SYMPLECTIC INSTANTON BUNDLES ON $\mathbb{P}^3$ AND 'T HOOFT INSTANTONS

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Abstract. We study the moduli space $I_{n,r}$ of rank-$2r$ symplectic instanton vector bundles on $\mathbb{P}^3$ with $r \geq 2$ and second Chern class $n \geq r+1,\ n-r \equiv 1(\mathrm{mod}2)$. We introduce the notion of tame symplectic instantons by excluding a kind of pathological monads and show that the locus $I^*_{n,r}$ of tame symplectic instantons is irreducible and has the expected dimension, equal to $4n(r+1) - r(2r+1)$. The proof is inherently based on a relation between the spaces $I^*_{n,r}$ and the moduli spaces of 't Hooft instantons.

1. Introduction

A *symplectic instanton vector bundle* of rank $2r$ and charge n on the projective 3-space $\mathbb{P}^3$ is an algebraic vector bundle $E = E_{2r}$ of rank $2r$ on $\mathbb{P}^3$ which is equipped with a symplectic structure $\phi :\ E \xrightarrow{\sim} E^\vee, \phi^\vee = -\phi$ and satisfies the vanishing conditions $h^0(E) = h^1(E \otimes \mathcal{O}_{\mathbb{P}^3}(-2)) = 0$. The Chern classes $c_1(E)$ and $c_3(E)$ vanish, and we also assume $c_2(E) = n \geq 1$. We shall denote by $I_{n,r}$ the moduli space of symplectic (n, r)-instantons.

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Rank r symplectic instantons on $\mathbb{P}^3$ relate in a natural manner with "physical" $\mathbf{Sp}(r)$ instantons on the four-sphere S^4, i.e., connections on principal $\mathbf{Sp}(r)$-bundles on S^4 with self-dual curvature [1]; the moduli spaces of the former are in a sense a complexification of the moduli spaces of the latter. This relation is expressed by the so-called Atiyah–Ward correspondence [3, 1], which relies on the fact that the projective space $\mathbb{P}^3$ is the twistor space of the four-sphere S^4. The present paper and its companion [7] are the first to study the geometry of the moduli spaces $I_{n,r}$. While [7] studied the case $n \equiv r \pmod 2$, with $n \geq r$, the present paper deals with the other case, $n \equiv r+1 \pmod 2$, with $n \geq r+1$. The main result of this paper is that a component $I^*_{n,r}$ of $I_{n,r}$ that is singled out by a certain open condition (which rules out some "badly behaved" monads) is irreducible.

We exploit as usual the monad method [8, 2, 4, 5, 6, 11, 12], which allows one to study instantons by means of hyperwebs of quadrics. Namely, we realize $I_{n,r}$ as the quotient space of a principal $GL(H_n)/\{\pm\mathrm{id}\}$-bundle $\pi_{n,r} : MI_{n,r} \to I_{n,r}$, where $MI_{n,r}$ is a locally closed subset of the vector space $\mathbf{S}_n$ of hyperwebs of quadrics (precise definitions will be given later on). The tame locus $I^*_{n,r}$ being open in $I_{n,r}$, its irreducibility is equivalent to that of $MI^*_{n,r} = \pi^{-1}_{n,r}(I^*_{n,r})$. The key ingredient of our approach is the reduction of the last problem to that of certain sets Z_{n-r+1} (see section 3). The sets Z_i as locally closed subsets of some vector spaces related to $\mathbf{S}_n$ were first defined in [9]. It is shown in [9, Section 9] that the Z_i can be interpreted essentially as open subsets of certain affine bundles over the monad spaces M^{tH}_{2i-1} of 't Hooft rank-2 mathematical instantons of charge $2i-1$—see more details in section 3.2. Thus the irreducibility of Z_{n-r+1}, hence that of $I^*_{n,r}$, is reduced to the irreducibility of the moduli spaces of 't Hooft instantons of fixed charge, which is well known; see references in [9]. This nontrivial relation between the spaces $I^*_{n,r}$ and the moduli of 't Hooft instantons is crucial for the results in this paper. Note that this process of reduction from $I^*_{n,r}$ to the moduli of 't Hooft instantons somewhat resembles Barth's approach in [5] to the proof of the irreducibility of the moduli space I_4 of instantons of charge 4. In that paper, Barth reduces the problem to the irreducibility of the space Q_n of commuting pairs of (good in some sense) pencils of quadrics for $n = 4$. In our case the role of the spaces Q_n is played by the moduli spaces of 't Hooft instantons.

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Notation and conventions. Throughout this paper, we consider an algebraically closed base field $\Bbbk$ of characteristic 0. All schemes will be Noetherian. By a general point of an irreducible (but not necessarily reduced) scheme $\mathcal{X}$ we mean a closed point of a dense open subset of $\mathcal{X}$. An irreducible scheme is generically reduced if it is reduced at all general points. We follow the notation of [9]. So, we fix an integer $n \geq 1$, and denote by H_n and V fixed vector spaces over $\Bbbk$ of dimension n and 4, respectively, and set $\mathbb{P}^3 = P(V)$. Furthermore, $\mathbf{S}_n$ (the *space of hyperwebs of quadrics*) will denote the vector space $S^2H_n^\vee \otimes \wedge^2 V^\vee$. A hyperweb of quadrics $A \in \mathbf{S}_n$ is a skew-symmetric homomorphism $A : H_n \otimes V \to H_n^\vee \otimes V^\vee$, and we denote by W_A the vector space $H_n \otimes V/\ker A$ and by c_A the canonical epimorphism $H_n \otimes V \twoheadrightarrow W_A$. A choice of A induces a skew symmetric isomorphism $q_A : W_A \xrightarrow{\sim} W_A^\vee$, and A is the composition $H_n \otimes V \overset{c_A}{\twoheadrightarrow} W_A \underset{\sim}{\overset{q_A}{\to}} W_A^\vee \overset{c_A^\vee}{\rightarrowtail} H_n^\vee \otimes V^\vee$.

