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ABSTRACT. Let $E/\mathbb{Q}$ be an elliptic curve which has split multiplicative reduction at a prime p and whose analytic rank r_{an} equals one. The main goal of this article is to relate the second order derivative of the Mazur-Tate-Teitelbaum p-adic L-function $L_p(E,s)$ attached to E to Nekovář's height pairing evaluated on natural elements arising from Beilinson-Kato elements. Our height formula allows us, among other things, to compare the order of vanishing of $L_p(E,s)$ at $s=1$ to its (complex) analytic rank r_{an}, assuming the non-triviality of the height pairing. This has strong consequences towards a conjecture of Mazur, Tate and Teitelbaum.

CONTENTS

1. INTRODUCTION

Fix a prime $p>3$ and an elliptic curve E defined over $\mathbb{Q}$ that has split multiplicative reduction at p. Let $L(E,s)$ (resp., $L_p(E,s)$) denote the complex Hasse-Weil L-function (resp., the Mazur-Tate-Teitelbaum p-adic L-function) attached to E. By the work of Wiles [Wil95], $L(E,s)$ is admits an analytic continuation to the whole complex plane. Let r_{an} denote the order of vanishing of $L(E,s)$ at $s=1$. As we have assumed that the elliptic curve E has split multiplicative reduction at p, the p-adic L-function $L_p(E,s)$

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has an exceptional zero at $s=1$, in the sense of Greenberg [Gre94], due to the vanishing of the interpolation factor $(1-p^{1-s})(1-p^{-s})$ at $s=1$. Mazur, Tate and Teitelbaum conjecture in this case that

$$\operatorname{ord}_{s=1} L_p(E,s) = 1 + r_{\text{an}}. \tag{1.1}$$

This is the conjecture that the title of this article refers to. Furthermore, they conjectured a formula for the first derivative of $L_p(E,s)$:

$$\frac{d}{ds}L_p(E,s)\Big|_{s=1} = \frac{\log_p(q_E)}{\operatorname{ord}_p(q_E)}\cdot\frac{L(E,1)}{\Omega_E^+}, \tag{1.2}$$

where Ω_E^+ is the real period of E and q_E is the Tate period of E (obtained via the p-adic uniformization of E) and $\log_p$ is the p-adic logarithm. Greenberg and Stevens [GS93] gave a proof of the assertion (1.2). The so-called *Saint-Etienne theorem* (formerly, a conjecture of Manin) proved in [BSDGP96] shows that $\log_p(q_E)\neq 0$. We therefore conclude that (1.1) holds true when $r_{\text{an}}=0$. As far as the author is aware, nothing substantial was known when $r_{\text{an}}>0$.

The conjecture of Birch and Swinnerton-Dyer (henceforth, abbreviated as BSD) predicts that the behavior of the Hasse-Weil L-function $L(E,s)$ controls on the algebraic side the (p-adic) Selmer group $\text{Sel}_p(E/\mathbb{Q})$ (see §2.1.1 below for a definition of the Selmer group). In particular, BSD predicts that $r_{\text{an}} = \operatorname{rank}_{\mathbb{Z}_p}(\text{Sel}_p(E/\mathbb{Q}))$ and further that the r_{an}-th derivative of $L(E,s)$ at $s=1$ should be expressed (among other things) in terms of a certain regulator calculated on $\text{Sel}_p(E/\mathbb{Q})$.

The conjectured equality (1.1) suggests that, in order to formulate the p-adic analog of BSD for $L_p(E,s)$ at $s=1$, one should replace the classical Selmer group with an extended Selmer group so as to compensate for the (conjectural) gap between the rank of $\text{Sel}_p(E/\mathbb{Q})$ and $\operatorname{ord}_{s=1} L_p(E,s)$. This has been carried out initially in [MTT86]; later Nekovář in [Nek06] defined his *extended* Selmer groups in a much more general framework. The purpose of this article is to express the first (resp., second) order derivative of $L_p(E,s)$ at $s=1$ when $r_{\text{an}}=0$ (resp., when $r_{\text{an}}=1$) in terms of Nekovářs's height pairings defined on his extended Selmer groups. When $r_{\text{an}}=0$, this allows us (in a rather *ad hoc* manner) to interpret Kobayashi's computations [Kob06] from the perspective offered by Nekovář's general theory. The main contribution of this article, however, concerns the case $r_{\text{an}}=1$. In this case, we reduce the conjecture (1.1) to the non-degeneracy of Nekovář's p-adic height pairing.

Before we explain the results of the current article in detail, let us introduce some notation. See also [Büy12] for an investigation along these lines when E is replaced by $\mathbb{G}_{\text{m}}$ and when the relevant p-adic L-function is the Kubota-Leopoldt p-adic L-function.

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1.1. **Notation and Hypotheses.** For any field K, fix a separable closure $\overline{K}$ of K and set $G_K = \text{Gal}(\overline{K}/K)$. Let $\mathbb{Q}_\infty/\mathbb{Q}$ denote the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ and let

$\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. We write $\rho_{\rm cyc}$ for the cyclotomic character $\rho_{\rm cyc} : \Gamma \xrightarrow{\sim} 1 + p\mathbb{Z}_p$. Let $\mathbb{Q}_n$ denote the unique sub-extension of $\mathbb{Q}_\infty/\mathbb{Q}$ of degree p^n over $\mathbb{Q}$, i.e., the fixed field of Γ^{p^n}. Let Φ_n be the completion of $\mathbb{Q}_n$ at the unique prime of $\mathbb{Q}_n$ above p, and set $\Phi_\infty = \cup\Phi_n$, the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}_p$. By slight abuse of notation $\mathrm{Gal}(\Phi_\infty/\mathbb{Q}_p)$ will be denoted by Γ as well. Let $\Gamma_n = \Gamma/\Gamma^{p^n} = \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$. We fix a topological generator γ of Γ. We also set $\Lambda = \mathbb{Z}_p[[\Gamma]]$ as the cyclotomic Iwasawa algebra and $J = \ker(\Lambda \to \mathbb{Z}_p)$ (where the arrow is the map induced from $\gamma \mapsto 1$) as the augmentation ideal.

Let $E/\mathbb{Q}$ be an elliptic curve that has split multiplicative reduction at p. Let $T = T_p(E)$ denote its p-adic Tate module and set $V = T \otimes \mathbb{Q}_p$. We have an exact sequence

$$0 \longrightarrow F_p^+T \longrightarrow T \longrightarrow F_p^-T \longrightarrow 0 \tag{1.3}$$

of $\mathbb{Z}_p[[G_{\mathbb{Q}_p}]]$-modules, where $F_p^+T \cong \mathbb{Z}_p(1)$ and $F_p^-T \cong \mathbb{Z}_p$. Let $T^* = \mathrm{Hom}(T, \mathbb{Z}_p(1))$ (resp., $V^* = T^* \otimes \mathbb{Q}_p$) and $F^\pm T^* = \mathrm{Hom}(F_p^\mp T, \mathbb{Z}_p(1))$, so that T^* fits in an exact sequence of $\mathbb{Z}_p[[G_{\mathbb{Q}_p}]]$-modules

$$0 \longrightarrow F_p^+T^* \longrightarrow T^* \longrightarrow F_p^-T^* \longrightarrow 0.$$

Note that the Weil pairing shows that there is an isomorphism $T \cong T^*$ of $\mathbb{Z}_p[[G_\mathbb{Q}]]$-modules. Let $\tan(E/\mathbb{Q}_p)$ denote the tangent space of $E/\mathbb{Q}_p$ at the origin and consider the Lie group exponential map

$$\exp_E : \tan(E/\mathbb{Q}_p) \longrightarrow E(\mathbb{Q}_p) \otimes \mathbb{Q}_p.$$

Fix a minimal Weierstrass model of E and let ω_E denote the corresponding holomorphic differential. The cotangent space $\mathrm{cotan}(E)$ is generated by the invariant differential ω_E, let $\omega_E^* \in \tan(E/\mathbb{Q}_p)$ be the corresponding dual basis. Then there is a dual exponential map

$$\exp_E^* : H^1(G_p, V^*) \longrightarrow \mathrm{cotan}(E) = \mathbb{Q}_p\omega_E$$

and an induced map

$$\exp_{\omega_E}^* = \omega_E^* \circ \exp_E^* : H^1(G_p, V^*) \longrightarrow \mathbb{Q}_p.$$

Let $E_p(s) = 1 - p^{-s}$ denote the Euler factor of $L(E, s)$ at p and define

$$\rho : \Gamma \xrightarrow{\rho_{\rm cyc}} 1 + p\mathbb{Z}_p \xrightarrow{E_p(1)^{-1}\log_p} \mathbb{Z}_p$$

to be a fixed normalization of $\rho_{\rm cyc}$.

1.2. **Statements of the results.** For $X = V, V^*$, let $\tilde{H}^1_f(\mathbb{Q}, X)$ denote Nekovář's extended Selmer group attached to X and let

$$\langle\, ,\, \rangle_{\rm Nek} : \tilde{H}^1_f(\mathbb{Q}, V) \otimes \tilde{H}^1_f(\mathbb{Q}, V^*) \longrightarrow \mathbb{Q}_p \otimes_{\mathbb{Z}_p} J/J^2 \tag{1.4}$$

denote Nekovář's height pairing; see §2.1 below for the definitions of these objects. Via the natural isomorphism $J/J^2 \xrightarrow{\sim} \Gamma$ (induced from $\gamma - 1 \mapsto \gamma$), the pairing (1.4) may be regarded to take values in $\mathbb{Q}_p \otimes \Gamma$. Let $\langle\, ,\, \rangle_{{\rm Nek},\rho}$ denote the compositum

$$\langle\, ,\, \rangle_{{\rm Nek},\rho} : \tilde{H}^1_f(\mathbb{Q}, V) \otimes \tilde{H}^1_f(\mathbb{Q}, V^*) \longrightarrow \mathbb{Q}_p \otimes_{\mathbb{Z}_p} J/J^2 \longrightarrow \mathbb{Q}_p \otimes \Gamma \xrightarrow{\rho} \mathbb{Q}_p.$$

Let $\mathfrak{z}_0^{\rm Kato} \in H^1(\mathbb{Q}, V^*)$ denote Kato's Beilinson element (whose basic properties are recalled in §3.2 below) and set $z^{\rm Kato} = \mathrm{loc}_p(\mathfrak{z}_0^{\rm Kato})$ to be the image of $\mathfrak{z}_0^{\rm Kato}$ under the localization map

$$\mathrm{loc}_p : H^1(\mathbb{Q}, V^*) \longrightarrow H^1(\mathbb{Q}_p, V^*).$$

When $r_{\text{an}} = 0$ or 1, one may define elements $[-\text{ord}_p(q_E)^{-1}] \in \tilde{H}^1_f(\mathbb{Q}, V)$ and $[\exp^*_{\omega_E}(z^{\text{Kato}})] \in \tilde{H}^1_f(\mathbb{Q}, V^*)$ of the extended Selmer groups, as in §3.3 below. We are now ready to state our first theorem.

