Max-Planck-Institut für Mathematik
Bonn

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Max-Planck-Institut für Mathematik
Preprint Series 2015 (19)

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MPIM 15-19

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Abstract. We investigate sections of arithmetic fundamental groups of hyperbolic curves over function fields. As a consequence we prove that the anabelian section conjecture of Grothendieck holds over all finitely generated fields over $\mathbb{Q}$ if it holds over all number fields, under the condition of finiteness of certain Shafarevich-Tate groups.

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§0. Introduction. Let k be a field of characteristic 0 and X a smooth, *projective*, and geometrically connected *hyperbolic* curve (i.e., $g(X) \geq 2$) over k. Let $\pi_1(X)$ be the *arithmetic étale fundamental* group of X which sits in the following exact sequence

$$1 \to \pi_1(\overline{X}) \to \pi_1(X) \xrightarrow{\text{pr}} G_k \overset{\text{def}}{=} \text{Gal}(\overline{k}/k) \to 1,$$

where $\overline{k}$ is an algebraic closure of k and $\overline{X} = X \times_k \overline{k}$. In this paper we investigate continuous group-theoretic sections $s : G_k \to \pi_1(X)$ of the projection $\text{pr} : \pi_1(X) \twoheadrightarrow G_k$, which we will refer to as *sections* of $\pi_1(X)$.

Sections of $\pi_1(X)$ arise naturally from k-rational points of X. More precisely, a *rational* point $x \in X(k)$ determines a decomposition subgroup $D_x \subset \pi_1(X)$, which is defined only modulo conjugation by the elements of $\pi_1(\overline{X})$, and which maps isomorphically to G_k via the projection $\text{pr} : \pi_1(X) \twoheadrightarrow G_k$. We will refer to such a section of $\pi_1(X)$ as *point-theoretic*, and say that it arises from the rational point $x \in X(k)$. We have a *set-theoretic* map

$$\varphi_X : X(k) \to \overline{\text{Sec}}_{\pi_1(X)}, \quad x \mapsto \varphi_X(x) = [s_x],$$

where $\overline{\text{Sec}}_{\pi_1(X)}$ is the set of *conjugacy classes* of sections of $\pi_1(X)$, modulo conjugation by the elements of $\pi_1(\overline{X})$, and $[s_x]$ denotes the image (i.e., conjugacy class) of a section s_x associated to $x \in X(k)$.

Definition 0.1. (i) We say that the **SC** (section conjecture) *holds* for X if the map $\varphi_X : X(k) \to \overline{\mathrm{Sec}}_{\pi_1(X)}$ is *bijective.*
(ii) We say that the **SC** *holds* over k if the **SC** holds for *every* smooth, projective, and geometrically connected hyperbolic curve X over k (cf. (i)).

In his seminal letter to Faltings, Grothendieck formulated the following conjecture (cf. [Grothendieck]).

Grothendieck's Anabelian Section Conjecture (GASC). *Assume that k is finitely generated over the prime field $\mathbb{Q}$. Then the* **SC** *holds over k.*

The injectivity of the map φ_X if k is finitely generated over $\mathbb{Q}$, or more generally if k is a sub-p-adic field, is well-known (cf. [Mochizuki], Theorem C). The statement of the **GASC** is thus equivalent to the surjectivity of the map φ_X, i.e., that *every section of $\pi_1(X)$ is point-theoretic* under the above assumptions on the field k. The **GASC**, even over number fields, is still wide open. More generally, one can ask: *what are all fields (of characteristic 0) for which the* **SC** *holds*? In this paper we investigate *the section conjecture over function fields (of curves) in characteristic* 0.

Given a smooth, *projective*, and connected curve C over a field k with function field $K \overset{\mathrm{def}}{=} k(C)$, an abelian variety A over K, define the *Shafarevich-Tate group*

$$Ш(A) \overset{\mathrm{def}}{=} \mathrm{Ker}\big(H^1(G_K, A) \to \prod_{c \in C^{\mathrm{cl}}} H^1(G_{K_c}, A_c)\big),$$

where $c \in C^{\mathrm{cl}}$ is a closed point, K_c is the completion of K at c, $A_c \overset{\mathrm{def}}{=} A \times_K K_c$, and the product is over *all* closed points of C.

Definition 0.2. Let k' be a field with $\mathrm{char}(k') = 0$. Consider the following conditions.
(i)(a) k' is *ℓ-cyclotomically full* for some prime integer ℓ, meaning that the ℓ-cyclotomic character $\chi_\ell : G_{k'} \to \mathbb{Z}_\ell^\times$ has open image.
(b) The ℓ'-cyclotomic character $\chi_{\ell'} : G_{k'} \to \mathbb{Z}_{\ell'}^\times$ is *non-Tate* in the sense of [Cadoret-Tamagawa] (§2, Definition), meaning that the 1 dimensional ℓ'-adic representation $\mathbb{Z}_{\ell'}(1)$ doesn't appear as a sub-representation of the representation arising from the ℓ'-adic Tate module of an abelian variety over k', $\forall$ prime integer ℓ'.
(ii) The **SC** holds over k'.
(iii) Given a function field $K \overset{\mathrm{def}}{=} k'(C)$ of a smooth, *projective*, and connected curve C over k', an abelian variety A over K, then $TШ(A) = 0$ (cf. Notations).
(iv) Given an abelian variety A over k', the group of k'-rational points $A(k')$, and a quotient $A(k') \twoheadrightarrow D$, the followings hold.
(a) The natural map $D \to D^\wedge$ (cf. Notations) is *injective.*
(b) The torsion group $D[N]$ is *finite* $\forall N \geq 1$, and $TD = 0$ (cf. Notations).
(v) Given a function field $K = k'(C)$ as in (iii), K admits a structure of Hausdorff topological field so that $X(K)$ becomes *compact* for any proper, smooth, and geometrically connected hyperbolic curve over K.
(vi) Given a smooth and connected (not necessarily projective) curve C over k' with function field $K \overset{\mathrm{def}}{=} k'(C)$, a finite *étale* morphism $\widetilde{C} \to C$, then the followings hold. If $\widetilde{C}_c(k'(c)) \neq \emptyset$, $\forall c \in C^{\mathrm{cl}}$ with residue field $k'(c)$, where $\widetilde{C}_c$ is the scheme-theoretic inverse image of c in $\widetilde{C}$, then $\widetilde{C}(K) \neq \emptyset$.

Given a field k with $\mathrm{char}(k) = 0$, we say that k *strongly* satisfies one of the conditions (i), (ii), (iii), (iv), (v), and (vi) above, if this condition is satisfied by any *finite* extension k'/k of k. We say that the field k *satisfies the condition* $(\star)$ if k *strongly* satisfies each of the conditions (i), (ii), (iii), (iv), (v), and (vi).

Conditions (i), (iv), (v), and (vi) above are satisfied by *finitely generated* fields over $\mathbb{Q}$. In this case, (i)(b) follows from the theory of weights, (iv) follows from the Mordell-Weil and Lang-Néron Theorems, (v) follows (for the discrete topology) from Mordell's conjecture: Faltings' Theorem and Néron specialisation Theorem, and (vi) follows from the Hilbertian property (cf. Lemma 4.1.5).

Definition 0.3. Let k be a field with $\mathrm{char}(k) = 0$, and $K = k(C)$ the function field of a smooth, projective, and connected curve C over k. Let X be a proper, smooth, and geometrically connected hyperbolic curve over K. We say that X *satisfies the condition* $(\star\star)$ if the followings hold.
(i) $X(K) \neq \emptyset$.
(ii) X admits a *stable* model $\mathcal{X} \to C$ over C such that for each closed point $c \in C^{\mathrm{cl}}$ with $\mathcal{X}_c \overset{\mathrm{def}}{=} \mathcal{X} \times_C \operatorname{Spec} k(c)$ singular it holds that *all* the irreducible components of $\mathcal{X}_c$ are smooth, geometrically connected, hyperbolic, and the singular points of $\mathcal{X}_c$ are *all* $k(c)$-rational.

Our main results in this paper are the following.

Theorem A. *Let k be a field with $\mathrm{char}(k) = 0$, and $K = k(C)$ the function field of a smooth, projective, and connected curve C over k. Assume that k strongly satisfies the conditions (i), (ii), (iv), (v), and (vi) in Definition 0.2. Let X be a proper smooth and geometrically connected hyperbolic curve over K, and $J \overset{\mathrm{def}}{=} \mathrm{Pic}^0_{X/K}$ its jacobian. Assume that X satisfies the condition $(\star\star)$ (cf. Definition 0.3), and $T\text{Ш}(J) = 0$. then the* **SC** *holds for X.*

Theorem B. *Let k be a field with $\mathrm{char}(k) = 0$, and $K = k(C)$ the function field of a smooth and connected curve C over k. Assume that k satisfies the condition $(\star)$ (cf. Definition 0.2). Let L/K be a finite extension. Then the* **SC** *holds over L, i.e., K strongly satisfies the condition (ii).*

In the case of *finitely generated fields* one obtains immediately from Theorem A, and Theorem B; respectively, the following corollaries.

Theorem A1. *Assume that the* **SC** *holds over all finitely generated fields over $\mathbb{Q}$ of transcendence degree $i \geq 0$. Let k be a field with $\mathrm{tr\,deg}_{\mathbb{Q}}\, k = i$, and $K = k(C)$ the function field of a smooth projective and connected curve C over k. Let X be a proper, smooth, and geometrically connected hyperbolic curve over K, and $J \overset{\mathrm{def}}{=} \mathrm{Pic}^0_{X/K}$ its jacobian. Assume that X satisfies the condition $(\star\star)$, and $T\text{Ш}(J) = 0$. then the* **SC** *holds for X.*

Theorem B1. *Assume that the* **SC** *holds over all number fields (i.e., all finite extensions of $\mathbb{Q}$) and that the condition (iii) (in Definition 0.2) holds for any field k' which is finitely generated over $\mathbb{Q}$. Then the* **SC** *holds over all finitely generated fields over $\mathbb{Q}$.*

Our method to prove Theorem A relies on a *local-global* approach and follows from a thorough investigation of sections of arithmetic fundamental groups of hyperbolic curves over *local fields of equal characteristic* 0, and over *function fields* *of curves in characteristic* 0. In §1 we establish some basic facts on *geometrically abelian* admissible fundamental groups and their sections. In §2 we investigate (under the assumption that condition (i) holds) sections of arithmetic fundamental groups of curves over *local fields of equal characteristic* 0. We observe that the section conjecture **SC** *doesn't hold over local fields of equal characteristic* 0 (cf. Lemma 2.1.3 and Proposition 2.3.1). We discuss those sections which are point-theoretic in the case of stable curves (cf. Lemma 2.1.4 and Lemma 2.2.2). In §3 we investigate sections of arithmetic fundamental groups of curves over *function fields* (of transcendence degree 1), and establish some of the basic techniques in order to investigate their point-theoreticity via a *local-global approach.* In §4 we prove Theorem A and explain how Theorem B can be derived from Theorem A.

Theorems A and B concern sections of arithmetic fundamental groups of *proper* curves over function fields. One can prove similar results for *non-cuspidal* sections of arithmetic fundamental groups of *affine* curves over function fields.

Acknowledgment. I would like to thank Akio Tamagawa for several discussions we had on the topic of this paper.

