THE GROWTH OF THE RANK OF ABELIAN VARIETIES UPON EXTENSIONS

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Abstract. We study the growth of the rank of elliptic curves and, more generally, Abelian varieties upon extensions of number fields.

First, we show that if L/K is a finite Galois extension of number fields such that $\mathrm{Gal}(L/K)$ does not have an index 2 subgroup and A/K is an Abelian variety, then $\operatorname{rk} A(L) - \operatorname{rk} A(K)$ can never be 1. We obtain more precise results when $\mathrm{Gal}(L/K)$ is of odd order, alternating, $\mathrm{SL}_2(\mathbb{F}_p)$ or $\mathrm{PSL}_2(\mathbb{F}_p)$. This implies a restriction on $\operatorname{rk} E(K(E[p])) - \operatorname{rk} E(K(\zeta_p))$ when E/K is an elliptic curve whose mod p Galois representation is surjective. Similar results are obtained for the growth of the rank in certain non-Galois extensions.

Second, we show that for every $n \geq 2$ there exists an elliptic curve E over a number field K such that $\mathbb{Q} \otimes \mathrm{End}_{\mathbb{Q}} \mathrm{Res}_{K/\mathbb{Q}} E$ contains a number field of degree 2^n. We ask whether every elliptic curve E/K has infinite rank over $K\mathbb{Q}(2)$, where $\mathbb{Q}(2)$ is the compositum of all quadratic extensions of $\mathbb{Q}$. We show that if the answer is yes, then for any $n \geq 2$, there exists an elliptic curve E/K admitting infinitely many quadratic twists whose rank is a positive multiple of 2^n.

1. Introduction

Let A be an Abelian variety over a number field K. By the Mordell–Weil theorem, the Abelian group $A(K)$ of K-rational points of A is finitely generated, so it is of the form $T \oplus \mathbb{Z}^r$, where T is the torsion group and r is the rank of A. In this paper, we study the growth of the rank of Abelian varieties upon extensions of number fields.

In Section 2, we consider an Abelian variety A over a number field K and a finite Galois extension L/K. We show that the structure of the group $G = \mathrm{Gal}(L/K)$ can impose restrictions on the growth of the rank of A under base extension from K to L. Suppose throughout this paragraph that $\operatorname{rk} A(L)$ is strictly greater than $\operatorname{rk} A(K)$. We prove that if G has odd order, then $\operatorname{rk} A(L) - \operatorname{rk} A(K) \geq p - 1$, where p is the smallest prime factor of $\#G$. If G is the alternating group A_n with $n \geq 5$, we obtain $\operatorname{rk} A(L) - \operatorname{rk} A(K) \geq n - 1$. If G is A_3 or A_4, then $\operatorname{rk} A(L) - \operatorname{rk} A(K) \geq 2$. If G is $\mathrm{SL}_2(\mathbb{F}_p)$ or $\mathrm{PSL}_2(\mathbb{F}_p)$ for a prime $p > 2$, then $\operatorname{rk} A(L) - \operatorname{rk} A(K) \geq \frac{p-1}{2}$. As a corollary, if E/K is an elliptic curve and $p > 2$ is a prime such that the mod p Galois representation of E is surjective, then $\operatorname{rk} E(K(E[p])) - \operatorname{rk} E(K(\zeta_p))$ is either zero or at least $\frac{p-1}{2}$. We prove that $\operatorname{rk} A(L) - \operatorname{rk} A(K) \geq 2$ whenever G does not have an index 2 subgroup.

Furthermore, for certain non-Galois extensions L/K with Galois closure M, we exhibit non-trivial relations between $\operatorname{rk} A(M) - \operatorname{rk} A(K)$ and $\operatorname{rk} A(L) - \operatorname{rk} A(K)$.

In Section 3, we build on the ideas of [1], where it was proved that $\mathbb{Q}$-curves of certain type have additional structure, which forces them to have even rank over their field of definition. The additional structure of these curves can be seen in

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the endomorphisms of their restrictions of scalars. In a similar way we construct, for arbitrarily large n, an elliptic curve E_n defined over a number field K_n such that the endomorphism algebra of the restriction of scalars $\operatorname{Res}_{K_n/\mathbb{Q}} E_n$ contains a cyclotomic field of degree 2^n.

Let $\mathbb{Q}(2)$ be the compositum of all quadratic extensions of $\mathbb{Q}$. We ask the following question: does every elliptic curve E over a number field K have infinite rank over $K\mathbb{Q}(2)$? We show that if the answer is yes, then an elliptic curve E_n/K_n as above has infinitely many quadratic twists whose rank is a positive multiple of 2^n.