Theorem A (Theorem 3.8 below)**.** *Suppose $r_{\text{an}} = 0$ or 1. Then,*

$$\frac{d}{ds} L_p(E,s)\Big|_{s=1} = \left\langle [-\text{ord}_p(q_E)^{-1}], [\exp^*_{\omega_E}(z^{\text{Kato}})] \right\rangle_{\text{Nek},\rho}.$$

This result should be compared to Benois' work in [Ben11a] and [Ben11b, Proposition 2.2.4].

Observe that when $r_{\text{an}} = 1$, the theorem of Greenberg-Stevens shows that the left hand side of the assertion in Theorem A equals 0. Kato's reciprocity law proved in [Kat04] shows that $[\exp^*_{\omega_E}(z^{\text{Kato}})] = 0$ as well. Hence, Theorem A says nothing when $r_{\text{an}} = 1$. In this case, we shall prove Theorem B below.

When $r_{\text{an}} \le 1$, a conjecture of Perrin-Riou (labeled by Conjecture 3.4 below) predicts that Kato's class $\mathfrak{z}_0^{\text{Kato}} \in H^1(\mathbb{Q}, V^*)$ is non-trivial. Shortly after posting the initial version of this article, the author was notified that R. Venerucci has (partially) proved this conjecture in his thesis*, by comparing Kato's class to a suitable Heegner point using a result of Bertolini and Darmon. Let Φ be a certain extension of $\mathbb{Q}_p$ (defined as in §3.4) and set $X_\Phi = X \otimes_{\mathbb{Q}_p} \Phi$ for $X = V, V^*$. Let $\tilde{\mathfrak{z}}^{\text{Kato}} \in \tilde{H}^1_f(\mathbb{Q}, V_\Phi) \cong \tilde{H}^1_f(\mathbb{Q}, V^*_\Phi)$ (where the identification is via the Weil pairing) denote the normalization of Kato's element $\mathfrak{z}_0^{\text{Kato}}$ given as in Definition 3.13. Finally, let $\gamma_0 \in \Gamma$ be a fixed generator satisfying $\log_p(\rho_{\text{cyc}}(\gamma_0)) = p$.

Theorem B (Theorem 3.18 below)**.** *Suppose $r_{\text{an}} = 1$ Then,*

$$\frac{1}{2}\left(\frac{d^2}{ds^2}(L_p(E,s))\Big|_{s=1}\right) \otimes (\gamma_0 - 1) = \langle \tilde{\mathfrak{z}}^{\text{Kato}}, \tilde{\mathfrak{z}}^{\text{Kato}} \rangle_{\text{Nek}},$$

where the equality takes place in $\Phi \otimes_{\mathbb{Z}_p} J/J^2$.

Remark 1.1. The reader might be concerned that the right hand side in Theorem B is independent of the choice of an isomorphism $\kappa : \Gamma \to 1 + p\mathbb{Z}_p$, whereas the choice of the element $\gamma_0 \in \Gamma$ relies on the choice $\kappa = \rho_{\text{cyc}}$. Note, however, that the definition of $L_p(E,s)$ (c.f., §3 below) also relies on the cyclotomic character ρ_{cyc} and the element $\left(\frac{d^2}{ds^2}(L_p(E,s))\big|_{s=1}\right) \otimes (\gamma_0 - 1)$ would remain unchanged if ρ_{cyc} was to be replaced by any other isomorphism $\kappa : \Gamma \to 1 + p\mathbb{Z}_p$.

Remark 1.2. A result similar to Theorem B above has also been obtained independently by R. Venerucci, see in particular Corollary 8.8 of his thesis [Ven13].

The key in the proofs of Theorems A and B is the description of the p-adic L-function $L_p(E,s)$ as the image of Kato's Beilinson elements under the Coleman map (as asserted in (3.4)); an explicit description of the Coleman map itself in terms of local units (c.f., §3.1) and a Rubin-style formula which reduces the calculation of Nekovář's heights to a computation of local Tate-pairings.

Theorem B has the following immediate corollary:

*We thank M. Bertolini for bringing Venerucci's work to our attention.

Corollary C (Corollary 3.19 below)**.** *Suppose $r_{\text{an}} = 1$. If Nekovář's p-adic height pairing is non-degenerate, then the Mazur-Tate-Teitelbaum conjecture* (1.1) *is true.*

Let $A/\mathbb{Q}$ be an elliptic curve with good ordinary reduction at p. When $\text{ord}_{s=1} L(A,s) = 1$, one may compare the order of vanishing of the Mazur-Tate-Teitelbaum p-adic L-function $L_p(A,s)$ to that of the complex Hasse-Weil L-function $L(A,s)$ (as in Corollary C), by making use of the results of [Sch85] and [PR93b], along with the recent proof of Skinner and Urban [SU10] of Mazur's main conjecture. Note however that this comparison would still require the non-degeneracy of a certain p-adic height pairing. Corollary C in this sense extends the results Schneider and Perrin-Riou to the case when the elliptic curve E in question has split multiplicative reduction at p (in which case the p-adic L-function attached to E possesses an exceptional zero).

We briefly outline the plan of the paper. In §2.1, we introduce Nekovář's Selmer complexes (whose cohomology yields his *extended Selmer groups*) and discuss their relation with various Selmer groups. In §2.2, we recall Nekovář's definition of height pairings in great generality. In §2.3, we carry out a local computation with the local Tate pairing (still in great generality) which is essential for the height calculations in §3. In §3.1 (resp., in §3.2), we define the Coleman map (resp., introduce Kato's Beilinson elements), which are used to define the elements of the extended Selmer groups on which we shall compute Nekovář's height pairing (and compare to the derivatives of the p-adic L-function $L_p(E,s)$). Once these elements are defined, we carry out the height computations in §3.3 in the case $r_{\text{an}} = 0$ and in §3.4 in the case $r_{\text{an}} = 1$.

2. Generalities on Nekovář's Theory of Selmer complexes

Let G be a profinite group (given the profinite topology) and let R be a complete discrete valuation ring with finite residue field of characteristic p. Let X be a free R-module of finite type on which G acts continuously. In this section we very briefly review Nekovář's theory of Selmer complexes and his definition of extended Selmer groups. Although the treatment in this section is far more general than what is needed for the purposes of this paper (e.g., from §3.3 on, K will be $\mathbb{Q}$ and the Galois module X considered below will be T or T^* (in degree zero)), it is still much less general than what is covered in [Nek06].

The G-module X is admissible in the sense of [Nek06, §3.2] and we can talk about the complex of *continuous* cochains $C^\bullet(G,X)$ as in §3.4 of loc.cit. Let K be a number field and for a finite set S of places of K, let S_f denote the subset of finite places within S. We denote by K_S the maximal subextension of $\overline{K}/K$ which is unramified outside S and set $G_{K,S}$ to be the Galois group $\text{Gal}(K_S/K)$. For all $w \in S_f$, we write K_w for the completion of K at w, and G_w for its absolute Galois group. Whenever it is convenient, we will identify G_w with a decomposition subgroup inside $G_K := \text{Gal}(\overline{K}/K)$. We will be interested in the cases when $G = G_{K,S}$ or $G = G_w$ and in the former case, S is chosen to contain all primes above p, all primes at which G representation X is ramified and all infinite places of K.

2.1. **Selmer complexes.**

Classical Selmer groups are defined as a subgroup of elements of the global cohomology group $H^1(G_{K,S},X)$ satisfying certain local conditions; see [MR04, §2.1] for the most general definition. The main idea of [Nek06] is to impose local conditions on the level of complexes. We go over basics of Nekovář's theory, for details see [Nek06].

Definition 2.1. *Local conditions* on X are given by a collection $\Delta(X) = \{\Delta_w(X)\}_{w\in S_f}$, where $\Delta_w(X)$ stands for a morphism of complexes of R-modules

$$i_w^+(X) : U_w^+ \longrightarrow C^\bullet(G_w, X)$$

for each $w \in S_f$.

Also set

$$U_v^-(X) = \text{Cone}\left(U_v^+(X) \xrightarrow{-i_v^+} C^\bullet(G_v, X)\right)$$

and

$$U_S^\pm(X) = \bigoplus_{w\in S_f} U_w^\pm(X); \quad i_S^+(X) = (i_w^+(X))_{w\in S_f}.$$

We also define

$$\text{res}_{S_f} : C^\bullet(G_{K,S}, X) \longrightarrow \bigoplus_{w\in S_f} C^\bullet(G_w, X)$$

as the canonical restriction morphism.

Definition 2.2. The *Selmer complex* associated with the choice of local conditions $\Delta(X)$ on X is given by the complex

$$\widetilde{C}_f^\bullet(G_{K,S}, X, \Delta(X)) := \text{Cone}(C^\bullet(G_{K,S}, X) \bigoplus U_S^+(X) \xrightarrow{\text{res}_{S_f} - i_S^+(X)} \bigoplus_{w\in S_f} C^\bullet(G_w, X))[-1]$$

where $[n]$ denotes a shift by n. The corresponding object in the derived category will be denoted by $\widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X, \Delta(X))$ and its cohomology by $\widetilde{H}_f^i(G_{K,S}, X, \Delta(X))$ (or simply by $\widetilde{H}_f^i(K, X)$ or by $\widetilde{H}_f^i(X)$ when there is no danger of confusion). The R-module $\widetilde{H}_f^1(X)$ will be called the *extended Selmer group*.

