PRIME (-1,1) AND JORDAN MONSTERS AND SUPERALGEBRAS OF VECTOR TYPE

SERGEY V. PCHELINTSEV AND IVAN P. SHESTAKOV

ABSTRACT. It is proved that the prime degenerate (-1,1) algebra constructed in [13] (the (-1,1)-monster) generates the same variety of algebras as the Grassman (-1,1)-algebra. Moreover, the same variety is generated by the Grassmann envelope of any simple nonassociative (-1,1)-superalgebra. The variety occurs to be the smallest variety of (-1,1)-algebras that contains prime nonassociative algebras.

Similar results are obtained for Jordan algebras. Thus, the Jordan monster (the prime degenerate algebra constructed in [13]) and the Grassmann envelope of the prime Jordan superalgebra of vector type have the same ideals of identities. It is also shown that the Jordan monster generates a minimal variety that contains prime degenerate Jordan algebras.

All the algebras and superalgebras are considered over a field of characteristic 0.

Keywords: (-1,1)-algebra, Jordan algebra, prime degenerate algebra, Grassmann envelope, superalgebra of vector type.

INTRODUCTION

An algebra is called *degenerate* if it contains nonzero absolute zero divisors. In [11, 13], the first examples of prime degenerate algebras in the varieties of alternative, Jordan and algebras of type (-1,1) were constructed, later received the name of *Pchelintsev Monsters* [8].

In [10], there were given another constructions of prime degenerate Jordan algebras, based on the Jordan superalgebra $J(\Phi[x], \frac{d}{dx})$ of vector fields on the line, and the superalgebra of Poisson brackets $J(\Phi[X, Y]; [\cdot, \cdot])$. Observe that the superalgebras $J(\Phi[x], \frac{d}{dx})$ and $J(\Phi[X, Y]; [\cdot, \cdot])$ have different identities.

In [21], the concept of a (-1,1)-superalgebra of vector type was introduced and it was proved that a Jordan superalgebra of vector type may be obtained as the (super)symmetrized algebra $A^{(+)}$ of a (-1,1)-superalgebra of vector type A. Furthermore, the (-1,1)-superalgebras

$A_{VF} := A(V, \Phi, \tau, \lambda)$ and the Jordan superalgebras $J_{VF} := J(V, \Phi, \tau, \lambda)$ of vector fields on a line associated with an additive homomorphism $\tau : V \to \Phi$ of abelian groups and partial map $\lambda : V \to V$ were introduced in [21]. It was proved that if $V \neq 0$ and the map τ is injective, then the superalgebras A_{VF} and J_{VF} are prime. These superalgebras were then used to construct prime degenerate (-1,1) and Jordan algebras.

In all the papers mentioned above, the degenerate prime algebras were constructed as free algebras in varieties generated by some auxiliary algebras. Thus, in [13] the auxiliary algebra was determined by a set of generators and relations, in [10] and [21] the auxiliary algebras appeared as the Grassmann envelopes of the superalgebras $J(\Phi[x], \frac{d}{dx})$, $J(\Phi[X, Y]; [\cdot, \cdot])$, and A_{VF}, J_{VF}, respectively.

We give now the main results of the article, noting that the definitions of all algebras in the statements are given in Sections 2 and 7. Below $G(X)$ denotes the Grassmann envelope of a superalgebra X.

Theorem A. *Let A_0 be the auxiliary algebra associated with the prime (-1,1)-monster [13]; $G_{(-1,1)}$ be the (-1,1)-Grassmann algebra; A_{VF} be the (-1,1)-superalgebra of vector fields on a line; B be a simple nonassociative (-1,1)-superalgebra. Then the algebras A_0, $G_{(-1,1)}$, $G(A_{VF})$, $G(B)$ over a field of characteristic 0 have the same ideals of identities.*

In particular, we show that the prime (-1,1)-algebras over a field of characteristic 0 constructed in [13] and [21] are isomorphic.

We prove Theorem A in the superalgebra setting, describing the free (-1,1)-superalgebra $\mathcal{F}_{(-1,1)}[\emptyset; x]$ on one odd generator and its central extensions. The case of simple superalgebras is based on the classification of simple nonassociative (-1,1)-superalgebras [23] and on the embedding theorem of a simple nonassociative (-1,1)-superalgebra into a suitable twisted superalgebra of vector type [32].

Additional properties of the variety $Var(A_0)$ are listed in the following theorem.

Theorem B. *Let $\mathcal{V}_0 = Var\, A_0$ and let $\mathcal{F}(A_0)$ be the free algebra of countable rank in the variety $\mathcal{V}_0$ over a field Φ of characteristic 0. Then*

a) $\mathcal{V}_0$ is the smallest variety of (-1,1)-algebras that contains prime nonassociative algebras;

b) The meta-ideals of finite index of the (-1,1)-monster $\mathcal{F}(A_0)$ have the same identities;

c) *A meta-ideal of the algebra* $\mathcal{F}(A_0)$ *which is contained in its commutant is not a free algebra of any variety of algebras.*

Note that in the proof of Theorem A the "minimal" quotient algebra $\overline{A}_0$, which has the same identities as the algebra A_0, is constructed.