2. Growth of the rank in extensions

In this section we study how the rank of an Abelian variety can grow upon finite extensions.

2.1. **Galois extensions.**

Let A be an Abelian variety over K, and let L be a finite Galois extension of K. We compare the ranks of $A(K)$ and $A(L)$ by looking at the action of $G = \operatorname{Gal}(L/K)$ on $\mathbb{Q} \otimes A(L)$. In particular, we are interested in the possible $\mathbb{Q}$-dimensions of $\mathbb{Q} \otimes (A(L)/A(K))$. This is controlled by the dimensions of the non-trivial irreducible $\mathbb{Q}$-linear representations of G. For example, the smallest non-trivial increase of the rank of A when going from K to L equals the smallest dimension of an irreducible non-trivial $\mathbb{Q}$-linear representation of $\operatorname{Gal}(L/K)$. We will study such constraints on the rank of $A(L)$ for various groups G.

Theorem 1. *Let $p > 2$ be a prime, let A be an Abelian variety defined over a number field K, and let L be a Galois extension of K such that $\operatorname{Gal}(L/K) \simeq \mathbb{Z}/p\mathbb{Z}$. Then*

$$\operatorname{rk} A(L) \equiv \operatorname{rk} A(K) \pmod{p-1}.$$

Proof. First note that the fixed subspace of $\mathbb{Q} \otimes A(L)$ under $\operatorname{Gal}(L/K)$ corresponds to $\mathbb{Q} \otimes A(K)$, the dimension of which is $\operatorname{rk} A(K)$.

Over $\mathbb{C}$, the irreducible representations of the group $\mathbb{Z}/p\mathbb{Z}$ are the trivial representation and the $p-1$ representations corresponding to the non-trivial characters of $\mathbb{Z}/p\mathbb{Z}$. However, the smallest field over which the non-trivial characters are defined is $\mathbb{Q}(\zeta_p)$. In fact, there are two irreducible $\mathbb{Q}$-linear representations: the trivial representation and a representation of dimension $p-1$. This implies that $\operatorname{rk} A(L) - \operatorname{rk} A(K) = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes (A(L)/A(K)))$ is a multiple of $p-1$. □

Corollary 2. *Let A be an Abelian variety defined over a number field K, and let L/K be a Galois extension of odd degree n. Then*

$$\operatorname{rk} A(L) = \operatorname{rk} A(K) \text{ or } \operatorname{rk} A(L) \geq \operatorname{rk} A(K) + p - 1,$$

where p is the smallest prime dividing n.

Proof. By the Feit–Thompson theorem [4], $\operatorname{Gal}(L/K)$ is solvable, so there is a sequence of intermediate fields $K = K_0 \subset K_1 \subset \cdots \subset K_r = L$ such that each extension K_i/K_{i-1} is cyclic of some prime degree $p_i \mid n$. Theorem 1 implies that $\operatorname{rk} A(L) - \operatorname{rk} A(K)$ is a linear combination of the $p_i - 1$, and the claim follows. □

We now turn our attention to the case where G is an alternating group.

Theorem 3. *Let A be an Abelian variety defined over a number field K. Let $n \geq 3$ and let L/K be a Galois extension with group A_n. Then*

$$\operatorname{rk} A(L) = \operatorname{rk} A(K) \text{ or } \operatorname{rk} A(L) \geq \operatorname{rk} A(K) + \begin{cases} 2 & \text{if } n = 4, \\ n-1 & \text{if } n = 3 \text{ or } n \geq 5. \end{cases}$$

Proof. As $A_3 \simeq \mathbb{Z}/3\mathbb{Z}$, the case $n = 3$ is already proved in Theorem 1.

The group A_4 has two non-trivial complex representations of dimension 1, but their images involve third roots of unity and are therefore are not defined over $\mathbb{Q}$. Hence all non-zero, non-trivial representations of A_4 over $\mathbb{Q}$ have dimension ≥ 2.

The group A_5 has two irreducible complex representations of dimension 3, but these involve fifth roots of unity. The minimal dimension of a non-trivial $\mathbb{Q}$-linear irreducible representation equals 4.

It is well known [6, Exercise 5.5] that the dimension of the smallest irreducible $\mathbb{C}$-linear (and hence also $\mathbb{Q}$-linear) representation of A_n is of dimension $n - 1$ for $n > 5$, completing the proof. □

Remark. In the setting of Theorem 3 for $n = 4$, as A_4 has one 2-dimensional and one 3-dimensional $\mathbb{Q}$-linear irreducible representation. As every positive integer apart from 1 can be written as the sum of multiples of 2 and 3, it follows that $\operatorname{rk} A(L) - \operatorname{rk} A(K)$ can a priori be any non-negative integer apart from 1.