The object in the derived category corresponding to the complex $C^\bullet(G_{K,S}, X)$ will be denoted by $\mathbf{R}\Gamma(G_{K,S}, X)$.

2.1.1. *Comparison with classical Selmer groups.* For each $w \in S_f$, suppose that we are given a submodule

$$H^1_{\mathcal{F}}(K_w, X) \subset H^1(K_w, X).$$

The data which $\mathcal{F}$ encodes is called a *Selmer structure* on M. Starting with $\mathcal{F}$, one defines the Selmer group as

$$H^1_{\mathcal{F}}(K, X) := \ker\left\{H^1(G_{K,S}, X) \longrightarrow \bigoplus_{w\in S_f} \frac{H^1(K_w, X)}{H^1_{\mathcal{F}}(K_w, X)}\right\}.$$

On the other hand, as explained in [Nek06, §6.1.3.1-2], there is an exact triangle

$$U_S^-(X)[-1] \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(G_{K,S}, X, \Delta(X)) \longrightarrow \mathbf{R}\Gamma(G_{K,S}, X) \longrightarrow U_S^-(X)$$

which gives rise to the following exact sequence in the level of cohomology that is used to compare Nekovář's extended Selmer groups to classical Selmer groups.

Proposition 2.3 ([Nek06, §0.8.0 and §9.6]). *For each i, the following sequence is exact:*

$$\ldots \longrightarrow H^{i-1}(U_S^-(X)) \longrightarrow \widetilde{H}^i_f(X) \longrightarrow H^i(G_{K,S}, X) \longrightarrow H^i(U_S^-(X)) \longrightarrow \ldots$$

When Nekovář's Selmer complex is given by a choice of *Greenberg local conditions*, the associated extended Selmer group compares to an appropriately defined *Greenberg Selmer groups*), whose definitions we now recall. For further details, see [Gre89, Gre94, Nek06].

Let I_w denote the inertia subgroup of G_w. Suppose we are given an $R[[G_w]]$-submodule F_w^+X of X for each place $w|p$ of K, set $F_w^-X = X/F_w^-X$. Then Greenberg's local conditions (on the complex level, i.e., in the sense of [Nek06, §6]) are given by

$$U_w^+ = \begin{cases} C^\bullet(G_w, F_w^+X) & \text{if } w|p, \\ C^\bullet(G_w/I_w, X^{I_w}) & \text{if } w \nmid p \end{cases}$$

with the obvious choice of morphisms

$$i_w^+(X) : U_w^+(X) \longrightarrow C^\bullet(G_w, X).$$

As in Definition 2.2, we then obtain a Selmer complex and an extended Selmer group, which we denote by $\widetilde{H}^1_f(X)$. Greenberg's local conditions are the only type of local conditions we will deal with from now on.

We now define the relevant Greenberg Selmer $\mathcal{F}$ on M:

Definition 2.4. The *canonical Selmer structure* $\mathcal{F}$ is given by

$$H^1_{\mathcal{F}}(K_w, X) = \begin{cases} \mathrm{im}\,(H^1(G_w, F_w^+X) \to H^1(G_w, X)) & \text{if } w|p, \\ \ker\,(H^1(G_w, X) \to H^1(I_w, X)) & \text{if } w \nmid p. \end{cases}$$

Remark 2.5. When $X = V$, it follows from [Rub00, Corollary 3.3(i)] and the proof of [Rub00, Proposition 6.7] that $H^1_{\mathcal{F}}(K_w, V) = 0$ for every $w \nmid p$.

Associated to the Selmer structure $\mathcal{F}$, we have the following Selmer group (which is called the *strict Selmer group* in [Nek06, §9.6.1] and denoted by $S_X^{\mathrm{str}}(K)$):

$$H^1_{\mathcal{F}}(K, X) = \ker\left(H^1(G_{K,S}, X) \longrightarrow \bigoplus_{w|p} H^1(G_w, F_w^-X) \oplus \bigoplus_{w\nmid p} H^1(I_w, X)\right). \tag{2.1}$$

Proposition 2.3 implies directly that:

Proposition 2.6. *The following sequence is exact:*

$$H^0(G_{K,S}, X) \longrightarrow \bigoplus_{w|p} H^0(G_w, F_w^-X) \longrightarrow \widetilde{H}^1_f(X) \longrightarrow H^1_{\mathcal{F}}(K, X) \longrightarrow 0.$$

When the coefficient ring R is an integral domain, we let F to be its field of fractions. Set $X_F = X \otimes F$ and $F_w^\pm X_F = (F_w^\pm X) \otimes F$. The true Selmer group $\mathrm{Sel}(K, X)$ is defined as

$$\mathrm{Sel}(K, X) = \ker\left(H^1(G_{K,S}, X) \longrightarrow \bigoplus_{w|p} H^1(I_w, F_w^-X_F) \oplus \bigoplus_{w\nmid p} H^1(I_w, X_F)\right).$$

We also define $H^1_{\mathcal{F}}(K, X_F) = H^1_{\mathcal{F}}(K, X) \otimes F$ and $\mathrm{Sel}(K, X_F) = \mathrm{Sel}(K, X) \otimes F$.

Remark 2.7. Note that in case $H^0(G_w, F_w^- X) = 0$ for all $w|p$, then the extended Selmer group $\widetilde{H}^1_f(X)$ coincides with the Selmer group $H^1_{\mathcal{F}}(K, X)$. However, if some $H^0(G_w, F_w^- X) \neq 0$, then $\widetilde{H}^1_f(X)$ is strictly larger than $H^1_{\mathcal{F}}(K, X)$ (under the assumption that X^{G_K}=0, say). This is the main feature of Nekovář's Selmer complexes: They reflect the existence of exceptional zeros, unlike classical Selmer groups.

Remark 2.8. In this remark, let $X = T$, $X_F = V$ and $K = \mathbb{Q}$. It is well-known (c.f., [CG96, Gre99]) that the Selmer group $H^1_{\mathcal{F}}(\mathbb{Q}, T)$ compares to the true Selmer group $\mathrm{Sel}_p(E/\mathbb{Q}) = \mathrm{Sel}(\mathbb{Q}, T)$ by the following exact sequence:

$$0 \longrightarrow H^1_{\mathcal{F}}(\mathbb{Q}, T) \longrightarrow \mathrm{Sel}_p(E/\mathbb{Q}) \longrightarrow H^1(G_p, F_p^- T)_{\mathrm{tor}} \oplus \left(\bigoplus_{\ell \in S_f - \{p\}} \mathfrak{t}_\ell \right)$$

where $\mathfrak{t}_\ell = \ker(H^1(G_\ell, T) \to H^1(I_\ell, V)) \big/ \ker(H^1(G_\ell, T) \to H^1(I_\ell, T))$. In our setting, the $\mathbb{Z}_p$-module $H^1(G_p, F_p^- T) = \mathrm{Hom}(G_p, \mathbb{Z}_p)$ is torsion free and the order of $\mathfrak{t}_\ell$ equals the p-part of the Tamagawa factor at ℓ. We therefore conclude at once that $H^1_{\mathcal{F}}(\mathbb{Q}, T)$ is a subgroup of $\mathrm{Sel}_p(E/\mathbb{Q})$ of finite index, and further infer that:

- $H^1_{\mathcal{F}}(\mathbb{Q}, T) = \mathrm{Sel}_p(E/\mathbb{Q})$ if
  - (i) p is prime to all Tamagawa factors of E, or if,
  - (ii) $\mathrm{Sel}_p(E/\mathbb{Q}) = 0$.
- In general, $H^1_{\mathcal{F}}(\mathbb{Q}, V) = \mathrm{Sel}_p(E/\mathbb{Q}) \otimes \mathbb{Q}_p$.

2.2. **Height pairings.** We now recall Nekovář's definition of height pairings on his extended Selmer groups. All the references in this section are to [Nek06, §11] unless otherwise stated. Until the end, we assume that $K = \mathbb{Q}$.

Let $X^* = \mathrm{Hom}(X, R)(1)$ (in Nekovář's language this is $\mathcal{D}(X)(1)$, the Grothendieck dual of X) and $X_F^* = \mathrm{Hom}(X_F, F)(1)$. Let Γ be the Galois group $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. Nekovář's height pairing

$$\langle \,, \rangle_{\mathrm{Nek}} : \ \widetilde{H}^1_f(X) \otimes_R \widetilde{H}^1_f(X^*) \longrightarrow R \otimes_{\mathbb{Z}_p} \Gamma$$

is defined in two steps:

(i) Apply the *Bockstein* morphism

$$\beta : \widetilde{\mathbf{R}\Gamma}_f(X) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(X)[1] \otimes_{\mathbb{Z}_p} \Gamma$$

See [Nek06, §11.1.3] for the original definition of β. Let β^1 denote the map induced on the level of cohomology:

$$\beta^1 : \ \widetilde{H}^1_f(X) \longrightarrow \widetilde{H}^2_f(X) \otimes_{\mathbb{Z}_p} \Gamma.$$

(ii) Use the *Poitou-Tate global duality* pairing

$$\langle \,, \rangle_{\mathrm{PT}} : \ \widetilde{H}^2_f(X) \otimes_R \widetilde{H}^1_f(X^*) \longrightarrow R$$

on the image of β^1 inside of $\widetilde{H}^2_f(X) \otimes \Gamma$. Here the global pairing comes from summing up the invariants of the local cup product pairing, see [Nek06, §6.3] for more details.

Any choice of a homomorphism $\kappa : \Gamma \to F$ induces an F-valued height pairing

$$\langle\,,\,\rangle_{\mathrm{Nek},\kappa} :\ \widetilde{H}^1_f(X_F) \otimes_R \widetilde{H}^1_f(X_F^*) \longrightarrow F\ .$$

2.3. **Computations with the local Tate pairing.** For X and X^* as above, we set $K = \mathbb{Q}$ and let $\langle\,,\,\rangle_{\mathrm{Tate}} : H^1(\Phi_n, X) \otimes H^1(\Phi_n, X^*) \to R$ denote the local Tate-pairing. Fix elements $\xi = \{\xi_n\} \in \varprojlim H^1(\Phi_n, X)$ and $\mathbf{z} = \{z_n\} \in \varprojlim H^1(\Phi_n, X^*(1))$ and define

$$\mathcal{L}_\xi^{(n)} = \sum_{\tau\in\Gamma_n} \langle \xi_n, z_n^\tau\rangle_{\mathrm{Tate}} \cdot \tau \in R[\Gamma_n]\,.$$

The elements $\mathcal{L}_\xi^{(n)}$ are compatible with respect to restriction maps $R[\Gamma_n] \to R[\Gamma_m]$ for $m \geq n$ and we may therefore define $\mathcal{L}_\xi = \lim \mathcal{L}_\xi^{(n)} \in R[[\Gamma]]$.