Notations. Given a scheme Y over a field L with algebraic closure $\overline{L}$ we write $Y_{\overline{L}} \overset{\text{def}}{=} Y \times_{\operatorname{Spec} L} \operatorname{Spec} \overline{L}$ for the geometric fibre of Y. Given a scheme C, a field L, $Y \to C$ and $\operatorname{Spec} L \to C$ morphisms of schemes, we write $Y_L \overset{\text{def}}{=} Y \times_C \operatorname{Spec} L$. For an algebraic group G over a field L of characteristic 0, with algebraic closure $\overline{L}$, we write $TG \overset{\text{def}}{=} \varprojlim_{N \geq 1} G[N](\overline{L})$ for the Tate module of G, where $G[N] \overset{\text{def}}{=} \operatorname{Ker}(G \xrightarrow{[N]} G)$ is the kernel of the homomorphism of multiplication by N. For a profinite group H we write H^{ab} for the maximal abelian quotient of H. For an abelian group D we write $D^\wedge \overset{\text{def}}{=} \varprojlim_{N \geq 1} D/ND$, where $ND \overset{\text{def}}{=} \{N.a \mid a \in D\}$. Given an integer $N \geq 1$, we write $D[N] \overset{\text{def}}{=} \{b \in D \mid N.b = 0\}$, and $TD \overset{\text{def}}{=} \varprojlim_{N \geq 1} D[N]$ for the Tate module of D.

§**1 Geometrically abelian admissible fundamental groups.** Both in 1.1, and 1.2, K will denote a field of characteristic 0.

1.1. Let $0 \to H \to B \to A \to 0$ be a *semi-abelian* scheme over K, where $A \to \operatorname{Spec} K$ is an *abelian variety* and $H = \mathbb{G}_{m,K}^r$ is a (split) *torus* over K of rank $r \geq 0$. Let η be a geometric point of B with value in the zero section, which induces a geometric point η of A and H. Then η determines an algebraic closure $\overline{K}$ of K. Write $\overline{\eta}$ for the geometric point of $B_{\overline{K}}$, $A_{\overline{K}}$, and $H_{\overline{K}}$; respectively, which is induced by η. We have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \pi_1(H_{\overline{K}},\overline{\eta}) & = & \pi_1(H_{\overline{K}},\overline{\eta}) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \pi_1(B_{\overline{K}},\overline{\eta}) & \longrightarrow & \pi_1(B,\eta) & \longrightarrow & G_K & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \text{id}\downarrow & & \\
1 & \longrightarrow & \pi_1(A_{\overline{K}},\overline{\eta}) & \longrightarrow & \pi_1(A,\eta) & \longrightarrow & G_K & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

where $G_K \overset{\text{def}}{=} \text{Gal}(\overline{K}/K)$ (cf. [Grothendieck1], Exposé IX, Théorème 6.1, for the exactness of the horizontal sequences. One verifies easily the exactness of the vertical sequences). Moreover, there is a natural identification of G_K-modules $\pi_1(B_{\overline{K}},\overline{\eta}) \overset{\sim}{\to} TB$, where TB is the Tate module of B. Thus, we have an exact sequence of G_K-modules $0 \to TH \to TB \to TA \to 0$. Further, there is a natural identification of G_K-modules $TH \overset{\sim}{\to} \hat{\mathbb{Z}}(1)^r$.

The Kummer exact sequences $0 \to B[N] \to B \overset{N}{\to} B \to 0$, $\forall\ N \geq 1$, induce a natural exact sequence; the so called *Kummer exact sequence*

$$0 \to B(K)^\wedge \to H^1(G_K, TB) \to TH^1(G_K, B) \to 0. \tag{1.1}$$

We will identify $B(K)^\wedge$ with its image in $H^1(G_K, TB)$ via the above Kummer map $B(K)^\wedge \to H^1(G_K, TB)$.

Lemma 1.1.1. *We have a commutative diagram of exact sequences*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H(K)^\wedge & \longrightarrow & B(K)^\wedge & \longrightarrow & A(K)^\wedge & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
H^0(G_K, TA) & \longrightarrow & H^1(G_K, TH) & \longrightarrow & H^1(G_K, TB) & \longrightarrow & H^1(G_K, TA) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & TH^1(G_K, B) & \longrightarrow & TH^1(G_K, A) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where the vertical sequences are the Kummer exact sequences.

Proof. Follows easily from the fact that $H^1(G_K, \mathbb{G}_m) = 0$, and the exact sequence $0 \to H(\overline{K}) \to B(\overline{K}) \to A(\overline{K}) \to 0$ of G_K-modules. □

Definition 1.1.2. Let $\eta \in H^1(G_K, TB)$.
(i) We say that η is *pro-geometric* if η lies in the subgroup $B(K)^\wedge$ of $H^1(G_K, TB)$ (cf. sequence (1.1)).
(ii) We say that η is *geometric* if η is in the image of the composite homomorphism $B(K) \to B(K)^\wedge \to H^1(G_K, TB)$.

1.2. Let $X \to \operatorname{Spec} K$ be a proper, smooth, and geometrically connected *hyperbolic curve* over K. Let ξ be a geometric point of X with value in its generic point. Thus, ξ determines an algebraic closure $\overline{K}$ of K. Write $\bar{\xi}$ for the geometric point of $X_{\overline{K}}$ which is induced by ξ. We have a natural exact sequence of étale fundamental groups

$$1 \to \pi_1(X_{\overline{K}}, \bar{\xi}) \to \pi_1(X, \xi) \xrightarrow{\text{pr}} G_K \to 1, \tag{1.2}$$

where $G_K \stackrel{\text{def}}{=} \operatorname{Gal}(\overline{K}/K)$. Write $\pi_1(X, \xi)^{(\text{ab})} \stackrel{\text{def}}{=} \pi_1(X, \xi)/\operatorname{Ker}(\pi_1(X_{\overline{K}}, \bar{\xi}) \twoheadrightarrow \pi_1(X_{\overline{K}}, \bar{\xi})^{\text{ab}})$. We will refer to $\pi_1(X, \xi)^{(\text{ab})}$ as the *geometrically abelian* quotient of $\pi_1(X, \xi)$. Assume that $X(K) \neq \emptyset$. Write $J \stackrel{\text{def}}{=} \operatorname{Pic}^0_{X/K}$ for the jacobian variety of X, and $\iota : X \to J$ for the embedding which maps a rational point $x_0 \in X(K)$ to the zero section of J. Then ι induces a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(X_{\overline{K}}, \bar{\xi})^{\text{ab}} & \longrightarrow & \pi_1(X, \xi)^{(\text{ab})} & \longrightarrow & G_K & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow \text{id} & & \\ 1 & \longrightarrow & \pi_1(J_{\overline{K}}, \bar{\xi}) & \longrightarrow & \pi_1(J, \xi) & \longrightarrow & G_K & \longrightarrow & 1 \end{array} \tag{1.3}$$

where the vertical maps are isomorphisms, hence a natural identification of G_K-modules $\pi_1(X_{\overline{K}}, \bar{\xi})^{\text{ab}} \xrightarrow{\sim} \pi_1(J_{\overline{K}}, \bar{\xi}) \xrightarrow{\sim} TJ$. Let

$$s : G_K \to \pi_1(X, \xi)$$

be a *section* of $\pi_1(X, \xi)$. Then s induces a *section*

$$s^{\text{ab}} : G_K \to \pi_1(X, \xi)^{(\text{ab})}$$

of the projection $\pi_1(X, \xi)^{(\text{ab})} \twoheadrightarrow G_K$. The set of splittings of the upper sequence in diagram (1.3) is, up to conjugation by the elements of $\pi_1(X_{\overline{K}}, \bar{\xi})^{\text{ab}}$, a torsor under the Galois cohomology group $H^1(G_K, \pi_1(X_{\overline{K}}, \bar{\xi})^{\text{ab}}) \xrightarrow{\sim} H^1(G_K, TJ)$. Fix a *base point* of the torsor of splittings of this exact sequence; for example the splitting arising from the zero section of J, i.e., the splitting arising from the rational point $x_0 \in X(K)$ (cf. above discussion). Then the above (conjugacy class of the) section s^{ab} corresponds to an element

$$s^{\text{ab}} \in H^1(G_K, TJ).$$

We will refer to s^{ab} as the *abelian portion* of the section s.

Recall the Kummer exact sequence:

$$0 \to J(K)^\wedge \to H^1(G_K, TJ) \to TH^1(G_K, J) \to 0. \tag{1.4}$$

Note that there exist natural maps $X(K) \xrightarrow{\iota} J(K) \to J(K)^\wedge$, where for $x \in X(K)$ the image $\iota(x)$ is the class $[x - x_0]$ of the degree 0 divisor $x - x_0$.

Definition 1.2.1. Let $\eta \in H^1(G_K, TJ)$. We say that η is *point-theoretic* if η is in the image of the composite map $X(K) \xrightarrow{\iota} J(K) \to J(K)^\wedge \to H^1(G_K, TJ)$.

The following Lemma follows easily from the various definitions.

Lemma 1.2.2. *Suppose that the section $s = s_x$, $x \in X(K)$, is point-theoretic (cf. §0). Then $s^{\mathrm{ab}} \in H^1(G_K, TJ)$ is point-theoretic (cf. Definition 1.2.1), and s^{ab} is the image of x via the composite map $X(K) \xrightarrow{\iota} J(K) \to J(K)^\wedge \to H^1(G_K, TJ)$. In particular, s^{ab} is pro-geometric and geometric (cf. Definition 1.1.2).*

1.3. In this section k is a field of characteristic 0, K is a complete discrete valuation field with residue field k, and $\mathcal{O}_K$ the valuation ring of K. Let $X \to \operatorname{Spec} \mathcal{O}_K$ be a proper, *stable*, and geometrically connected (relative) curve over $\mathcal{O}_K$, with X_K *smooth*. Assume that the irreducible components $\{X_i\}_{i \in I}$ of $X_k = \sum_{i \in I} X_i$ are *smooth*, *geometrically connected*, and the singular points $\{x_j\}_{j \in J}$ of X_k are all *k-rational*. Let ξ (resp. ξ') be a geometric point of X with value in its generic point (resp. with value in the generic point of some irreducible component X_{i_0} of X_k). Thus, ξ (resp. ξ') determines an algebraic closure $\overline{K}$ (resp. $\overline{k}$) of K (resp. k). We have exact sequences of *arithmetic "admissible" fundamental groups*

$$1 \to \pi_1(X_{\overline{K}}, \overline{\xi}) \to \pi_1(X_K, \xi) \to G_K \to 1,$$

and

$$1 \to \pi_1(X_{\overline{k}}, \overline{\xi}')^{\mathrm{adm}} \to \pi_1(X_k, \xi')^{\mathrm{adm}} \to G_K \to 1,$$

where $G_K \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{K}/K)$, and the geometric point $\overline{\xi}$ (resp. $\overline{\xi}'$) is naturally induced by ξ (resp. ξ'). Here, the superscript "adm" means *admissible fundamental group* (cf. [Mochizuki1], §2, for more details on the definition of π_1^{adm}). Moreover, we have a commutative diagram of exact sequences

$$(1.5) \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(X_{\overline{K}}, \overline{\xi}) & \longrightarrow & \pi_1(X_K, \xi) & \longrightarrow & G_K & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \mathrm{Sp}} & & \downarrow & & \\ 1 & \longrightarrow & \pi_1(X_{\overline{k}}, \overline{\xi}')^{\mathrm{adm}} & \longrightarrow & \pi_1(X_k, \xi')^{\mathrm{adm}} & \longrightarrow & G_K & \longrightarrow & 1 \end{array}$$

where the middle and left vertical maps are continuous homomorphisms of *specialisation*, which are isomorphisms since $\mathrm{char}(k) = 0$, and are defined up to inner conjugation (cf. loc. cit.).