Here is a similar result about extensions with Galois group $\mathrm{SL}_2(\mathbb{F}_p)$ or $\mathrm{PSL}_2(\mathbb{F}_p)$.

Theorem 4. *Let A be an Abelian variety over a number field K, and let L be a finite Galois extension with group $\mathrm{SL}_2(\mathbb{F}_p)$ or $\mathrm{PSL}_2(\mathbb{F}_p)$ for some prime $p > 2$. Then*

$$\operatorname{rk} A(L) = \operatorname{rk} A(K) \ \text{ or } \ \operatorname{rk} A(L) \geq \operatorname{rk} A(K)) + \frac{p-1}{2}.$$

Proof. The minimal dimension of an irreducible non-trivial $\mathbb{Q}$-linear representation of $\mathrm{SL}_2(\mathbb{F}_p)$ is $(p-1)/2$ [6, Chapter 5.2, pages 71–73]. Similarly, the smallest dimension of a non-trivial irreducible $\mathbb{Q}$-linear representation of $\mathrm{PSL}_2(\mathbb{F}_p)$ is $(p-1)/2$ [6, Exercise 5.10, page 71]. □

Corollary 5. *Let E be an elliptic curve over a number field K, and let $p > 2$ be a prime such that the Galois representation*

$$\rho_p\colon\ \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{F}_p).$$

coming from the action of $\mathrm{Gal}(\overline{K}/K)$ on $E[p]$ is surjective. Let L be a subfield of $K(E[p])$ with $[K(E[p]) : L] \leq 2$. Then

$$\operatorname{rk} E(L) = \operatorname{rk} E(K(\zeta_p)) \ \text{ or } \ \operatorname{rk} E(L) \geq \operatorname{rk} E(K(\zeta_p)) + \frac{p-1}{2}$$

Proof. We note that ρ_p factors through $\mathrm{Gal}(K(E[p])/K)$. By the properties of the Weil pairing, $K(E[p])$ contains $K(\zeta_p)$, and ρ_p identifies $\mathrm{Gal}(K(E[p])/K(\zeta_p))$ with $\mathrm{SL}_2(\mathbb{F}_p)$. If $[K(E[p]) : L] = 2$, then L is the fixed field of $-I$, the unique element of order 2 in $\mathrm{SL}_2(\mathbb{F}_p)$, and ρ_p identifies $\mathrm{Gal}(L/K(\zeta_p))$ with $\mathrm{PSL}_2(\mathbb{F}_p)$. Hence $\mathrm{Gal}(K(E[p])/K(\zeta_p))$ is isomorphic to $\mathrm{SL}_2(\mathbb{F}_p)$ if $L = K(E[p])$, and to $\mathrm{PSL}_2(\mathbb{F}_p)$ if $[K(E[p]) : L] = 2$. The claim now follows from Theorem 4. □

Remark. By Serre's open image theorem [13], if E does not have complex multiplication, then the map ρ_p is surjective for all but finitely many primes p.

In view of the results proved in this section, it is natural to wonder what characterizes the groups G such that for Galois extensions L/K with group G, one can give a non-trivial lower bound on $\operatorname{rk} A(L) - \operatorname{rk} A(K)$, whenever $\operatorname{rk} A(L) - \operatorname{rk} A(K) > 0$. The following theorem answers this question.

Theorem 6. *Let L/K be a finite Galois extension of number fields such that $G = \mathrm{Gal}(L/K)$ does not contain a subgroup of index 2. Then for any Abelian variety A over K, either $\operatorname{rk} A(L) = \operatorname{rk} A(K)$ or $\operatorname{rk} A(L) \geq \operatorname{rk} A(K) + 2$.*

Proof. As G has no subgroup of index 2, there is no non-trivial homomorphism $G \to \mathbb{Q}^\times_{\mathrm{tors}} = \{1, -1\}$. Therefore G has no non-trivial irreducible representation of dimension 1 over $\mathbb{Q}$. $\square$

Note that none of the groups G considered above has an index 2 subgroup. If L/K is a finite Galois extension for which $\mathrm{Gal}(L/K)$ does have an index 2 subgroup, then L contains $K(\sqrt{d})$ for some $d \in K^\times \setminus (K^\times)^2$. If A is an Abelian variety over K, we cannot exclude that $\mathrm{rk}\, A(L) = \mathrm{rk}\, A(K(\sqrt{d})) > \mathrm{rk}\, A(K)$. Now $\mathrm{rk}\, A(K(\sqrt{d})) = \mathrm{rk}\, A(K) + \mathrm{rk}\, A^d(K)$, where A^d is the quadratic twist of A by d; in general, we cannot prove any restrictions on $\mathrm{rk}\, A^d(K)$. So index 2 subgroups form an obstruction for results of the type of Theorems 1, 3 and 4; in this sense, Theorem 6 is best possible.

