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ON REAL FORMS OF A BELYI ACTION OF ALTERNATING GROUPS

C. BAGIŃSKI, J. J. ETAYO, G. GROMADZKI, E. MARTÍNEZ

Abstract. In virtue of the Belyi Theorem a complex algebraic curve can be defined over the algebraic numbers if and only if the corresponding Riemann surface can be uniformized by a subgroup of a Fuchsian triangle group. Such surfaces are known as Belyi surfaces. Here we study certain natural actions of the alternating groups A_n on them. We show that they are symmetric and we calculate the number of connected components, called ovals, of the corresponding real forms. It will be obvious that all symmetries with ovals are conjugate and we shall calculate the number of purely imaginary real forms both in case of A_n considered here and S_n considered in an earlier paper of the last three authors.

1. Introduction

In virtue of the Belyi Theorem [1] an algebraic curve can be defined over the algebraic numbers if and only if the corresponding Riemann surface can be uniformized by a subgroup of a Fuchsian triangle group. Such surfaces are known as Belyi surfaces and, by results of Köck, Lau and Singerman [5] and [6] they are symmetric if and only if these algebraic numbers can be simultaneously real. An important class of Belyi surfaces is formed by the Riemann surfaces with so called large groups of automorphisms, and necessary and sufficient algebraic conditions for them to be symmetric were found by Singerman in [9]. In [3], the third author has developed an algebraic method to calculate the number of connected components of the real forms corresponding to the symmetries given by the above theorem of Singerman. In [2] that method was successfully applied to study certain canonical actions of symmetric groups on Riemann surfaces.

Here we study the alternating groups actions within described above framework. Namely, we take a certain canonical and, considered by the group theorists as one

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of the most, natural action of the alternating groups on Belyi Riemann surfaces given by Propositions 3.3 and 3.4. We show that such surfaces are symmetric and then we calculate the number of connected components of the corresponding real forms for alternating groups. It will be obvious that all symmetries with ovals are conjugate. The importance of A_n in this context follows from the Cayley embedding theorem which gives that an arbitrary finite group acts as a group of birational automorphisms on some algebraic curve defined over algebraic reals.

The last section is devoted to purely imaginary complex algebraic curves. We show there that their number for a symmetric quasi-platonic Riemann surface X with the action of $G = \mathrm{Aut}(X)$ corresponding to a pair of generating cycles α, β for which the application $\alpha \mapsto \alpha^{-1}, \beta \mapsto \beta^{-1}$ extends to an automorphism of G does not depend on this pair if $G = \mathrm{S}_n$, while for $G = \mathrm{A}_n$ it depends on it exactly up to such extent up to which it forces $\mathrm{Aut}^{\pm}(X)$ to be S_n or $\mathrm{A}_n \times \mathrm{Z}_2$ which turn out to be the only cases that can happen for the actions considered in this paper.

2. Preliminaries and known results

We shall use the combinatorial method based on the Riemann uniformization theorem and on the theory of Fuchsian groups. Following them, a compact Riemann surface X of genus $g \geq 2$ can be represented as the orbit space $\mathcal{H}/\Gamma$ of the hyperbolic plane $\mathcal{H}$, with respect to the action of some Fuchsian surface group Γ being a discrete and cocompact subgroup of the group of isometries of $\mathcal{H}$, isomorphic to the fundamental group of the surface. Furthermore, the group of conformal automorphisms of the surface given in such a way can be represented as the factor group $G = \Delta/\Gamma$ for some other Fuchsian group Δ. Thus we can write the faithful action of a finite group G as a group of automorphisms of a Riemann surface X of genus $g \geq 2$ by a smooth epimorphism $\theta : \Delta \to G$, which means that its kernel Γ is torsion free or, equivalently, it preserves the orders of the canonical elliptic generators of Δ. However since we shall deal with surfaces with large groups of automorphisms such a group can be assumed to be a triangle group Δ with signature $(0; k, l, m)$ and so, since such a group is unique up to conjugacy in the group of all isometries of $\mathcal{H}$, the corresponding surface is determined, up to isomorphism, by a pair of generators a, b of orders k and l whose product is an element of order m and with these notations we have the, mentioned above, result of Singerman from [9].

Theorem 2.1. *Let X be a Riemann surface with the full large group of automorphisms G, corresponding to a generating pair (a, b), where a, b and ab have orders k, l and m respectively. Then X is symmetric if and only if the mapping*

$\varphi(a) = a^{-1}, \varphi(b) = b^{-1}$ or $\varphi(a) = b^{-1}, \varphi(b) = a^{-1}$ induces an automorphism of G.
□

The set of fixed points of a symmetry of a Riemann surface of genus $g \geq 2$ is homeomorphic to the set of $\mathbb{R}$-rational points of a real form of the complex algebraic curve corresponding to this surface and its symmetry. In turn, the latter consists of k disjoint Jordan curves called *ovals* for some k ranging between 0 and $g+1$ in virtue of the classical Harnack theorem [4].

Now given an automorphism φ of G, two elements x, $y \in G$ are said to be *φ-conjugate* ($x \sim_\varphi y$) if $x = wy\varphi(w)^{-1}$ for some $w \in G$. Observe that for $\varphi = 1$ this coincides with the ordinary notion of conjugacy $\sim$. Recall also that *the isotropy group* of φ is the subgroup consisting of all elements of G fixed by φ. With these notations we have the following result from [3] which describes the number of ovals of the conjugacy classes of symmetries from Theorem 2.1.

Theorem 2.2. *Let a and b be a generating pair of elements of G of orders $k = 2k'+1$ and $l = 2l'+1$ respectively so that ab has order $m = 2m'$. Then the corresponding Riemann surface X has at most two types of symmetries: one with and one without ovals. Symmetries with ovals always exist and a symmetry without ovals exists if and only if $\varphi(g) = g^{-1}$ for some $g \in G$ not φ-conjugate to 1. Furthermore all symmetries with fixed points are conjugate and they have N/M ovals, where N is the order of the isotropy group of φ in G and $M/2$ is the order of $(ab)^{m'}a^{-k'}b^{-l'}(ab)^{m'}b^{l'}a^{k'}$.*

Proof. For $x = ab, y = b^{-1}$, we have a generating pair for G of elements of orders m, l whose product $xy = a$ has order k. So by Theorem 4.1 in [3], the only symmetry up to conjugacy with fixed points has N/M ovals, where N is the order of the isotropy group of φ in G, where $\varphi(x) = x^{-1}, \varphi(y) = y^{-1}$ and $M/2$ is the order of $x^{m'}(xy)^{-k'}y^{l'}x^{m'}y^{-l'}(xy)^{k'} = (ab)^{m'}a^{-k'}b^{-l'}(ab)^{m'}b^{l'}a^{k'}$. □

Theorem 2.3. *Let a and b be a generating pair of elements of G of order $k = 2k'+1$ and $l = 2l'+1$ respectively so that ab has order $m = 2m'+1$. Then the corresponding Riemann surface X has at most two types of symmetries: one with and one without ovals. Symmetries with ovals always exist while a symmetry without ovals exists if and only if $\varphi(g) = g^{-1}$ for some $g \in G$ not φ-conjugate to 1. Furthermore all symmetries with fixed points are conjugate and they have N/M ovals, where N is the order of the isotropy group of φ in G and M is the order of $(ab)^{-m'}b^{l'}a^{k'}$.*

Proof. This is the case with three odd parameters which can be found directly in [3]. □

3. Generating sets of the alternating groups

We are going to consider three sets of generators of the alternating group A_n. These sets are already known but we shall use through the paper the relationship between them. The starting point goes back to Moore in 1897 who gave in [8] a complete presentation of A_n by means of defining generators and relations.

Proposition 3.1. *The elements $a_i = (1,2)(i,i+1)$ for $2 \le i \le n-1$ generate A_n, and a complete presentation of this group is*

$$\langle a_i \mid a_2^3, a_i^2 \text{ for } i \ge 3,\ (a_i a_{i+1})^3, (a_i a_j)^2 \text{ for } |i-j| \ge 2\rangle.$$

□

Proposition 3.2. *For $i \ge 3$ the elements $v_i = (1,2,i)$ generate A_n.*

Proof. For $i \ge 3$ we have

$$v_i^2 v_{i+1} v_i^2 = (1,i,2)(1,2,i+1)(1,i,2) = (1,2)(i,i+1) = a_i.$$

Thus elements a_i for $i \ge 3$ can be expressed by the elements v_i and since $a_2 = v_3^2$ the set of v_i generates the group A_n. □

Proposition 3.3. *Let n be odd. The elements $\alpha = (1,2,3)$ and $\beta = (1,2,\ldots,n)$ generate A_n.*

Proof. First of all let us observe that $v_3 = \alpha$. Now we have

$$\begin{aligned}(\alpha^{-1}\beta)v_i(\beta^{-1}\alpha) &= (1,3,2)(1,2,\ldots,n)(1,2,i)(1,n,n-1,\ldots,2)(1,2,3)\\ &= (1,2,i+1)\\ &= v_{i+1}.\end{aligned}$$

Hence by induction on i all the elements v_i are generated by α and β and so is generated the group A_n. □

Proposition 3.4. *For even n, $\alpha = (1,2,3)$ and $\gamma = (2,3,\ldots,n)$ generate A_n.*

Proof. Here

$$\begin{aligned}(\gamma\alpha)v_i^2(\gamma\alpha)^{-1} &= (1,2,3)(2,3,\ldots,n)(1,i,2)(2,n,n-1,\ldots,3)(1,3,2)\\ &= (1,2,i+1)\\ &= v_{i+1}\end{aligned}$$

and so as in the previous case, the elements $\alpha = v_3$ and γ generate A_n. □

4. The actions

This Section is devoted to construct an automorphism φ of A_n satisfying the condition in Theorem 2.1.

First let n be odd. Consider $\alpha' = (1,2)$ and $\beta' = (1,2,\ldots,n)$ which is a pair of generators of the group S_n. The mapping φ' defined by $\varphi'(\alpha') = \alpha'^{-1} = \alpha'$, $\varphi'(\beta') = \beta'^{-1}$, is an automorphism of S_n. When n is odd the mapping φ' is the conjugation by

$$f = (1,2)(3,n)(4,n-1)\ldots((n+1)/2,(n+1)/2+2),$$

see [2, Theorem 3.1].

Theorem 4.1. *Let n be odd. The mapping φ defined by $\varphi(\alpha) = \alpha^{-1}$, $\varphi(\beta) = \beta^{-1}$ is an automorphism of A_n which is the restriction of $\beta^{-1}\varphi'\beta$ to A_n.*

Proof. Recall that $\alpha = (1,2,3)$, $\beta = (1,2,3,\ldots,n)$. Then, with the above α', β' we have $\alpha'\beta'\alpha'\beta'^{-1} = \alpha$. Hence

$$\begin{aligned} \varphi'(\alpha) &= \varphi'(\alpha'\beta'\alpha'\beta'^{-1}) \\ &= \alpha'\beta'^{-1}\alpha'\beta' \\ &= \beta'^{-1}(\beta'\alpha'\beta'^{-1}\alpha')\beta' \\ &= \beta'^{-1}(\alpha'\beta'\alpha'\beta'^{-1})^{-1}\beta' \\ &= \beta'^{-1}\alpha^{-1}\beta' \\ &= \beta^{-1}\varphi(\alpha)\beta \end{aligned} \qquad \begin{aligned} \varphi'(\beta) &= \varphi'(\beta') \\ &= \beta'^{-1} \\ &= \beta^{-1} \\ &= \beta^{-1}\beta^{-1}\beta \\ &= \beta^{-1}\varphi(\beta)\beta. \end{aligned}$$

This way $\varphi(\alpha) = \beta^{-1}\varphi'(\alpha)\beta$, and $\varphi(\beta) = \beta^{-1}\varphi'(\beta)\beta$, and so φ is the restriction of $\beta^{-1}\varphi'\beta$ to the group A_n. □

Consider now the case n even. Conjugating suitably α and γ from Proposition 3.4, we can get new generators $(1,2,n)$ and $(1,2,\ldots,n-1)$ of A_n which we denote by α and β and with them we have

Theorem 4.2. *The mapping φ defined by $\varphi(\alpha) = \alpha^{-1}$, $\varphi(\beta) = \beta^{-1}$ induces an automorphism of A_n which is a conjugation in S_n.*

Proof. It suffices to check that $\delta\alpha\delta = (1, n, 2) = \alpha^{-1}$ and $\delta\beta\delta = (1, n-1, n-2, \dots, 3, 2) = \beta^{-1}$ for $\delta = (1,2)(3, n-1)(4, n-2)\dots(n/2, n/2+2)$. □

Hence for any value of n we have a pair of generators α and β of A_n, and an automorphism φ of A_n satisfying $\varphi(\alpha) = \alpha^{-1}$ and $\varphi(\beta) = \beta^{-1}$. Now we are looking for the orders of the isotropy groups in Theorems 2.2 and 2.3. From the proof of Theorems 4.1 and 4.2 we have

Corollary 4.3. *If n is odd, $G_\varphi = \mathrm{C}_{\mathrm{S}_n}(\beta f) \cap \mathrm{A}_n$, if n is even, $G_\varphi = \mathrm{C}_{\mathrm{S}_n}(\delta) \cap \mathrm{A}_n$, where C_G means the centralizer in the group G.* □

Because of corollary 4.3, in order to determine the order of G_φ we are going to consider the relevant centralizers.

For n even, $\delta = (1,2)(3, n-1)(4, n-2)\dots(n/2, n/2+2)$. Evidently, $\sigma = (1,2) \in \mathrm{C}_{\mathrm{S}_n}(\delta)$; and for n odd,

$$\begin{aligned}
\beta f &= (1,2,3,\dots,n)(1,2)(3,n)(4,n-1)(5,n-2)\dots \\
&\quad ((n-1)/2, (n-1)/2+4)((n+1)/2, (n+1)/2+2) \\
&= (1,3)(4,n)(5,n-1)(6,n-2)\dots((n-1)/2,(n-1)/2+5) \\
&\quad ((n+1)/2,(n+1)/2+3)((n+3)/2,((n+3)/2+1).
\end{aligned}$$

So $\sigma' = (1,3) \in \mathrm{C}_{\mathrm{S}_n}(\beta f)$.

We have seen that in both cases the centralizer contains elements of S_n not belonging to A_n.

Let us denote by $x!!$ the even factorial (or double factorial) of x, that is to say, $x!! = x(x-2)(x-4)\dots$

Theorem 4.4. *G_φ has order $(n-2)!!$ or $(n-1)!!/2$ for n even and odd, respectively.*

Proof. Let n be even. Then $G_\varphi = \mathrm{C}_{\mathrm{S}_n}(\delta) \cap \mathrm{A}_n$. Since δ is a product of $n/2-1$ transpositions with two fixed points, its centralizer in S_n has order $2 \cdot 2^{n/2-1}(n/2-1)! = 2(n-2)!!$, (see [7, Prop. 23, page 133]). That centralizer contains elements not belonging to A_n and hence $|G_\varphi| = \frac{1}{2}|\mathrm{C}_{\mathrm{S}_n}(\delta)| = (n-2)!!$

For n odd, $G_\varphi = \mathrm{C}_{\mathrm{S}_n}(\beta f) \cap \mathrm{A}_n$. Since βf is a product of $(n-1)/2$ transpositions with one fixed point, its centralizer in S_n has order $2^{(n-1)/2}((n-1)/2)! = (n-1)!!$, and as above $|G_\varphi| = (n-1)!!/2$. □

5. The number of ovals

In this Section we find the number of connected components, that is to say ovals, of real forms corresponding to the symmetries of Riemann surfaces with the action of A_n given by the generating pair (α, β) from Theorems 4.1 and 4.2. We shall use Theorems 2.2 and 2.3, as well as the results of Section 3.

Let n be odd, we have $\alpha = (1, 2, 3)$ and $\beta = (1, 2, 3, \ldots, n)$. The elements α and β generate A_n. These generators are of $(3, n, n)$ type.

Theorem 5.1. *Let X be a symmetric Riemann surface corresponding to the generating pair (α, β) of a finite $(3, n, n)$ group A_n, where n is odd. Then X has a symmetry without ovals and a symmetry with $(n-1)!!/4$ ovals.*

Proof. The group A_n is generated by $a = \alpha = (1, 2, 3)$, $b = \beta = (1, 2, 3, \ldots, n)$ and so $ab = (1, 3, 4, \ldots, n-1, n, 2)$. Then, in terms of Theorem 2.3, $k = 3$, $l = m = n$, and so $k' = 1$, $l' = m' = (n-1)/2$.

First we show that the surface has a symmetry without ovals. Let $g = (1, 3)(4, n)$. Then $\varphi(g) = \beta\varphi'(g)\beta^{-1} = \beta f g f^{-1}\beta^{-1} = \beta f g(\beta f)^{-1} = g = g^{-1}$. Now $g\beta f$ is a composition of $(n-5)/2$ transpositions, whilst βf is a composition of $(n-1)/2$ transpositions. Hence $g\beta f$ is not conjugate to βf, and so g is not φ-conjugate to 1.

Now we calculate the number of ovals of the symmetry with ovals. By using Theorems 2.3 and 4.4, $N = (n-1)!!/2$. On the other hand M is the order of the element $(ab)^{-(n-1)/2} b^{(n-1)/2} a$ which is equal to

$$(1, 3, 4, \ldots, n-1, n, 2)^{-(n-1)/2}(1, 2, \ldots, n)^{(n-1)/2}(1, 2, 3) = (2, 3)((n+3)/2, (n+5)/2).$$

So $M = 2$ and hence $N/M = (n-1)!!/4$. □

Let now $n > 4$ be even. Then $\alpha = (1, 2, n)$ and $\beta = (1, 2, \ldots, n-1)$ generate A_n. These generators are of $(3, n-1, n-2)$ type.

Theorem 5.2. *Let X be a symmetric Riemann surface corresponding to the generating pair (α, β) of a finite $(3, n-1, n-2)$ group A_n, where n is even. Then X*

has a symmetry without ovals and a symmetry with $(n-4)!!$ *or* $(n-4)!!/2$ *ovals depending on* n *to be a multiple of* 4 *or not.*

Proof. The group A_n is generated by $a=\alpha=(1,2,n)$, $b=\beta=(1,2,3,\ldots,n-1)$, and $ab=(1,n)(2,3,\ldots,n-1)$. Then in terms of Theorem 2.2, $k=3$, $l=n-1$, $m=n-2$, and so $k'=1$, $l'=m'=n/2-1$.

As in the previous Theorem the surface has a symmetry without ovals. Let $g=(1,2)(3,n-1)$. Then $\varphi(g)=\delta g\delta=g=g^{-1}$. Since $g\delta$ is a composition of $n/2-3$ transpositions, and δ is a composition of $n/2-1$ transpositions, $g\delta$ is not a conjugate of δ, and so g is not φ-conjugate to 1.

Take now the symmetry with ovals. By Theorem 4.4, the value of N in Theorem 2.2 is equal to $(n-2)!!$. On the other hand $M/2$ is the order of the element

$$(ab)^{n/2-1}a^{-1}b^{-(n/2-1)}(ab)^{n/2-1}b^{n/2-1}a$$

which is equal to the product

$$(1,n)^{n/2-1}(2,3,\ldots,n-1)^{n/2-1}(1,n,2)(1,2,3,\ldots,n-1)^{-(n/2-1)}$$
$$(1,n)^{n/2-1}(2,3,\ldots,n-1)^{n/2-1}(1,2,3,\ldots,n-1)^{n/2-1}(1,2,n).$$

When n is multiple of 4, the last permutation is

$$(1,3,4,\ldots,n/2)(n/2+2,n,n-1,\ldots,n/2+3),$$

and hence has order $n/2-1$, whilst in the other case, it is

$$(1,3,4,\ldots,n/2,n,n-1,n-2,\ldots,n/2+2)(2,n/2+1),$$

and so has order $n-2$.

Therefore M is respectively $n-2$ and $2(n-2)$, and so M/N is respectively $(n-4)!!$ and $(n-4)!!/2$ as claimed. □

6. Purely imaginary forms

¿From Theorems 2.2 and 2.3 we know necessary and sufficient conditions for Riemann surfaces described there to admit a symmetry without ovals which correspond to purely imaginary forms. Now we shall deal with the number of conjugacy classes of such symmetries of Riemann surfaces corresponding both to the action of $G=\mathrm{A}_n$ considered here and to the one for $G=\mathrm{S}_n$ which was considered in [2]. By [9] (se also [3] for explicite statement), $G^{\pm} = \mathrm{Aut}^{\pm}(X) = G \rtimes \mathrm{Z}_2 = \langle a, b\rangle \rtimes \langle t\rangle$, where $tgt = \varphi(g)$. With these notations we have

Theorem 6.1. *Two elements g_1, g_2 of G give rise to nonconjugate fixed point free symmetries if and only if*

(*a*) (*a*) $\varphi(g_i) = g_i^{-1}$ *and* $g_i \not\sim_{\varphi} 1$,
(*b*) $g_1 \not\sim_{\varphi} g_2$ *and* $g_1 \not\sim_{\varphi} \varphi(g_2)$.

Proof. Clearly each element of $G^{\pm} \setminus G$ has the form gt for some $g \in G$. Now for a symmetry, we have $1 = (gt)^2 = g\varphi(g)$ and gt has ovals if and only if $gt \sim_{G^{\pm}} t$ which in turn means $gt = wtw^{-1} = wtw^{-1}tt$. Consequently $g \sim_{\varphi} 1$ and so (a). Now $g_1 t \sim_{G^{\pm}} g_2 t$ if and only if $g_1 t = w(g_2 t)w^{-1} = wg_2\varphi(w)t$ or $g_1 t = (wt)(g_2 t)(wt)^{-1} = w\varphi(g_2)\varphi(w)^{-1}t$ for some $w \in G$ which gives (b). □

We shall not only find the number of purely imaginary real forms of surfaces considered in this paper but we shall show, actually, that this number for symmetric quasi-platonic Riemann surfaces with the action of G depends on α and β only up to a certain extent. For effective use of this theorem for our actions we need some preparation. The first lemma is rather easy.

Lemma 6.2. *Let γ be a cycle of length m and let ξ, η be involutions such that $\gamma = \xi\eta$, and $\mathrm{supp}(\xi), \mathrm{supp}(\eta) \subseteq \mathrm{supp}(\gamma)$.*

(*a*) *If m is odd then both ξ and η are products of $(m-1)/2$ disjoint transpositions.*
(*b*) *If m is even then one of the involutions ξ, η is a product of $m/2$ disjoint transpositions and the other one is a product of $m/2 - 1$ disjoint transpositions.*

□

Let $G \in \{\mathrm{A}_n,\ \mathrm{S}_n\}$. For $G = \mathrm{A}_n$, let α and β be its generating cycles given in Propositions 3.3 and 3.4. For $G = \mathrm{S}_n$, let $\alpha = (1,\ 2), \beta = (1,\ 2,\ 3, \ldots,\ n)$ be a pair of generating cycles defining the action of S_n studied in [2].

Proposition 6.3. *Let φ be an automorphism of $G \in \{\mathrm{A}_n,\ \mathrm{S}_n\}$ such that $\alpha^{\varphi} = \alpha^{-1}$ and $\beta^{\varphi} = \beta^{-1}$. Let $\tilde{G} = G \rtimes \langle\varphi\rangle$.*

(*a*) *If $G = \mathrm{A}_n$ and $n \equiv 1, 2 \pmod 4$ or $G = \mathrm{S}_n$, then $\tilde{G} \cong G \times \mathrm{Z}_2$, where Z_2 is a cyclic group of order 2.*
(*b*) *If $G = \mathrm{A}_n$ and $n \equiv 0, 3 \pmod 4$, then $\tilde{G} \cong \mathrm{S}_n$.*

Proof. (a) Let $G = \mathrm{A}_n$. If $n = 4k + l$, where $l \in \{1, 2\}$, then β has length $4k + 1$ and is a product of two involutions which are products of $2k$ disjoint transpositions, by Lemma 6.2. So they are even involutions. By the Lemmata 4.1 an 4.2 and [2, Theorem 3.1] we can choose one of them in such a way that a previously fixed automorphism of A_n acts as a conjugation by one of these involutions, say τ. Hence in the group $\tilde{G}$ we have $\tau\varphi = \varphi\tau$. It is obvious that $\tau\varphi$ has order 2 and centralizes A_n. Therefore $\tilde{G} = G\langle\tau\varphi\rangle = G \times \langle\tau\varphi\rangle$. We have an analogous proof for $G = \mathrm{S}_n$.

(b) Let again $G = \mathrm{A}_n$. If $n = 4k+l$, where $l \in \{0, 3\}$, then β has length $4(k-1)+3$ or $4k+3$, and then it is a product of odd involutions being products of $2k-1$ or $2k+1$ disjoint involutions. Moreover there are not even involutions with product equal to β. It is well known that S_n is a semidirect product of A_n and a subgroup generated by an odd involution acting on the set $\{1, 2, \ldots, n\}$ and $\tilde{G}$ is such a group. □

Now observe that for a fixed positive even integer m in the range $1 \leq m \leq n$, the set of all involutions $\tau \in \mathrm{A}_n$ such that $|\mathrm{supp}(\tau)| = m$ form one conjugacy class of A_n and S_n as well. The symmetric group S_n has $\lfloor n/2 \rfloor$ different conjugacy classes of involutions while the alternating group A_n has $\lfloor n/4 \rfloor$ such classes. So in particular we get

Lemma 6.4. *Let $\tilde{G}$ be as in Proposition 6.3.*

- (a) *If $G = \mathrm{S}_n$ then the number of conjugacy classes of involutions from $\tilde{G} - G$ is equal to $\lfloor n/2 \rfloor$.*
- (b) *Let $G = \mathrm{A}_n$. If $\tilde{G} \cong G \times \mathrm{Z}_2$ then the number of conjugacy classes from $\tilde{G} - G$ is equal to $\lfloor n/4 \rfloor$. If $\tilde{G} \cong \mathrm{S}_n$ then the number of conjugacy classes of involutions from $\tilde{G} - G$ is equal to $\lfloor n/2 \rfloor - \lfloor n/4 \rfloor$.*

Proof. (a) By Proposition 6.3, $\tilde{G} - G = G\tau$, where τ is an involution centralizing G. If C_γ is the conjugacy class of γ in G then $C_\gamma\tau$ is a conjugacy class of $\gamma\tau$ in $\tilde{G}$. Hence, the number of conjugacy classes of involutions of G is the same as the number of conjugacy classes of involutions of $\tilde{G}$ contained in $\tilde{G} - G$.

(b) For the case $\tilde{G} \cong G \times \mathrm{Z}_2$ the proof is the same as for (a). Let $\tau \in \tilde{G} - G$ be a fixed involution. Let $\tilde{G} = \mathrm{S}_n$. If $\tau \in \mathrm{A}_n$ is an involution, then the conjugacy class of τ in S_n is equal to the conjugacy class of this element in A_n. Hence the number of conjugacy classes of involutions of $\tilde{G} - G$ is equal to $\lfloor n/2 \rfloor - \lfloor n/4 \rfloor$. □

Suppose now that α and β are fixed arbitrary cycles generating $G \in \{\mathrm{A}_n, \mathrm{S}_n\}$, such that $\tau\alpha\tau = \alpha^{-1}$ and $\tau\beta\tau = \beta^{-1}$ for an involution τ. From the proof of Theorem

6.1, it follows that in order to calculate the number of purely imaginary forms we have to find in $\tilde{G} = G \rtimes \langle\tau\rangle$ the number of conjugacy classes of involutions of $\tilde{G}$ which are not in G and which are not conjugated to $\alpha\tau$, τ and $\beta\tau$. Observe however that for $G = \mathrm{S}_n$ with $|\mathrm{supp}(\beta)|$ odd, the elements τ and $\beta\tau$ are conjugated but τ and $\alpha\tau$ not, as $|\mathrm{supp}(\alpha)|$ is even. If both $|\mathrm{supp}(\alpha)|$ and $|\mathrm{supp}(\beta)|$ are even then $\alpha\tau$ is conjugated to $\beta\tau$ but not conjugated to τ. If $G = \mathrm{A}_n$ then the three elements τ, $\alpha\tau$, $\beta\tau$ are conjugated with each other. As a consequence we obtain our final theorem.

Theorem 6.5. *Let G be the symmetric group S_n or the alternating group A_n generated by two cycles α, β, so that the correspondence $\varphi(\alpha) = \alpha^{-1}$ and $\varphi(\beta) = \beta^{-1}$ induces an automorphism of G. Then the complex algebraic curve corresponding to α, β has*

$$\begin{array}{ll} \lfloor n/2 \rfloor - 2 & \text{if } G = \mathrm{S}_n, \\ \lfloor n/2 \rfloor - \lfloor n/4 \rfloor - 1 & \text{if } \tilde{G} \cong \mathrm{S}_n, \\ \lfloor n/4 \rfloor - 1 & \text{if } \tilde{G} \cong \mathrm{A}_n \times Z_2, \end{array}$$

purely imaginary forms. □

Remark 6.6. If α and β giving the action of G are not cycles, and $\tau\alpha\tau = \alpha^{-1}$ and $\tau\beta\tau = \beta^{-1}$ for some involution τ, then in all three cases the elements τ, $\alpha\tau$, $\beta\tau$ may lie in one, two or three conjugacy classes.

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