Let $J \overset{\mathrm{def}}{=} \mathrm{Pic}^0_{X_K/K}$ be the jacobian of X_K. Consider the exact sequence of K-algebraic groups

$$0 \to \Lambda \to B \to J \to 0,$$

where $B \twoheadrightarrow J$ is the *rigid analytic uniformisation* of J, $0 \to H \to B \to A \to 0$ is the generic fibre of an $\mathcal{O}_K$-semi-abelian scheme $0 \to \mathcal{H} \to \mathcal{B} \to \mathcal{A} \to 0$, where $\mathcal{A}$ is an $\mathcal{O}_K$-abelian scheme, $\mathcal{H} \xrightarrow{\sim} \mathbb{G}^r_{m, \mathcal{O}_K}$ is a split $\mathcal{O}_K$-torus of rank $r \geq 0$, and $\Lambda \subset B(K)$ is a K-lattice of rank r (cf. [Fresnel-Van der Put], 6.7).

Lemma 1.3.1. *The exact sequence $0 \to \Lambda \to B \to J \to 0$ induces an exact sequence of G_K-modules $0 \to TB \to TJ \to \Lambda^\wedge \to 0$, where $\Lambda^\wedge \overset{\mathrm{def}}{=} \Lambda \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ is a trivial G_K-module.*

Proof. Follows easily from the exact sequence $0 \to \Lambda \to B(\overline{K}) \to J(\overline{K}) \to 0$ of G_K-modules, and the fact that Λ is torsion free. $\square$

Thus, we have a commutative diagram of exact sequences

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & TA & \longrightarrow & TJ/TH & \longrightarrow & \Lambda^\wedge & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & {\scriptstyle \mathrm{id}}\uparrow & & \\
0 & \longrightarrow & TB & \longrightarrow & TJ & \longrightarrow & \Lambda^\wedge & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & TH & = & TH & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{1.6}
$$

The fundamental group $\pi_1(X_{\overline{k}}, \bar{\xi}')$ is the maximal quotient of $\pi_1(X_{\overline{k}}, \bar{\xi}')^{\mathrm{adm}}$ which classifies admissible covers of $X_{\overline{k}}$ which are étale above $X_{\overline{k}}$, let $\pi_1(X_{\overline{K}}, \bar{\xi})^{\mathrm{et}}$ be the corresponding quotient of $\pi_1(X_{\overline{K}}, \bar{\xi})$ (cf. diagram (1.5)). Write $\pi_1(X_{\overline{k}}, \bar{\xi}')^{\mathrm{cs}}$ for the maximal quotient of $\pi_1(X_{\overline{k}}, \bar{\xi}')$ which classifies finite étale covers of $X_{\overline{k}}$ which arise from finite topological coverings of the intersection graph associated to $X_{\overline{k}}$ (the quotient $\pi_1(X_{\overline{k}}, \bar{\xi}')^{\mathrm{cs}}$ factorizes through $\pi_1(X_{\overline{k}}, \bar{\xi}')^{\mathrm{ab}}$), and $\pi_1(X_{\overline{K}}, \bar{\xi})^{\mathrm{cs}}$ for the corresponding quotient of $\pi_1(X_{\overline{K}}, \bar{\xi})^{\mathrm{et}}$ (which is in fact a quotient of $\pi_1(X_{\overline{K}}, \bar{\xi})^{\mathrm{et,ab}}$).

Lemma 1.3.2. *Under the identification $\pi_1(X_{\overline{K}}, \bar{\xi})^{\mathrm{ab}} \overset{\sim}{\to} TJ$, the quotient $\pi_1(X_{\overline{K}}, \bar{\xi})^{\mathrm{ab}} \twoheadrightarrow \pi_1(X_{\overline{K}}, \bar{\xi})^{\mathrm{cs}}$ is identified with the quotient $TJ \twoheadrightarrow \Lambda^\wedge$, and the quotient $\pi_1(X_{\overline{K}}, \bar{\xi})^{\mathrm{ab}} \twoheadrightarrow \pi_1(X_{\overline{K}}, \bar{\xi})^{\mathrm{et,ab}}$ is identified with the quotient $TJ \twoheadrightarrow TJ/TH$. Thus, $TH \overset{\sim}{\to} \hat{\mathbb{Z}}(1)^r$ is naturally identified with the subgroup of $\pi_1(X_{\overline{k}}, \bar{\xi}')^{\mathrm{adm,ab}}$ which is generated by the inertia subgroups at the double points of $X_{\overline{k}}$. Moreover, TA is naturally identified (as a G_K-module) with $\prod_{i \in I} TJ_i$ where $J_i \overset{\mathrm{def}}{=} \mathrm{Pic}^0_{X_i/k}$ is the jacobian of the irreducible component X_i of X_k (G_K acts on TJ_i via its quotient $G_K \twoheadrightarrow G_k \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$).*

Proof. Follows easily from the various definitions (cf. [Fresnel-Van der Put], 6.7.4). $\square$

§2. Sections of arithmetic fundamental groups of curves over local fields of equal characteristic 0. In this section K is a complete discrete valuation field of *equal characteristic* 0, $\mathcal{O}_K$ its valuation ring, and k its residue field. We use the notations introduced in §0 and §1. Moreover, we assume that k satisfies the condition (i)(b) in Definition 0.2, unless we specify otherwise.

Both in 2.1 and 2.2 we will use the notations and assumptions in 1.3. Thus, X is a proper, *stable*, and geometrically connected $\mathcal{O}_K$-curve with X_K smooth as in 1.3. We assume that X_K is *hyperbolic*, i.e., $g(X_K) \geq 2$. We have a natural exact sequence $1 \to I_K \to G_K \to G_k \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k) \to 1$ where I_K is the *inertia* group. Moreover, there exists a natural isomorphism $I_K \overset{\sim}{\to} \hat{\mathbb{Z}}(1)$ where the "(1)" is a Tate twist.

2.1. The good reduction case. Assume that X is *smooth*. Recall the notations in 1.3. In this case we have a commutative diagram of exact sequence

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X_{\overline{K}},\bar{\xi}) & \longrightarrow & \pi_1(X_K,\xi) & \longrightarrow & G_K & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{Sp}} & & \downarrow & & \\
1 & \longrightarrow & \pi_1(X_{\overline{k}},\bar{\xi}') & \longrightarrow & \pi_1(X_k,\xi') & \longrightarrow & G_k & \longrightarrow & 1
\end{array}
\tag{2.1}
$$

where the middle vertical map is the surjective homomorphism of *specialisation* (defined up to conjugation, cf. [Grothendieck1], Exposé X, §2), the left vertical map is an isomorphism (since $\mathrm{char}(k) = 0$), and the right vertical map is a surjection.

Lemma 2.1.1. *The followings hold.*
(i) The right square in diagram (2.1) is cartesian.
(ii) The projection $\pi_1(X_K, \xi) \twoheadrightarrow G_K$ *induces a natural isomorphism* $\mathrm{Ker}(\mathrm{Sp}) \xrightarrow{\sim} I_K$.

Proof. The proof of (i) is similar to the proof of Lemma 3.3.2 in [Saïdi]. Assertion (ii) is clear in light of (i). □

Let

$$s : G_K \to \pi_1(X_K, \xi)$$

be a *section* of $\pi_1(X_K, \xi)$ which induces, by composing with the specialisation homomorphism $\mathrm{Sp} : \pi_1(X_K, \xi) \to \pi_1(X_k, \xi')$, a continuous homomorphism $s' \overset{\mathrm{def}}{=} \mathrm{Sp} \circ s : G_K \to \pi_1(X_k, \xi')$.

Lemma 2.1.2. *The equality* $\mathrm{Ker}(s') = I_K$ *holds. In particular,* s' *factorizes through* G_k *and induces a section* $\tilde{s} : G_k \to \pi_1(X_k, \xi')$ *of* $\pi_1(X_k, \xi')$.

Proof. It suffices to show that the image $s'(I_K)$ of the inertia subgroup in $\pi_1(X_k, \xi')$ is trivial. This image is contained in $\pi_1(X_{\overline{k}}, \bar{\xi}')$ by diagram (2.1). A standard (well-known) weight argument, using the fact that k satisfies the condition (i)(b) in Definition 0.2, shows that this image must be trivial (cf. [Hoshi-Mochizuki], Lemma 1.6). □

Assume that the section s is *point-theoretic*, i.e., $s = s_x : G_K \to \pi_1(X_K, \xi)$ is associated to a *rational point* $x \in X(K)$ (cf. §0). Let $\overline{x} \in X(k)$ be the *specialisation* of the point x. Then one verifies easily that the section $\tilde{s} : G_k \to \pi_1(X_k, \xi')$ of $\pi_1(X_k, \xi')$, which is induced by s (cf. Lemma 2.1.2), is *point-theoretic* and arises from the k-rational point $\overline{x}$, i.e., $[\tilde{s}] = [s_{\overline{x}}]$ holds in $\overline{\mathrm{Sec}}_{\pi_1(X_k,\xi')}$.

Lemma 2.1.3. *We use the same notations as above. Let* $x' \in X(K)$ *be a rational point which specialises in* $\overline{x}$. *Then* $[s_x] = [s_{x'}]$. *In particular, the natural map* $\varphi_X : X(K) \to \overline{\mathrm{Sec}}_{\pi_1(X_K,\xi)}$ *(cf. §0) is not injective.*

Proof. Indeed, it follows immediately from lemma 2.1.1(i) and Lemma 2.1.2 that a section $s : G_K \to \pi_1(X_K, \xi)$ of $\pi_1(X_K, \xi)$ is uniquely determined by the continuous homomorphism $s' \overset{\mathrm{def}}{=} \mathrm{Sp} \circ s : G_K \twoheadrightarrow G_K/I_K \to \pi_1(X_k, \xi')$ that it induces. In particular, all rational points $x' \in X(K)$ which specialise in $\overline{x}$ (there are infinitely many such points x') give rise to the same conjugacy class of sections of $\pi_1(X_K, \xi)$, from which the second assertion follows. □

Conversely we have the following.

Lemma 2.1.4. *Assume that the section $\tilde{s}: G_k \to \pi_1(X_k, \xi')$ of $\pi_1(X_k, \xi')$ which is induced by s (cf. Lemma 2.1.2) is point-theoretic, i.e., $\tilde{s} = s_{\overline{x}}$ for some k-rational point $\overline{x} \in X(k)$. Then the section s is point-theoretic, i.e., $s = s_x$ for some (non unique) $x \in X(K)$ which specialises in the point $\overline{x}$.*

Proof. Let $x \in X(K)$ be a rational point which specialises in $\overline{x}$ (such a point x exists since X is smooth). Then $[s] = [s_x]$ holds by the same argument used in the proof of Lemma 2.1.3. $\square$

2.2. The bad reduction case. In this section, and in addition to our assumptions, we will assume that k satisfies the condition (iv)(a) in Definition 0.2.