Remark. The above results rely on the decomposition of $A(L)$ into irreducible representations of $G = \mathrm{Gal}(L/K)$. More conceptually, they can be interpreted via a decomposition of the Weil restriction $B = \mathrm{Res}_{L/K} A_L$ in the category of Abelian varieties over K up to isogeny, namely

$$B \sim \bigoplus_{\rho} B_\rho,$$

where ρ ranges over the irreducible $\mathbb{Q}$-linear representations of G and the group algebra $\mathbb{Q}[G]$ acts on B_ρ through a simple quotient algebra R_ρ; see [3, § 3.4] and [8, Theorem 4.5]. Our results can then be explained by the fact that in the situations we consider, R_ρ is strictly larger than $\mathbb{Q}$ for all non-trivial ρ.

2.2. **Non-Galois extensions.** We start by recalling a bit of representation theory. Let G be a finite group, and let $H \subseteq G$ a subgroup. For finite-dimensional $\mathbb{Q}$-linear representations V of G, we are interested in non-trivial relations between the dimensions of the $\mathbb{Q}$-vector spaces $V^G \subseteq V^H \subseteq V$.

Let C_G denote the set of conjugacy classes of G, and let χ_V denote the character of the representation V, viewed as a function on C_G. It is well known, as a special case of Schur's orthogonality relations, that the dimension of V^G equals

$$\begin{aligned} d_G(\chi_V) &= \frac{1}{\#G} \sum_{g \in G} \chi_V(\text{conjugacy class of } g) \\ &= \frac{1}{\#G} \sum_{c \in C_G} \#c \cdot \chi_V(c). \end{aligned}$$

We can write $\chi_V = \sum_{\chi \in X(G)} n_\chi \chi$, where $X(G)$ is the set of characters of irreducible $\mathbb{Q}$-linear representations of G and the n_χ are non-negative integers. Then we have

$$\begin{aligned} \dim V^G &= n_{\mathbf{1}}, \\ \dim V^H &= \sum_{\chi \in X(G)} n_\chi d_H(\chi), \\ \dim V &= \sum_{\chi \in X(G)} n_\chi d_{\{\mathrm{id}\}}(\chi). \end{aligned}$$

In the above notation, $d_{\{\mathrm{id}\}}(\chi) = \chi(\{\mathrm{id}\})$ is the dimension of the irreducible representation of G with character χ. The results obtained above for Galois extensions L/K with group G are explained by the fact that in all the cases we considered, $d_{\{\mathrm{id}\}}(\chi) > 1$ for all non-trivial irreducible representations χ of G over $\mathbb{Q}$.

The explanation of the following theorem is that the pairs (G, H) we consider have the property that $d_{\{\mathrm{id}\}}(\chi)$ is strictly greater than $d_H(\chi)$ for all non-trivial χ.

Namely, we note that

$$\dim V^H - \dim V^G = \sum_{\chi \neq \mathbf{1}} n_\chi d_H(\chi),$$
$$\dim V - \dim V^H = \sum_{\chi \in X(G)} n_\chi (d_{\{\mathrm{id}\}}(\chi) - d_H(\chi)).$$

If $\dim V^H > \dim V^G$, then n_χ is non-zero for some $\chi \neq \mathbf{1}$, and the contribution of this χ in the formula for $\dim V - \dim V^H$ shows that $\dim V > \dim V^H$.

We apply the above observations to the Galois group of a normal closure of a non-Galois extension L/K of number fields. For simplicity, we assume $[L : K] \leq 4$.