Definition 2.9. Suppose $\xi_0 = 0$. In this case, we define

$$\begin{aligned}\mathrm{Der}_{\rho_{\mathrm{cyc}}}(\mathcal{L}_\xi)(z_0) &:= \lim_{n\to\infty} \sum_{\tau\in\Gamma_n} \log_p(\rho_{\mathrm{cyc}}(\tau^{-1})) \cdot \langle \xi_n^\tau, z_n\rangle_{\mathrm{Tate}} \\ &= -\lim_{n\to\infty} \sum_{\tau\in\Gamma_n} \log_p(\rho_{\mathrm{cyc}}(\tau)) \cdot \langle \xi_n^\tau, z_n\rangle_{\mathrm{Tate}}.\end{aligned}$$

Here we make sense of $\rho_{\mathrm{cyc}}(\tau)$ as follows for $\tau \in \Gamma_n$. Choose any lift $\tilde{\tau} \in \Gamma$ of τ and set $\rho_{\mathrm{cyc}}(\tau) = \rho_{\mathrm{cyc}}(\tilde{\tau})$. The value of $\log_p(\rho_{\mathrm{cyc}}(\tau))$ is therefore well-defined modulo p^n, but the limit above clearly does not depend on the choice of lifts $\tilde{\tau}$. See [Büy12, Lemma 5.9] for a proof that this limit exists.

Lemma 2.10. *Suppose $\xi_0 = 0$. There is an element $\xi' = \{\xi'_n\} \in \varprojlim H^1(\Phi_n, X)$ such that $\xi = \frac{(\gamma-1)}{\log_p(\rho_{\mathrm{cyc}}(\gamma))} \cdot \xi'$. Furthermore, ξ' is uniquely determined when the Λ-module $\varprojlim H^1(\Phi_n, X)$ has no $(\gamma-1)$-torsion.*

Proof. This follows at once from the exactness of the sequence

$$0 \longrightarrow H^1(\mathbb{Q}_p, X\otimes\Lambda)[\gamma-1] \longrightarrow H^1(\mathbb{Q}_p, X\otimes\Lambda) \xrightarrow{\gamma-1} H^1(\mathbb{Q}_p, X\otimes\Lambda) \longrightarrow H^1(\mathbb{Q}_p, T)$$

and using the identification $\varprojlim H^1(\Phi_n, X) = H^1(\mathbb{Q}_p, X\otimes\Lambda)$; where $H^1(\mathbb{Q}_p, X\otimes\Lambda)[\gamma-1]$ stands for the $(\gamma-1)$-torsion submodule of $H^1(\mathbb{Q}_p, X\otimes\Lambda)$. □

Note that ξ'_0 does not depend on the choice of γ.

Lemma 2.11. *Suppose $\xi_0 = 0$ and let $\xi' = \{\xi'_n\}$ is the element whose existence was proved in Lemma 2.10. Then $\mathrm{Der}_{\rho_{\mathrm{cyc}}}(\mathcal{L}_\xi)(z_0) = \langle \xi'_0, z_0\rangle_{\mathrm{Tate}}$.*

Proof. Observe that

$$\begin{aligned}
\log_p(\rho_{\mathrm{cyc}}(\gamma)) \sum_{\tau\in\Gamma_n} \log_p(\rho_{\mathrm{cyc}}(\tau^{-1})) \cdot \xi_n^\tau &= \sum_{\tau\in\Gamma_n} \log_p(\rho_{\mathrm{cyc}}(\tau^{-1})) \cdot (\xi_n')^{\tau(\gamma-1)} \\
&= \sum_{\tau\in\Gamma_n} \left(\log_p(\rho_{\mathrm{cyc}}(\tau^{-1}))(\xi_n')^{\tau\gamma} - \log_p(\rho_{\mathrm{cyc}}(\tau^{-1}))(\xi_n')^{\tau}\right) \\
&= \sum_{\sigma\in\Gamma_n} \left(\log_p(\rho_{\mathrm{cyc}}(\sigma^{-1}))(\xi_n')^{\sigma} + \log_p(\rho_{\mathrm{cyc}}(\gamma))(\xi_n')^{\sigma}\right) \\
&\qquad\qquad - \sum_{\tau\in\Gamma_n} \log_p(\rho_{\mathrm{cyc}}(\tau^{-1}))(\xi_n')^{\tau} \\
&= \log_p(\rho_{\mathrm{cyc}}(\gamma)) \sum_{\sigma\in\Gamma_n} (\xi_n')^{\sigma},
\end{aligned}$$

where all the equalities take place in R/p^nR, and the third equality is obtained by setting $\sigma = \tau\gamma$. This shows that $\sum_{\tau\in\Gamma_n} \log_p(\rho_{\mathrm{cyc}}(\tau^{-1})) \cdot \xi_n^\tau = \sum_{\sigma\in\Gamma_n} (\xi_n')^{\sigma}$ (in $R/p^{n-1}R$). By the commutativity of the diagram

$$\begin{array}{ccccc}
H^1(\Phi_n, X) & \times & H^1(\Phi_n, X^*) & \xrightarrow{\langle,\rangle_{\mathrm{Tate}}} & R \\
\uparrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle cor} & & \\
H^1(\mathbb{Q}_p, X) & \times & H^1(\mathbb{Q}_p, X^*) & \xrightarrow{\langle,\rangle_{\mathrm{Tate}}} & R
\end{array}$$

and the fact that both $\{\xi_n'\}$ and $\{z_n\}$ are norm-coherent, we conclude that

$$\left\langle \sum_{\tau\in\Gamma_n} \log_p(\rho_{\mathrm{cyc}}(\tau^{-1})) \cdot \xi_n^\tau\,,\, z_n \right\rangle_{\mathrm{Tate}} = \langle \xi_0', z_0\rangle_{\mathrm{Tate}}$$

in $R/p^{n-1}R$. Proof of the Lemma follows by letting $n \to \infty$. □

Definition 2.12. Suppose $\xi_0 = 0$ and let $\xi' = \{\xi_n'\}$ be as above. Define

$$\mathcal{L}_\xi' := \mathcal{L}_{\xi'} = \left\{ \sum_{\tau\in\Gamma_n} \langle \xi_n', z_n^\tau\rangle_{\mathrm{Tate}} \cdot \tau \right\} \in \Lambda.$$

Observe that this element depends both on the choice of γ and the choice of ξ'.

Let $J = \ker(\Lambda \to \mathbb{Z}_p)$ denote the augmentation ideal. We have an isomorphism

$$R \otimes_{\mathbb{Z}_p} J/J^2 \xrightarrow{\sim} R \otimes_{\mathbb{Z}_p} \Gamma \xrightarrow{\sim} R$$

given by $1 \otimes (\gamma - 1 \mod J^2) \mapsto \frac{1}{p}\log_p(\rho_{\mathrm{cyc}}(\gamma))$. Let $1 \otimes (\gamma_0 - 1) \in J/J^2$ denote the image of $1 \in R$ under the inverse of this composition.

Lemma 2.13. $\dfrac{(\gamma-1)}{\log_p(\rho_{\mathrm{cyc}}(\gamma))}\mathcal{L}_\xi' \equiv \mathcal{L}_\xi \mod J^2.$

Proof. The proof of this is identical to the proof of Lemma 2.11. □

The exact sequence

$$0 \longrightarrow X \otimes J/J^2 \longrightarrow X \otimes \Lambda/J^2 \xrightarrow{j} X \otimes J/J^2 \longrightarrow 0 \tag{2.2}$$

where j stands for the map induced from multiplication by $(\gamma-1)/\log_p(\rho_{\rm cyc}(\gamma))$. Consider the commutative diagram

$$\begin{array}{ccc} H^1(\mathbb{Q}_p, X\otimes\Lambda) & \xrightarrow{j} & J\cdot H^1(\mathbb{Q}_p, X\otimes\Lambda) \\ {\scriptstyle \rm red}\downarrow & & \downarrow{\scriptstyle \mathcal{D}} \\ H^1(\mathbb{Q}_p, X\otimes\Lambda/J^2) & \xrightarrow{j} & H^1(\mathbb{Q}_p, X\otimes J/J^2) \end{array}$$

where the vertical map on the left is the reduction map and the lower horizontal map is induced from (2.2). The map $\mathcal{D}$ is obtained by completing the square; note that it is well-defined as when $x\in H^1(\mathbb{Q}_p, X\otimes\Lambda)[\gamma-1]$, one has $(\gamma-1)\cdot\text{red}(x)=0$. If $\xi_0=0$, then as explained in Lemma 2.10, the element ξ is in the image of the map $H^1(\mathbb{Q}_p, X\otimes\Lambda)\xrightarrow{j} H^1(\mathbb{Q}_p, X\otimes\Lambda)$ and therefore one can define an element $\mathcal{D}(\xi)\in H^1(\mathbb{Q}_p, X\otimes J/J^2)$. Since the G_p-action on J/J^2 is trivial, we have $H^1(\mathbb{Q}_p, X\otimes J/J^2) = H^1(\mathbb{Q}_p, X)\otimes J/J^2$. In case $H^1(\mathbb{Q}_p, X\otimes\Lambda)[\gamma-1]=0$, observe that $\mathcal{D}(\xi)=\xi_0'\otimes(\gamma_0-1)$ where ξ_0' is as in Lemma 2.11. If we let

$$\langle\, ,\,\rangle_{J/J^2} \,:\, \big(H^1(\mathbb{Q}_p, X)\otimes J/J^2\big)\otimes H^1(\mathbb{Q}_p, X^*) \longrightarrow R\otimes J/J^2$$

denote the pairing induced from the local Tate pairing, we also have (compare to Lemma 2.11)

$$\mathrm{Der}_{\rho_{\rm cyc}}(\mathcal{L}_\xi)(z_0)\otimes(\gamma_0-1) = \langle \mathcal{D}(\xi), z_0\rangle_{J/J^2} \in R\otimes J/J^2. \tag{2.3}$$

3. The height formulas

Fix a generator $\{\zeta_{p^n}\}$ of $\mathbb{Z}_p(1) = \varprojlim_n \boldsymbol{\mu}_{p^n}$. Let $E/\mathbb{Q}$ be an elliptic curve that has split multiplicative reduction at p. Then E is a Tate curve at p, i.e., it admits a uniformization

$$\mathbb{C}_p^\times/q_E^{\mathbb{Z}} \xrightarrow{\sim} E(\mathbb{C}_p)$$

for some $q_E\in\mathbb{Q}_p^\times$. The following theorem that was formerly known as Manin's conjecture was proved in [BSDGP96]:

Theorem 3.1 (*Saint-Etienne Theorem*)**.** $\log_p(q_E)\neq 0$.