Next, suppose that X_k is *singular*. Recall that the irreducible components $\{X_i\}_{i\in I}$ of $X_k = \sum_{i\in I} X_i$ are *smooth*, *geometrically connected*, and the singular points $\{x_j\}_{j\in J}$ of X_k are all *k-rational* (cf. 1.3). Let X_i be an irreducible component of X_k and $D_{X_i} \subset \pi_1(X_k, \xi')^{\text{adm}}$ a *decomposition group* associated to X_i. Thus, D_{X_i} is the decomposition group of an irreducible component of the special fibre of a universal admissible cover $\widetilde{X}$ of X which lies above the component X_i, and D_{X_i} is only defined up to conjugation (cf. [Mochizuki1], §4). Let $\overline{X}_i$ be the (unique, since X_i is geometrically connected) irreducible component of $X_{\overline{k}}$ above X_i. Then we have a commutative diagram

$$
\begin{array}{ccccccccc}
 & 1 & \longrightarrow & \Delta_{\overline{X}_i} & \longrightarrow & D_{X_i} & \longrightarrow & G_K & \longrightarrow & 1 \\
 & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \text{id}} & & \\
(2.2) & 1 & \longrightarrow & \pi_1(\overline{X}_i, *)^{\text{adm}} & \longrightarrow & \pi_1(X_i, *)^{\text{adm}} & \longrightarrow & G_K & \longrightarrow & 1 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & 1 & \longrightarrow & \pi_1(\overline{U}_i, *)^{\text{tame}} & \longrightarrow & \pi_1(U_i, *)^{\text{tame}} & \longrightarrow & G_k & \longrightarrow & 1
\end{array}
$$

Here, $\Delta_{\overline{X}_i}$ is defined so that the upper horizontal sequence is exact, $*$ denote base points, and "$\pi_1(\ )^{\text{adm}}$" denote the admissible fundamental group of X_i, and $\overline{X}_i$, which are marked by the *cusps*, i.e., the double points of X_k (resp. $X_{\overline{k}}$) lying on X_i (resp. $\overline{X}_i$), and $U_i \overset{\text{def}}{=} X_i \setminus \{\text{cusps}\}$, $\overline{U}_i \overset{\text{def}}{=} \overline{X}_i \setminus \{\text{cusps}\}$, respectively. The superscript "tame" means *tame* fundamental group. The left and middle upper vertical maps are natural isomorphisms (cf. loc. cit.), and the lower right square is cartesian. Note that $\pi_1(\overline{U}_i, *)^{\text{tame}} = \pi_1(\overline{U}_i, *)$, and $\pi_1(U_i, *)^{\text{tame}} = \pi_1(U_i, *)$, since $\text{char}(k) = 0$.

Let

$$s : G_K \to \pi_1(X_K, \xi)$$

be a *section* of $\pi_1(X_K, \xi)$, which induces a section $s' \overset{\text{def}}{=} \text{Sp} \circ s : G_K \to \pi_1(X_k, \xi')^{\text{adm}}$ of the projection $\pi_1(X_k, \xi')^{\text{adm}} \twoheadrightarrow G_K$ (cf. diagram (1.5)).

Suppose that X is *regular* and s is *point-theoretic*, i.e., $s = s_x$ arises from a K-rational point $x \in X(K)$ (cf. §0). Then the K-rational point x *specialises* in a k-rational point $\overline{x} \in X(k)$ which is a *smooth* point of X_k and lies on a unique irreducible component X_i of X_k (cf. [Liu], Corollary 9.1.32). Moreover, it follows from the various definitions that $s(G_K) \subset D_{X_i} \subset \pi_1(X_k, \xi')^{\text{adm}}$ holds, where D_{X_i} is a *decomposition group* associated to X_i (cf. above discussion). In particular, the section $s = s_x$ induces a section $s_i : G_K \to \pi_1(X_i, *)^{\text{adm}}$ of the projection $\pi_1(X_i, *)^{\text{adm}} \twoheadrightarrow G_K$, and a continuous homomorphism $\tilde{s}_i : G_K \to \pi_1(U_i, *)^{\text{tame}}$ (cf. diagram (2.2)).

Lemma 2.2.1. *The followings hold.*
(i) The section s_i is unramified, i.e., $\tilde{s}_i(I_K) = \{1\}$. In particular, $\tilde{s}_i$ induces a section $\overline{s}_i : G_k \to \pi_1(U_i, *)^{\text{tame}}$ of $\pi_1(U_i, *)^{\text{tame}}$.
(ii) The section $\overline{s}_i : G_k \to \pi_1(U_i, *)^{\text{tame}}$ in (i) is point-theoretic and arises from the k-rational point $\overline{x} \in U_i(k)$. In particular, the section $\overline{s}_i$ is non-cuspidal, i.e., $\overline{s}_i(G_k)$ is not contained in a cuspidal decomposition group associated to a cusp. □

Proof. Note that U_i is hyperbolic since X is stable. Assertion (i) follows from the assumptions (i)(b) and (iv)(a). First, assumption (iv)(a) implies that the closed points of X_i are uniquely determined by their corresponding (conjugacy classes of) decomposition groups in $\pi_1(U_i, *)^{\text{tame}}$, such a decomposition group is self-normalising in $\pi_1(U_i, *)^{\text{tame}}$, and no non-cuspidal decomposition group is contained in a cuspidal decomposition group (cf. the arguments used in the proof of Theorem 1.3 in [Mochizuki2], and [Tamagawa], Proposition 2.8(i)). Second, if $\tilde{s}_i(I_K)$ is non trivial then one shows, using the condition (i)(b), that $\tilde{s}_i(I_K) \subseteq \pi_1(\overline{U}_i, *)^{\text{tame}}$ would be a non-trivial (necessarily torsion free) procyclic group contained in an inertia group I_y at a cusp $y \in X_i \setminus U_i$ (cf. [Hoshi-Mochizuki], Lemma 1.6), hence $\tilde{s}_i(G_K)$ will be contained in a decomposition group associated to y, and the latter would contain a decomposition group associated to $\overline{x}$ which is a contradiction (cf. above discussion). Assertion (ii) follows easily. □

Conversely, suppose that the section s satisfies $s(G_K) \subset D_{X_i}$, i.e., the image of s is contained in a decomposition group associated to an irreducible component X_i of X_s. Thus, s induces a section $s_i : G_K \to \pi_1(X_i, *)^{\text{adm}}$ of the projection $\pi_1(X_i, *)^{\text{adm}} \twoheadrightarrow G_K$, which induces a homomorphism $\tilde{s}_i : G_K \to \pi_1(U_i, *)^{\text{tame}}$ (cf. diagram (2.2)). Assume further that s_i is *unramified*, i.e., $\tilde{s}_i(I_K) = \{1\}$. Then $\tilde{s}_i$ induces naturally a section $\overline{s}_i : G_k \to \pi_1(U_i, *)^{\text{tame}}$ of $\pi_1(U_i, *)^{\text{tame}}$ (cf. diagram (2.2)).

Lemma 2.2.2. *With the same notations/assumptions as above suppose that the section $\overline{s}_i : G_k \to \pi_1(U_i, *)^{\text{tame}}$ is point-theoretic and arises from a k-rational point $\overline{x} \in U_i(k)$. Then the section s is point-theoretic, and s arises from a (non unique) rational point $x \in X(K)$ which specialises in the point $\overline{x} \in U_i(k)$.*

Proof. Similar to the proof of Lemma 2.1.4. □

2.3. In what follows we provide examples of sections of arithmetic fundamental groups of hyperbolic curves over local fields of equal characteristic 0 which are *not point-theoretic*. We use the notations and assumptions in 2.2.

Let X be a *regular and stable $\mathcal{O}_K$-curve* satisfying the conditions in 2.2. Let $x_j \in X(k)$ be a k-rational double point of X_k and write $D_{x_j} \subset \pi_1(X_k, \xi')^{\text{adm}}$ for the *decomposition group* of x_j. Thus, D_{x_j} is the decomposition group of a closed point of the special fibre of a universal admissible cover $\widetilde{X}$ of X which lies above the double point x_j, and D_{x_j} is only defined up to conjugation (cf. [Mochizuki1], §5 and §6). We have an exact sequence $1 \to \Delta_{x_j} \to D_{x_j} \to G_K \to 1$, where Δ_{x_j} is the kernel of the projection $D_{x_j} \twoheadrightarrow G_K$. Moreover, there exists a natural isomorphism $\Delta_{x_j} \xrightarrow{\sim} \hat{\mathbb{Z}}(1)$ (cf. loc. cit., the discussion in page 22). The profinite group D_{x_j} is isomorphic to the admissible fundamental group $\pi_1^{\mathrm{adm}}(\mathcal{X})$ of $\mathcal{X} \overset{\mathrm{def}}{=} \operatorname{Spec} \frac{\mathcal{O}_K[[S,T]]}{(ST-\pi)}$, where π is a uniformiser of $\mathcal{O}_K$ (cf. loc. cit.). The above exact sequence splits. Indeed, the (admissible) covers $\mathcal{Y}_N \to \mathcal{X}$ defined generically by extracting an N-th root of S with $\mathcal{Y}_N$ normal, for all integers $N \geq 1$, define a splitting of this sequence. Such a splitting induces naturally a section $s_{x_j} : G_K \to \pi_1(X_K, \xi) \overset{\sim}{\to} \pi_1(X_k, \xi')^{\mathrm{adm}}$ of $\pi_1(X_K, \xi)$. The section s_{x_j} is *not point-theoretic*. Indeed, if s_{x_j} arises from a rational point $x \in X(K)$, then x specialises in a smooth non-cuspidal point of an irreducible component X_i (cf. Lemma 2.2.1 and the discussion before) which is necessarily adjacent to an irreducible component $X_{i'}$ passing through x_j (cf. [Hoshi-Mochizuki1], Corollary 1.15(iv)). Let $s_i : G_k \to \pi_1(U_i, *)^{\mathrm{tame}}$ be the section of $\pi_1(U_i, *)^{\mathrm{tame}}$ which is induced by s_{x_j} (cf. Lemma 2.2.1.). This section would then be cuspidal which contradicts Lemma 2.2.1(ii).

Proposition 2.3.1. *Let K be a complete discrete valuation field with residue field k of characteristic 0. Assume that k satisfies the conditions (i)(b) and (iv)(a) (cf. Definition 0.2). Then there exists a proper, smooth, geometrically connected hyperbolic curve C over K, and a section $s : G_K \to \pi_1(C, *)$ of $\pi_1(C, *)$ which is not point-theoretic.*

Proof. Write $\mathcal{O}_K$ for the valuation ring of K. Using formal patching techniques one can construct a proper, stable, and regular $\mathcal{O}_K$-curve X satisfying the assumptions in 2.2. In particular, $C \overset{\mathrm{def}}{=} X_K$ is smooth, hyperbolic, geometrically connected, and the double points of X_k are k-rational (compare with [Saïdi1], Lemma 6.3). Then as in the above discussion (before Proposition 2.3.1) let $s_{x_j} : G_K \to \pi_1(C, *)$ be a section arising from a double point x_j of the special fibre X_s of X. Then s_{x_j} is not point-theoretic as explained above. □

Despite the somehow "*negative*" result in Proposition 2.3.1 showing the existence of non point-theoretic sections we prove the following "*positive*" result which will be used in the proof of Theorem A.