For any morphism of $\mathcal{O}_X$-sheaves $f : \mathcal{F} \to \mathcal{F}'$ we denote by the same letter f the induced morphism $id \otimes f : U \otimes \mathcal{F} \to U \otimes \mathcal{F}'$, and analogously, for any homomorphism $f : U \to U'$ of $\Bbbk$-vector spaces, the induced morphism $f \otimes id : U \otimes \mathcal{F} \to U' \otimes \mathcal{F}$. For $A \in \mathbf{S}_n$ we denote by a_A the composition $H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{u} H_n \otimes V \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{c_A} W_A \otimes \mathcal{O}_{\mathbb{P}^3}$, where u is the tautological subbundle morphism. By abuse of notation, we denote by the same symbol a $\Bbbk$-vector space, say U, and the associated affine space $\mathbf{V}(U^\vee) = \mathrm{Spec}(\mathrm{Sym}^* U^\vee)$.

2. Explicit construction of symplectic instantons

In this section we provide some examples and recall some facts about $MI_{n,r}$, in particular, its relation with the moduli space $I_{n,r}$ of symplectic instantons, see [7, Section 3]. Let us consider the *set of (n,r)-instanton hyperwebs of quadrics*

$$MI_{n,r} := \left\{ A \in \mathbf{S}_n \;\middle|\; \begin{array}{rl} \text{(i)} & \mathrm{rk}(A : H_n \otimes V \to H_n^\vee \otimes V^\vee) = 2n+2r, \\ \text{(ii)} & \text{the morphism } a_A^\vee : W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \to H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \text{ is} \\ & \text{surjective,} \\ \text{(iii)} & h^0(E_{2r}(A)) = 0, \text{ where } E_{2r}(A) := \ker(a_A^\vee \circ q_A)/\operatorname{Im} a_A. \end{array} \right\} \tag{1}$$

Theorem 2.1. *(i) For each $n \geq 1$, the space $MI_{n,r}$ of (n,r)-instanton nets of quadrics is a locally closed subscheme of the vector space $\mathbf{S}_n$, given locally at any point $A \in MI_{n,r}$ by*

$$\binom{2n-2r}{2} = 2n^2 - n(4r+1) + r(2r+1) \tag{2}$$

equations obtained as the rank condition (i) in (1).

(ii) The natural morphism

$$\pi_{n,r} : MI_{n,r} \to I_{n,r},\ A \mapsto [E_{2r}(A)], \tag{3}$$

is a principal $GL(H_n)/\{\pm\mathrm{id}\}$-bundle in the étale topology. Hence $I_{n,r}$ is a quotient stack $MI_{n,r}/(GL(H_n)/\{\pm\mathrm{id}\})$, and is therefore an algebraic space.

The fibre $F_{[E]} = \pi_n^{-1}([E])$ over a point $[E] \in I_{n,r}$ is a principal homogeneous space of $GL(H_n)/\{\pm\mathrm{id}\}$, so that the irreducibility of $(I_{n,r})_{red}$ amounts to the irreducibility of the scheme $(MI_{n,r})_{red}$. Besides, (2) yields

$$\dim_A MI_{n,r} \geq \dim \mathbf{S}_n - (2n^2 - n(4r+1) + r(2r+1)) = n^2 + 4n(r+1) - r(2r+1) \tag{4}$$

at all points $A \in MI_{n,r}$. Thus, $\dim_{[E]} I_{n,r} \geq 4n(r+1) - r(2r+1)$ at all points $[E] \in I_{n,r}$, as $MI_{n,r} \to I_{n,r}$ is an étale principal $GL(H_n)/\{\pm\mathrm{id}\}$-bundle.

2.1. **Symplectic $(n+1,n)$-instantons.** We give a construction of symplectic $(n+1,n)$-instantons and describe their relation to usual rank-2 instantons with second Chern class $c_2 = 2n$. This will be established at the level of spaces of hyperwebs of quadrics $MI_{n+1,n}$ and $MI_{2n,1}$, regarded as spaces of monads.

Denote by $\mathrm{Isom}_{n+1,n-1}$ the set of all isomorphisms

$$\zeta : H_{n+1} \oplus H_{n-1} \xrightarrow{\sim} H_{2n}. \tag{5}$$

This is the principal homogeneous space of the group $GL(2n)$. Moreover, for any $\zeta \in \mathrm{Isom}_{n+1,n-1}$, let $p_\zeta : \mathbf{S}_{2n} \twoheadrightarrow \mathbf{S}_{n+1}$ be the induced epimorphism, and, for any monomorphism $i : H_n \hookrightarrow H_{n+1}$, let $pr_{(i)} : \mathbf{S}_{n+1} \to \mathbf{S}_n$ be the induced epimorphism.

Note that $MI_{2n,1}$ is irreducible [10, Theorem 1.1], and one has the following result [10, Theorem 3.1].

Theorem 2.2. *There exists a dense open subset $MI^*_{2n,1}$ of $MI_{2n,1}$ such that, for any hyperweb $A \in MI^*_{2n,1}$ and a general $\zeta \in \mathrm{Isom}_{n+1,n-1}$ the rank of the homomorphism $B = p_\zeta(A) : H_{n+1} \otimes V \to H^\vee_{n+1} \otimes V^\vee$ coincides with the rank of $A : H_{2n} \otimes V \to H^\vee_{2n} \otimes V^\vee$:*

$$\mathrm{rk}B = \mathrm{rk}A = 4n+2. \tag{6}$$

Set $W_{4n+2} := H_{2n} \otimes V/\ker A$ and define the skew-symmetric isomorphism $q_A :\ W_{4n+2} \xrightarrow{\sim} W^\vee_{4n+2}$ and the morphism of sheaves $a_A :\ H_{2n} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to W_{4n+2} \otimes \mathcal{O}_{\mathbb{P}^3}$ with H_{2n} and W_{4n+2} taken instead of H_n and W_A, respectively. The morphism a_A and its transpose ${}^t a_A = a^\vee_A \circ q_A :\ W_{4n+2} \otimes \mathcal{O}_{\mathbb{P}^3} \to H^\vee_{2n} \otimes \mathcal{O}_{\mathbb{P}^3}(1)$ yield a monad

$$\mathcal{M}_A :\quad 0 \to H_{2n} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_A} W_{4n+2} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{{}^t a_A} H^\vee_{2n} \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0$$

with cohomology sheaf $E(A)$, $[E(A)] \in I_{2n,1}$, see Theorem 2.1.