Let $L(E/\mathbb{Q}, s)$ denote the Hasse-Weil L-function attached to E. It is known thanks to [Wil95, BCDT01] that $L(E/\mathbb{Q}, s)$ is an entire function, let $r_{\rm an} := \mathrm{ord}_{s=1}L(E/\mathbb{Q}, s)$ be the order of vanishing at $s=1$.

Attached to E, there is an element $\mathcal{L}_E\in\Lambda$ (the *Mazur-Tate-Teitelbaum p-adic L-function*) constructed in [MTT86] and characterized by the interpolation formula

$$\chi(\mathcal{L}_E) = \tau(\chi)\frac{L(E,\chi^{-1},1)}{\Omega_E^+}$$

for every non-trivial character χ of Γ of finite order, where $\tau(\chi)=\sum_{\delta\in\Delta_n}\chi(\delta)\zeta_{p^{n+1}}^{\delta}$ is the Gauss sum and where n is the smallest integer such that χ factors through $\Delta_n :=$

Γ/Γ^{p^n}. Furthermore, the Mazur-Tate-Teitelbaum's p-adic L-function vanishes at the trivial character $\mathbf{1}$, namely, $\mathbf{1}(\mathcal{L}_E) = 0$. Setting

$$L_p(E,s) := \rho_{\text{cyc}}^{s-1}(\mathcal{L}_E)\,,$$

we conclude in this case that $L_p(E,1) = 0$. A theorem of Greenberg-Stevens [GS93] expresses the derivative of the p-adic L-function $L_p(E,s)$ at $s=1$ in terms of the L-value:

$$\frac{d}{ds}L_p(E,s)\Big|_{s=1} = \frac{\log_p(q_E)}{\text{ord}_p(q_E)}L(E,1)/\Omega_E^+. \tag{3.1}$$

We therefore conclude when $r_{\text{an}} = 0$ or 1, the order of vanishing of $L_p(E,s)$ at $s=1$ is at least $1+r_{\text{an}}$. Our goal is to express $\frac{d}{ds}L_p(E,s)\big|_{s=1}$ (resp., $\frac{d^2}{ds^2}L_p(E,s)\big|_{s=1}$) when $r_{\text{an}} = 0$ (resp., when $r_{\text{an}} = 1$) in terms of Nekovář's height pairings, evaluated on elements obtained from Kato's Euler system and the Coleman map, whose basic properties we outline below.

Remark 3.2. By a slight abuse, we will denote the measure on Γ associated to an element $\mathcal{L} \in \Lambda$ also by $\mathcal{L}$. Then for any continuous character $\psi : \Gamma \to \mathbb{C}_p$, we will have $\int_\Gamma \psi \cdot d\mathcal{L} = \psi(\mathcal{L})$. For example, we will sometimes prefer to write $L_p(E,s) = \int_\Gamma \rho_{\text{cyc}}^{s-1} \cdot d\mathcal{L}_E$.

3.1. **The (explicit) Coleman map for a Tate Curve.** We review here the definition of the Coleman map following [Rub98] and [Kob03, Section 8]. Let $\mathfrak{O}_n$ denote the ring of integers of Φ_n and let $\mathfrak{m}_n$ denote the maximal ideal of $\mathfrak{O}_n$ and $\pi_n \in \mathfrak{m}_n$ a fixed uniformizer. Denote 1-units of $\mathfrak{O}_n$ by U_n^1. For a fixed generator $\{\zeta_{p^n}\}$ of $\mathbb{Z}_p(1)$, one constructs elements $c_n \in \widehat{\mathbb{G}}_m(\mathfrak{m}_n)$ so that the elements $d_n := 1 + c_n \in U_n^1$ are norm compatible as n varies and d_n generates $(U_n^1)^{\mathbf{N}=1}$ where $\mathbf{N}$ stands for the absolute norm from Φ_n to $\mathbb{Q}_p$. Let

$$d_\infty = \{d_n\} \in \varprojlim \Phi_n^\times \,\widehat{\otimes}\, \mathbb{Z}_p \cong \varprojlim H^1(\Phi_n, \mathbb{Z}_p(1)) \cong H^1(\mathbb{Q}_p, \mathbb{Z}_p(1)\otimes\Lambda),$$

where the first isomorphism hollows from Kummer theory and second from [Col98, Proposition II.1.1]. As $\mathbf{N}(d_n) = 1$ by construction, it follows that d_∞ is in the kernel of the augmentation map:

$$d_\infty \in \ker(H^1(\mathbb{Q}_p, \mathbb{Z}_p(1)\otimes\Lambda) \longrightarrow H^1(\mathbb{Q}_p, \mathbb{Z}_p(1))) = (\gamma-1)H^1(\mathbb{Q}_p, \mathbb{Z}_p(1)\otimes\Lambda).$$

Let

$$\mathfrak{C}_\infty = \{\mathfrak{C}_n\} \in H^1(\mathbb{Q}_p, \mathbb{Z}_p(1)\otimes\Lambda) = \varprojlim \Phi_n^\times \,\widehat{\otimes}\, \mathbb{Z}_p \tag{3.2}$$

be the element chosen such that

$$d_\infty = \frac{(\gamma-1)}{\log_p(\rho_{\text{cyc}}(\gamma))} \cdot \mathfrak{C}_\infty\ .$$

It is straightforward to verify that the element $\mathfrak{C}_0$ does not depend on the choice of γ. As we have assumed the elliptic curve E has split multiplicative reduction mod p, it follows that E is locally a Tate curve, namely that $E_{/\mathbb{Q}_p} = E_q$ where

$$E_q : y^2 + xy = x^3 + a_4(q)x + a_6(q)$$

with $q = q_E \in \mathbb{Q}_p^\times$ satisfying $\mathrm{ord}_p(q) > 0$ and

$$a_4(q) = -\sum_{n\geq 1} \frac{n^3 q^n}{1-q^n} \quad , \quad a_6(q) = -\frac{5}{12}\sum_{n\geq 1} \frac{n^3 q^n}{1-q^n} + \frac{7}{12}\sum_{n\geq 1} \frac{n^5 q^n}{1-q^n} \; .$$

Then E_q admits a Tate uniformization

$$\phi : \mathbb{C}_p^\times / q^{\mathbb{Z}} \xrightarrow{\sim} E_q(\mathbb{C}_p).$$

This isomorphism induces an isomorphism of formal groups

$$\widehat{\phi} : \widehat{\mathbb{G}}_m \xrightarrow{\sim} \widehat{E}.$$

Via this isomorphism, we regard the element $c_n \in \widehat{\mathbb{G}}_m(\mathfrak{m}_n)$ as an element of $\widehat{E}(\mathfrak{m}_n)$, and by the Kummer map also an element of $H^1(\Phi_n, T)$. Using the local duality pairing

$$\langle \, , \, \rangle_{\mathrm{Tate},E} : H^1(\Phi_n, T) \times H^1(\Phi_n, T^*) \longrightarrow \mathbb{Z}_p,$$

we obtain $\mathbb{Z}_p[\Gamma_n]$-linear maps

$$\begin{aligned} \mathrm{Col}_n : H^1(\Phi_n, T^*) &\to \mathbb{Z}_p[\Gamma_n] \\ z &\mapsto \sum_{\tau\in\Gamma_n} \langle c_n^\tau, z\rangle_{\mathrm{Tate},E} \cdot \tau \end{aligned}$$

which are compatible as n varies with respect to corestriction maps and natural projections. Hence these maps yield in the limit a Λ-equivariant map

$$\mathrm{Col} : \varprojlim H^1(\Phi_n, T^*) \cong H^1(\mathbb{Q}_p, T^* \otimes \Lambda) \longrightarrow \Lambda.$$

As explained in [Kob06, §4],

$$\mathrm{Col}_n(z) = \sum_{\tau\in\Gamma} \langle d_n^\tau, \mathrm{loc}_p^s(z_n)\rangle_{\mathrm{Tate}} \cdot \tau \tag{3.3}$$

where $\mathrm{loc}_p^s : H^1(\Phi_n, T^*) \to H^1(\Phi_n, F_p^- T^*)$ is the projection on to the singular quotient so that we obtain a map (which we still denote by Col) in the limit

$$\mathrm{Col} : \varprojlim H^1(\Phi_n, F_p^- T^*) \cong H^1(\mathbb{Q}_p, F_p^- T^* \otimes \Lambda) \longrightarrow \Lambda.$$

3.2. **Kato's Beilinson elements.** Given an elliptic curve E, Kato has constructed an element

$$\mathfrak{z}_\infty^{\mathrm{Kato}} = \{\mathfrak{z}_n^{\mathrm{Kato}}\} \in \varprojlim H^1(\mathbb{Q}_n, T^*) \otimes \mathbb{Q}_p = H^1(\mathbb{Q}, T^* \otimes \Lambda) \otimes \mathbb{Q}_p$$

which has the property that

$$\mathrm{Col}(\mathrm{loc}_p(\mathfrak{z}_\infty^{\mathrm{Kato}})) = \mathcal{L}_E, \tag{3.4}$$

where $\mathrm{loc}_p : H^1(\mathbb{Q}_n, -) \to H^1(\Phi_n, -)$ is the localization at p. For simplicity, we set $z_n^{\mathrm{Kato}} = \mathrm{loc}_p(\mathfrak{z}_n^{\mathrm{Kato}})$ and write z^{Kato} in place of $z_0^{\mathrm{Kato}} \in H^1(\mathbb{Q}_p, T^*) \otimes \mathbb{Q}_p$. For each $n \geq 0$, let

$$\mathrm{loc}_p^s : H^1(\Phi_n, T^*) \otimes \mathbb{Q}_p \longrightarrow H^1(\Phi_n, F_p^- T^*) \otimes \mathbb{Q}_p$$

denote the natural projection map.