Proposition 2.3.2. *Let K be a complete discrete valuation field with residue field k of characteristic 0. Assume that k satisfies the conditions (i) and (ii) in Definition 0.2. Let C be a proper, smooth, and geometrically connected hyperbolic K-curve, and $s : G_K \to \pi_1(C, *)$ a section of $\pi_1(C, *)$. Assume that $C(K) \neq \emptyset$, and C admits a stable model X over the ring of integers $\mathcal{O}_K$ of K which satisfies the conditions in 2.2 and such that the irreducible components $\{X_i\}_{i \in I}$ of X_k are all hyperbolic. Then the abelian portion s^{ab} of s is pro-geometric (cf. Definition 1.1.2(i) and the discussion in 1.2).*

Proof. First, if C has good reduction over $\mathcal{O}_K$, i.e., if X is smooth, then the assertion follows immediately from Lemma 2.1.4 and Lemma 1.2.2. Next, we assume that $X_k = \sum_{i \in I} X_i$ is singular. Let $\widetilde{X} \to X$ be the pro-universal admissible cover of X. Then the following holds (cf. [Mochizuki-Hoshi1], Corollary 1.15(iii)): the set of irreducible components $\widetilde{X}_{i_0} \subset \widetilde{X}_k$ such that $s(G_K) \subseteq D_{\widetilde{X}_{i_0}}$ is *nonempty.* (Here one uses condition (i)(a) which implies that G_K is ℓ-cyclotomically full for some prime integer ℓ.) The quotient $\pi_1(C_{\overline{K}}, *)^{\mathrm{cs}}$ of $\pi_1(C_{\overline{K}}, *)$ (cf. §1, the discussion before Lemma 1.3.2) is a quotient of $\pi_1(C, *)$ (since it is naturally a quotient of $\pi_1(X, *)$), and is naturally identified with $\Lambda^\wedge$ (cf. Lemma 1.3.2 and the notations therein). We have a commutative diagram of exact sequence

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Delta_C & \longrightarrow & \Pi_C & \longrightarrow & G_K & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(C_{\overline{K}},*) & \longrightarrow & \pi_1(C,*) & \longrightarrow & G_K & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \pi_1(C_{\overline{K}},*)^{\mathrm{cs}} & = & \pi_1(C_{\overline{K}},*)^{\mathrm{cs}} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

where Δ_C and Π_C are defined so that the vertical sequences are exact. The image of the decomposition group of an irreducible component of $\widetilde{X}_k$ in $\pi_1(C_{\overline{K}},*)^{\mathrm{cs}}$ is trivial (by the very definition of $\pi_1(C_{\overline{K}},*)^{\mathrm{cs}}$). Thus, the image $s(G_K)$ of the section s maps trivially to $\pi_1(C_{\overline{K}},*)^{\mathrm{cs}}$ via the lower right vertical map in the above diagram (by the above mentioned result), and s induces a section $s : G_K \to \Pi_C$ of the projection $\Pi_C \twoheadrightarrow G_K$. Write $X^{\mathrm{cs}} \to X$ for the sub-cover of $\widetilde{X} \to X$ corresponding to the quotient $\pi_1(C,*) \twoheadrightarrow \pi_1(C_{\overline{K}},*)^{\mathrm{cs}}$, and $C^{\mathrm{cs}} \to C$ for the corresponding cover on the generic fibres.

We have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_C & \longrightarrow & \Pi_C & \longrightarrow & G_K & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \mathrm{id}\downarrow & & \\
1 & \longrightarrow & \Delta_C^{\mathrm{et}} & \longrightarrow & \Pi_C^{\mathrm{et}} & \longrightarrow & G_K & \longrightarrow & 1
\end{array}$$

where Π_C^{et} is the maximal quotient of Π_C which corresponds to étale covers of C^{cs} which extend to étale covers above X^{cs}, and Δ_C^{et} (which is defined so that the lower sequence is exact) is naturally identified with the free product $\star_{i\in I}\pi_1(X_{i,\overline{k}},*)$. Thus, Π_C^{et} is identified with the fibre product $\star_{i\in I}(\pi_1(X_i,*)\times_{G_k} G_K)$ over G_K. Moreover, s induces a section $s^{\mathrm{et}} : G_K \to \Pi_C^{\mathrm{et}}$ of the projection $\Pi_C^{\mathrm{et}} \twoheadrightarrow G_K$, which corresponds to a family of sections $\overline{s}_i : G_k \to \pi_1(X_i,*)$ of $\pi_1(X_i,*)$, $\forall i \in I$ (cf. Lemma 2.1.2). The sections $\overline{s}_i$ arise from (uniquely determined) rational points $\overline{x}_i \in X_i(k)$ by our assumption that k satisfies (ii), $\forall i \in I$. Now, recall the commutative diagram (1.6) in §1. The exact sequence of G_K-modules $0 \to TB \to TJ \to \Lambda^\wedge \to 0$ (where $J \overset{\mathrm{def}}{=} \mathrm{Pic}^0_{C/K}$ (cf. 1.2 and the discussion before Lemma 1.3.1)) induces an exact cohomology sequence $\Lambda^\wedge \to H^1(G_K,TB) \to H^1(G_K,TJ) \to \mathrm{Hom}(G_K,\Lambda^\wedge)$. The image of $s^{\mathrm{ab}} \in H^1(G_K,TJ)$ in $\mathrm{Hom}(G_K,\Lambda^\wedge)$ is trivial (cf. above discussion), hence s^{ab} is the image of an element $\eta \in H^1(G_K,TB)$. The image of η in $H^1(G_K,TA) \overset{\sim}{\to} \prod_{i\in I} H^1(G_K,TJ_i)$ (cf. the diagram in Lemma 1.1.1) coincides with $(\overline{s}_i^{\mathrm{ab}})_{i\in I} \in \prod_{i\in I} H^1(G_k,TJ_i) \hookrightarrow \prod_{i\in I} H^1(G_K,TJ_i)$, where $\overline{s}_i^{\mathrm{ab}} \in H^1(G_k,TJ_i)$ is the abelian portion of the section $\overline{s}_i : G_k \to \pi_1(X_i,*)$ which is point-theoretic $\forall i \in I$ (where

$J_i \overset{\text{def}}{=} \mathrm{Pic}^0_{X_i/k}$, cf. Lemma 1.3.2 and Lemma 1.2.2). (Indeed, this image coincides with the abelian portion of the section $s^{\text{et}} : G_K \to \Pi_C^{\text{et}}$ of the projection $\Pi_C^{\text{et}} \twoheadrightarrow G_K$ which is induced by s.) In particular, the image of η in $H^1(G_K, A)$ is pro-geometric, i.e., lies in $A(K)^\wedge \overset{\sim}{\to} \prod_{i\in I} J_i(k)^\wedge$ (cf. Lemma 3.3 for this latter isomorphism). It follows then immediately from the diagram in Lemma 1.1.1 that η is pro-geometric, i.e., lies in $B(K)^\wedge$. From this we deduce that s^{ab} lies in the image of $B(K)^\wedge$ in $H^1(G_K, TJ)$ via the above map $H^1(G_K, TB) \to H^1(G_K, TJ)$, this image is contained in $J(K)^\wedge$. $\square$

§**3. Sections of arithmetic fundamental groups of curves over function fields in characteristic** 0**.** We use the notations introduced in §1 and §2. In this section k is a field of characteristic 0 which *strongly satisfies* the condition (i)(b) in Definition 0.2. Let C be a *projective*, smooth, connected algebraic curve over k, and $K \overset{\text{def}}{=} k(C)$ the function field of C.

Let $X \to \operatorname{Spec} K$ be a *proper*, smooth, and geometrically connected *hyperbolic curve* over K. Consider a *model* $\mathcal{X}$ of X over C, i.e., $\mathcal{X} \to C$ is a flat and proper morphism with $\mathcal{X}_K = X$. Let $U \subseteq C$ be the largest nonempty open subscheme of C such that the fibres of $\mathcal{X}$ over U are *smooth*, and $\mathcal{X}_U \overset{\text{def}}{=} \mathcal{X} \times_C U$. Let $c \in U^{\text{cl}}$ be a closed point and $\mathcal{X}_c \overset{\text{def}}{=} \mathcal{X}_{k(c)}$ the fibre of $\mathcal{X}$ at c. Let ξ (resp. ξ_c) be a geometric point of $\mathcal{X}$ with value in its generic point (resp. with value in the generic point of $\mathcal{X}_c$). Then ξ (resp. ξ_c) determines an algebraic closure $\overline{K}$ (resp. $\overline{k(c)}$) of K (resp. of the residue field $k(c)$ of C at c).

Lemma 3.1. *For $c \in U^{\text{cl}}$, there exists a commutative diagram*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X_{\overline{K}}, \overline{\xi}) & \longrightarrow & \pi_1(X, \xi) & \longrightarrow & G_K & \longrightarrow & 1 \\
 & & \mathrm{id}\big\downarrow & & \big\downarrow & & \big\downarrow & & \\
1 & \longrightarrow & \pi_1(X_{\overline{K}}, \overline{\xi}) & \longrightarrow & \pi_1(\mathcal{X}_U, \xi) & \longrightarrow & \pi_1(U, \xi) & \longrightarrow & 1 \\
 & & \big\uparrow & & \big\uparrow & & \big\uparrow & & \\
1 & \longrightarrow & \pi_1(\mathcal{X}_{\overline{k(c)}}, \overline{\xi_c}) & \longrightarrow & \pi_1(\mathcal{X}_c, \xi_c) & \longrightarrow & G_{k(c)} & \longrightarrow & 1
\end{array} \tag{3.1}$$

where $G_K = \mathrm{Gal}(\overline{K}/K)$, $G_{k(c)} = \mathrm{Gal}(\overline{k(c)}/k(c))$, $\overline{\xi}$ (resp. $\overline{\xi_c}$) are geometric points induced by ξ (resp. ξ_c), the middle and right upper vertical maps are natural continuous surjective homomorphisms, the middle lower vertical map is defined up to conjugation, the right lower vertical map is injective, and both squares on the right are cartesian.

Proof. Follows from the functoriality of fundamental groups and the homotopy exact sequence for π_1 (cf. [Grothendieck1], Éxposé XIII, §4). $\square$

Let

$$s : G_K \to \pi_1(X, \xi)$$

be a *section* of $\pi_1(X, \xi)$.

Lemma 3.2. *There exists a section $s_U : \pi_1(U, \xi) \to \pi_1(\mathcal{X}_U, \xi)$ of the projection $\pi_1(\mathcal{X}_U, \xi) \twoheadrightarrow \pi_1(U, \xi)$ which extends the section s. Moreover, for each closed point*

$c \in U^{\mathrm{cl}}$ the section s_U restricts to a section $s_c : G_{k(c)} \to \pi_1(\mathcal{X}_c, \xi_c)$ of $\pi_1(\mathcal{X}_c, \xi_c)$, and we have a commutative diagram

$$
\begin{array}{ccc}
G_K & \xrightarrow{s} & \pi_1(X,\xi) \\
\downarrow & & \downarrow \\
\pi_1(U,\xi) & \xrightarrow{s_U} & \pi_1(\mathcal{X}_U,\xi) \\
\uparrow & & \uparrow \\
G_{k(c)} & \xrightarrow{s_c} & \pi_1(\mathcal{X}_c,\xi_c)
\end{array} \tag{3.2}
$$

where the vertical maps are the ones in diagram (3.1).