It is an open question what identities define the variety $\mathcal{V}_0$? The conjecture first formulated in [12] that $\mathcal{V}_0$ coincides with the variety $\mathcal{S}t$ of *strongly (-1,1)* algebras, that is, defined by the identity $[[x,y],z] = 0$, is still open.

The technique developed in the proof of Theorem A, allows us to obtain certain analogues of these results for Jordan algebras.

Theorem C. *Let* $J = J(\Gamma, \delta)$ *be a prime Jordan superalgebra of vector type where* $\Gamma = \Gamma_0$, *$J[Z;x]$ be a central extension of the free Jordan superalgebra generated by one odd element (see Section 7). Then the algebras*

$$J_0 = A_0^{(+)},\ G(J),\ (G_{(-1,1)})^{(+)},\ G(J[Z;x]),\ G(J_{VT})$$

over a field of characteristic 0 have the same ideals of identities.

In addition, the variety $\mathcal{JV}_0$ *generated by any of these algebras is a minimal variety containing prime degenerate algebras.*

As a corollary, it turns out that over a field of characteristic 0 the prime degenerate special Jordan algebras constructed in [10, 21] are isomorphic to the Jordan monster $\mathcal{F}(J_0)$. In addition, $\mathcal{F}(J_0)$ coincides with the subalgebra of the Jordan algebra generated by a countable set of free generators X of the algebra $\mathcal{F}(A_0)$.

The simple alternative superalgebras were classified in [31, 22]. Over a field of characteristic 3, these are the superalgebras $B(1,2)$, $B(4,2)$, and the twisted superalgebra of vector type $B(\Gamma, D, \gamma)$. The free algebras in the varieties generated by the Grassmann envelopes of finite-dimensional superalgebras $B(1,2)$ and $B(4,2)$ can not be prime (see [10, page 9]), while for the infinite-dimensional superalgebra $B(\Gamma, D, \gamma)$ this question is open.

Concerning the Grassmann algebras, we recall that they may be defined as the algebras of skew-symmetric functions or as certain subalgebras of Grassmann envelopes of free superalgebras generated by one odd element.

For the first time such an algebra was considered by G.V.Dorofeev in the variety of solvable of index two alternative algebras; it is the classical *Dorofeev example* [1]. He used also the nilpotent of index 7 alternative Grassmann algebra to prove that any alternative algebra with three generators satisfies certain identities that do not hold in all alternative algebras [2].

In [24, 26], additive bases for free superalgebras with one odd generator in the varieties of Malcev and alternative superalgebras were constructed, and new elements in the radicals of free algebras were found. In particular, bases for Malcev and alternative Grassmann algebras were obtained there.

Identities of Grassmann algebras in some varieties of alternative algebras over a field of characteristic 3 and of right alternative metabelian algebras over a field of characteristic different from 2 and 3, have been studied in [14, 15, 16].

In this paper we also construct a base of the (-1,1) Grassmann algebra and prove that the free unital (-1,1)-superalgebra with one odd generator is isomorphic to the (-1,1)-superalgebra of vector type $B(\Phi[t_0, t_1, \ldots], D, t_0)$ where $D(t_i) = t_{i+1}$, $i = 0, 1, \ldots$.

The basic concepts related with identities and varieties, can be found in [33].

1. The Basic Notions

1.1. $(-1, 1)-$algebras and Jordan algebras. Recall that an algebra A is called *right alternative* if it satisfies the identity

$$(x, y, y) = 0,$$

where $(x, y, z) = (xy)z - x(yz)$ is the associator of the elements x, y, z. A right alternative algebra A is called a *(-1,1)-algebra* if it satisfies the identity

$$(1) \qquad (x, y, z) + (y, z, x) + (z, x, y) \quad = \quad 0,$$

and it is called a *strongly (-1,1)-algebra* if it satisfies

$$(2) \qquad [[x, y], z] = 0,$$

where $[x, y] = xy - yx$ is the commutator of the elements x, y.

In any right alternative algebra hold the identities:

$$(3) \qquad (ab, x, y) + (a, b, [x, y]) \quad = \quad a(b, x, y) + (a, x, y)b,$$
$$(4) \qquad (a, x, y)x \quad = \quad (a, x, xy)$$

Moreover, any algebra satisfies the identities

$$(5) \qquad (xy, z, t) - (x, yz, t) + (x, y, zt) = x(y, z, t) + (x, y, z)t,$$
$$(6) \qquad [xy, z] - x[y, z] - [x, z]y = (x, y, z) - (x, z, y) + (z, x, y).$$

A commutative algebra is called *Jordan algebra* if it satisfies the identity

$$(x^2, y, x) = 0.$$

For an algebra A denote by $A^{(+)}$ the associated symmetrized algebra, with the same vector space and the multiplication $x \odot y = \frac{1}{2}(xy + yx)$. If A is a right alternative algebra, then the algebra $A^{(+)}$ is a special Jordan algebra (see [33]).