Remark 3.3. It may be proved that Kato's Beilinson elements are locally integral, namely that

$$\mathrm{loc}_p(\mathfrak{z}_n^{\mathrm{Kato}}) \in H^1(\Phi_n, T^*).$$

Furthermore, in case $E(\mathbb{Q})[p] = 0$, Kato's elements are globally integral as well, i.e.,

$$\mathfrak{z}_\infty^{\mathrm{Kato}} = \{\mathfrak{z}_n^{\mathrm{Kato}}\} \in H^1(\mathbb{Q}, T^* \otimes \Lambda).$$

Perrin-Riou in [PR93a, §3.3.2] proposes the following:

Conjecture 3.4. The element $\mathfrak{z}_0^{\mathrm{Kato}} \in H^1(\mathbb{Q}, T^*) \otimes \mathbb{Q}_p$ is non-trivial iff $\mathrm{ord}_{s=1} L(E,s) \leq 1$.

In this article, we need the "if" part of this conjecture and this has been established by Venerucci in his forthcoming thesis:

Theorem 3.5 (Venerucci)**.** *If* $\mathrm{ord}_{s=1} L(E,s) \leq 1$, *then the element* $\mathfrak{z}_0^{\mathrm{Kato}} \in H^1(\mathbb{Q}, T^*) \otimes \mathbb{Q}_p$ *is non-trivial.*

3.3. **The case $r_{\mathbf{an}} = 0$.**

Proposition 3.6 (Kato, Kolyvagin)**.** *Suppose* $L(E,1) \neq 0$. *Then*

(i) $\mathrm{Sel}_p(E/\mathbb{Q})$ *is finite,*
(ii) $H^1_{\mathcal{F}}(\mathbb{Q}, V) = 0$.

Using Proposition 2.6, we obtain isomorphisms

$$H^0(G_p, F_p^- V) \xrightarrow{\sim} \tilde{H}^1_f(V) \tag{3.5}$$

$$H^0(G_p, F_p^- V^*) \xrightarrow{\sim} \tilde{H}^1_f(V^*) \tag{3.6}$$

Let $\alpha \in H^0(G_p, F_p^- V)$ and $\alpha^* \in H^0(G_p, F_p^- V^*)$. Denote their respective images under the isomorphisms (3.5) and (3.6) by $[\alpha]$ and $[\alpha^*]$. The exact sequence (1.3) yields an injection

$$\partial_p : H^0(G_p, F_p^- V) \hookrightarrow H^1(G_p, F_p^+ V). \tag{3.7}$$

Let $z : G_{\mathbb{Q}} \twoheadrightarrow \Gamma$ be the tautological homomorphism. Letting $G_{\mathbb{Q}}$ act trivially on Γ, one may view z as an element of $H^1(\mathbb{Q}, \Gamma) = \mathrm{Hom}(G_{\mathbb{Q}}, \Gamma)$. Its localization $z_p \in H^1(G_p, \Gamma)$ also corresponds to the tautological homomorphism $G_p \twoheadrightarrow \Gamma$, where we now view Γ as the decomposition group of p inside $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$.

Proposition 3.7. *Let* $z_p \cup \alpha_p^* \in H^1(G_p, \mathbb{Q}_p \otimes \Gamma) = H^1(G_p, \mathbb{Q}_p) \otimes \Gamma$ *be the cup-product of* z_p *and* α_p^*. *Then we have the following equality in* $\mathbb{Q}_p \otimes \Gamma$*:*

$$\langle [\alpha], [\alpha^*] \rangle_{\mathrm{Nek}} = \langle \partial_p(\alpha_p), -z_p \cup \alpha_p^* \rangle_{\mathrm{Tate}} .$$

Proof. This follows from [Nek06, Corollary 11.4.7], along with the remark 11.3.5.3 of loc.cit. □

Recall Kato's local element $z^{\mathrm{Kato}} \in H^1(\mathbb{Q}_p, T^*)$ and the element $\mathfrak{C}_0 \in H^1(G_p, F_p^+ T) \cong \widehat{\mathbb{Q}_p^\times}$ obtained using the explicit description of Coleman map. The first part of the Theorem below should be thought of as a "Rubin-style formula", although it doesn't seem to follow from Nekovář's version [Nek06, 11.5.11] of it. The second part expresses the leading coefficient of the p-adic L-function in terms of Nekovář's height pairing. Recall the homomorphism $\rho : \Gamma \to \mathbb{Z}_p$, which is the compositum of the maps

$$\rho : \Gamma \xrightarrow{\rho_{\mathrm{cyc}}} 1 + p\mathbb{Z}_p \xrightarrow{E_p(1)^{-1}\log_p} \mathbb{Z}_p ,$$

where $E_p(s) = 1 - p^{-s}$ is the Euler factor at p.

Theorem 3.8. (i) $\left\langle [\mathrm{ord}_p(q_E)^{-1}], [\exp^*_{\omega_E}(z^{\mathrm{Kato}})] \right\rangle_{\mathrm{Nek},\rho} = \langle \mathfrak{C}_0, \mathrm{loc}_p^s(z^{\mathrm{Kato}}) \rangle_{\mathrm{Tate}} .$

(ii) $\frac{d}{ds} L_p(E,s)\Big|_{s=1} = \left\langle [-\mathrm{ord}_p(q_E)^{-1}], [\exp^*_{\omega_E}(z^{\mathrm{Kato}})] \right\rangle_{\mathrm{Nek},\rho}.$

Proof. Write $q = q_E = p^{\mathrm{ord}_p(q)} \cdot u_q$ and let χ_p be the compositum

$$\chi_p : \widehat{\mathbb{Q}_p^\times} \xrightarrow{\text{rec}} G_p^{\text{ab}} \longrightarrow \Gamma \xrightarrow{\rho} \mathbb{Z}_p\,.$$

Since the image of $1 \in \mathbb{Q}_p = H^0(G_p, F_p^- V)$ under (3.7) is q, it follows from Prop. 3.7 that

$$\begin{aligned}
\big\langle [\mathrm{ord}_p(q)^{-1}], [\exp^*_{\omega_E}(z^{\text{Kato}})] \big\rangle_{\text{Nek},\rho} &= \Big\langle q^{\mathrm{ord}_p(q)^{-1}}, -\exp^*_{\omega_E}(z^{\text{Kato}}) \cdot \chi_p \Big\rangle_{\text{Tate}} \\
&= -\mathrm{ord}_p(q)^{-1} \exp^*_{\omega_E}(z^{\text{Kato}}) \, \langle u_q, \chi_p \rangle_{\text{Tate}} && (3.8) \\
&= -(1-1/p)\,\mathrm{ord}_p(q)^{-1} \log_p(u_q) \exp^*_{\omega_E}(z^{\text{Kato}}) && (3.9) \\
&= \langle \mathfrak{C}_0, \mathrm{loc}_p^s(z^{\text{Kato}}) \rangle_{\text{Tate}} && (3.10)
\end{aligned}$$

where the equality (3.8) is because the homomorphism z_p factors through the inertia subgroup of G_p^{ab}, (3.9) follows thanks to our normalization of Nekovář's height and (3.10) is the main calculation carried out in [Kob06, §4]. This completes the proof of (i).

To prove (ii), observe that $\frac{d}{ds}\rho_{\text{cyc}}^{s-1} = \log_p \rho_{\text{cyc}} \cdot \rho_{\text{cyc}}^{s-1}$, hence

$$\begin{aligned}
\frac{d}{ds} L_p(E,s)\Big|_{s=1} &= \int_\gamma \log_p \rho_{\text{cyc}} \cdot d\mathcal{L}_E \\
&= \lim_{n\to\infty} \sum_{\tau \in \Gamma_n} \log_p \rho_{\text{cyc}}(\tau) \, \big\langle d_n^\tau \,,\, \mathrm{loc}_p^s(z_\infty^{\text{Kato}}) \big\rangle_{\text{Tate}} \\
&= \lim_{n\to\infty} \Big\langle \sum_{\tau \in \Gamma_n} \log_p \rho_{\text{cyc}}(\tau) \cdot d_n^\tau \,,\, \mathrm{loc}_p^s(z_\infty^{\text{Kato}}) \Big\rangle_{\text{Tate}}
\end{aligned}$$

where the second equality follows from the explicit description of the Coleman map (essentially (3.3), see also [Kob06, p. 572]). By Lemma 2.11 applied with $X = F_p^+ T$, $X^* = F_p^- T^*$, $\xi = d_\infty$ (so that $\xi_0' = \mathfrak{C}_0$) and $\mathbf{z} = \mathrm{loc}_p^s(z_\infty^{\text{Kato}})$,

$$\lim_{n\to\infty} \Big\langle \sum_{\tau \in \Gamma_n} \log_p \rho_{\text{cyc}}(\tau) \cdot d_n^\tau \,,\, \mathrm{loc}_p^s(z_\infty^{\text{Kato}}) \Big\rangle_{\text{Tate}} = -\langle \mathfrak{C}_0, \mathrm{loc}_p^s(z_0^{\text{Kato}}) \rangle_{\text{Tate}}$$

and (ii) now follows from (i). □

3.4. **The case $r_{\text{an}} = 1$.** Until the end of this article, suppose that $r_{\text{an}} = 1$. Assume in addition that $E(\mathbb{Q})[p] = 0$. As we have noted in Remark 3.3, this assumption implies that Kato's elements are integral:

$$\mathfrak{z}_\infty^{\text{Kato}} = \{\mathfrak{z}_n^{\text{Kato}}\} \in H^1(\mathbb{Q}, T \otimes \Lambda).$$

Note that we had introduced Kato's elements $\mathfrak{z}_0^{\text{Kato}}$ within $H^1(\mathbb{Q}, T^*)$. Using the natural isomorphism $T \cong T^*$, we may regard Kato's elements as cohomology classes for T as well. Recall the localization of Kato's element $z^{\text{Kato}} := \mathrm{loc}_p(\mathfrak{z}_0^{\text{Kato}}) \in H^1(\mathbb{Q}_p, T)$.