Proof. Follows easily from Lemma 2.1.2 and Lemma 3.1. $\square$

Next, for a closed point $c \in C^{\mathrm{cl}}$ write K_c for the completion of K at c, $\mathcal{O}_c$ the ring of integers of K_c, and $\hat{\mathcal{X}}_c \overset{\mathrm{def}}{=} \mathcal{X} \times_C \operatorname{Spec} \mathcal{O}_c$. Note that $\hat{\mathcal{X}}_c \times_{\operatorname{Spec} \mathcal{O}_c} \operatorname{Spec} k(c) = \mathcal{X}_c$, and $\hat{\mathcal{X}}_c \times_{\operatorname{Spec}(\mathcal{O}_c)} \operatorname{Spec} K_c = X_{K_c}$. Let η_c be a geometric point of $\hat{\mathcal{X}}_c$ with value in its generic point which determines an algebraic closure $\overline{K_c}$ of K_c.

Let $c \in U^{\mathrm{cl}}$. There exists a commutative diagram of exact sequences

$$
\begin{array}{ccccc}
\pi_1(\hat{\mathcal{X}}_c,\eta_c) & \longrightarrow & \pi_1(\mathcal{O}_c,\eta_c) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \\
\pi_1(\mathcal{X},\xi) & \longrightarrow & \pi_1(U,\xi) & \longrightarrow & 1 \\
\uparrow & & \uparrow & & \\
\pi_1(\mathcal{X}_c,\xi_c) & \longrightarrow & G_{k(c)} & \longrightarrow & 1
\end{array} \tag{3.3}
$$

where the upper vertical maps are defined up to conjugation (the lower vertical maps are as in diagram (3.1)), as well as a commutative digram

$$
\begin{array}{ccc}
\pi_1(\mathcal{X}_c,\xi_c) & \longrightarrow & \pi_1(\hat{\mathcal{X}}_c,\eta_c) \\
\downarrow & & \downarrow \\
\pi_1(\mathcal{X},\xi) & \xrightarrow{\mathrm{id}} & \pi_1(\mathcal{X},\xi)
\end{array} \tag{3.4}
$$

where the vertical maps are as in diagram (3.3), the upper horizontal map is an isomorphism, and which commutes with the various projections to $G_{k(c)}$, $\pi_1(\mathcal{O}_c, \eta_c)$, and $\pi_1(U, \xi)$; respectively, i.e., commutes with the induced commutative diagram

$$
\begin{array}{ccc}
G_{k(c)} & \longrightarrow & \pi_1(\mathcal{O}_c,\eta_c) \\
\downarrow & & \downarrow \\
\pi_1(U,\xi) & \xrightarrow{\mathrm{id}} & \pi_1(U,\xi)
\end{array}
$$

Moreover, the squares in diagram (3.3) are cartesian. In particular, the section $s_U : \pi_1(U, \xi) \to \pi_1(\mathcal{X}_U, \xi)$ (cf. Lemma 3.2) induces naturally a section

$$\tilde{s}_c : \pi_1(\mathcal{O}_c, \eta_c) \to \pi_1(\hat{\mathcal{X}}_c, \eta_c)$$

of the projection $\pi_1(\hat{\mathcal{X}}_c,\eta_c)\twoheadrightarrow\pi_1(\mathcal{O}_c,\eta_c)$, which induces naturally by pull back via the natural cartesian diagram

$$\begin{array}{ccccc}
\pi_1(X_{K_c},\eta_c) & \longrightarrow & G_{K_c} & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \\
\pi_1(\hat{\mathcal{X}}_c,\eta_c) & \longrightarrow & \pi_1(\mathcal{O}_c,\eta_c) & \longrightarrow & 1
\end{array}$$

a section

$$\hat{s}_c : G_{K_c} \to \pi_1(X_{K_c},\eta_c)$$

of $\pi_1(X_{K_c},\eta_c)$, $\forall c\in U^{\mathrm{cl}}$, where $G_{K_c}=\mathrm{Gal}(\overline{K_c}/K_c)$. Further, recall the digram (2.1):

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X_{\overline{K_c}},\overline{\eta_c}) & \longrightarrow & \pi_1(X_c,\eta_c) & \longrightarrow & G_{K_c} & \longrightarrow & 1 \\
 & & \downarrow & & {\scriptstyle\mathrm{Sp}}\downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(\mathcal{X}_{\overline{k(c)}},\overline{\xi}_c) & \longrightarrow & \pi_1(\mathcal{X}_c,\xi_c) & \longrightarrow & G_{k(c)} & \longrightarrow & 1
\end{array}$$

where the vertical specialisation map is deduced, by pull back, from the upper horizontal map in diagram (3.4) via the natural surjective map $\pi_1(X_c,\eta_c)\twoheadrightarrow\pi_1(\hat{\mathcal{X}}_c,\eta_c)$. Then it follows that the above section $\hat{s}_c$ of $\pi_1(X_{K_c},\eta_c)$ induces, via the above diagram, the section s_c of $\pi_1(\mathcal{X}_c,\xi_c)$ in Lemma 3.2 (cf. Lemma 2.1.2).

From now on we assume that $X(K)\neq\emptyset$. Let $\mathcal{J}\overset{\mathrm{def}}{=}\mathrm{Pic}^0_{\mathcal{X}_U/U}\to U$ be the (relative) jacobian of the (relative) smooth curve $\mathcal{X}_U\to U$, $J\overset{\mathrm{def}}{=}\mathcal{J}_K$ the jacobian variety of X, $\mathcal{J}_c\overset{\mathrm{def}}{=}\mathcal{J}_{k(c)}$ the jacobian variety of $\mathcal{X}_c$, and $\hat{\mathcal{J}}_c\overset{\mathrm{def}}{=}\mathcal{J}\times_U\mathrm{Spec}\,\mathcal{O}_c$ the (relative) jacobian of $\hat{\mathcal{X}}_c$. For $c\in C^{\mathrm{cl}}$ write $J_c\overset{\mathrm{def}}{=}J_{K_c}$ for the jacobian variety of X_c.

For $c\in U^{\mathrm{cl}}$ the above diagram induces a commutative diagram of exact sequences

(3.5)
$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X_{\overline{K_c}},\overline{\eta_c})^{\mathrm{ab}} & \longrightarrow & \pi_1(X_{K_c},\eta_c)^{(\mathrm{ab})} & \longrightarrow & G_{K_c} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(\mathcal{X}_{\overline{k(c)}},\overline{\xi}_c)^{\mathrm{ab}} & \longrightarrow & \pi_1(\mathcal{X}_c,\xi_c)^{(\mathrm{ab})} & \longrightarrow & G_{k(c)} & \longrightarrow & 1
\end{array}$$

whose right square is *cartesian*. Further, we have a natural identification $TJ_c\overset{\sim}{\to}T\mathcal{J}_c$ between Tate modules (since $\mathrm{char}\,k(c)=0$). We will identify $\pi_1(X_{\overline{K_c}},\overline{\eta_c})^{\mathrm{ab}}$ and $\pi_1(\mathcal{X}_{\overline{k(c)}},\overline{\xi}_c)^{\mathrm{ab}}$ via the isomorphism in diagram (3.5), and will further identify them (as Galois modules) with the Tate modules TJ_c, and $T\mathcal{J}_c$, respectively.

Lemma 3.3. *Let $c\in U^{\mathrm{cl}}$. Then we have a commutative diagram of exact sequences*
(3.6)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & J(K_c)^\wedge & \longrightarrow & H^1(G_{K_c},TJ_c) & \longrightarrow & TH^1(G_{K_c},J_c) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{J}_c(k(c))^\wedge & \longrightarrow & H^1(G_{k(c)},T\mathcal{J}_c) & \longrightarrow & TH^1(G_{k(c)},\mathcal{J}_c) & \longrightarrow & 0
\end{array}$$

where the middle vertical map is the inflation map (deduced from diagram (3.5)), the left vertical map is an isomorphism, the middle and right vertical maps are injective, and the horizontal sequences are the Kummer exact sequences (cf. §1).

Proof. Follows from the fact that there exist isomorphisms $H^1(G_{k(c)}, T\mathcal{J}_c) \xrightarrow{\sim} H^1(\mathrm{Gal}(K_c^{\mathrm{ur}}/K_c), TJ_c)$ and $TH^1(G_{k(c)}, \mathcal{J}_c) \xrightarrow{\sim} TH^1(\mathrm{Gal}(K_c^{\mathrm{ur}}/K_c), J_c)$, where K_c^{ur}/K_c denotes the maximal unramified subextension of $\overline{K_c}/K_c$, and the fact that the kernel of the specialisation map $J_c(K_c) \twoheadrightarrow \mathcal{J}_c(k(c))$ is uniquely divisible (cf. [Lang-Tate], Proposition 8). (See also the commutative diagram in loc. cit. page 675.) □

Next, for $c \in U^{\mathrm{cl}}$, we fix compatible *base points* of the torsor of splittings of the horizontal sequences in diagram (3.5). The sections $\hat{s}_c : G_{K_c} \to \pi_1(X_{K_c}, \eta_c)$ and $s_c : G_{k(c)} \to \pi_1(\mathcal{X}_c, \xi_c)$ give rise naturally to sections $\hat{s}_c^{\mathrm{ab}} : G_{K_c} \to \pi_1(X_{K_c}, \eta_c)^{(\mathrm{ab})}$ and $s_c^{\mathrm{ab}} : G_{k(c)} \to \pi_1(\mathcal{X}_c, \xi_c)^{(\mathrm{ab})}$ of the projections $\pi_1(X_{K_c}, \eta_c)^{(\mathrm{ab})} \twoheadrightarrow G_{K_c}$ and $\pi_1(\mathcal{X}_c, \xi_c)^{(\mathrm{ab})} \twoheadrightarrow G_{k(c)}$; respectively, which correspond to elements $\hat{s}_c^{\mathrm{ab}} \in H^1(G_{K_c}, TJ_c)$ and $s_c^{\mathrm{ab}} \in H^1(G_{k(c)}, T\mathcal{J}_c)$, respectively (cf. §1). It follows from the various definitions that s_c^{ab} maps to $\hat{s}_c^{\mathrm{ab}}$ via the middle vertical map in diagram (3.6). The following follows from the various definitions (cf. Lemma 1.2.2).

Lemma 3.4. *Let $c \in U^{\mathrm{cl}}$. Assume that the section s_c is point-theoretic (in particular, $\hat{s}_c$ is point-theoretic by Lemma 2.1.4). Then s_c^{ab} and $\hat{s}_c^{\mathrm{ab}}$ are point-theoretic (cf. Definition 1.2.1). In particular, s_c^{ab} and $\hat{s}_c^{\mathrm{ab}}$ are pro-geometric (cf. Definition 1.1.2(i) and Lemma 1.2.2).*

Let $c \in C^{\mathrm{cl}} \setminus U^{\mathrm{cl}}$. We have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X_{\overline{K_c}}, \overline{\eta_c}) & \longrightarrow & \pi_1(X_{K_c}, \eta_c) & \longrightarrow & G_{K_c} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(X_{\overline{K}}, \overline{\xi}) & \longrightarrow & \pi_1(X, \xi) & \longrightarrow & G_K & \longrightarrow & 1
\end{array}$$

where the middle vertical map is defined up to conjugation and the right square is *cartesian.* In particular, the section $s : G_K \to \pi_1(X, \xi)$ induces via the above diagram a section $\hat{s}_c : G_{K_c} \to \pi_1(X_{K_c}, \eta_c)$ of $\pi_1(X_{K_c}, \eta_c)$.