Let

$$i_\zeta : H_{n+1} \hookrightarrow H_{2n}$$

be the monomorphism defined by the isomorphism (5). The composition $a_B : H_{n+1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \overset{i_\zeta}{\hookrightarrow} H_{2n} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \overset{a_A}{\to} W_{4n+2} \otimes \mathcal{O}_{\mathbb{P}^3}$ and its transpose ${}^t a_B = a_B^\vee \circ q_A$ yield a monad

$$\mathcal{M}_B : \quad 0 \to H_{n+1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \overset{a_B}{\to} W_{4n+2} \otimes \mathcal{O}_{\mathbb{P}^3} \overset{{}^t a_B}{\to} H_{n+1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0$$

with the cohomology sheaf

$$E_{2n}(B) := \ker {}^t a_B / \mathrm{im} a_B, \quad c_2(E_{2n}(B)) = n+1.$$

The symplectic isomorphism $q_A : \; W_{4n+2} \overset{\sim}{\to} W_{4n+2}^\vee$ induces a symplectic structure on $E_{2n}(B)$,

$$\phi_B : E_{2n}(B) \overset{\sim}{\to} E_{2n}(B)^\vee. \tag{7}$$

Moreover, (6) implies an isomorphism $H_{n+1} \otimes V / \ker B \simeq W_{4n+2}$, hence a monomorphism of spaces of sections $h^0({}^t a_B) : W_{4n+2} \otimes \mathcal{O}_{\mathbb{P}^3} \overset{{}^t a_B}{\to} H_{n+1}^\vee V^\vee$ in the monad $\mathcal{M}_B$. Hence for this monad one has $h^0(E_{2n}(B)) = 0$. This together with (7) means that $E_{2n}(B)$ is a symplectic instanton:

$$[E_{2n}(B)] \in I_{n+1,n}.$$

Note that by construction the monads $\mathcal{M}_A$ and $\mathcal{M}_B$ fit into the commutative diagram (8)

$$\begin{tikzcd}
0 \arrow[r] & H_{n+1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r, "a_B"] \arrow[d, hook, "i_\zeta"] & W_{4n+2} \otimes \mathcal{O}_{\mathbb{P}^3} \arrow[r, "q_A", "\cong"'] \arrow[d, equal, "\cong"'] & W_{4n+2}^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \arrow[r, "a_B^\vee"] \arrow[d, equal, "w^\vee"', "\cong"] & H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \arrow[r] & 0 \\
0 \arrow[r] & H_{2n} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r, "a_A"] & W_{4n+2} \otimes \mathcal{O}_{\mathbb{P}^3} \arrow[r, "q_A", "\cong"'] & W_{4n+2}^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \arrow[r, "a_A^\vee"] & H_{2n}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \arrow[r] \arrow[u, "i_\zeta^\vee"] & 0,
\end{tikzcd}$$

In view of (7) and the canonical isomorphism $H_{2n}/i_\zeta(H_{n+1}) \simeq H_{n-1}$, this diagram yields the quotient monad

$$\mathcal{M}_{A,B} : \quad 0 \to H_{n-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \overset{a_{A,B}}{\to} E_{2n}(B) \underset{\simeq}{\overset{\phi_B}{\to}} E_{2n}(B)^\vee \overset{a_{A,B}^\vee}{\to} H_{n-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0$$

whose cohomology sheaf is

$$E_2(A) = \ker(a_{A,B}^\vee \circ \phi_B) / \mathrm{im}\ a_A.$$

2.2. **A special family of symplectic** $(2n-r+1, r)$**-instantons.** For any integer r, $2 \leq r \leq n-1$, with $n \geq 3$, consider a monomorphism

$$\tau : H_{2n-r+1} \hookrightarrow H_{2n} \tag{9}$$

such that

$$\tau(H_{2n-r+1}) \supset i_\zeta(H_{n+1}). \tag{10}$$

The image of $A \in MI_{2n,1}$ under the projection $\mathbf{S}_{2n} \twoheadrightarrow \mathbf{S}_{2n-r+1}$ induced by τ produces a hyperweb of quadrics

$$A_\tau \in \mathbf{S}_{2n-r+1}.$$

This corresponds to a monad

$$\mathcal{M}_\tau: \quad 0 \to H_{2n-r+1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_\tau} W_{4n+2} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{a_\tau^\vee \circ q_A} H_{2n-r+1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0,$$

whose cohomology is the rank $2r$ bundle

$$E_{2r}(A_\tau) = \ker(a_\tau^\vee \circ q_A)/\mathrm{im}\, a_\tau. \tag{11}$$

where $a_\tau := a_A \circ \tau$. The bundle $E_{2r}(A_\tau)$ has a natural symplectic structure

$$\phi_r: \ E_{2r}(A_\tau) \xrightarrow{\sim} E_{2r}(A_\tau)^\vee \tag{12}$$

induced by the antiselfduality of the monad $\mathcal{M}_\tau$. Moreover by (10) the monad $\mathcal{M}_\tau$ can be included into diagram (8) as a middle row, thus obtaining a three-row commutative, anti-self-dual diagram. Thus, in addition to the monad $\mathcal{M}_{A,B}$, we also have the monads