Theorem 7. *Let L/K be an extension of number fields, let $n = [L : K]$, let M/K be a normal closure of L/K, and let $G = \mathrm{Gal}(M/K)$. Let A be an Abelian variety over K.*

(1) *If $n = 3$ and $G \simeq S_3$, then $\mathrm{rk}\, A(M) - \mathrm{rk}\, A(K) \geq 2(\mathrm{rk}\, A(L) - \mathrm{rk}\, A(K))$.*
(2) *If $n = 4$ and $G \simeq A_4$, then $\mathrm{rk}\, A(M) - \mathrm{rk}\, A(K) \geq 3(\mathrm{rk}\, A(L) - \mathrm{rk}\, A(K))$, and $\mathrm{rk}\, A(L)$ and $\mathrm{rk}\, A(M)$ have the same parity.*
(3) *If $n = 4$ and $G \simeq S_4$, then $\mathrm{rk}\, A(M) - \mathrm{rk}\, A(K) \geq 3(\mathrm{rk}\, A(L) - \mathrm{rk}\, A(K))$.*

Proof. Let $H = \mathrm{Gal}(M/L) \subseteq G$, so that $[G : H] = n$. We identify G with a transitive subgroup of S_n acting on $\{1, 2, \ldots, n\}$, in such a way that H is the stabilizer of 1. We put $V = A(M)$, so that $A(L) = V^H$ and $A(K) = V^G$.

First let L/K be a non-cyclic extension of degree 3, so that $G = S_3$ and $H = \{\mathrm{id}, (23)\} \subset G$. The group S_3 has three irreducible representations over $\mathbb{Q}$ (the situation is the same as over $\mathbb{C}$): the trivial representation $\mathbf{1}$, the sign representation ϵ, and a unique two-dimensional representation ρ, namely the obvious permutation representation of S_3 on $\{(x_1, x_2, x_3) \in \mathbb{Q}^3 \mid x_1 + x_2 + x_3 = 0\}$. One can check easily that the H-invariant subspaces of $\mathbf{1}$, ϵ, ρ are of dimension 1, 0, 1, respectively. This implies that if

$$V \simeq n_{\mathbf{1}} \cdot \mathbf{1} \oplus n_\epsilon \cdot \epsilon \oplus n_\rho \cdot \rho,$$

then

$$\dim V^G = n_{\mathbf{1}},$$
$$\dim V^H = n_{\mathbf{1}} + n_\rho,$$
$$\dim V = n_{\mathbf{1}} + n_\epsilon + 2n_\rho.$$

This is equivalent to (1).

Let V_4 denote the unique normal subgroup of order 4 of S_4; more concretely,

$$V_4 = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\} \subset A_4 \subset S_4.$$

Let us now consider $G = A_4$. Then we have $H = \langle (234) \rangle$ and $G = V_4 \rtimes H$. The group A_4 has three irreducible representations over $\mathbb{Q}$: the trivial representation $\mathbf{1}$, the direct sum of the two non-trivial one-dimensional representations ϵ and $\bar{\epsilon}$ of A_4/V_4 (each of which is defined over $\mathbb{Q}(\zeta_3)$), and the standard 3-dimensional representation ρ_3. One checks that the H-invariant subspaces of $\mathbf{1}$, $\epsilon + \bar{\epsilon}$, ρ_3 are of dimension 1, 0, 1, respectively. This proves (2).

Finally, we consider $G = S_4$. Then we have $H \simeq S_3$ and $G = V_4 \rtimes H$. The group S_4 has five irreducible representations, both over $\mathbb{C}$ and over $\mathbb{Q}$: the trivial representation $\mathbf{1}$, the sign representation ϵ, a two-dimensional representation ρ_2 arising via the surjection $S_4 \to S_3$ from the two-dimensional representation ρ of S_3, the standard 3-dimensional representation ρ_3, and the 3-dimensional representation $\epsilon \otimes \rho_3$. One checks that the H-invariant subspaces of $\mathbf{1}$, ϵ, ρ_2, ρ_3, $\epsilon \otimes \rho_3$ are of dimension 1, 0, 0, 1, 0, respectively. This proves (3). □

Finally, we note a curious property of certain quadratic twists of Abelian varieties over quadratic extensions of number fields.

Proposition 8. *Let L/K be a quadratic extension of number fields, and let A be an Abelian variety of dimension g over L. Let $B = \operatorname{Res}_{L/K} A$, and assume that $\mathbb{Q} \otimes \operatorname{End}_K B$ contains a number field of some degree n. Let $\delta \in K^\times$, and let M be the Galois closure of $L(\sqrt{\delta})$ over K.*

(1) *The rank of $A(L)$ is divisible by n.*
(2) *If $\operatorname{Gal}(M/K) \simeq V_4$, then $\operatorname{rk} A^\delta(L)$ is divisible by n.*
(3) *If $\operatorname{Gal}(M/K) \simeq D_4$, then $2 \operatorname{rk} A^\delta(L)$ is divisible by n.*

Proof. Our assumption on $\operatorname{End}_K B$ implies that $\operatorname{rk} A(L) = \operatorname{rk} B(K)$ is divisible by n, so (1) is clear.