By $A^\sharp$ we will denote the *unital hull* of an algebra A, that is, the algebra obtained from A by adjoining the external unit element.

For an element $a \in A$ we denote by R_a and L_a the operators of right and left multiplication on a: $R_a : x \mapsto xa$, $L_a : x \mapsto ax$. We use the common notation T_a for any of R_a, L_a. We set also $R_{a,b} = R_a R_b - R_{ab}$.

1.2. Superalgebras.
Recall that a *superalgebra* $A = A_0 \oplus A_1$ is a $\mathbb{Z}_2$-graded algebra. A typical example of a superalgebra is the associative Grassmann algebra $G = G_0 \oplus G_1$, with the generators $1, e_1, e_2, \ldots$; $e_i e_j = -e_j e_i$, and the standard Z_2-grading. For a superalgebra $A = A_0 \oplus A_1$, the *Grassmann envelope* $G(A)$ is defined via $G(A) = A_0 \otimes G_0 \oplus A_1 \otimes G_1$. Let $\mathcal{V}$ be a variety of algebras. A superalgebra $B = B_0 \oplus B_1$ is called a $\mathcal{V}$-*superalgebra* if its Grassmann envelope $G(B)$ is a $\mathcal{V}$-algebra [31]. In particular, if $A \in \mathcal{V}$, then $A \otimes G = A_0 \otimes G_0 \oplus A_1 \otimes G_1$ is a $\mathcal{V}$-superalgebra. We will denote by $F_\mathcal{V}[X; Y]$ the free $\mathcal{V}$-superalgebra on the sets X and Y of even and odd generators, respectively.

It follows easily from the definition that a superalgebra $A = A_0 \oplus A_1$ is a *strongly (-1,1)-superalgebra* if and only if for any homogeneous elements x, y, z, the following identities hold:

$$(x, y, z) + (-1)^{yz}(x, z, y) = 0,$$
$$[[x, y]_s, z]_s = 0,$$

where $(-1)^{xy} = (-1)^{|x||y|}$, $|x|$ means the parity of a homogeneous element x, i.e., $|x| = i$ if $x \in A_i$; $[x, y]_s = xy - (-1)^{xy} yx$ is a *supercommutator* of homogeneous elements x, y.

1.3. Centers.
Following A.Thedy [29], consider in a right alternative algebra A the following centers:

$K(A) = \{k \in A \mid (\forall x) [k, x] = 0\}$, *the commutative center*;

$V(A) = \{v \in A \mid (\forall x) (x, x, v) = 0\}$, *the left alternative center*;

$N(A) = \{n \in A \mid (\forall x, y) (x, y, n) = (x, n, y) = (n, x, y) = 0\}$, *the associative center*;

$Z(A) = K(A) \cap N(A)$, *the full center*.

It is known and is easy to see that all the mentioned centers, except $V(A)$, are subalgebras of A and $K(A) \subseteq V(A)$.

An important role in the theory of (-1,1)-algebras is played by the commutative center $K(A)$, whose elements we will call *central*. Remind the basic properties of $K(A)$ (see [3]):

a) A is an associative bimodule over $K(A)$;
b) $K(A)$ is invariant under the operators $R_{a,b}$;
c) if $k \in K(A)$, $a, x, y, z \in A$, then

(7)
$$(k, x, y) = 2(x, k, y),$$

(8)
$$[kx, y] = k[x, y] + \tfrac{3}{2}(k, x, y),$$

(9)
$$(x, y, zk) = (x, y, z)k + (x, y, k)z,$$

(10)
$$(k, a^2, x) = 2(k, a, x)a,$$

(11)
$$(k^2, x, y) = 2(k, x, y)k.$$

Note that identities (7) - (11) verified also in a (-1,1)-superalgebra A for every $k \in K(A) \cap A_0$, $a \in A_0$, and arbitrary homogeneous elements x, y, z.

1.4. **The Grassmann algebra of a variety $\mathcal{V}$.** Consider the free superalgebra $F_{\mathcal{V}}[\emptyset; x] = F_0 \oplus F_1$ in a variety $\mathcal{V}$ on one odd generator x. Following [25, 26], we define the $\mathcal{V}$-*Grassmann algebra* $G_{\mathcal{V}}[X]$ as the subalgebra of the Grassmann envelope $G(F_{\mathcal{V}}[\emptyset; x])$ generated by the set $X = \{x \otimes e_i, \ i = 1, 2, \ldots\}$.