$$\mathcal{M}'_\tau: \quad 0 \to H_{n-r} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a'_\tau} E_{2n}(B) \underset{\simeq}{\xrightarrow{\phi}} E_{2n}(B)^\vee \xrightarrow{a'^\vee_\tau} H_{n-r}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \tag{13}$$

with cohomology

$$E_{2r}(A_\tau) = \ker(a'^\vee_\tau \circ \phi)/\mathrm{im}\, a'_\tau,$$

and

$$\mathcal{M}''_\tau: \quad 0 \to H_{r-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a''_\tau} E_{2r}(A_\tau) \underset{\simeq}{\xrightarrow{\phi_\tau}} E_{2r}(A_\tau)^\vee \xrightarrow{a''^\vee_\tau} H_{r-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \tag{14}$$

with cohomology

$$E_2(A) = \ker(a''^\vee_\tau \circ \phi_\tau)/\mathrm{im}\, a''_\tau.$$

Since $E_{2n}(B)$ is a symplectic instanton, $h^0(E_{2n}(B)) = h^i(E_{2n}(B)(-2)) = 0$, and the monad $\mathcal{M}'_\tau$ yields

$$h^0(E_{2r}(A_\tau)) = h^i(E_{2r}(A_\tau)(-2)) = 0, \ i \geq 0, \qquad c_2(E_{2r}(A_\tau)) = 2n - r + 1.$$

This, together with (12), means that

$$[E_{2r}(A_\tau)] \in I_{2n-r+1,r}. \tag{15}$$

Remark 2.3. The maps τ lie in the set

$$N_{n,r} := \{\tau \in \mathrm{Hom}(H_{2n-r+1}, H_{2n}) |\ \tau \text{ is injective and } \mathrm{im}\, \tau \supset \mathrm{im}\, i_\zeta\}$$

which, for fixed $A \in MI_{2n,1}(\zeta)$, parameterizes a family of hyperwebs A_τ from $MI_{2n-r+1,r}$. Now, $N_{n,r}$ is a principal $GL(H_{2n-r+1})$-bundle over an open subset of the Grassmannian $Gr(n-r, n-1)$, so it is irreducible. As a result, the family of the three-row extensions of the diagram (8) is parameterized by the irreducible variety $MI_{2n,1}(\zeta) \times N_{n,r}$. This in turn implies that the family $D_{n,r}$ of isomorphism classes of symplectic rank-$2r$ bundles obtained from these diagrams by (11) is an irreducible, locally closed subset of $I_{2n-r+1,r}$. It is not clear a priori if the closure of $D_{n,r}$ in $I_{2n-r+1,r}$ is an irreducible component of $I_{2n-r+1,r}$. △

Let $2 \leq r \leq n-1$. For every monomorphism $i : H_n \hookrightarrow H_{2n-r+1}$, denote by $B(A,i)$ the image of $A \in MI_{2n-r+1,r}$ under the projection $\mathbf{S}_{2n-r+1} \twoheadrightarrow \mathbf{S}_n$ induced by i. It may be regarded as a homomorphism $B(A,i) : H_n \otimes V \to H_n^\vee \otimes V^\vee$.

Definition 2.4. *We say that $A \in MI_{2n-r+1,r}$ satisfies property (*) if there exists a monomorphism $i : H_n \hookrightarrow H_{2n-r+1}$ such that $B(A,i)$ is invertible.*

This is an open condition on A. By Theorem 2.1, $\pi_{2n-r+1,r} : MI_{2n-r+1,r} \to I_{2n-r+1,r}$ is a principal bundle, so that, if an element $A \in \pi^{-1}_{2n-r+1,r}([E_{2r}])$ satisfies (*), then any other point $A' \in \pi^{-1}_{2n-r+1,r}([E_{2r}])$ satisfies (*). A symplectic instanton E_{2r} from $I_{2n-r+1,r}$ is said to be *tame* if some (hence all) $A \in \pi^{-1}_{2n-r+1,r}([E_{2r}])$ satisfies property (*). This is an open condition on $[E_{2r}] \in I_{2n-r+1,r}$.

Remark 2.5. Using (10), we see that any $[E_{2r}] \in D_{n,r}$ is tame. We define

$$I^*_{2n-r+1,r} := I_{(1)} \cup \ldots \cup I_{(k)},$$

where $I_{(1)}, \ldots, I_{(k)}$ are the irreducible components of $I_{2n-r+1,r}$ whose general points are tame symplectic instantons. As $D_{n,r} \subset I^*_{2n-r+1,r}$ by definition, $I^*_{2n-r+1,r}$ is nonempty. If we define $MI^*_{2n-r+1,r} = \pi^{-1}_{2n-r+1,r}(I^*_{2n-r+1,r})$, then the map $\pi_{2n-r+1,r} : MI^*_{2n-r+1,r} \to I^*_{2n-r+1,r}$ is a principal $GL(H_{2n-r+1})/\{\pm 1\}$-bundle. △

3. Irreducibility of $I^*_{2n-r+1,r}$

3.1. **A dense open subset of $MI^*_{2n-r+1,r}$.** We want to obtain the irreducibility of $I^*_{n,r}$ by reducing it to that of $X_{n,r}$, a dense open subset of $MI^*_{2n-r+1,r}$. The subset $X_{n,r}$ is a locally closed subset of the product of an affine space and an affine cone over a Grassmannian. Given an integer $n \geq 1$, we define the dense open subset of $\mathbf{S}_n$

$$\mathbf{S}^0_n := \{A \in \mathbf{S}_n \mid A : H_n \otimes V \to H_n^\vee \otimes V^\vee \text{ is an invertible map}\}.$$

We need some more notation. By definition, an element $B \in \mathbf{S}^0_n$ is an invertible anti-self-dual map $H_n \otimes V \to H_n^\vee \otimes V^\vee$. Its inverse $B^{-1} : H_n^\vee \otimes V^\vee \to H_n \otimes V$ is also anti-self-dual. Consider the vector space $\mathbf{\Sigma}_{n,r} := H^\vee_{n-r+1} \otimes H_n^\vee \otimes \wedge^2 V^\vee$. An element $C \in \mathbf{\Sigma}_{n,r}$ can be viewed as a linear map $C : H_{n-r+1} \otimes V \to H_n^\vee \otimes V^\vee$, and its dual $C^\vee : H_n \otimes V \to H^\vee_{n-r+1} \otimes V^\vee$.

As the composition $C^\vee \circ B^{-1} \circ C$ is anti-self-dual, we can consider it as an element of $\wedge^2(H_{n-r+1}^\vee \otimes V^\vee) \simeq \mathbf{S}_{n-r+1} \oplus \wedge^2 H_{n-r+1}^\vee \otimes S^2V^\vee$ Thus the condition

$$D - C^\vee \circ B^{-1} \circ C \in \mathbf{S}_{n-r+1}, \quad D \in \wedge^2(H_{n-r+1}^\vee \otimes V^\vee)$$

makes sense.

Under an arbitrary direct sum decomposition

$$\xi : H_n \oplus H_{n-r+1} \xrightarrow{\sim} H_{2n-r+1} \tag{16}$$

we can represent the hyperweb $A \in \mathbf{S}_{2n-r+1}$, regarded as a homomorphism $A : H_n \otimes V \oplus H_{n-r+1} \otimes V \to H_n^\vee \otimes V^\vee \oplus H_{n-r+1}^\vee \otimes V^\vee$, as the $(8n-4r+4) \times (8n-4r+4)$-matrix of homomorphisms

$$A = \begin{pmatrix} A_1(\xi) & A_2(\xi) \\ -A_2(\xi)^\vee & A_3(\xi) \end{pmatrix}, \tag{17}$$

where

$$A_1(\xi) \in \mathbf{S}_n, \quad A_2(\xi) \in \mathbf{\Sigma}_{n,r} := \mathrm{Hom}(H_n, H_{n-r+1}^\vee) \otimes \wedge^2 V^\vee, \quad A_3(\xi) \in \mathbf{S}_{n-r+1}.$$

With this notation, the decomposition (16) induces an isomorphism

$$\tilde{\xi} : \ \mathbf{S}_{2n-r+1} \xrightarrow{\sim} \mathbf{S}_n \oplus \mathbf{\Sigma}_{n,r} \oplus \mathbf{S}_{n-r+1}, \quad A \mapsto (A_1(\xi), A_2(\xi), A_3(\xi)). \tag{18}$$

Let $\mathrm{Isom}_{n,r}$ be the set of all isomorphisms ξ in (16). According to Definition 2.4, there exists $\xi \in \mathrm{Isom}_{n,r}$ such that the set

$$MI_{2n-r+1,r}^*(\xi) := \{A \in MI_{2n-r+1,r} \mid A \text{ satisfies property } (*) \text{ for the monomorphism}$$
$$i_\xi : H_n \hookrightarrow \ H_{2n-r+1} \text{ determined by } \xi\}$$

is a dense open subset of $MI_{2n-r+1,r}^*$. Now take $A \in MI_{2n-r+1,r}^*(\xi)$ and consider A as a matrix of homomorphisms as in (17). By definition, the submatrix $A_1(\xi)$ is invertible. By a suitable elementary transformation we reduce the matrix A to an equivalent matrix $\tilde{A}$ of the form

$$\tilde{A} = \begin{pmatrix} \mathrm{id}_{H_n \otimes V} & A_1(\xi)^{-1} \circ A_2(\xi) \\ 0 & A_2(\xi)^\vee \circ A_1(\xi)^{-1} \circ A_2(\xi) + A_3(\xi) \end{pmatrix}.$$

Since $\mathrm{rk}\tilde{A} = \mathrm{rk}A = 2(2n-r+1) + 2r = 4n+2$, we obtain the following relation between the matrices $A_1(\xi),\ A_2(\xi)$ and $A_3(\xi)$:

$$\mathrm{rk}(A_2(\xi)^\vee \circ A_1(\xi)^{-1} \circ A_2(\xi) + A_3(\xi)) = 2. \tag{19}$$

Consider the embedding of the Grassmannian

$$G := Gr(2, H_{n-r+1}^\vee \otimes V^\vee) \hookrightarrow P(\wedge^2(H_{n-r+1}^\vee \otimes V^\vee)),$$

and let $KG \subset \wedge^2(H^\vee_{n-r+1} \otimes V^\vee)$ be the affine cone over G. Set $KG^* := KG \smallsetminus \{0\}$. We can now rewrite (19) as

$$A_2(\xi)^\vee \circ A_1(\xi)^{-1} \circ A_2(\xi) + A_3(\xi) \in KG^*, \tag{20}$$

where

$$A_2(\xi)^\vee \circ A_1(\xi)^{-1} \circ A_2(\xi) \in \wedge^2(H^\vee_{n-r+1} \otimes V^\vee), \quad A_3(\xi) \in \mathbf{S}_{n-r+1}. \tag{21}$$

Now consider the set

$$\widetilde{X}_{n,r} := \{(B,C,D) \in \mathbf{S}^0_n \times \mathbf{\Sigma}_{n,r} \times KG^* \mid D - C^\vee \circ B^{-1} \circ C \in \mathbf{S}_{n-r+1}\}. \tag{22}$$

Since for an arbitrary point $y = (B,C,D) \in \tilde{X}_n$ the point $\tilde{\xi}^{-1}(B,C,D - C^\vee \circ B^{-1} \circ C)$ lies in $\mathbf{S}_{2n-r+1}$, it may be considered as a homomorphism $A_y : \ H_{2n-r+1} \otimes V \to H^\vee_{2n-r+1} \otimes V^\vee$ of rank $4n+2$, and we have a well-defined $(4n+2)$-dimensional vector space $W_{4n+2}(y) := H_{2n-r+1} \otimes V/\ker A_y$ together with a canonical epimorphism $c_y : \ H_{2n-r+1} \otimes V \twoheadrightarrow W_{4n+2}(y)$ and an induced skew-symmetric isomorphism $q_y : \ W_{4n+2}(y) \xrightarrow{\sim} W_{4n+2}(y)^\vee$ such that $A_y = c^\vee_y \circ q_y \circ c_y$. Now, similarly to the morphism $a_A : \ H_{2n-r+1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to W_{4n+2} \otimes \mathcal{O}_{\mathbb{P}^3}$ (see subsection 2.1), a morphism of sheaves

$$a_y = c_y \circ u : \ H_{2n-r+1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to W_{4n+2}(y) \otimes \mathcal{O}_{\mathbb{P}^3} \tag{23}$$

is defined, together with its transpose ${}^t a_y = a^\vee_y \circ q_y : \ W^\vee_{4n+2}(y) \otimes \mathcal{O}_{\mathbb{P}^3} \to H^\vee_{2n-r+1} \otimes \mathcal{O}_{\mathbb{P}^3}(1)$. We now introduce an open subset $X_{n,r}$ of the set $\widetilde{X}_{n,r}$,

$$X_{n,r} := \left\{ y \in \widetilde{X}_{n,r} \ \middle|\ \begin{array}{l} (i)\ {}^t a_y \text{ is epimorphic,} \\ (ii)\ [\ker {}^t a_y / \mathrm{im} a_y] \in I^*_{2n-r+1,r} \end{array} \right\}. \tag{24}$$

Since the conditions (i) and (ii) on a point $y \in \widetilde{X}_{n,r}$ in (24) are open, from (20) and (21) we obtain the following result.

Proposition 3.1. *There exist a decomposition $\xi \in \mathrm{Isom}_{n,r}$, a dense open subset $MI^*_{2n-r+1,r}(\xi)$ of $MI^*_{2n-r+1,r}$ and an isomorphism of reduced schemes*

$$f_{n,r} : MI^*_{2n-r+1,r}(\xi) \xrightarrow{\sim} X_{n,r}, \ A \mapsto (A_1(\xi), A_2(\xi), A_3(\xi)).$$

The inverse isomorphism is given by the formula

$$f^{-1}_{n,r} : \ X_{n,r} \xrightarrow{\sim} MI^*_{2n-r+1,r}(\xi) : \ (B,C,D) \mapsto \ \tilde{\xi}^{-1}(B, \ C, \ D - C^\vee \circ B^{-1} \circ C),$$

where $\widetilde{\xi}$ is defined in (18).

The following theorem will be proved in Subsection 3.2.

Theorem 3.2. *$X_{n,r}$ is irreducible of dimension* $(2n-r+1)^2 + 4(2n-r+1)(r+1) - r(2r+1)$.

Proposition 3.1 and Theorem 3.2 imply that $MI^*_{2n-r+1,r}$ is irreducible of dimension $(2n-r+1)^2+4(2n-r+1)(r+1)-r(2r+1)$ for any $n \leq 3$ and $2 \leq r \leq n-1$. Thus, for these values of n and r, the space $I^*_{2n-r+1,r}$ is irreducible and has dimension $4(2n-r+1)(r+1)-r(2r+1)$. Substituting $2n-r+1 \mapsto n$, we obtain the main result of this paper.

Theorem 3.3. *For any integer $r \geq 2$ and for any integer $n \geq r-1$ such that $n \equiv r-1(\mathrm{mod}2)$, the moduli space $I^*_{n,r}$ of tame symplectic instantons is an open subset of an irreducible component of $I_{n,r}$ of dimension $4n(r+1)-r(2r+1)$.*

3.2. **Proof of the irreducibility of $X_{n,r}$.** We prove now Theorem 3.2. Consider the set $\widetilde{X}_{n,r}$ defined in (22). Since $X_{n,r}$ is an open subset of $\widetilde{X}_{n,r}$, it is enough to prove the irreducibility of $\widetilde{X}_{n,r}$. In view of the isomorphism $\mathbf{S}^0_n \xrightarrow{\sim} (\mathbf{S}^\vee_n)^0 :\ B \mapsto B^{-1}$, we rewrite $\widetilde{X}_{n,r}$ as

$$\widetilde{X}_{n,r} = \{(B, C, D) \in (\mathbf{S}^\vee_n)^0 \times \mathbf{\Sigma}_{n,r} \times KG^* \mid D - C^\vee \circ B \circ C \in \mathbf{S}_{n-r+1}\}.$$

If a direct sum decomposition

$$H_n \xrightarrow{\sim} H_{n-r+1} \oplus H_{r-1}$$

has been fixed, any linear map

$$C \in \mathbf{\Sigma}_{n,r} = \mathrm{Hom}(H_{n-r+1}, H^\vee_n \otimes \wedge^2 V^\vee), \quad C : H_{n-r+1} \otimes V \to H^\vee_n \otimes V^\vee,$$

can be represented as a homomorphism

$$C :\ H_{n-r+1} \otimes V \to H^\vee_{n-r+1} \otimes V^\vee \ \oplus\ H^\vee_{r-1} \otimes V^\vee,$$

and also as a block matrix

$$C = \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \tag{25}$$

with

$$\phi \in \mathrm{Hom}(H_{n-r+1}, H^\vee_{n-r+1}) \otimes \wedge^2 V^\vee = \mathbf{\Phi}_{n-r+1}, \quad \psi \in \mathbf{\Psi}_{n,r} := \mathrm{Hom}(H_{n-r+1}, H^\vee_{r-1}) \otimes \wedge^2 V^\vee.$$

In the same way, any $D \in (\mathbf{S}^\vee_n)^0 \subset \mathbf{S}^\vee_n = S^2 H_n \ \otimes \wedge^2 V \subset \mathrm{Hom}(H^\vee_n \otimes V^\vee, H_n \otimes V)$ can be represented as

$$B = \begin{pmatrix} B_1 & \lambda \\ -\lambda^\vee & \mu \end{pmatrix}, \tag{26}$$

with

$$B_1 \in \mathbf{S}^\vee_{n-r+1} \subset \mathrm{Hom}(H^\vee_{n-r+1} \otimes V^\vee, H_{n-r+1} \otimes V), \tag{27}$$

$$\lambda \in \mathbf{L}_{n,r} := \mathrm{Hom}(H^\vee_r, H_{n-r+1}) \otimes \wedge^2 V, \quad \mu \in \mathbf{M}_{r-1} := S^2 H_{r-1} \otimes \wedge^2 V.$$

By (25) and (26) the composition

$$C^\vee \circ B \circ C : H_{n-r+1} \otimes V \to H_{n-r+1}^\vee \otimes V^\vee \quad (C^\vee \circ B \circ C \in \wedge^2(H_{n-r+1}^\vee \otimes V^\vee))$$

can be written in the form

$$C^\vee \circ B \circ C = \phi^\vee \circ B_1 \circ \phi + \phi^\vee \circ \lambda \circ \psi - \psi^\vee \circ \lambda^\vee \circ \phi + \psi^\vee \circ \mu \circ \psi. \tag{28}$$

In view of (25)-(27) we have

$$\mathbf{S}_n^\vee \times \mathbf{\Sigma}_{n,r} = \mathbf{S}_{n-r+1}^\vee \times \mathbf{\Phi}_{n-r+1} \times \mathbf{\Psi}_{n,r} \times \mathbf{L}_{n,r} \times \mathbf{M}_{r-1},$$

and well-defined morphisms

$$\tilde{p} : \widetilde{X}_{n,r} \to \mathbf{L}_{n,r} \times \mathbf{M}_r \times KG, \ (B_1, \phi, \psi, \lambda, \mu, D) \mapsto (\lambda, \mu, D).$$

and

$$p := \tilde{p}|\overline{X}_{n,r} : \overline{X}_{n,r} \to \mathbf{L}_{n,r} \times \mathbf{M}_{r-1} \times KG.$$

Here $\overline{X}_{n,r}$ is the closure of $\widetilde{X}_{n,r}$ in $(\mathbf{S}_n^\vee)^0 \times \mathbf{\Sigma}_{n,r} \times KG$. Moreover, we have:

Proposition 3.4. *Let $n \geq 2$. For any $B \in (\mathbf{S}_n^\vee)^0$ and for a general choice of the decomposition $H_n \simeq H_{n-r+1} \oplus H_{r-1}$, the block B_1 of B in* (26) *is nondegenerate.*

Proof. By applying Proposition 7.3] in [9] r times, one obtains a decomposition $H_n \xrightarrow{\sim} H_{n-r+1} \oplus H_{r-1}$ such that $B_1 : H_{n-r+1}^\vee \otimes V^\vee \to H_{n-r+1} \otimes V$ in (26) is nondegenerate, that is, $B_1 \in (\mathbf{S}_{n-r+1}^\vee)^0$. □

If $\mathcal{X}$ is any irreducible component of $X_{n,r}$, taken with its reduced structure, and $\overline{\mathcal{X}}$ is its closure in $\overline{X}_{n,r}$, we pick up a point $z = (B_1, \phi, \psi, \lambda, \mu, D) \in \mathcal{X}$ not lying in the components of $X_{n,r}$ different from $\mathcal{X}$, and such that the decomposition $H_n \simeq H_{n-r+1} \oplus H_{r-1}$ is general. Then, by Proposition 3.4, $B_1 \in (\mathbf{S}_{n-r+1}^\vee)^0$. Consider the morphism

$$f : \ \mathbb{A}^1 \to \overline{\mathcal{X}}, \ t \mapsto (B_1, t^2\phi, t\psi, t\lambda, t^2\mu, t^4 D), \quad f(1) = z.$$

This is well defined as a consequence of (28). The point $f(0) = (B_1, 0, 0, 0, 0, 0)$ lies in the fibre $p^{-1}(0,0,0)$, so that $p^{-1}(0,0,0) \cap \overline{\mathcal{X}} \neq \varnothing$. In different terms,

$$\rho^{-1}(0,0,0) \neq \varnothing, \quad \text{where} \quad \rho := p|\overline{\mathcal{X}}. \tag{29}$$

By (28) and the definition of $\widetilde{X}_{n,r}$, one has

$$\tilde{p}^{-1}(0,0,0) = \{(B_1, \phi, \psi) \in (\mathbf{S}_{n-r+1}^\vee)^0 \times \mathbf{\Phi}_{n-r+1} \times \mathbf{\Psi}_{n,r} \mid \phi^\vee \circ B_1 \circ \phi \in \mathbf{S}_{n-r+1}\}. \tag{30}$$

Now for each $i \geq 1$ consider the set Z_i mentioned in the introduction. This set Z_i is defined in [9, Section 7] as

$$Z_i = \{(B, \phi) \in (\mathbf{S}_i^\vee)^0 \times \mathbf{\Phi}_i \mid \phi^\vee \circ B \circ \phi \in \mathbf{S}_i\}, \tag{31}$$

and has a natural structure of closed subscheme of $(\mathbf{S}_i^\vee)^0 \times \mathbf{\Phi}_i$ The key point in the sequel is the fact that Z_i is an integral scheme of dimension $4i(i+2)$—see [9, Theorem 7.2]. This statement is based on the following relation between Z_i for $i \geq 2$ and the moduli space of 't Hooft instantons of charge $2i-1$. Fix a monomorphism $j : H_{i-1} \hookrightarrow H_i$. For an arbitrary point $z = (B, \phi) \in Z_i$, let E_{2i} be a symplectic vector bundle of rank $2i$ defined as a cokernel of a morphism of sheaves $\tilde{B} : H_i \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to H_i^\vee \otimes \Omega_{\mathbb{P}^3}(1)$ naturally induced by B. Let $s(z) : H_i \to H^0(E_{2i}(1))$ be the composition of ϕ understood as a homomorphism $H_i \to H_i^\vee \otimes \wedge^2 V^\vee$ and of the evaluation map $H_i^\vee \otimes \wedge^2 V^\vee \to H^0(E_{2i}(1))$, and let s_z be the composition $s_z : H_i \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \stackrel{s(z)}{\to} H^0(E_{2i}(1)) \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \stackrel{ev}{\to} E_{2i}$, where ev is the evaluation morphism. Using the symplecticity of E_{2i}, one obtains an antiselfdual monad $M(z) : 0 \to H_{i-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \stackrel{s_z \circ j}{\to} E_{2i} \stackrel{^t(s_z \circ j)}{\to} H_{i-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0$ with a rank-2 cohomology vector bundle $E_2(z)$ with $c_1 = 0$ and $c_2 = 2i-1$. A standard diagram chase yields a monomorphism $H_i/j(H_{i-1}) \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to E_2(z)$ showing that $h^0(E_2(z)(1)) \neq 0$, i. e. that $E_2(z)$ is a 't Hooft instanton vector bundle. Thus the association $z \rightsquigarrow M(z)$ yields a morphism of Z_i to the space M_{2i-1}^{tH} of the 't Hooft monads, which is irreducible since the moduli space of 't Hooft instantons of charge $2i-1$ is known to be irreducible. It is shown in [9, Section 9] that this morphism $Z_i \to M_{2i-1}^{tH}$ is a composition of a dense open embedding and the structure map of an affine bundle over M_{2i-1}^{tH}. This implies the irreducibility of Z_i.

Now, comparing (31) for $i = n-r+1$ with (30), we obtain scheme-theoretic inclusions

$$\rho^{-1}(0,0,0) \subset p^{-1}(0,0,0) \subset \tilde{p}^{-1}(0,0,0) = Z_{n-r+1} \times \mathbf{\Psi}_{n,r}. \tag{32}$$

By the above, Z_{n-r+1} is an integral scheme of dimension $4(n-r+1)(n-r+3)$. This together with (32) implies that

$$\dim \rho^{-1}(0,0,0) \leq \dim p^{-1}(0,0,0) \leq \dim Z_{n-r+1} + \dim \mathbf{\Psi}_{n,r} = 4(n-r+1)(n-r+3) \tag{33}$$
$$+6(r-1)(n-r+1) = (n-r+1)(4n+2r+6).$$

Hence, in view of (29),

$$\begin{aligned} \dim \overline{\mathcal{X}} &\leq \dim \rho^{-1}(0,0,0) + \dim \mathbf{L}_{n,r} + \dim \mathbf{M}_{r-1} + \dim KG \\ &\leq (n-r+1)(4n+2r+6) + 6(r-1)(n-r+1) + 3(r-1)r + (8n-8r+5) \\ &= (2n-r+1)^2 + 4(2n-r+1)(r+1) - r(2r+1). \end{aligned} \tag{34}$$

On the other hand, formula (4)—with n replaced by $2n-r+1$—and Proposition 3.1 show that, for any point $x \in \mathcal{X}$ such that $A := f_{n,r}^{-1}(x) \in MI_{2n-r+1,r}^*(\xi)$,

$$(2n-r+1)^2 + 4(2n-r+1)(r+1) - r(2r+1) \leq \dim_A MI_{2n-r+1,r}^*(\xi) = \dim \overline{\mathcal{X}}. \tag{35}$$

Comparing (34) with (35), we see that all the inequalities in (33)–(35) are equalities. In particular,

$$\dim \rho^{-1}(0,0) = \dim(Z_{n-r+1} \times \mathbf{\Psi}_{n,r}) = \dim \overline{\mathcal{X}} - \dim(\mathbf{L}_{n,r} \times \mathbf{M}_{r-1} \times KG). \tag{36}$$

Since, by Theorem [9, Theorem 7.2], the scheme Z_{n-r+1} is integral and so $Z_{n-r+1} \times \mathbf{\Psi}_{n,r}$ is integral as well, (32) and (36) yield the coincidence of the integral schemes

$$\rho^{-1}(0,0,0) = p^{-1}(0,0,0) = \tilde{p}^{-1}(0,0,0) = Z_{n-r+1} \times \mathbf{\Psi}_{n,r}. \tag{37}$$

We need now the following easy Lemma, which is a slight generalization of Lemma 7.4 from [9].

Lemma 3.5. *Let $f : X \to Y$ be a morphism of reduced schemes, with Y an integral scheme. Assume that there exists a closed point $y \in Y$ such that, for any irreducible component X' of X,*

(a) $\dim f^{-1}(y) = \dim X' - \dim Y$*,*

(b) the scheme-theoretic inclusion of fibres $(f|_{X'})^{-1}(y) \subset f^{-1}(y)$ is an isomorphism of integral schemes.

Then

(i) there exists an open subset U of Y containing y such that the morphism $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is flat, and

(ii) X is integral.

By applying this lemma to $X = X_{n,r},\ X' = \mathcal{X},\ Y = \mathbf{L}_{n,r} \times \mathbf{M}_{r-1} \times KG,\ y = (0,0), f = p$, also in view of (36) and (37), one obtains that $X_{n,r}$ is integral and is of dimension

$$(2n - r + 1)^2 + 4(2n - r + 1)(r + 1) - r(2r + 1).$$

Theorem 3.2 is thus proved.

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