Proposition 3.9. *Under the running assumptions, $z^{\text{Kato}} \neq 0$.*

Proof. Assume on the contrary that

$$z^{\text{Kato}} = \mathrm{loc}_p(\mathfrak{z}_0^{\text{Kato}}) = 0. \tag{3.11}$$

Let $\mathcal{F}_{\text{str}}$ denote the Selmer structure on T given by

- $H_{\mathcal{F}_{\text{str}}}(\mathbb{Q}_\ell, T) = H_{\mathcal{F}}(\mathbb{Q}_\ell, T)$, if $\ell \neq p$,

- $H_{\mathcal{F}_{\mathrm{str}}}(\mathbb{Q}_p, T) = 0$.

so that (3.11) amounts to saying $\mathfrak{z}_0^{\mathrm{Kato}} \in H_{\mathcal{F}_{\mathrm{str}}}(\mathbb{Q}, T)$. As $\mathfrak{z}_0^{\mathrm{Kato}}$ is non-torsion thanks to our running assumptions and Theorem 3.5, it follows that $\mathrm{rank}_{\mathbb{Z}_p}(H^1_{\mathcal{F}_{\mathrm{str}}}(\mathbb{Q}, T)) \geq 1$.

Let $\mathcal{F}_{\mathrm{str}}$ denote also the propagation of the Selmer structure (in the sense of [MR04]) to T/p^nT. For any positive integer n, identify the quotient T/p^nT with $E[p^n]$. By [MR04, Lemma 3.7.1], we have an injection

$$H_{\mathcal{F}_{\mathrm{str}}}(\mathbb{Q}, T)/p^n H_{\mathcal{F}_{\mathrm{str}}}(\mathbb{Q}, T) \hookrightarrow H_{\mathcal{F}_{\mathrm{str}}}(\mathbb{Q}, T/p^nT) = H_{\mathcal{F}_{\mathrm{str}}}(\mathbb{Q}, E[p^n])$$

induced from the projection $T \to T/p^nT$. This shows that

$$\mathrm{length}_{\mathbb{Z}_p}\left(H_{\mathcal{F}_{\mathrm{str}}}(\mathbb{Q}, E[p^n])\right) \geq n. \tag{3.12}$$

Let now $\mathcal{F}_{\mathrm{can}}$ denote the canonical Selmer structure on T, given by

- $H_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q}_\ell, T) = H_{\mathcal{F}}(\mathbb{Q}_\ell, T)$, if $\ell \neq p$,
- $H_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q}_p, T) = H^1(\mathbb{Q}_p, T)$.

Let $\mathcal{F}^*_{\mathrm{can}}$ denote the dual Selmer structure on $\mathrm{Hom}(T, \boldsymbol{\mu}_{p^\infty}) \cong E[p^\infty]$, where the isomorphism is obtained via the Weil-pairing. The propagation of $\mathcal{F}^*_{\mathrm{can}}$ on $E[p^\infty]$ to its submodule $E[p^n]$ will also be denoted by $\mathcal{F}^*_{\mathrm{can}}$. It follows from [Rub00, Lemma I.3.8(i)] (together with the discussion in [MR04, §6.2]) that we have an inclusion

$$H_{\mathcal{F}_{\mathrm{str}}}(\mathbb{Q}_\ell, E[p^n]) \subset H_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}_\ell, E[p^n])$$

for every ℓ, which in turn shows that together with (3.12) that

$$\mathrm{length}_{\mathbb{Z}_p}\left(H_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, E[p^n])\right) \geq n. \tag{3.13}$$

On the other hand, as $\mathfrak{z}_0^{\mathrm{Kato}} \neq 0$, it follows from [MR04, Cor. 5.2.13] that $H_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, E[p^\infty])$ is finite. This however shows that the length of

$$H_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, E[p^n]) \cong H_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, E[p^\infty])[p^n]$$

(where the isomorphism is thanks to [MR04, Lemma 3.5.3], which holds true here owing to our assumption that $E(\mathbb{Q})[p] = 0$) is bounded independently of n. This contradicts (3.13) and shows that our assumption (3.11) is wrong. $\square$

Remark 3.10. In this remark we elaborate on the "only if" part of Conjecture 3.4. Suppose $\mathfrak{z}_0^{\mathrm{Kato}} \in H^1(\mathbb{Q}, T^*)$ is non-torsion[‡]. It follows by the theory of Euler systems that the strict Selmer group

$$H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, V/T) := \ker(H^1_{\mathcal{F}}(\mathbb{Q}, V/T) \longrightarrow H^1(\mathbb{Q}_p, V/T))$$

is finite. It then follows from global duality (c.f., Theorem 5.2.15 and Corollary 5.2.6 of [MR04]) that

$$\mathrm{rank}_{\mathbb{Z}_p}(H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q}, T^*)) = \dim_{\mathbb{Q}_p}(V^*)^- = 1, \tag{3.14}$$

where $(V^*)^-$ stands for the -1-eigenspace of V^* of a fixed complex conjugation in $G_{\mathbb{Q}}$. This in turn shows that $\mathrm{rank}_{\mathbb{Z}_p}(\mathrm{Sel}(\mathbb{Q}, T^*)) \leq 1$. The conjecture of Birch and Swinnerton-Dyer then predicts the assertion of Conjecture 3.4.

Suppose now that $\mathrm{rank}_{\mathbb{Z}_p}(\mathrm{Sel}(\mathbb{Q}, T^*)) = 0$. As explained in (3.14), the $\mathbb{Z}_p$-module $H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q}, T^*)$ is of rank 1 and that $\mathrm{loc}_p^s(\mathfrak{z}_0^{\mathrm{Kato}}) \neq 0$. Kato's reciprocity law implies in this case that $L(E, 1) \neq 0$, *unconditionally*.

[‡]Under the assumption that $E(\mathbb{Q})[p] = 0$, the $\mathbb{Z}_p$-module $H^1(\mathbb{Q}, T^*)$ is torsion-free. Hence, our assumption amounts to asking that $\mathfrak{z}_0^{\mathrm{Kato}} \neq 0$

In the case $\mathrm{rank}_{\mathbb{Z}_p}(\mathrm{Sel}(\mathbb{Q},T^*)) = 1$, unfortunately we are not able to go this far. As $\mathrm{rank}_{\mathbb{Z}_p}(\mathrm{Sel}(\mathbb{Q},T^*)) = 1$, we conclude by (3.14) that $H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q},T^*)\otimes\mathbb{Q}_p = \mathrm{Sel}(\mathbb{Q},T^*)\otimes\mathbb{Q}_p$ and hence $\mathfrak{z}_0^{\mathrm{Kato}} \in \mathrm{Sel}(\mathbb{Q},T^*)\otimes\mathbb{Q}_p$. One would then expect to relate the height of $\mathfrak{z}_0^{\mathrm{Kato}}$ to $L'(E,1)$[†] and conclude this way that $L'(E,1)\neq 0$. This, however, seems untractable at this stage[§]. When p is a good-ordinary prime, Perrin-Riou in [PR93a] shows that the p-adic height of $\mathfrak{z}_0^{\mathrm{Kato}}$ is related to the derivative of the Mazur-Tate-Teitelbaum p-adic L-function. Our Theorem 3.18 extends this to the case where p is a prime of split multiplicative reduction.

Observe that $\mathrm{loc}_p^s(\mathfrak{z}_0^{\mathrm{Kato}}) = 0$ as we have assumed $\mathrm{ord}_{s=1} L(E,s) = 1$. We therefore conclude that $\mathrm{loc}_p(\mathfrak{z}_0^{\mathrm{Kato}}) = z^{\mathrm{Kato}} \in H^1_f(\mathbb{Q}_p,T)$. Consider the diagram with exact rows:

$$\begin{tikzcd}
H^0(\mathbb{Q}_p,F_p^-T) \arrow[r, "\partial"] & H^1(\mathbb{Q}_p,F_p^+T) \arrow[r, "\phi"] & H^1(\mathbb{Q}_p,T) \\
\mathbb{Z}_p \arrow[u, "\sim"] \arrow[r, "\partial"'] & \widehat{\mathbb{Q}_p^\times} \arrow[u, "\sim"'] \arrow[r, "\phi"'] & \widehat{\mathbb{Q}_p^\times}/q_E^{\mathbb{Z}_p} \arrow[u, hook, "\psi"']
\end{tikzcd}$$

We note that $\partial(1) = q_E$ and $\mathrm{im}(\phi) = E(\mathbb{Q}_p)\otimes\mathbb{Z}_p = H^1_f(\mathbb{Q}_p,T)$ is the isomorphic image of $\widehat{\mathbb{Q}_p^\times}/q_E^{\mathbb{Z}_p}$ under the map ψ. Let $\mathfrak{C}_0 \in \widehat{\mathbb{Q}_p^\times}$ be the explicit element defined as in (3.2).

Lemma 3.11. *For any $\alpha\in\mathbb{Z}_p$, $\log_p(\mathfrak{C}_0^\alpha) = 0$.*

Proof. As above, let U_n^1 denote the 1-units of the extension Φ_n and $(U_n^1)^{\mathbf{N=1}} \subset U_n^1$ be the submodule of absolute norm 1. Let $\pi_n \in \Phi_n$ be the uniformizer defined as in [Kob06, pp. 570], so that the collection $\{\pi_n\}$ is norm-compatible. It follows from the discussion in [Kob06, pp. 570] that $\mathfrak{C}_n = \pi_n^e u_n$ for some $e\in\mathbb{Z}_p$ and $u_n \in (U_n^1)^{\mathbf{N=1}}$, where e is explicitly determined in Proposition 2.2 of loc.cit. This shows that $\mathfrak{C}_0^\alpha = \mathbf{N}(\mathfrak{C}_n)^\alpha = p^{\alpha e}$. ☐

As the $\mathbb{Q}_p$-vector space $H^1_f(\mathbb{Q}_p,T)\otimes\mathbb{Q}_p$ is of dimension one, we define $\lambda\in\mathbb{Q}_p$ (which we call the *local normalization factor*) to be the unique element which verifies

$$\psi\circ\phi(\mathfrak{C}_0) = \lambda\cdot z^{\mathrm{Kato}} \tag{3.15}$$

inside of the isomorphic image $H^1_f(\mathbb{Q}_p,V)$ of $H^1(\mathbb{Q}_p,F_p^+T)\otimes\mathbb{Q}_p$ under ϕ.

Lemma 3.12. *The normalization factor λ is non-zero.*

Proof. It follows from Theorem 3.1 and Lemma 3.11 that $\phi(\mathfrak{C}_0) \in \widehat{\mathbb{Q}_p^\times}/q_E^{\mathbb{Z}_p}$ is non-torsion. The proof the lemma now follows as the map ψ is injective. ☐

Let $\Phi = \mathbb{Q}_p(\sqrt{\lambda})$ and let $\mathcal{O}$ be the ring of integers of Φ. For $X = T, V, T^*$ or V^*, define $X_\Phi = X\otimes_{\mathbb{Z}_p}\mathcal{O}$.

Definition 3.13. We define the normalization of Kato's element to be

$$\tilde{\mathfrak{z}}^{\mathrm{Kato}} = \lambda^{1/2}\cdot\mathfrak{z}_0^{\mathrm{Kato}} \in \mathrm{Sel}(\mathbb{Q},V_\Phi).$$

[†]As a matter of fact, as $\mathrm{Sel}(\mathbb{Q},T^*)$ is rank one, one would expect to relate $\mathfrak{z}_0^{\mathrm{Kato}}$ to Heegner points. This indeed is the content of Perrin-Riou's conjecture. The Gross-Zagier formula would then express $L'(E,1)$ in terms of the height of $\mathfrak{z}_0^{\mathrm{Kato}}$.

[§]See, however, Venerucci's thesis for progress in this direction.

Set $\Xi_n = \mathrm{loc}_p^s(\mathfrak{z}_n^{\mathrm{Kato}}) \in H^1(\Phi_n, F_p^- T^*)$ and $\Xi = \{\Xi_n\} \in H^1(\mathbb{Q}_p, F_p^- T^* \otimes \Lambda)$. Note that we are once again implicitly identifying T with T^*. As our running assumptions show that $\Xi_0 = 0$, this allows us to choose $\Xi' = \{\Xi'_n\} \in H^1(\mathbb{Q}_p, F_p^- T^* \otimes \Lambda)$ as in Lemma 2.10 (applied with $X = F_p^- T^*$).

Definition 3.14. Let $\mu_E \in \Lambda$ be the element defined as

$$\mu_E = \left\{ \sum_{\tau \in \Gamma_n} \langle \mathfrak{C}_n^\tau, \Xi'_n \rangle_{\mathrm{Tate}} \cdot \tau \right\} \in \varprojlim \mathbb{Z}_p[\Gamma_n]\,.$$

Although μ_E depends on the choice of Ξ' and γ, the value

$$\int_\Gamma \mathbf{1} \cdot d\mu_E = \mathbf{1}(\mu_E) = \langle \mathfrak{C}_0, \Xi'_0 \rangle_{\mathrm{Tate}} \tag{3.16}$$

does not, as shown by Lemma 2.11.

Recall the augmentation ideal $J = \ker(\Lambda \to \mathbb{Z}_p)$.

Proposition 3.15. $\dfrac{(\gamma-1)^2}{\log_p(\rho_{\mathrm{cyc}}(\gamma))^2}\, \mu_E \equiv \mathcal{L}_E \mod J^3$.

Proof. Let $\mathcal{L}'_E \in \Lambda$ be the element

$$\mathcal{L}'_E = \left\{ \sum_{\tau \in \Gamma_n} \langle \mathfrak{C}_n^\tau, \Xi_n \rangle_{\mathrm{Tate}} \cdot \tau \right\}$$

and recall that $\mathcal{L}_E = \left\{ \sum_{\tau \in \Gamma_n} \langle d_n^\tau, \Xi_n \rangle_{\mathrm{Tate}} \cdot \tau \right\}$, as explained in [Kob06, §4]. Lemma 2.13 shows that

$$\frac{(\gamma-1)}{\log_p(\rho_{\mathrm{cyc}}(\gamma))}\, \mathcal{L}'_E \equiv \mathcal{L}_E \mod J^2,$$

and also that

$$\frac{(\gamma-1)}{\log_p(\rho_{\mathrm{cyc}}(\gamma))}\, \mu_E \equiv \mathcal{L}'_E \mod J^2.$$

□

Recall $L_p(E,s) = \rho_{\mathrm{cyc}}^{s-1}(\mathcal{L}_E)$ and the generator $\gamma_0 \in \Gamma$ that satisfies $\log_p(\rho_{\mathrm{cyc}}(\gamma_0)) = p$.

Proposition 3.16. $\dfrac{d^2}{ds^2}\,(L_p(E,s))\big|_{s=1} = 2 \cdot \langle \mathfrak{C}_0, \Xi'_0 \rangle_{\mathrm{Tate}}$.

Proof. This follows from Proposition 3.15 and (3.16). □

Remark 3.17. The equality proved in Proposition 3.16 should be considered as the extension of the displayed equality (2) in [Kob06, p. 574], to the case $r_{\mathrm{an}} = 1$.

Theorem 3.18. *We have the following equality in $\Phi \otimes_{\mathbb{Z}_p} J/J^2$:*

$$\frac{1}{2}\left(\frac{d^2}{ds^2}(L_p(E,s))\Big|_{s=1} \right) \otimes (\gamma_0 - 1) = \langle \tilde{\mathfrak{z}}^{\mathrm{Kato}}, \tilde{\mathfrak{z}}^{\mathrm{Kato}} \rangle_{\mathrm{Nek}}\,.$$

Proof. Recall that $\Xi_n := \mathrm{loc}_p^s(\mathfrak{z}_n^{\mathrm{Kato}})$ and $\Xi := \{\xi_n\} \in H^1(\mathbb{Q}_p, F_p^- T^* \otimes \Lambda)$. As we have already observed above, Ξ is in the kernel of the augmentation map

$$H^1(\mathbb{Q}_p, F_p^- T^* \otimes \Lambda) \longrightarrow H^1(\mathbb{Q}_p, F_p^- T^*)$$

and we therefore have an element (thanks to the discussion in §2.3)

$$\mathcal{D}(\Xi) \in H^1(\mathbb{Q}_p, F_p^- T^*) \otimes J/J^2.$$

It follows from [Nek06, Proposition 11.5.11] † and Remark 2.5 that

$$\langle \lambda \cdot \mathfrak{z}^{\mathrm{Kato}}, \mathfrak{z}^{\mathrm{Kato}} \rangle_{\mathrm{Nek}} = -\langle \mathfrak{C}_0, \mathcal{D}(\Xi) \rangle_{J/J^2}\,, \tag{3.17}$$

where the pairing on the right hand is the $\Phi \otimes J/J^2$-valued local Tate pairing

$$\langle\, , \,\rangle_{J/J^2} \,:\, H^1(\mathbb{Q}_p, F_p^+ V) \otimes \left(H^1(\mathbb{Q}_p, F_p^- V^*) \otimes J/J^2\right) \longrightarrow J/J^2\,.$$

Furthermore, we have

$$\langle \mathfrak{C}_0, \mathcal{D}(\Xi) \rangle_{J/J^2} = -\mathrm{Der}_{\rho_{\mathrm{cyc}}}(\mathcal{L}_\Xi)(\mathfrak{C}_0) \otimes (\gamma_0 - 1) \tag{3.18}$$

$$= \langle \mathfrak{C}_0, \Xi_0' \rangle_{\mathrm{Tate}} \otimes (\gamma_0 - 1) \tag{3.19}$$

$$= \frac{1}{2}\left(\frac{d^2}{ds^2}(L_p(E,s))\Big|_{s=1}\right) \otimes (\gamma_0 - 1) \tag{3.20}$$

where (3.18) follows from (2.3) (applied with the choices $X = F_p^- T^*$, $X^* = F_p^+ T$ and $\Xi = \xi$, $z_0 = \mathfrak{C}_0$); the equality (3.19) from Lemma 2.11 and (3.20) from Proposition 3.16. The proof now follows from (3.17) and the Φ-linearity of Nekovář's height pairing. □

Corollary 3.19. *Assuming Nekovář's height pairing is non-degenerate,*

$$\mathrm{ord}_{s=1} L_p(E,s) = 1 + r_{\mathrm{an}}$$

when $r_{\mathrm{an}} = 0, 1$.

Proof. The assertion is due to Greenberg-Stevens [GS93] (without the assumption on Nekovář's heights) when $r_{\mathrm{an}} = 0$. The case $r_{\mathrm{an}} = 1$ follows from Theorem 3.18 and [Nek06, Proposition 11.4.9], which reduces the non-degeneracy of the height pairing $\langle\, , \,\rangle_{\mathrm{Nek}}$ to the non-degeneracy of its restriction to $\mathrm{Sel}_p(\mathbb{Q}, V) \otimes \mathrm{Sel}_p(\mathbb{Q}, V^*)$, where both $\mathrm{Sel}_p(\mathbb{Q}, V)$ and $\mathrm{Sel}_p(\mathbb{Q}, V^*)$ are $\mathbb{Q}_p$-vector spaces of dimension one. □

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†where the element $[y_v^+] \in H^1(\mathbb{Q}_v, Y_v^+)$ that appears in loc.cit. is the element $\mathfrak{C}_0 \in H^1(\mathbb{Q}_p, F_p^+ V)$ here (thanks to the choice of our normalization factor λ as in (3.15)); and $\mathcal{D}(\Xi) \in H^1(\mathbb{Q}_p, F_p^- V^*) \otimes J/J^2$ is the element denoted by $[(D^1 x_{\mathrm{Iw}})_v] \in H^1(\mathbb{Q}_v, X_v^-) \otimes J/J^2$ in loc.cit.

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