Lemma 1.1. [25] *Let B be a base of $F_{\mathcal{V}}[\emptyset; x]$. Then the $\mathcal{V}$-Grassmann algebra $G_{\mathcal{V}}[X]$ has a base formed by the elements $u(x) \otimes e_{i_1} \ldots e_{i_n}$, where $u(x) \in B$, $n = \deg u$, $i_1 < i_2 < \cdots i_n$.*

Note that the Grassmann algebra can be defined by generators and relations as an algebra of skew (alternative) functions. Let $F_{\mathcal{V}}[X]$ be the free algebra in the variety $\mathcal{V}$ on a set $X = \{x_i \mid i = 1, 2, \ldots\}$ of free generators. Consider the ideal I, generated by polynomials of the form: $v, w + w^{(ij)}$, where v and w are monomials such that v has degree ≥ 2 with respect to at least one variable, and w is a multilinear monomial in the variables $x_1, \ldots, x_n \in X, 1 \leq i < j \leq n$; (ij) is a transposition of i and j. Denote the quotient algebra $F_{\mathcal{V}}[X]/I$ as $Skew_{\mathcal{V}}[X]$ and call it *the algebra of skew functions of the variety $\mathcal{V}$*; here the generators x_n are identified with their images $x_n + I$ under the canonical homomorphism.

Proposition 1.2. [25] *The algebras $Skew_{\mathcal{V}}[X]$ and $G_{\mathcal{V}}[X]$ are isomorphic.*

Lemma 1.3. [24, 25] *Let f be a multilinear polynomial of degree n. If $f(x, \ldots, x) = 0$ in the free $\mathcal{V}$-superalgebra $F_{\mathcal{V}}[\emptyset; x]$, then the identity*

$$\sum_{\sigma \in S_n} (-1)^\sigma f(x_{1\sigma}, \ldots, x_{n\sigma}) = 0$$

holds in $\mathcal{V}$.

Corollary 1.4. *Let f be a multilinear polynomial of degree n. If $f(x_1, \ldots, x_n) = 0$ in $G_{\mathcal{V}}[X]$, then the identity*

$$\sum_{\sigma \in S_n} (-1)^{\sigma} f(x_{1\sigma}, \ldots, x_{n\sigma}) = 0$$

holds in $\mathcal{V}$.

2. The auxiliary superalgebras

2.1. **The superalgebras of vector type.** Let $\Gamma = \Gamma_0 \oplus \Gamma_1$ be an associative (super)commutative superalgebra, D be a nonzero even derivation of Γ, $\gamma \in \Gamma_0$. Denote by $\bar{\Gamma}$ an isomorphic copy of the vector space Γ with respect to the isomorphism $a \mapsto \bar{a}$ and set $B(\Gamma, D, \gamma) = \Gamma \oplus \bar{\Gamma}$ with the multiplication

$$a \times b = ab,$$
$$a \times \bar{b} = (-1)^b (\bar{a} \times b) = \overline{ab},$$
$$\bar{a} \times \bar{b} = (-1)^b (\gamma ab + 2D(a)b + aD(b)),$$

where $a, b \in \Gamma_0 \cup \Gamma_1$, ab is the product of the elements a and b in Γ; and with the grading

$$B(\Gamma, D, \gamma)_{\bar{0}} = \Gamma_0 + \bar{\Gamma}_1, \quad B(\Gamma, D, \gamma)_{\bar{1}} = \Gamma_1 + \bar{\Gamma}_0.$$

The superalgebra $B(\Gamma, D, \gamma)$ is called *the twisted superalgebra of vector type* [21]. It is a strongly (-1,1)-superalgebra which is simple if and only if the algebra Γ does not contain proper D-invariant ideals (i.e. is D-simple).

The adjoint symmetrized superalgebra $B(\Gamma, D, \gamma)^{(+)}$ with the supersymmetric multiplication

$$x \bullet y = \frac{1}{2}(x \times y + (-1)^{xy} y \times x)$$

is isomorphic to the Jordan algebra of vector type $J(\Gamma, \delta)$, i.e., has the following multiplication:

$$a \bullet b = ab, \qquad a \bullet \bar{b} = (-1)^b \bar{a} \bullet b = \overline{ab}, \qquad \bar{a} \bullet \bar{b} = (-1)^b (a^{\delta} b - a b^{\delta}),$$

where $a^{\delta} = \frac{1}{2} D(a)$. The superalgebras $J(\Gamma, \delta)$ were introduced by K.McCrimmon [7].

When $\Gamma = \Gamma_0$ is an algebra, we will call the superalgebras $B(\Gamma, D, \gamma)$ and $J(\Gamma, \delta)$ of *even vector type*.

2.2. The superalgebras A_{VF} and J_{VF} of vector fields on a line.

In [21], the second author introduced (-1,1)-superalgebras of vector fields on a line $A_{VF} := A(V, \Phi, \tau, \lambda)$. Recall that V is an additive commutative semigroup; $\tau : V \to \Phi$ is an additive homomorphism; $\lambda : V \to V$ is a partial map defined everywhere except possibly the neutral element 0 that satisfies the condition $\lambda(u + v) = \lambda(u) + v$ in its domain.

Consider the graded vector space $A = A_0 \oplus A_1$ with the bases $\{a_v \mid v \in V\}$ and $\{x_v \mid v \in V\}$ for A_0 and A_1, respectively. The multiplication on the superalgebra $A_{VF} = A$ is defined by the rules:

$$a_u \cdot a_v = a_{u+v}, \ a_u \cdot x_v = x_u \cdot a_v = x_{u+v}, \ x_u \cdot x_v = (4\tau(u) + 2\tau(v)) a_{\lambda(u+v)}.$$

The Jordan superalgebra J_{VF} is obtained from A_{VF} by supersymmetrization: $J_{VF} = A_{VF}^{(+)}$. Each of the superalgebras A_{VF} and J_{VF} is a superalgebra of even vector type. Also, if $V \neq 0$ and τ is injective, then the superalgebras A_{VF} and J_{VF} are prime [21, Theorem 1].

2.3. The algebra A_0.

Following [13], we denote by A_0 the strongly (-1,1)-algebra defined by the generators $z, e_1, \ldots, e_n, \ldots$ and the relations:

$$e_i e_j + e_j e_i = 0, \quad (e_i, e_j, e_p) = 0, \quad ([e_i, e_j], e_p, e_q) = 0,$$

$$((z, e_i, e_j), e_p, e_q) = 0, \qquad [z, f(z, e_i, \ldots, e_j)] = 0$$

for any polynomial $f(z, e_i, \ldots, e_j)$. It is known [13] that the algebra A_0 has an additive basis consisting of the elements:

$$(12) \qquad z^n (z, g_1, g_2) \cdots (z, g_{2m-1}, g_{2m})(h_1 \cdots h_p),$$

where n, m, p are nonnegative integers with $n + m + p > 0$, g_i, h_i are elements of the set $E = \{e_1, \ldots, e_n, \ldots\}$, which is ordered by its indices and $g_1 < \cdots < g_{2m} < h_1 < \cdots < h_p$. Observe that the elements of the form (z, e, g) and eg, where $e, g \in E$, are contained in the full center $Z(A_0)$ of the algebra A_0. Note also that the subalgebra generated by the set E is an associative Grassmann algebra (without the identity) with the standard generators $e_1, \ldots, e_n, \ldots$.

The algebra A_0 can be constructed through superalgebras. Consider the twisted superalgebra of even vector type $B = B\left(\Phi[z, s], s\frac{d}{dz}, 1\right)$. Observe that

$$\bar{1} \times \bar{1} = 1, \ (z, \bar{1}, \bar{1}) = 2s, \ (s, \bar{1}, \bar{1}) = 0.$$

Now the subalgebra of the Grassmann envelope $G(B)$ generated by the elements $z \otimes 1, \ \bar{1} \otimes e_i, \ i = 1, 2, \ldots$, is isomorphic to the algebra A_0.

2.4. **The auxiliary algebras $\overline{A}_0$ and $\overline{J}_0$.** Denote by $\overline{A}_0$ the algebra defined in the variety of strongly (-1,1)-algebras by the generators $z, e_1, \ldots, e_n, \ldots$ and the relations:

$$e_i e_j = 0, \qquad ((z, e_i, e_j), e_p, e_q) = 0, \qquad [z, f(z, e_i, \ldots, e_j)] = 0$$

for any polynomial $f(z, e_i, \ldots, e_j)$. It is known [13] that the algebra $\overline{A}_0$ has an additive basis consisting of the elements:

$$(13) \qquad z^n (z, g_1, g_2) \ldots (z, g_{2m-1}, g_{2m}) g_{2m+1},$$

where n and m are nonnegative integers, $n + m > 0$, $g_i \in E = \{e_1, \ldots, e_n, \ldots\}$ and $g_1 < \cdots < g_{2m} < g_{2m+1}$. Note that the elements of the form (z, e, g), where $e, g \in E$ are contained in the full center of the algebra $Z\left(\overline{A}_0\right)$.

Similarly to A_0, the algebra $\overline{A}_0$ can be constructed via superalgebras, using the twisted superalgebra of vector type $B\left(\Phi\left[z, s\right], s\frac{d}{dz}, 0\right)$.

By analogy with the $\overline{A}_0$, denote by $\overline{J}_0$ the algebra defined in the variety of Jordan algebras by the generators $z, e_1, \ldots, e_n, \ldots$ and the relations:

$$e_i e_j = 0, \ ((z, e_i, e_j), e_p, e_q) = 0, \ (z, f, f) = 0$$

for any $f = f(z, e_i, \ldots, e_j)$. It is clear that the algebra $\overline{J}_0$ has an additive basis consisting of the elements of the form (13). In addition, it is easy to see that

a) $(z, e, g) \in Z\left(\overline{J}_0\right)$ for any $e, g \in E$;

b) $\overline{J}_0 = \overline{A}_0^{(+)}$.

3. THE FREE (-1,1)-SUPERALGEBRA $F_{(-1,1)}[\emptyset; x]$ AND ITS CENTRAL EXTENSION $F[Z; x]$

3.1. **The free (-1,1)-superalgebra $F_{(-1,1)}[\emptyset; x]$.** Let $\Phi[T]$ be the algebra of polynomials on a countable set of variables $T = \{t_0, \ldots, t_n, \ldots\}$, and let D be the derivation of $\Phi[T]$ defined by the condition $D(t_i) = t_{i+1}$, $i = 0, 1, \ldots$. Consider the twisted superalgebra of vector type $B(\Phi[T], D, t_0)$. By [22, Theorem 4], the superalgebra $B(\Phi[T], D, t_0)$ is prime. Let $\Phi_0[T]$ denotes the subalgebra of polynomials without constant terms, then the subspace $\Phi_0[T] \oplus \overline{\Phi[T]}$ is a subsuperalgebra of $B(\Phi[T], D, t_0)$ which we will denote as $B_0(\Phi[T], D, t_0)$. It is clear that $B(\Phi[T], D, t_0) = (B_0(\Phi[T], D, t_0))^\sharp$, the unital hull. Clearly, $B_0(\Phi[T], D, t_0)$ is also prime. One can easily check that it is generated by the odd element $\overline{1}$.

Theorem 3.1. *Let Φ be a field of characteristic $\neq 2, 3$. The free $(-1,1)$-superalgebra $F_{(-1,1)}[\emptyset; x]$ over Φ on one odd generator x is isomorphic to the superalgebra $B_0(\Phi[T], D, t_0)$ with the free generator $\bar{1}$.*

Denote $F := F_{(-1,1)}[\emptyset; x]$. We prove first two lemmas.

Lemma 3.2. $x^2 \in K(F)$.

Proof. By superized associator Jacoby identity (1), we have

$$3(x, x, x) = 0, \ (x^2, x, x) - (x, x^2, x) + (x, x, x^2) = 0,$$

which implies

$$(14) \qquad\qquad (x, x, x) \ = \ [x^2, x] = 0,$$
$$(15) \qquad\qquad (x^2, x, x) \ = \ 2(x, x^2, x).$$

By associator identity (5), we have

$$
\begin{aligned}
(x^2 \cdot x, x, x) - (x^2, x^2, x) + (x^2, x, x^2) &= x^2(x, x, x) + (x^2, x, x)x, \\
(x^2, x, x^2) - (x, x^2, x^2) + (x, x, x \cdot x^2) &= x(x, x, x^2) + (x, x, x)x^2, \\
(x^2, x^2, x) - (x, x \cdot x^2, x) + (x, x, x^2 \cdot x) &= x(x, x^2, x) + (x, x, x^2)x,
\end{aligned}
$$

which implies

$$
\begin{aligned}
(x^3, x, x) &= 2(x^2, x^2, x) + (x^2, x, x)x, \\
(x, x, x^3) &= (x^2, x^2, x) + x(x, x, x^2), \\
(x^2, x^2, x) &= [x, (x, x^2, x)].
\end{aligned}
$$

Now, the superized associator Jacoby identity (1) implies

$$(x^3, x, x) + 2(x, x, x^3) = 0,$$

hence by (15) we have $4(x^2, x^2, x) + 2[(x, x^2, x), x] = 0$ and eventually

$$(16) \qquad\qquad (x^2, x^2, x) = 0.$$

In particular, $[x^3, x^2] = (x^2, x, x^2) = 0$.

Let now $u(x)$ be a monomial on x of degree $n > 3$, consider $[u(x), x^2]$. We may assume that $[v(x), x^2] = 0$ for every monomial $v(x)$ of degree less then n. Recall that a (-1,1)-algebra A satisfies the identity $[[a, b], D(A)] = 0$ where $D(A)$ is the associator ideal of A (see [20]). Therefore, in F we have $[2x^2, D(F)] = [[x, x]_s, D(F)] = 0$, and hence we may write

$$[u(x), x^2] = [x^2(x^2 v(x)), x^2],$$

where $\deg v(x) \geq 0$. Denote $a = x^2$, $v = v(x)$. By (6) and the induction assumption we have

$$
\begin{aligned}
[u(x), x^2] &= [a(av), a] = a[av, a] + [a, a](av) \\
&\quad + (a, av, a) - (a, a, av) + (a, a, av) \\
&= -(a, a, av).
\end{aligned}
$$

Since $a = x^2$ is even, by (4) we have $(a, a, av) = (a, a, v)a$. The above arguments show that $(a, a, v) = -[av, a] = 0$, proving the lemma.   □

Lemma 3.3. $[[F, F]_s, F]_s = 0$, *that is, F is a strongly (-1,1)-superalgebra.*

Proof. Evidently, it suffices to prove that $u(x) \in K(F)$ for every monomial u of even degree. We will use induction on $\deg u$. A base of the induction is given by Lemma 3.2.

Since the variety of (-1,1)-algebras is a 2-variety (the square of an ideal is an ideal), we may assume that u has form vx or xv, where v is a monomial of odd degree (see, for instance, [34]). Similarly, we may assume that $v = wx$ or $v = xw$ for some even $w \in F$. The inclusion $x^2 \in K(F)$ implies, due to [33, theorem 13.10], that

$$(17) \qquad yz + zy \in K(F) \text{ for any odd } y, z \in F.$$

In particular, $vx + xv \in K(F)$, and it suffices to prove that $(wx)x$, $x(xw) \in K(F)$. By induction, $w \in K(F)$, hence $wx^2, x^2w \in K(F)$, and we have by (8)

$$
\begin{aligned}
(wx)x &= wx^2 + (w, x, x) \equiv (w, x, x) = \tfrac{2}{3}([wx, x]_s - w[x, x]_s) \\
&= \tfrac{2}{3}((wx)x + x(wx) - 2wx^2) \overset{(17)}{\equiv} 0 \pmod{K(F)}
\end{aligned}
$$

Similarly, by (7),

$$x(xw) = x^2w - (x, x, w) \equiv -(x, x, w) \equiv \tfrac{1}{2}(w, x, x) \equiv 0 \pmod{K(F)}.$$

□

Proof of the theorem. Denote $A = F_0$, then $F_1 = A^\sharp x$, $F = A \oplus A^\sharp x$, where by 1.4.a) A is an associative and commutative Φ-algebra and $A^\sharp x$ is a commutative and associative A-module generated by x. Furthermore, by (11), the application $D : a \mapsto \tfrac{1}{2}(a, x, x)$ is a derivation of A. We claim that A coincides with the Φ-subalgebra A_0 generated by the set $\{x^2, D^i(x^2), \ i = 1, 2, \ldots\}$. It is equivalent to say that $F = A_0 + A_0^\sharp x$, and since $x \in A_0 + A_0^\sharp x$, it suffices to prove that $A_0 + A_0^\sharp x$ is a subsuperalgebra of F. Since $A_0 \subseteq K(F)$, we have $a \cdot bx = bx \cdot a = (ab)x$ for any $a, b \in A$.

Now, let us calculate the product of elements from $A^\sharp x$:

$$
\begin{aligned}
ax \cdot bx &= (a, x, bx) + a(x \cdot bx) = (a, bx, x) + a(x \cdot xb) \\
&= (ab \cdot x)x - a(bx \cdot x) + a \cdot x^2 b - a(x, x, b) \\
&= (ab, x, x) + (ab)x^2 - a(b, x, x) - a(bx^2) + (ab)x^2 - a(x, x, b) \\
&= (\text{by } (7)) = 2D(ab) - 2aD(b) + aD(b) + (ab)x^2 \\
&= 2D(a)b + aD(b) + (ab)x^2.
\end{aligned}
$$

Therefore, $A_0 + A_0^\sharp x$ is a subsuperalgebra of F and $F = A_0 + A_0^\sharp x$, $A_0 = A = F_0$. Moreover, the obtained relation shows that the superalgebra F is a homomorphic image of the superalgebra $B_0(A^\sharp, D, x^2)$ under the homomorphism $\pi : a + \bar{b} \mapsto a + xb$. Consider the homomorphism $\varphi : \Phi_0[T] \to A$, $t_i \mapsto D^i(x^2)$. Clearly, it is a homomorphism of differential algebras which can be extended to a homomorphism of superalgebras

$$
\tilde{\varphi} : B_0(\Phi[T], D, t_0) \to B_0(A^\sharp, D, x^2).
$$

Now the composition $\pi \circ \tilde{\varphi}$ maps surjectively $B_0(\Phi[T], D, t_0)$ onto F. Since F is a free superalgebra generated by x, this map is an isomorphism. $\qquad\square$

Corollary 3.4. $(F_{(-1,1)}[\emptyset; x])^\sharp \cong B(\Phi[T], D, t_0)$.

Corollary 3.5. *The variety* $Var\left(F_{(-1,1)}[\emptyset; x]\right)$ *can not be generated by a finite dimensional superalgebra.*

Proof. Suppose by contradiction that $Var\left(F_{(-1,1)}^s[\emptyset; x]\right) = Var(B)$ where $\dim B = m < \infty$. Consider the Grassmann envelope $G(B)$. Let $x_1, \ldots, x_k, \ldots, x_m$ be a base of B with $x_1, \ldots, x_k$ be a base of B_0, then any element $u \in [G(B), G(B)]$ can be written in the form $u = \sum_{i,j=1}^{k} a_{ij} \otimes [x_i, x_j] + \sum_{i,j=1}^{m} g_i g_j \otimes x_i x_j$, where $a_{ij} \in G_0$, $g_i \in G_1$ or $g_j \in G_1$. Clearly, $(g_i g_j)^2 = 0$ for every pair i, j. Besides, by the identity of Kleinfeld $[x, y]^3 = 0$ valid in every strongly (-1,1)-algebra [6], we have $[x_i, x_j]^3 = 0$ for $i, j \leq k$. Since $[G(B), G(B)] \subseteq Z(G(B))$, we see that the element u is nilpotent of degree $m^3 + 1$. In other words, the subspace $[G(B), G(B)] \subseteq Z(G(B))$ satisfies the identity $x^{m^3+1} = 0$. Linearizing this identity, in view of associativity and commutativity of $Z(G(B))$ we get that $[G(B), G(B)]$ satisfies the identity $(m^3 + 1)! \, x_1 x_2 \ldots x_{m^3+1} = 0$. Since $char\,\Phi = 0$, this implies that $G(B)$ satisfies the identity

$$
[x_1, y_1][x_2, y_2] \cdots [x_{m^3+1}, y_{m^3+1}] = 0.
$$

Therefore, the superalgebra B satisfies the identity

$$
[x_1, y_1]_s [x_2, y_2]_s \cdots [x_{m^3+1}, y_{m^3+1}]_s = 0.
$$

Since $F_{(-1,1)}[\emptyset; x] \in Var\, B$, it should satisfies this identity as well, which is impossible since $[x, x]_s^k = (2x^2)^k \neq 0$ for any k. $\qquad\square$

3.2. **The central extension** $F[Z; x]$. Let $F_{St}[Z; x]$ be the free strongly (-1,1)-superalgebra on a set Z of even generators and on an odd generator x. Denote by $F[Z; x]$ the quotient superalgebra $F_{St}[Z; x]/I$ where I is the ideal generated by the set $\{[z, f],\ z \in Z,\ f \in F_{St}[Z; x]\}$. In other words, $F[Z; x]$ is a strongly (-1,1)-superalgebra freely generated by a set of even central variables Z and an odd variable x. Clearly, $F[Z; x]$ satisfies the following universal property:
for any strongly (-1,1)-superalgebra B, any odd $b \in B_1$, and for any mapping $\varphi : Z \rightarrow K(B)$ there exists a unique homomorphism $\tilde{\varphi} : F[Z; x] \rightarrow B$ such that $\tilde{\varphi}|_Z = \varphi$, $\tilde{\varphi}(x) = b$.

Consider the polynomial ring $\Phi[T_Z]$, where $T_Z = T_0 \cup (\cup_{z \in Z} T_z)$, $T_0 = \{t_0, t_1, \ldots\}, T_z = \{z = z_0, z_1, z_2, \ldots\}$, $z \in Z$, and let D be the derivation of $\Phi[T_Z]$ defined by $D(t_i) = t_{i+1}$, $D(z_i) = z_{i+1}$.

Proposition 3.6. *Let Φ be a field of characteristic $\neq 2, 3$. The super-algebra $F[Z; x]$ is isomorphic to the superalgebra $B_0(\Phi[T_Z], D, t_0)$.*

Proof. Denote $F = F[Z; x]$, $K = K(F)$, and let A be a Φ-subalgebra of F_0 generated by the set $\{D^i(x^2), D^j(z) \,|\, z \in Z; i, j = 0, 1, \ldots\}$, where $D(a) = \frac{1}{2}(a, x, x)$. Let us prove that $F = A + A^\sharp x$. Observe first that since F is strongly (-1,1), we have $0 = [F, [x, x]_s] = 2[F, x^2]$, hence $x^2 \in K$. Furthermore, by (11) D is a derivation of K, hence $A \subseteq K$. Since $Z \cup \{x\} \subseteq A + A^\sharp x$, it suffices to prove that $A + A^\sharp x$ is a subsuperalgebra of F.

As above, we have for any $a, b \in A^\sharp$

$$a \cdot bx = ab \cdot x, \ ax \cdot bx = 2D(a)b + aD(b) + (ab)x^2.$$

This proves that $F = A + A^\sharp x$. Moreover, this also shows that the superalgebra F is a homomorphic image of the superalgebra $B_0(A^\sharp, D, x^2)$ under the homomorphism $\pi : a + \bar{b} \mapsto a + xb$. