_1, z_2, \dots$ and do the first element not to be equal to zero. If the statement holds for the reenumerated set, then it holds for the original set $z_1, z_2, \dots$). Let us denote $y_i = z_i z_1^{-1}$. Then the automorphism δ acts on the element y_i by the rule

$$\delta(y_i) = \delta(z_i z_1^{-1}) = \delta(z_i)\delta(z_1^{-1}) = \alpha a_i z_i \alpha^{-1} a_1^{-1} z_1^{-1} = a_i a_1^{-1} z_i z_1^{-1} = a_i a_1^{-1} y_i$$

Since the transcendence degree of F over $\mathbb{Q}$ is finite, then there exists a maximal subset of algebraically independent over $\mathbb{Q}$ elements in the set $y_2, y_3, \dots$, i.e. there exists such a finite set $y_{i_1}, y_{i_2}, \dots, y_{i_k}$ of algebraically independent over $\mathbb{Q}$ elements, that the set $y_{i_1}, y_{i_2}, \dots, y_{i_k}, y_j$ is algebraically dependent over $\mathbb{Q}$ for every j.

Without loosing of generality we can consider that the set $y_2, \dots, y_k$ is a maximal subset of algebraically independent over $\mathbb{Q}$ elements in the set $y_1, y_2, \dots$

If $n > k$ is a positive integer, then the elements $y_2, \dots, y_k, y_n \in F$ satisfy the conditions of the lemma 4. Therefore $y_n = 0$ for all $n > k$ and since $y_n = z_n z_1^{-1}$ then $z_n = 0$ for all $n > k$ and the only non-zero elemets are $z_1, z_2, \dots, z_k$. $\square$

Let us remind some facts about Chevalley groups. We use definitions and denotations from [4].

Let Φ be an indecompasable root system of rang l with the subsystem of simple roots Δ, $|\Delta| = l$. The elementary Chevalley group $\Phi(F)$ of the type Φ over the field F is a subgroup in the automorphism group of the simple Lie algebra $\mathcal{L}$ of the type Φ, which is generated by the elementary root elements $x_\alpha(t)$, $\alpha \in \Phi$, $t \in F$. The dimension of the Lie algebra $\mathcal{L}$ is equal to $|\Phi| + |\Delta|$ and therefore group $\Phi(F)$ can be considered as a subgroup in the group of all $(|\Phi| + |\Delta|) \times (|\Phi| + |\Delta|)$ invertible matrices.

In the elementary Chevalley group, we consider the following important elements $n_\alpha(t) = x_\alpha(t) x_{-\alpha}(-t^{-1}) x_\alpha(t)$, $h_\alpha(t) = n_\alpha(t) n_\alpha(-1)$, $t \in F^*$, $\alpha \in \Phi$.

For an arbitrary Chevalley group G of the type Φ over the field F we have the following short exact sequence of groups

$$1 \to Z(G) \to G \to \Phi(F) \to 1,$$

where $Z(G)$ is a center of the group G, and by the lemma 2 we are mostly interested in the study of the R_∞-property for elementary Chevalley groups.

Detailed information on the automorphism group of Chevalley groups can be found in [43, 31]. Every Chevalley group has the following automorphisms

1. Inner automorphism φ_g, induced by the element $g \in G$

$$\varphi_g : x \mapsto g x g^{-1}.$$

2. Diagonal automorphism φ_h

$$\varphi_h : x \mapsto h x h^{-1},$$

where the element h can be presented as a diagonal $(|\Phi| + |\Delta|) \times (|\Phi| + |\Delta|)$ matrix. If F is an algebraically closed field then any diagonal automorphism is inner [43, Lemma 4].

3. Field automorphism $\overline{\delta}$
$$\overline{\delta} : x = (x_{ij}) \mapsto (\delta(x_{ij})),$$
where δ is an automorphism of the field F.

4. Graph automorphism $\overline{\rho}$, which acts on the generators of the group G by the rule
$$\overline{\rho} : x_\alpha(t) \mapsto x_{\rho(\alpha)}(t),$$
where ρ is a symmetry of Dynkin diagram. An order of the graph automorphism is equal to 2 or to 3.

Any field automorphism commutes with any graph automorphism. All the diagonal automorphisms form a normal subgroup in the group which is generated by diagonal, graph and field automorphisms.

Theorem of Steinberg says that for any automorphism φ of the elementary Chevalley group $G = \Phi(F)$ there exists an inner automorphism φ_g, a diagonal automorphism φ_h, a graph automorphism $\overline{\rho}$ and a field automorphism $\overline{\delta}$, such that $\varphi = \overline{\rho}\overline{\delta}\varphi_h\varphi_g$ [31].

3 Chevalley groups

In this paragraph we extend the following result from [43, Theorem 1].