Next, consider the following commutative diagram

(3.7)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J(K)^\wedge & \longrightarrow & H^1(G_K, TJ) & \longrightarrow & TH^1(G_K, J) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \prod_c J_c(K_c)^\wedge & \longrightarrow & \prod_c H^1(G_{K_c}, TJ_c) & \longrightarrow & \prod_c TH^1(G_{K_c}, J_c) & \longrightarrow & 0
\end{array}$$

where the horizontal sequences are the Kummer exact sequences, the vertical maps are the diagonal mappings, and the product in the lower sequence is over *all* closed points $c \in C^{\mathrm{cl}}$. For $c \in U^{\mathrm{cl}}$, the above vertical map $H^1(G_K, TJ) \to H^1(G_{K_c}, TJ_c)$ factorizes as $H^1(G_K, TJ) \to H^1(G_{k(c)}, T\mathcal{J}_c) \to H^1(G_{K_c}, TJ_c)$ where the map $H^1(G_K, TJ) \to H^1(G_{k(c)}, T\mathcal{J}_c)$ is deduced from diagram (3.1), and the latter map $H^1(G_{k(c)}, T\mathcal{J}_c) \to H^1(G_{K_c}, TJ_c)$ is as in Lemma 3.3. For $c \in C^{\mathrm{cl}} \setminus U^{\mathrm{cl}}$ the vertical map $H^1(G_K, TJ) \to H^1(G_{K_c}, TJ_c)$ is deduced from the map $\pi_1(X_{K_c}, \eta_c) \to \pi_1(X, \xi)$ discussed above.

Fix a *base point* of the torsor of splittings of the exact sequence $1 \to \pi_1(X_{\overline{K}}, \overline{\xi})^{\text{ab}} \to \pi_1(X, \xi)^{(\text{ab})} \to G_K \to 1$, the corresponding base points of the torsors of splittings of the sequences $1 \to \pi_1(X_{\overline{K_c}}, \overline{\eta_c})^{\text{ab}} \to \pi_1(X_{K_c}, \eta_c)^{(\text{ab})} \to G_{K_c} \to 1$, $\forall c \in C^{\text{cl}} \setminus U^{\text{cl}}$, deduced from the maps $\pi_1(X_{K_c}, \eta_c) \to \pi_1(X, \xi)$ above, and for $c \in U^{\text{cl}}$ the corresponding base points of the torsors of splittings of the lower and upper sequence in diagram (3.5) (cf. diagram (3.1) and Lemma 3.2). The section $s : G_K \to \pi_1(X, \xi)$ gives rise naturally to an element $s^{\text{ab}} \in H^1(G_K, TJ)$ whose image in $H^1(G_{K_c}, TJ_c)$ coincides with the element $\hat{s}_c^{\text{ab}} \in H^1(G_{K_c}, TJ_c)$ which arises from the section $\hat{s}_c : G_{K_c} \to \pi_1(X_c, \xi)$ which is induced by s, $\forall c \in C^{\text{cl}}$ (cf. above discussion).

Recall (cf. §0)

$$Ш(J) \overset{\text{def}}{=} \operatorname{Ker}\Big(H^1(G_K, J) \to \prod_{c \in C^{\text{cl}}} H^1(G_{K_c}, J_c)\Big).$$

Note that the kernel of the map $TH^1(G_K, J) \to \prod_{c \in C} TH^1(G_{K_c}, J_c)$ in diagram (3.7) is naturally identified with the Tate module $TШ(J)$ of the Shafarevich-Tate group $Ш(J)$. The following is immediate from the various definitions.

Lemma 3.5. *Suppose that $\hat{s}_c^{\text{ab}} \in H^1(G_{K_c}, TJ_c)$ is pro-geometric (cf. Definition 1.1.2(i)) $\forall c \in C^{\text{cl}}$, and $TШ(J) = 0$. Then $s^{\text{ab}} \in J(K)^\wedge$ is pro-geometric.*

§4. Proofs of Theorems A and B.

In this section we prove Theorems A and B.

4.1. Proof of Theorem A. Recall the notations introduced in §3 that we will use throughout. Let k, C, $K = k(C)$, $X \to \operatorname{Spec} K$, be as in §3. We assume that X satisfies the condition $(\star\star)$ (cf. Definition 0.3). In particular, $X(K) \neq \emptyset$ by $(\star\star)$ (i). Let $\mathcal{X} \to C$ be the stable model of X over C (which satisfies the condition $(\star\star)$(ii)) and $U \subseteq C$ the largest nonempty open subscheme of C such that the fibres of $\mathcal{X}$ over U are *smooth*. Assume that k strongly satisfies the conditions (i), (ii), (iv), (v), and (vi), in Definition 0.2. Let $J \overset{\text{def}}{=} \operatorname{Pic}^0_{X/K}$ be the jacobian of X, and assume that $TШ(J) = 0$. We will show that the map (cf. §0)

$$\varphi_X : X(K) \to \overline{\operatorname{Sec}}_{\pi_1(X,\xi)}, \qquad x \mapsto \varphi_X(x) = [s_x],$$

is *bijective.*

First, we prove φ_X is *injective.* Let $x_1, x_2 \in X(K)$ such that $[s_1 \overset{\text{def}}{=} s_{x_1}] = [s_2 \overset{\text{def}}{=} s_{x_2}]$ holds in $\overline{\operatorname{Sec}}_{\pi_1(X,\xi)}$. Thus, $[s_{1,c}] = [s_{2,c}]$ in $\overline{\operatorname{Sec}}_{\pi_1(\mathcal{X}_c,\xi_c)}$, $\forall c \in U^{\text{cl}}$, and $s_{i,c} = s_{x_{i,c}}$ is point-theoretic, where $x_{i,c} \in \mathcal{X}_c(k(c))$ is uniquely determined by $s_{i,c}$ for $i \in \{1,2\}$ by our assumption that k strongly satisfies (ii). The map $\varphi_{\mathcal{X}_c} : \mathcal{X}_c(k(c)) \to \overline{\operatorname{Sec}}_{\pi_1(\mathcal{X}_c,\xi_c)}$ is bijective by assumption, hence $x_{1,c} = x_{2,c}$, $\forall c \in U^{\text{cl}}$. Moreover, $x_{i,c}$ is the specialisation of x_i in $\mathcal{X}_c$ (cf. discussion before Lemma 2.1.3). From this it follows that $x_1 = x_2$ and φ_X is injective. Indeed, the natural specialisation map $X(K) \to \prod_{c \in U^{\text{cl}}} \mathcal{X}_c(k(c))$ is injective.

Next, we prove that φ_X is *surjective.* Let

$$s : G_K \to \pi_1(X, \xi)$$

be a *section* of $\pi_1(X, \xi)$. We will show that s is *point-theoretic* under the above assumptions.

First, it follows from the condition (ii) that the section s_c (cf. diagram (3.2)) is point-theoretic and arises from a unique rational point $x_c \in \mathcal{X}(k(c))$, $\forall c \in U^{\text{cl}}$. In particular, $\hat{s}_c^{\text{ab}} \in H^1(G_{K_c}, TJ_c)$ is point-theoretic in this case (cf. discussion before Lemma 3.4, Definition 1.2.1, and Lemma 2.1.4). Moreover, it follows from the conditions (i) and (ii) in Definition 0.2, as well as the condition $(\star\star)$(ii), that $\hat{s}_c^{\text{ab}} \in J(K_c)^\wedge \subset H^1(G_{K_c}, TJ_c)$ is pro-geometric in the sense of Definition 1.1.2(i), $\forall c \in C^{\text{cl}}$ (cf. Lemma 1.2.2, Proposition 2.3.2, and the discussion before Lemma 3.5). Then it follows from the assumption $T\text{Ш}(J) = 0$ that $s^{\text{ab}} \in J(K)^\wedge \subset H^1(G_K, TJ)$ is pro-geometric (cf. Lemma 3.5).

Lemma 4.1.1. *The natural homomorphism $J(K) \to J(K)^\wedge$ is injective and $s^{\text{ab}} \in J(K) \subseteq J(K)^\wedge$ is geometric.*

Proof. There exist closed points $c_1, c_2 \in U^{\text{cl}}$ such that the natural specialisation homomorphism $J(K) \to \mathcal{J}_{c_1}(k(c_1)) \times \mathcal{J}_{c_2}(k(c_2))$ is injective (cf. [Poonen-Voloch], Proposition 2.4). We have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J(K) & \longrightarrow & H \overset{\text{def}}{=} \mathcal{J}_{c_1}(k(c_1)) \times \mathcal{J}_{c_2}(k(c_2)) & \longrightarrow & H/J(K) & \longrightarrow & 0 \\ & & \downarrow & & \phi \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J(K)^\wedge & \xrightarrow{\psi} & H^\wedge = \mathcal{J}_{c_1}(k(c_1))^\wedge \times \mathcal{J}_{c_2}(k(c_2))^\wedge & \longrightarrow & (H/J(K))^\wedge & \longrightarrow & 0 \end{array}$$

where the right and middle vertical maps are injective homomorphisms by the assumption (iv)(a), and the maps ψ and ϕ are the natural ones. (The exactness of the lower sequence in the above diagram follows easily from the assumption (iv)(b).) In particular, the left vertical map is injective, and the equality $J(K) = \phi(H) \cap \psi(J(K)^\wedge)$ holds inside $H^\wedge$. The image of $s^{\text{ab}} \in J(K)^\wedge$ in $H^\wedge$ via the map ψ is the element $(s_{c_1}^{\text{ab}}, s_{c_2}^{\text{ab}}) \in H^\wedge \subset H^1(G_{k(c_1)}, T\mathcal{J}_{c_1}) \times H^1(G_{k(c_2)}, T\mathcal{J}_{c_2})$ associated to the sections $s_{c_i} : G_{k(c_i)} \to \pi_1(\mathcal{X}_{c_i}, \xi_{c_i})$, for $i \in \{1, 2\}$ (induced by the section s) which are point-theoretic by the assumption (ii). Hence $s_{c_i}^{\text{ab}} \in \mathcal{J}_{c_i}(k(c_i))$ is geometric (cf. Lemma 1.2.2), and $s^{\text{ab}} \in J(K)$ is geometric by the above discussion. □

Let $\mathcal{X}_U \xrightarrow{\iota} \mathcal{J}$ be an embedding mapping an element of $\mathcal{X}_U(U) = X(K)$ (which is nonempty by our assumptions) to the zero section.