If $\operatorname{Gal}(M/K) \simeq V_4$, the field M is equal to $L(\sqrt{\delta})$, and M is also of the form $L(\sqrt{e})$ with $e \in K$. Hence

$$\begin{aligned} \operatorname{rk} A^\delta(L) &= \operatorname{rk} A(M) - \operatorname{rk} A(L) \\ &= \operatorname{rk} B(K(\sqrt{e})) - \operatorname{rk} B(K). \end{aligned}$$

By assumption, both terms on the right-hand side are divisible by n, implying (2).

If $\operatorname{Gal}(M/K) \simeq D_4$, the field M is a quadratic extension of $L(\sqrt{\delta})$. Let M_0 be the unique V_4-extension of K contained in M. By looking at the irreducible representations of D_4, one can show that there are non-negative integers a, b, c, d, e such that

$$\begin{aligned} \operatorname{rk} A(M) &= a + b + c + d + 2e, \\ \operatorname{rk} A(M_0) &= a + b + c + d, \\ \operatorname{rk} A(L(\sqrt{\delta})) &= a + c + e, \\ \operatorname{rk} A(L) &= a + c. \end{aligned}$$

We note that L, M_0, M (but not $L(\sqrt{\delta})$) are all of the form $L \otimes_K N$ for some number field N. This implies that the ranks of $A(L)$, $A(M)$, and $A(M)$ are all divisible by n. Therefore $2e$ is divisible by n. Furthermore,

$$\begin{aligned} \operatorname{rk} A^\delta(L) &= \operatorname{rk} A(L(\delta)) - \operatorname{rk} A(L) \\ &= e, \end{aligned}$$

which proves (3). □

3. $\mathbb{Q}$-curves and ranks of twists

The question whether the rank of an elliptic curve over $\mathbb{Q}$ can be arbitrarily large is one of the most important open problems concerning elliptic curves. Somewhat similar questions are: how large can the rank of a twist of a fixed elliptic curve $E/\mathbb{Q}$ be, and what is the largest n such that E has infinitely many twists with rank at least n? The best known result about the latter question for an arbitrary $E/\mathbb{Q}$ is that there exist infinitely many twists of E with rank at least 2 [9]. There exist elliptic curves with infinitely many twists of rank at least 4 [10, 11]. If one assumes the parity conjecture, then there are also elliptic curves over $\mathbb{Q}$ with infinitely many quadratic twists of rank 5 [11].

In this section, for arbitrarily large n, we construct elliptic curves E_n over number fields K_n such that the endomorphism ring of the Weil restriction of scalars $\operatorname{Res}_{K_n/\mathbb{Q}} E_n$ contains an order in a number field of degree 2^n. We also study the problem of constructing, for arbitrarily large n, elliptic curves over number fields admitting infinitely many quadratic twists whose rank is a positive multiple of 2^n. We ask the question whether every elliptic curve E/K has infinite rank over $K\mathbb{Q}(2)$, where $\mathbb{Q}(2)$ is the compositum of all quadratic extensions of $\mathbb{Q}$. A positive answer would imply that the elliptic curves E_n/K_n just mentioned have infinitely many quadratic twists whose rank is a positive multiple of 2^n.

The ideas are inspired by [1], where it was proved that every elliptic curve E with a point of order 13 or 18 over a quadratic field K has even rank. The reason for this is that the endomorphism ring of $\operatorname{Res}_{K/\mathbb{Q}} E$ contains $\mathbb{Z}[\sqrt{d}]$, where d is not a square. This forces $(\operatorname{Res}_{K/\mathbb{Q}} E)(\mathbb{Q}) \simeq E(K)$ to be a $\mathbb{Z}[\sqrt{d}]$-module. Hence, $E(K)$ is of even rank. The result mentioned in the previous paragraph shows that one can similarly construct elliptic curves over number fields whose rank is divisible by integers larger than 2.

A $\mathbb{Q}$-*curve* is an elliptic curve E over $\overline{\mathbb{Q}}$ that is $\overline{\mathbb{Q}}$-isogenous to ${}^\sigma E$ for all $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. An interesting property of $\mathbb{Q}$-curves is the fact that the rich structure of these curves has consequences for their rank. For example, the proof that all elliptic curves over quadratic fields with a point of order 13 or 18 have even rank [1] uses the fact that all such curves are in fact $\mathbb{Q}$-curves. A good and thorough account of the properties of the endomorphism algebras of the restrictions of scalars of $\mathbb{Q}$-curves can be found in [12].