Theorem 1. *Let G be a Chevalley group of the type Φ over the field F of zero characteristic and the transcendence degree of F over $\mathbb{Q}$ is finite. Then*

1. *If Φ is a root system of the type $A_l (l \geq 7)$, $B_l (l \geq 4)$, E_8, F_4, G_2, then G possesses the R_∞-property.*

2. *If the equation $T^k = a$ can be solved in the field F for any element a, then G possesses the R_∞-property in the case of the root systems $A_l (l = 2, 3, 4, 5, 6)$, $B_l (l = 2, 3)$, $C_l (l \geq 3)$, $D_l (l \geq 4)$, E_6, E_7, where k is a positive integer from the table*

Φ	A_l	B_l	C_l	D_l	E_6	E_7
k	$l+1$	2	2	2	6	2

In particular, this theorem says that if F is an algebraically closed field of zero characteristic, such that the transcendence degree of F over $\mathbb{Q}$ is finite, then a Chevalley group of any normal type over the field F possesses the R_∞-property.

Here we exclude the condition of solvability of equations from the second item of the theorem 1. We prove the following result.

Theorem 2. *Let G be a Chevalley group of the type Φ over the field F of zero characteristic. If the transcendence degree of F over $\mathbb{Q}$ is finite, then G possesses the R_∞-property.*

Proof. Since $G/Z(G) \cong \Phi(F)$ then by the lemma 2 it is sufficient to prove that the elementary Chevalley group $\Phi(F)$ possesses the R_∞-property. Consider that $G = \Phi(F)$.

Let us consider an arbitrary automorphism φ of the group G and prove that the number of φ-conjugacy classes is infinite. By the theorem of Steinberg there exists an inner automorphism φ_g, a diagonal automorphism φ_h, a graph automorphism $\overline{\rho}$ and a field automorphism $\overline{\delta}$, such that $\varphi = \overline{\rho}\overline{\delta}\varphi_h\varphi_g$. By the lemma 1 the Reidemeister number $R(\varphi)$ is infinite if and only if the Reidemeister number $R(\varphi\varphi_{g^{-1}})$ is infinite, and we can consider that $\varphi = \overline{\rho}\overline{\delta}\varphi_h$.

Suppose that $R(\varphi) < \infty$ and consider the following elements of the group G

$$g_i = h_{\alpha_1}(p_{i1})h_{\alpha_2}(p_{i2})\dots h_{\alpha_l}(p_{il}), \quad i = 1, 2, \dots,$$

where $p_{11} < p_{12} < \dots < p_{1l} < p_{21} < p_{22} < \dots$ are prime numbers. In the matrix representation the element g_i has diagonal form

$$g_i = diag(a_{i1}, a_{i2}, \dots, a_{i|\Phi|}, \underbrace{1, \dots, 1}_{|\Phi|}),$$

for certain rational numbers a_{ij}, such that $\nu(a_{ij}) \neq \varnothing$ and $\nu(a_{ij}) \cap \nu(a_{rs}) = \varnothing$ for $i \neq r$ since $\nu(a_{ij}) \subseteq \{p_{i1}, \dots, p_{il}\}$ (see [43]).

Since $R(\varphi) < \infty$ then there exists an infinite subset of φ-conjugated elements in the set $g_1, g_2, \dots$. Without loosing of generality we can consider that all the elements $g_1, g_2, \dots$ belong to the φ-conjugacy class $[g_1]_\varphi$ of the element g_1. It means that there exists an infinite set of matrices $Z_2, Z_3, \dots$ from G such that

$$g_1 = Z_i g_i \varphi(Z_i^{-1}), \quad i = 2, 3, \dots$$

Acting on this equalities by degrees of the automorphism φ we have

$$\begin{array}{rcl} g_1 & = & Z_i g_i \varphi(Z_i^{-1}), \\ \varphi(g_1) & = & \varphi(Z_i)\varphi(g_i)\varphi^2(Z_i^{-1}), \\ \varphi^2(g_1) & = & \varphi^2(Z_i)\varphi^2(g_i)\varphi^3(Z_i^{-1}), \quad i = 2, 3, \dots \\ & \dots & \\ \varphi^5(g_1) & = & \varphi^5(Z_i)\varphi^5(g_i)\varphi^6(Z_i^{-1}). \end{array}$$

If we multiply all of these equalities we conclude, that

$$g_1\varphi(g_1)\dots\varphi^5(g_1) = Z_i g_i \varphi(g_i)\dots\varphi^5(g_i)\varphi^6(Z_i^{-1}) \tag{1}$$

Since the matrix g_i has a diagonal form and the automorphism φ_h acts as a conjugation by the diagonal matrix then $\varphi_h(g_i) = g_i$. Since the matrix g_i has rational entries, then $\overline{\delta}(g_i) = g_i$ and therefore $\varphi(g_i) = \overline{\rho}(g_i)$. If we denote $\widetilde{g}_i = g_i\varphi(g_i)\dots\varphi^5(g_i) = g_i\overline{\rho}(g_i)\dots\overline{\rho}^5(g_i)$ then

$$\widetilde{g}_i = diag(b_{i1}, b_{i2}, \dots, b_{i|\Phi|}, \underbrace{1, \dots, 1}_{l}), \quad i = 1, 2, \dots$$

since $\overline{\rho}$ permutes elements on the diagonal of the matrix g_i. Moreover, $\nu(b_{ij}) \neq \varnothing$ and $\nu(b_{ij}) \cap \nu(b_{rs}) = \varnothing$ for $i \neq r$, since $\nu(b_{ij}) \subseteq \nu(a_{i1}) \cup \dots \cup \nu(a_{i|\Phi|})$.

Since graph and field automorphisms commute and diagonal automorphisms form a normal subgroup in the group, which is generated by graph, field and diagonal automorphisms, then for a certain diagonal automorphism $\varphi_{\widetilde{h}}$ we have $\varphi^6 = (\overline{\rho}\overline{\delta}\varphi_h)^6 = \varphi_{\widetilde{h}}\overline{\delta}^6\overline{\rho}^6$. Since an order of the automorphism $\overline{\rho}$ is equal to 2 or to 3, then $\overline{\rho}^6 = id$ and $\varphi^6 = \varphi_{\widetilde{h}}\overline{\delta}^6$. Then the equality (1) can be rewritten

$$\widetilde{g}_1 = Z_i\widetilde{g}_i\varphi^6(Z_i^{-1}) = Z_i\widetilde{g}_i\varphi_{\widetilde{h}}\overline{\delta}^6(Z_i^{-1}) = Z_i\widetilde{g}_i\widetilde{h}\overline{\delta}^6(Z_i^{-1})\widetilde{h}^{-1}, \quad i = 2, 3, \dots$$

If we multiply this equality by the element $\widetilde{h}$ on the right and denote $\hat{g}_i = \widetilde{g}_i\widetilde{h}$ then we have

$$\hat{g}_1 = Z_i\hat{g}_i\overline{\delta}^6(Z_i^{-1}), \quad i = 2, 3, \dots \tag{2}$$

From this equality we have

$$\overline{\delta}^6(Z_i) = \hat{g}_1^{-1}Z_i\hat{g}_i, \quad i = 2, 3, \dots \tag{3}$$

If we denote $\widetilde{h} = diag(c_1, c_2, \dots, c_{|\Phi|}, \underbrace{1, \dots, 1}_{l})$, then

$$\hat{g}_i = \widetilde{g}_i\widetilde{h} = diag(b_{i1}c_1, b_{i2}c_2, \dots, b_{i|\Phi|}c_{|\Phi|}, \underbrace{1, \dots, 1}_{l}), \quad i = 2, 3, \dots$$

Let $Z_i = \left(\begin{array}{c|c} Q_i & R_i \\ \hline S_i & T_i \end{array}\right)$, where $Q_i = (q_{i,mn})$ is a $|\Phi| \times |\Phi|$ matrix, $R_i = (r_{i,mn})$ is a $|\Phi| \times |\Delta|$ matrix, $S_i = (s_{i,mn})$ is a $|\Delta| \times |\Phi|$ matrix, $T_i = (t_{i,mn})$ is a $|\Delta| \times |\Delta|$ matrix.

Then by the equality (3) for all $m = 1, \dots, |\Phi|,\ n = 1, \dots, |\Phi|$ we have

$$\delta^6(q_{i,mn}) = (b_{1m}c_m)^{-1}b_{in}c_nq_{i,mn} = d_{mn}b_{in}q_{i,mn}, \quad i = 2, 3, \dots,$$

where $d_{mn} = (b_{1m}c_m)^{-1}c_n$. Since $\nu(b_{in}) \neq \varnothing$ and $\nu(b_{in}) \cap \nu(b_{jn}) = \varnothing$ for $i \neq j$, then we can apply the lemma 5 to the set $q_{2,mn}, q_{3,mn}, \dots$ Therefore by the lemma 5 there exists a positive integer N_{mn}, such that $q_{i,mn} = 0$ for every $i > N_{mn}$.

If we denote by N the value

$$N = \max_{n,m=1,\dots,|\Phi|} N_{mn},$$

then for every $i > n$ we have $Q_i = O_{|\Phi|\times|\Phi|}$. Using the same arguments to the matrices $S_2, S_3, \dots$ we conclude that for sufficiently large indexes i all the matrices S_i are the matrices with zero entries only, and therefore the matrix Z_i has the form

$$Z_i = \begin{pmatrix} O_{|\Phi|\times|\Phi|} & R_i \\ O_{|\Delta|\times|\Phi|} & T_i \end{pmatrix}.$$

The determinant of this matrix is equal to zero and it means that Z_i can not belong to G. This contradiction proves the theorem.

□

4 Linear algebraic groups

If G is a linear algebraic group over an algebraically closed field, then it has a unique maximal solvable normal subgroup $R(G)$, called the radical of G. A connected linear algebraic group G is called reductive if its radical is a torus, or, equivalently, if it can be decomposed $G = G'T'$ with G' a semisimple group and T' a central torus [51, §6.5].

The quotient group $G/R(G)$ has a trivial radical, i.e. is a semisimple group [30, §19.5].

Theorem 3. *Let F be such an algebraically closed field of zero characteristic that the transcendence degree of F over $\mathbb{Q}$ is finite. If the reductive linear algebraic group G over the field F has a nontrivial quotient group $G/R(G)$, then G possesses the R_∞-property.*

Proof. For the group G we have the following short exact sequence of groups

$$1 \to R(G) \to G \to G/R(G) \to 1,$$

Since G is reductive, then the radical $R(G)$ is a central torus and therefore is a characteristic subgroup of G. Hence by the lemma 2 it is sufficient to prove that the semisimple group $G/R(G)$ possesses the R_∞-property and we can consider that G is a semisimple linear algebraic group. Since G is a semisimple linear algebraic group, then it is a product, with some amalgamation of (finite) centers, of its simple subgroups $H_1, H_2, \dots, H_k$ [30, §14.2]

$$G = H_1 \circ \cdots \circ H_k.$$

Every simple linear algebraic group H_i is a Chevalley group of (normal) type Φ_i over the field F. Factoring the group G by its center we have the following short exact sequence of groups

$$1 \to Z(H_1 \circ \dots \circ H_k) \to H_1 \circ \dots \circ H_k \to \Phi_1(F) \times \dots \times \Phi_k(F) \to 1,$$

where $\Phi_i(F)$ is an elementary Chevalley group of the type Φ_i over the field F. Hence, by the lemma 2 we can consider that $G = \Phi_1(F) \times \dots \times \Phi_k(F)$ and prove that this group possesses the R_∞-property. Permute the groups $\Phi_1(F), \dots, \Phi_k(F)$ so that all the groups with the same root system form blocks

$$G = \underbrace{\Phi_1(F) \times \dots \Phi_1(F)}_{k_1} \times \underbrace{\Phi_2(F) \times \dots \Phi_2(F)}_{k_2} \times \dots \times \underbrace{\Phi_r(F) \times \dots \Phi_r(F)}_{k_r},$$

where $k_1 + k_2 + \dots + k_r = k$. Denote $G_i = \underbrace{\Phi_i(F) \times \dots \Phi_i(F)}_{k_i}$.

Every group G_i is a characteristic subgroup of $G = G_1 \times \dots \times G_r$. It is obvious that if some group is a direct product of its characteristic subgroups and at least one of this subgroups possesses the R_∞-property then the group itself possesses the R_∞-property. Therefore it is sufficient to prove that the group

$$G = \Phi(F) \times \dots \times \Phi(F) = \Phi(F)^k$$

possesses the R_∞-property.

Every element $g \in G = \Phi(F)^k$ can be presented as a direct sum of k matrices $g_1, \dots, g_k$ of the size $(|\Phi| + |\Delta|) \times (|\Phi| + |\Delta|)$ each of which belongs to $\Phi(F)$.

An automorphism group of G has the form

$$\mathrm{Aut}(G) = (\mathrm{Aut}(\Phi(F)))^k \leftthreetimes S_k, \tag{4}$$

where S_k is a permutation group on k symbols.

To prove that the group $G = \Phi(F)^k$ possesses the R_∞-property consider an arbitrary automorphism φ of the group G and prove that $R(\varphi) = \infty$. By the equality (4) the automorphism φ can be written in the following form

$$\varphi = (\varphi_1, \dots, \varphi_k, \sigma),$$

where $\varphi_1, \dots, \varphi_k \in \mathrm{Aut}(\Phi(\mathrm{F}))$, $\sigma \in S_k$, and φ acts on the group G by the rule

$$\varphi : x_1 \oplus x_2 \oplus \dots \oplus x_k \mapsto \varphi_{1^\sigma}(x_{1^\sigma}) \oplus \varphi_{2^\sigma}(x_{2^\sigma}) \oplus \dots \oplus \varphi_{k^\sigma}(x_{k^\sigma}), \tag{5}$$

where i^σ denotes an image of i by the permutation σ.

Every automorphism $\varphi_i \in \mathrm{Aut}(\Phi(F))$ can be presented as a product of the inner automorphism φ_{g_i}, the diagonal automorphism φ_{h_i}, the graph automorphism

$\overline{\rho}_i$ and the field automorphism $\overline{\delta}_i$. Since F is an algebraically closed field, then every diagonal automorphism φ_{h_i} is inner [43, Lemma 4], hence for every i we can consider that $\varphi_i = \varphi_{x_i}\overline{\rho}_i\overline{\delta}_i$. Then the automorphism φ can be presented as a product of two automorphism

$$\varphi = (\varphi_{x_{1^\sigma}}, \varphi_{x_{2^\sigma}}, \dots, \varphi_{x_{k^\sigma}}, id)(\overline{\rho}_1\overline{\delta}_1, \overline{\rho}_2\overline{\delta}_2, \dots, \overline{\rho}_k\overline{\delta}_k, \sigma),$$

where $(\varphi_{x_{1^\sigma}}, \varphi_{x_{2^\sigma}}, \dots, \varphi_{x_{k^\sigma}}, id)$ is an inner automorphism. By the lemma 1 we can consider that $\varphi_i = \overline{\rho}_i\overline{\delta}_i$ and

$$\varphi = (\overline{\rho}_1\overline{\delta}_1, \overline{\rho}_2\overline{\delta}_2, \dots, \overline{\rho}_k\overline{\delta}_k, \sigma).$$

Using the induction on r prove that

$$\varphi^r : g_1 \oplus \dots \oplus g_k \mapsto \psi_1(x_{1^{\sigma^r}}) \oplus \dots \oplus \psi_k(x_{k^{\sigma^r}}), \tag{6}$$

where $\psi_i = \varphi_{i^\sigma}\varphi_{i^{\sigma^2}} \dots \varphi_{i^{\sigma^r}}$.

The basis of the induction ($r = 1$) is obvious (the equality (5)). If we suppose that the equality (6) holds for some r, then

$$\begin{aligned}
\varphi^{r+1}(g_1 \oplus \dots \oplus g_k) &= \varphi(\varphi^r(g_1 \oplus \dots \oplus g_k)) \\
&= \varphi(\psi_1(x_{1^{\sigma^r}}) \oplus \dots \oplus \psi_k(x_{k^{\sigma^r}})) \\
&= \varphi_{1^\sigma}\psi_{1^\sigma}(x_{1^{\sigma^{r+1}}}) \oplus \dots \oplus \varphi_{k^\sigma}\psi_{k^\sigma}(x_{k^{\sigma^{r+1}}}).
\end{aligned}$$

Note the equality $\varphi_{i^\sigma}\psi_{i^\sigma} = \varphi_{i^\sigma}\varphi_{i^{\sigma^2}} \dots \varphi_{i^{\sigma^{r+1}}}$, what we wanted to prove.

Consider the set of elements $g_1, g_2, \dots$ of the group $\Phi(F)$ from the theorem 2

$$g_i = h_{\alpha_1}(p_{i1})h_{\alpha_2}(p_{i2}) \dots h_{\alpha_l}(p_{il}), \quad i = 1, 2, \dots,$$

where $p_{11} < p_{12} < \dots < p_{1l} < p_{21} < p_{22} < \dots$ are prime integers. This elements are presented by the diagonal matrices

$$g_i = diag(a_{i1}, a_{i2}, \dots, a_{i|\Phi|}, \underbrace{1, \dots, 1}_{l}), \quad i = 1, 2, \dots$$

where a_{ij} are rational numbers, such that $\nu(a_{ij}) \neq \varnothing$ and $\nu(a_{ij}) \cap \nu(a_{rs}) = \varnothing$ for $i \neq r$.

As we already shown in the theorem 2 for every automorphism $\varphi_j = \overline{\rho}_j\overline{\delta}_j$ we have $\varphi_j(g_i) = \overline{\rho}_j(g_i)$.

Let us consider the set of elements $\widetilde{g}_1, \widetilde{g}_2, \dots$ of the group $G = \Phi(F)^k$, where $\widetilde{g}_i = g_i \oplus \dots \oplus g_i$. Then by the arguments above $\varphi(\widetilde{g}_i) = \overline{\rho}_{1^\sigma}(g_i) \oplus \dots \oplus \overline{\rho}_{k^\sigma}(g_i)$.

Suppose that $R(\varphi) < \infty$. Then there is an infinite subset of φ-conjugated elements in the set $\widetilde{g}_1, \widetilde{g}_2, \dots$. Without loosing of generality we can consider that all the matrices $\widetilde{g}_1, \widetilde{g}_2, \dots$ belong to the φ-conjugacy class $[\widetilde{g}_1]_\varphi$ of the element $\widetilde{g}_1$. Then for certain matrices $Z_2, Z_3, \dots$ we have

$$\widetilde{g}_1 = Z_i \widetilde{g}_i \varphi(Z_i^{-1}), \quad i = 2, 3, \dots$$

Denote by s an order of the permutation σ and act on this equality by the degrees of the automorphism φ

$$\begin{array}{rcl} \widetilde{g}_1 & = & Z_i \widetilde{g}_i \varphi(Z_i^{-1}) \\ \varphi(\widetilde{g}_1) & = & \varphi(Z_i)\varphi(\widetilde{g}_i)\varphi^2(Z_i^{-1}) \\ \vdots & \vdots & \vdots \\ \varphi^{6s-2}(\widetilde{g}_1) & = & \varphi^{6s-2}(Z_i)\varphi^{6s-2}(\widetilde{g}_i)\varphi^{6s-1}(Z_i^{-1}) \\ \varphi^{6s-1}(\widetilde{g}_1) & = & \varphi^{6s-1}(Z_i)\varphi^{6s-1}(\widetilde{g}_i)\varphi^{6s}(Z_i^{-1}) \end{array}$$

If we multiply all of these equalities, we obtain the equality

$$\widetilde{g}_1 \varphi(\widetilde{g}_1)\varphi^2(\widetilde{g}_1)\dots\varphi^{6s-1}(\widetilde{g}_1) = Z_i \widetilde{g}_i \varphi(\widetilde{g}_i)\varphi^2(\widetilde{g}_i)\dots\varphi^{6s-1}(\widetilde{g}_i)\varphi^{6s}(Z_i^{-1}). \tag{7}$$

The element $\widetilde{g}_i\varphi(\widetilde{g}_i)\varphi^2(\widetilde{g}_i)\dots\varphi^{6s-1}(\widetilde{g}_i)$ can be rewritten in details

$$\begin{array}{l} \widetilde{g}_i\varphi(\widetilde{g}_i)\varphi^2(\widetilde{g}_i)\dots\varphi^{6s-1}(\widetilde{g}_i) = \\ = (g_i \oplus \dots \oplus g_i)(\overline{\rho}_{1^\sigma}(g_i) \oplus \dots \oplus \overline{\rho}_{k^\sigma}(g_i))\dots(\overline{\rho}_{1^{\sigma^{6s-1}}}(g_i) \oplus \dots \oplus \overline{\rho}_{k^{\sigma^{6s-1}}}(g_i)) = \\ = g_i\overline{\rho}_{1^\sigma}(g_i)\dots\overline{\rho}_{1^{\sigma^{6s-1}}}(g_i) \oplus \dots \oplus g_i\overline{\rho}_{k^\sigma}(g_i)\dots\overline{\rho}_{k^{\sigma^{6s-1}}}(g_i)) = \widehat{g}_{i1} \oplus \dots \oplus \widehat{g}_{ik}, \end{array}$$

where $\widehat{g}_{ij} = g_i\overline{\rho}_{j^\sigma}(g_i)\dots\overline{\rho}_{j^{\sigma^{6s-1}}}(g_i)$.

Since every graph automorphism $\overline{\rho}_j$ permutes elements on the diagonal of the matrix g_i then for every $j = 1, \dots, k$, $i = 1, 2, \dots$ we have

$$\widehat{g}_{ij} = diag(b_{ij1}, b_{ij2}, \dots, b_{ij|\Phi|}, \underbrace{1, \dots, 1}_{l}), \tag{8}$$

where, $\nu(b_{ijr}) \neq \varnothing$ and $\nu(b_{ijr}) \cap \nu(b_{uvw}) = \varnothing$ for $i \neq u$ since $\nu(b_{ijr}) \subseteq \{p_{i1}, \dots, p_{il}\}$.

By the equality (6) we have $\varphi^s = (\psi_1, \psi_2, \dots, \psi_k, id)$, where $\psi_i = \varphi_{i^\sigma}\varphi_{i^{\sigma^2}}\dots\varphi_{i^{\sigma^r}}$. Since all the automorphisms $\varphi_1, \varphi_2, \dots, \varphi_k$ are the products of graph and field automorphism ($\varphi_i = \overline{\rho}_i\overline{\delta}_i$) and graph and field automorphisms commute then every automorphism ψ_i is a product of graph and field automorphisms $\psi_i = \overline{\xi}_i\overline{\theta}_i$ for certain $\overline{\xi}_i$, $\overline{\theta}_i$. Therefore

$$\varphi^{6s} = (\varphi^s)^6 = (\overline{\xi}_1\overline{\theta}_1, \dots \overline{\xi}_k\overline{\theta}_k, id)^6 = (\overline{\xi}_1^6\overline{\theta}_1^6, \dots \overline{\xi}_k^6\overline{\theta}_k^6, id) = (\overline{\theta}_1^6, \dots \overline{\theta}_k^6, id)$$

Using this fact, denoting the matrix $Z_i = Z_{i1} \oplus \dots \oplus Z_{ik}$ projecting the equality (7) to the first group $\Phi(F)$ we obtain the equality

$$\widehat{g}_{11} = Z_{i1}\widehat{g}_{i1}\overline{\theta}_1^6(Z_{i1}), \quad i = 2, 3, \dots$$

This equality is the same as the equality (2) from the theorem 2. Using the same arguments as in the theorem 2 we conclude that for sufficiently large coefficient N the matrix Z_{iN} is degenerated and therefore the matrix Z_N is degenerated but it contradicts to the fact that Z_N belongs to G. $\square$

We use the fact that the group G is a reductive linear algebraic group in order to say that the radical $R(G)$ is a characteristic subgroup of G. Even the theorem 3 holds for every connected linear algebraic group such that the radical $R(G)$ is a characteristic. For example, if any automorphism of the group G is a morphism of the group G (as of an affine manifold) then the radical $R(G)$ is characteristic [51, Theorem 7.1(c)] and the theorem 3 holds for such groups.

5 Finite Reidemeister number in linear groups

Following [46], we define the *Reidemeister spectrum of* G as

$$Spec(G) = \{R(\varphi) \mid \varphi \in \mathrm{Aut}(G)\}.$$

In particular, G possesses the R_∞-property if and only if $Spec(G) = \{\infty\}$.

It is easy to see that $Spec(\mathbb{Z}) = \{2\} \cup \{\infty\}$, and, for $n \geq 2$, the spectrum of $\mathbb{Z}^n$ is full, i.e. $Spec(\mathbb{Z}^n) = \mathbb{N} \cup \{\infty\}$. For free nilpotent groups we have the following: $Spec(N_{22}) = 2\mathbb{N} \cup \{\infty\}$ (N_{22} is the discrete Heisenberg group) [36, 10, 46], $Spec(N_{23}) = \{2k^2 \mid k \in \mathbb{N}\} \cup \{\infty\}$ [46] and $Spec(N_{32}) = \{2k-1 \mid k \in \mathbb{N}\} \cup \{4k \mid k \in \mathbb{N}\} \cup \{\infty\}$ [46].

Recently, in [5] it was proven that the group N_{rc} ($r > 1$) admits an automorphism with finite Reidemeister number if and only if $c < 2r$.

In [24], examples of polycyclic non-virtually nilpotent groups which admit automorphisms with finite Reidemeister numbers have been described. In this examples G is a semidirect product of $\mathbb{Z}^2$ and $\mathbb{Z}$ by Anosov automorphism defined by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. The group G is solvable and of the exponential growth. The automorphism φ with finite Reidemeister number is defined by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $\mathbb{Z}^2$ and as $-id$ on $\mathbb{Z}$.

Metabelian (therefore, solvable) finitely generated, non-polycyclic groups have quite interesting Reidemeister spectrum [15]: for example, if the homomorphism $\theta : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}[1/p]^2)$ is such that $\theta(1) = \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$, then we have the following cases:

a) If $r = s = \pm 1$ then $Spec(\mathbb{Z}[1/p]^2 \rtimes_\theta \mathbb{Z}) = \{2n \mid n \in \mathbb{N},\ (n,p) = 1\} \cup \{\infty\}$, where (n,p) denote the greatest common divisor of n and p.

b) If $r = -s = \pm 1$ then $Spec(\mathbb{Z}[1/p]^2 \rtimes_\theta \mathbb{Z}) = \{2p^l(p^k \pm 1), 4p^l \mid l, k > 0\} \cup \{\infty\}$.

c) If $rs = 1$ and $|r| \neq 1$ then $Spec(\mathbb{Z}[1/p]^2 \rtimes_\theta \mathbb{Z}) = \{2(p^l \pm 1), 4 \mid l > 0\} \cup \{\infty\}$.

d) If either r or s does not equal to ± 1, and $rs \neq 1$ then $Spec(\mathbb{Z}[1/p]^2 \rtimes_\theta \mathbb{Z}) = \{\infty\}$.

In the paper [32] Jabara proved that if residually finite group G admits an automorphism of prime order p with finite Reidemeister number, then G is virtually nilpotent group of class bounded by a function of p.

From another side, we have described in an Introduction a lot of classes of non-solvable, finitely generated, residually finite groups which have the R_∞-property. All together was a motivation for the following conjecture

Conjecture R (A. Fel'shtyn, E. Troitsky [18, Conjecture R]) Every infinite, residually finite, finitely generated group either possesses the R_∞-property or is a virtually solvable group.

Here we study this question for infinite linear groups.

Proposition 1. *Let G be a reductive linear algebraic over the field F of zero characteristic and finite transcendence degree over $\mathbb{Q}$. If G possesses an automorphism φ with finite Reidemeister number then G is a torus.*

Proof Since G possesses an automorphism φ with finite Reidemeister number, then by the theorem 3, it has trivial quotient group $G/R(G)$, therefore $G = R(G)$ and hence G is a central torus (therefore, is solvable). $\square$

6 Groups with property S_∞

Suppose that $\Psi \in \mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$. Let $\mathfrak{S}(\Psi)$ be the set of isogredience classes of Ψ. Then $\mathfrak{S}(\mathrm{Id})$ can be identified with the set of conjugacy classes of $G/Z(G)$ (see [18]).

The definition of the similarity (isogredience) from Introduction goes back to Jacob Nielsen. He observed (see [33]) that conjugate lifting of homeomorphism of surface have similar dynamical properties. This led Nielsen to the definition of the isogredience of liftings in this case. Later Reidemeister and Wecken succeeded in generalizing the theory to continuous maps of compact polyhedra (see [33]).

The set of isogredience classes of automorphisms representing a given outer automorphism and the notion of index $\mathrm{Ind}(\Psi)$ defined via the set of isogredience classes are strongly related to important structural properties of Ψ (see [19]).

One of the main results of [38] is that for any non-elementary hyperbolic group and any Ψ the set $\mathfrak{S}(\Psi)$ is infinite, i. e. $S(\Psi) = \infty$. Thus, this result says: any non-elementary hyperbolic group is an S_∞-group. On the other hand, finite and finitely generated Abelian groups are evidently non S_∞-groups.

Two representatives of Ψ have the forms $\varphi_s a$, $\varphi_q a$ for some $s, q \in G$ and fixed $a \in \Psi$. They are isogredient if and only if

$$\varphi_q a = \varphi_g \varphi_s a \varphi_g^{-1} = \varphi_g \varphi_s \varphi_{a(g^{-1})} a,$$

$$\varphi_q = \varphi_{gsa(g^{-1})}, \qquad q = gsa(g^{-1})c, \quad c \in Z(G)$$

(see [38, p. 512]). So, the following statement is proved.

Lemma 6. *[18, Lemma 3.3] Let $\varphi \in \Psi$ be an automorphism of the group G and $\overline{\varphi}$ be an automorphism of the group $G/Z(G)$ which is induced by φ. Then the number $S(\Psi)$ is equal to the number of $\overline{\varphi}$-conjugacy classes in the group $G/Z(G)$.*

Since $Z(G)$ is a characteristic subgroup, we obtain the following statement

Theorem 4. *[18, Theorem 3.4] Suppose, $|Z(G)| < \infty$. Then G is an R_∞-group if and only if G is an S_∞-group.*

A more advanced example of a non S_∞-group is the Osin's group [45]. This is a non-residually finite exponential growth group with two conjugacy classes. Since it is simple, it is not S_∞ group (see [18]).

Theorem 5. *Let F be such an algebraically closed field of zero characteristic that the transcendence degree of F over $\mathbb{Q}$ is finite. If the reductive linear algebraic group G over the field F has a nontrivial quotient group $G/R(G)$, then G possesses the S_∞-property.*

Proof. Since $R(G)$ is a characteristic subgroup of G then by the lemma 3 it is sufficient to prove the theorem for semisimple group $G/R(G)$. The result follows immediately from the theorem 3 and the theorem 4 and from the fact that semisimple linear algebraic group has finite center. $\square$

Proposition 2. *Let G be a reductive linear algebraic group over the field F of zero characteristic and finite transcendence degree over $\mathbb{Q}$. If G possesses an outer automorphism Ψ with finite number $S(\Psi)$ then G is a torus.*

Proof Since G possesses an outer automorphism Ψ with finite number $S(\Psi)$, then by the theorem 5, it has a trivial quotient $G/R(G)$, therefore $G = R(G)$ and is a central torus. $\square$

The conjecture of Fel'shtyn and Troitsky from the section 5 can be rewritten in terms of S_∞-property by the following way.

Conjecture S Every infinite, residually finite, finitely generated group either possesses the S_∞-property or is a virtually solvable group.

Really, if $S(\varphi\mathrm{Inn}(G)) < \infty$ for some automorphism $\varphi \in \mathrm{Aut}(G)$ then by the lemma 6 we have $R(\overline{\varphi}) < \infty$, where $\overline{\varphi}$ is an automorphism of the group $G/Z(G)$ induced by φ. Since G is residually finite finitely generated group, then $G/Z(G)$ is also finitely generated and residually finite and by the conjecture R is a virtually solvable group.

It means that there exists a solvable subgroup $\overline{H} \leq G/Z(G)$ of finite index. Let n be a derived length of $\overline{H}$, i. e. $\overline{H}^{(n)} = 1$. Let H be a preimage of $\overline{H}$ under the canonical homomorphism $G \to G/Z(G)$. Then $H^{(n)} \leq Z(G)$ and $H^{(n+1)} = 1$, therefore H is a solvable group. Since $G/H \simeq (G/Z(G))/(H/Z(G)) = (G/Z(G))/\overline{H}$, then the index of H in G is equal to the index of $\overline{H}$ in $G/Z(G)$, i. e. is finite, therefore G is solvable by finite.

We have proven in all that [18, Conjecture S] can be formulated without the restriction that the group under consideration has finite center.

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