Lemma 4.1.2. *The element $s^{\text{ab}} \in \iota(X(K)) \subset J(K)$ is point-theoretic.*

Proof. For each closed point $c \in U^{\text{cl}}$ the element $s_c^{\text{ab}} \in \mathcal{J}(k(c))^\wedge \subset H^1(G_{k(c)}, T\mathcal{J}_c)$ corresponding to the section $s_c : G_{k(c)} \to \pi_1(\mathcal{X}_c, \xi_c)$ lies in the subset $\iota(\mathcal{X}(k(c))) \subset \mathcal{J}(k(c)) \subset \mathcal{J}(k(c))^\wedge$, since the section s_c is point-theoretic by (ii). We view $s^{\text{ab}} \in J(K)$ as a rational section of $\mathcal{J} \to U$, in fact $s^{\text{ab}} : U \to \mathcal{J}$ is a morphism since U is a smooth curve. For each closed point $c \in U^{\text{cl}}$ the image $s^{\text{ab}}(c)$ is a closed point of $\mathcal{X}_c \subset \mathcal{J}_c$, where we view $\mathcal{X}_c$ as a closed subscheme of $\mathcal{J}_c$ via the closed immersion $\mathcal{X}_c \xrightarrow{\iota_c} \mathcal{J}_c$ induced by ι. From this it follows that the morphism $s^{\text{ab}} : U \to \mathcal{J}$ factorizes as $s^{\text{ab}} : U \to \mathcal{X}_U \xrightarrow{\iota} \mathcal{J}$ and s^{ab} belongs to the subset $\iota(X(K)) \subseteq J(K)$. □

Let $\tilde{x} \in X(K)$ such that $\iota(\tilde{x}) = s^{\text{ab}}$. For $c \in U^{\text{cl}}$ let $\tilde{x}_c$ be the specialisation of $\tilde{x}$ in $\mathcal{X}_c$.

Lemma 4.1.3. *The equality $\tilde{x}_c = x_c$ holds in $\mathcal{X}(k(c))$, $\forall c \in U^{\mathrm{cl}}$.*

Proof. First, the equality $s_c^{\mathrm{ab}} = s_{\tilde{x}_c}^{\mathrm{ab}} = s_{x_c}^{\mathrm{ab}}$ holds in $H^1(G_{k(c)}, T\mathcal{J}_c)$ (cf. diagram (3.5)). The equality $\tilde{x}_c = x_c$ follows then from the injectivity of the maps $\iota(\mathcal{X}_c) \hookrightarrow \mathcal{J}_c(k(c)) \hookrightarrow \mathcal{J}_c(k(c))^\wedge \to H^1(G_{k(c)}, T\mathcal{J}_c)$, for $c \in U^{\mathrm{cl}}$ (see the assumption (iv)(a)). $\square$

Next, and in order to show that the section s is point-theoretic it suffices to show, by a well-know *limit argument* in anabelian geometry (cf. [Tamagawa], Proposition 2.8 (iv)), using the assumption (v) (cf. Definition 0.2), the following. Let $H \subseteq \pi_1(X, \xi)$ be an open subgroup such that $s(G_K) \subset H$, corresponding to an étale cover $Y \to X$, then $Y(K) \neq \emptyset$ holds. There is a natural identification $H = \pi_1(Y, \xi)$. Moreover, the cover $Y \to X$ extends to an étale cover $\mathcal{Y}_U \to \mathcal{X}_U$ and $\mathcal{Y}_U$ is a smooth model of Y over U (cf. Lemma 3.1 and Lemma 3.2). Let $\mathcal{Y} \to \mathcal{X}$ be the normalisation of $\mathcal{X}$ in $\mathcal{Y}_U$, and for a closed point $c \in C^{\mathrm{cl}}$ write $\mathcal{Y}_c \overset{\mathrm{def}}{=} \mathcal{Y}_{k(c)}$. We will show that $Y(K) \neq \emptyset$.

Let $s' : G_K \to \pi_1(Y, \xi)$ be the section of $\pi_1(Y, \xi) = H$ induced by s, which extends to a section $s'_U : \pi_1(U, \xi) \to \pi_1(\mathcal{Y}_U, \xi)$ of the projection $\pi_1(\mathcal{Y}_U, \xi) \twoheadrightarrow \pi_1(U, \xi)$, and further induces a section $s'_c : G_{k(c)} \to \pi_1(\mathcal{Y}_c, \xi_c)$ of $\pi_1(\mathcal{Y}_c, \xi_c)$, $\forall c \in U^{\mathrm{cl}}$ (cf. loc. cit.). Note that s'_c is induced by the section s_c. The section s'_c is point-theoretic and arises from a unique rational point $y_c \in \mathcal{Y}_c(k(c))$. Moreover, x_c is the image of y_c in $\mathcal{X}_c$ via the morphism $\mathcal{Y}_c \to \mathcal{X}_c$ (cf. condition (ii), the fact that s_c is point-theoretic and arises from x_c, and s'_c is induced by s_c). View $\tilde{x} \in X(K) = \mathcal{X}(U)$ as a section $\tilde{x} : U \to \mathcal{X}_U$, and let $\mathcal{Y}_{\tilde{x}}$ be the scheme-theoretic inverse image of $\tilde{x}(U)$ in $\mathcal{Y}_U$ via the above étale map $\mathcal{Y}_U \to \mathcal{X}_U$. Thus, $\mathcal{Y}_{\tilde{x}} \to \tilde{x}(U)$ is a finite étale map. We have $y_c \in \mathcal{Y}_{\tilde{x}}(k(c))$, $\forall c \in U^{\mathrm{cl}}$, as follows from the various definitions. Then $\mathcal{Y}_{\tilde{x}}(K) \neq \emptyset$ by the assumption (vi), and a fortiori $\mathcal{Y}_{\tilde{x}}(K) \subseteq \mathcal{Y}_U(K) = Y(K) \neq \emptyset$.

Thus, we proved that $[s] = [s_x]$ holds in $\overline{\mathrm{Sec}}_{\pi_1(X,\xi)}$ for a (unique) $x \in X(K)$. The following follows from Lemma 4.1.3 (cf. above proof that φ_X is injective).

Lemma 4.1.4. *The equality $x = \tilde{x}$ holds.*

This finishes the proof of Theorem A. $\square$

Finally, we show that Hilbertian fields satisfy the condition (vi).

Lemma 4.1.5. *Let k be a Hilbertian field. Then k strongly satisfies the condition (vi).*

Proof. Let k'/k be a finite extension and C a smooth and connected curve over k' with function field $K \overset{\mathrm{def}}{=} k'(C)$. Let $\widetilde{C} \to C$ be a finite étale cover with $\widetilde{C}_c(k'(c)) \neq \emptyset$, $\forall c \in C^{\mathrm{cl}}$ with residue field $k'(c)$. Note that k' is Hilbertian (cf. [Serre], 9.5). We show $\widetilde{C}(K) \neq \emptyset$. Assume that $\widetilde{C}(K) = \emptyset$. Then for each connected component $\widetilde{C}_\alpha$ of $\widetilde{C}$ the degree of the morphism $\widetilde{C}_\alpha \to C$ is ≥ 2. Hilbert's irreducibility theorem (cf. [Serre], 9.2) implies that there exist points $c \in C^{\mathrm{cl}}$ such that the fibre of c in each connected component of $\widetilde{C}$ is irreducible. This contradicts the assumption that $\widetilde{C}_c(k(c)) \neq \emptyset$, $\forall c \in C^{\mathrm{cl}}$. Thus, $\widetilde{C}(K) \neq \emptyset$ holds. $\square$

In the course of proving Theorem A we proved the following "*adelic Mordell-Lang*" statement.

Proposition 4.1.6. *With the same notations as above, assume that k only satisfies the condition (iv) in Definition 0.2 (where we take $k' = k$), and $\mathcal{X} \to C$ is only a* *proper and flat model of X over C. Then the map $J(K)^\wedge \to \prod_{c\in U^{\text{cl}}} J_c(K_c)^\wedge \xrightarrow{\sim} \prod_{c\in U^{\text{cl}}} \mathcal{J}_c(k(c))^\wedge$ (cf. diagram (3.7) and Lemma 3.3) is injective. Further, inside $\prod_{c\in U^{\text{cl}}} \mathcal{J}_c(k(c))^\wedge$ the equality $J(K)^\wedge \bigcap (\prod_{c\in U^{\text{cl}}} \iota(\mathcal{X}_c(k(c)))) = \iota(X(K))$ holds.*

Proof. See the statements and proofs of lemma 4.1.1, and Lemma 4.1.2. □

4.2. Proof of Theorem B. In this section we briefly explain how Theorem B can be derived from Theorem A. Let k be a field with $\text{char}(k) = 0$, and $K = k(C)$ the function field of a smooth and connected curve C over k. Assume that k satisfies the condition $(\star)$ (cf. Definition 0.2). Let L/K be a finite extension. Then we prove the **SC** holds over L. We can without loss of generality assume that $L = K$. Let X be a proper smooth and hyperbolic curve over K, we need to prove that the **SC** holds for X, i.e., that the map (cf. §0)

$$\varphi_X : X(K) \to \overline{\text{Sec}}_{\pi_1(X,\xi)}, \qquad x \mapsto \varphi_X(x) = [s_x],$$

is *bijective*. The injectivity of the map φ_X follows as in the proof of Theorem A, where one only uses the fact that k strongly satisfies the condition (ii) (cf. loc. cit.)

Next, we prove that φ_X is *surjective*. Let

$$s : G_K \to \pi_1(X, \xi)$$

be a *section* of $\pi_1(X, \xi)$. We need to show that s is *point-theoretic*, under the above assumptions. For this purpose we can, in the course of the proof, replace K by a finite extension L'/K. Indeed, let L'/K be a finite Galois extension, $s' : G_{L'} \overset{\text{def}}{=} \text{Gal}(\overline{K}/L') \to \pi_1(X_{L'}, \xi)$ the section of $\pi_1(X_{L'}, \xi)$ which is induced by s, and assume that $s' = s_{x'}$ is point-theoretic where $x' \in X(L')$. Then $s'(G_{L'})$ is self-normalising in $\pi_1(X_{L'}, \xi)$, and $s(G_K)$ is contained in the normaliser of $s'(G_{L'})$ in $\pi_1(X, \xi)$ which coincides with a decomposition group associated to the image x of x' in X (this follows from condition $(\star)(iv)(a)$, cf. proof of Lemma 2.2.1 and the references therein). The point x is then necessarily K-rational. We can also, without loss of generality, replace X by a *neighbourhood* of the section s, i.e., an étale cover $Y \to X$ corresponding to an open subgroup $H \subseteq \pi_1(X, \xi)$ containing $s(G_K)$. Now one verifies easily that there exists a finite extension L'/K, the corresponding section $s' : G_{L'} \overset{\text{def}}{=} \text{Gal}(\overline{K}/L') \to \pi_1(X_{L'}, \xi)$ of $\pi_1(X_{L'}, \xi)$ which is induced by s, and a *neighbourhood* of the section s', i.e., an étale cover $Y \to X_{L'}$ corresponding to an open subgroup $H \subseteq \pi_1(X_{L'}, \xi)$ containing $s'(G_{L'})$, such that Y satisfies the condition $(\star\star)$ (cf. Definition 0.3). The section $s' : G_{L'} \to H = \pi_1(Y, \xi)$ of $\pi_1(Y, \xi)$ is then point-theoretic by Theorem A. From this it follows easily that s is point theoretic (cf. above discussion).

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