Proposition 9. *For every integer $n \geq 2$, there exists an elliptic curve E over a number field K such that $\mathbb{Q}\otimes\operatorname{End}_{\mathbb{Q}}(\operatorname{Res}_{K/\mathbb{Q}} E)$ contains a number field of degree 2^n.*

Proof. Most of the work needed for this proposition has already been done in [12, pages 309–312]. Let

$$E_a \colon y^2 = x^3 - 3\sqrt{a}(4+5\sqrt{a})x + 2\sqrt{a}(2+14\sqrt{a}+11a), \tag{1}$$

where a is a non-square rational number, be a member of the family of $\mathbb{Q}$-curves parametrized by $X^*(3)$ (the quotient of $X_0(3)$ by the Atkin–Lehner involution w_3); see [12, page 309]. Let p be a prime such that $p \equiv 2 \pmod 3$ and $p \equiv 1 \pmod{2^n}$; there exists infinitely many such primes by the Chinese remainder theorem and Dirichlet's theorem on primes in arithmetic progressions. Note also that this prime satisfies $p \equiv 5 \pmod{12}$.

Let us write $\nu = \operatorname{ord}_2(p-1)$. Let ϵ be a Dirichlet character of order 2^ν and conductor $4p$. Such a character exists, since $(\mathbb{Z}/4p\mathbb{Z})^\times \simeq \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/(p-1)\mathbb{Z}$ and an element $(1,t)$, where t is of order 2^ν, written in additive notation, has the desired properties. We write $K = K_\epsilon(\sqrt{-p})$, where K_ϵ is the splitting field of ϵ.

As explained in [12, page 312 (d)], under these assumptions, there exists an element $\gamma \in K^\times$ such that E_{-p}^γ satisfies

$$\mathbb{Q}\otimes\operatorname{End}_{\mathbb{Q}}\operatorname{Res}_{K/\mathbb{Q}} E_{-p}^\gamma \simeq \mathbb{Q}(\zeta_{2^{\nu+1}}, \sqrt{3}).$$

Note that $\nu \geq n$, so $[\mathbb{Q}(\zeta_{2^{\nu+1}}, \sqrt{3}) : \mathbb{Q}] \geq 2\phi(2^{n+1}) = 2^{n+1}$. The claim follows. $\square$

Remark. Let $\mathbb{Q}(2)$ be the compositum of all quadratic extensions of $\mathbb{Q}$. Note that the elliptic curve E_a from (1) is defined over a quadratic field. By [7, Theorem 5], $E_a(\mathbb{Q}(2))$ has infinite rank. (The statement of loc. cit. is that $E_a(\mathbb{Q}^{\mathrm{ab}})$ has infinite rank, but the proof shows in fact that already $E_a(\mathbb{Q}(2))$ has infinite rank.) This implies that there exist infinitely many quadratic twists E_a^d with d a rational integer, pairwise non-isomorphic over $\mathbb{Q}(\sqrt{a})$, such that $E_a^d(\mathbb{Q}(\sqrt{a}))$ has positive rank. Let S be the set of such integers d. Since for any finite extension $F/\mathbb{Q}(\sqrt{a})$ the set of $d \in \mathbb{Q}$ with $\sqrt{d} \in F$ is finite, it follows that also over F there are infinitely many $d \in F^\times/(F^\times)^2$ such that $E_a^d(F)$ has positive rank.

If E is a $\mathbb{Q}$-curve and $K \subset \overline{\mathbb{Q}}$ is a number field, we say that E is *completely defined over* K if E and all isogenies $E \to {}^\sigma E$, for all $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, are defined over K.

Proposition 10. *Let E be a $\mathbb{Q}$-curve completely defined over a number field K such that $\mathbb{Q} \otimes \operatorname{End}_{\mathbb{Q}}(\operatorname{Res}_{K/\mathbb{Q}} E)$ contains a number field B. For every number field N which can be written as $K \otimes_{\mathbb{Q}} N'$ for some number field N', $\mathbb{Q} \otimes \operatorname{End}_{\mathbb{Q}}(\operatorname{Res}_{N/\mathbb{Q}} E)$ is a B-vector space.*

Proof. Let N' be a number field such that $N = K \otimes_{\mathbb{Q}} N'$ is also a field. Then

$$E(N) = E(K \otimes_{\mathbb{Q}} N') \simeq \operatorname{Res}_{K/\mathbb{Q}} E(N').$$

As $\mathbb{Q} \otimes \operatorname{End}_{N'}(\operatorname{Res}_{K/\mathbb{Q}} E)$ contains B, it follows that $\mathbb{Q} \otimes E(N) \simeq \mathbb{Q} \otimes \operatorname{Res}_{K/\mathbb{Q}} E(N')$ has a natural B-vector space structure. $\square$

Let $\mathbb{Q}(2)$ be the compositum of all quadratic extensions of $\mathbb{Q}$. The following question is a variant of [7, Question 2].

Question 11. Does every elliptic curve over a number field K have infinite rank over $K\mathbb{Q}(2)$?

Remark. Suppose that every Abelian variety over $\mathbb{Q}$ has infinite rank over $\mathbb{Q}(2)$; this is a variant of [5, page 127, Problem]. Then by taking the Weil restriction, we obtain a positive answer to Question 11.

Theorem 12. *Suppose that Question 11 has a positive answer. Let n be a positive integer. There exist a number field K and an elliptic curve E over K possessing infinitely many twists E^d over K such that $\operatorname{rk} E^d(K)$ is a positive multiple of 2^n.*

Proof. As shown in Proposition 9, there exists an elliptic curve E over a number field K such that $\mathbb{Q} \otimes \operatorname{End}_{\mathbb{Q}}(\operatorname{Res}_{K/\mathbb{Q}} E)$ contains a number field B of degree 2^n. It follows that the rank of $E(K) = (\operatorname{Res}_{K/\mathbb{Q}} E)(\mathbb{Q})$ is divisible by 2^n.

By assumption, E has infinitely many (pairwise non-isomorphic) quadratic twists E^d, with d a square-free integer, such that $\operatorname{rk} E^d(K) > 0$. Let E^d be such a twist, with $\sqrt{d} \notin K$. As $K \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{d}) \simeq K(\sqrt{d})$ is a field (see for example [2, Theorem 2.2]), it follows from Proposition 10 that the rank of $E(K(\sqrt{d})) = (\operatorname{Res}_{K/\mathbb{Q}} E)(\mathbb{Q}(\sqrt{d}))$ is divisible by 2^n. But

$$\operatorname{rk} E(K(\sqrt{d})) = \operatorname{rk} E(K) + \operatorname{rk} E^d(K),$$

from which it follows that $\operatorname{rk} E^d(K)$ is divisible by 2^n. $\square$

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References

[1] J. Bosman, P. Bruin, A. Dujella and F. Najman, *Ranks of elliptic curves with prescribed torsion over number fields*, preprint.

[2] P. M. Cohn, *On the decomposition of a field as a tensor product of fields*, Glasgow Math. J. **20** (1979), 141–145.

[3] C. Diem and N. Naumann, *On the structure of Weil restrictions of abelian varieties*, J. Ramanujan Math. Soc. **18** (2003), no. 2, 153–174.

[4] W. Feit and J. G. Thompson, *Solvability of groups of odd order*, Pacific J. Math. **13** (1963), 775–1029.

[5] G. Frey and M. Jarden, *Approximation theory and the rank od Abelian varieties over large algebraic fields*, Proc. London. Math. Soc. **28** (1974), 112–128.

[6] W. Fulton and J. Harris, Representation theory: A first course, Springer-Verlag, New York, 1993.

[7] B.-H. Im and M. Larsen, *Infinite rank of elliptic curves over* $\mathbb{Q}^{\mathrm{ab}}$, `http://arxiv.org/abs/1202.1187`.

[8] B. Mazur, K. Rubin, and A. Silverberg, *Twisting commutative algebraic groups*, J. Algebra **314** (2007), 419–438.

[9] J.-F. Mestre. *Rang de courbes elliptiques d'invariant donné*, C. R. Acad. Sci. Paris **314** (1992), 919–922.

[10] J.-F. Mestre, *Rang de certaines familles de courbes elliptiques d'invariant donné*, C. R. Acad. Sci. Paris **327** (1998), 763–764.

[11] K. Rubin and A. Silverberg, *Twists of elliptic curves of rank at least four*, in: Ranks of elliptic curves and random matrix theory, J. B. Conrey et al., eds., London Math. Soc. Lect. Notes **341**, Cambridge University Press (2007), 177–188.

[12] J. Quer, $\mathbb{Q}$-*curves and abelian varieties of* GL_2-*type*, Proc. London Math. Soc. **81** (2000), 285–317.

[13] J. P. Serre, *Propriétés galoisiennes des points d'ordre fini des courbes elliptiques*, Invent. Math. **15** (1972) 259–311.

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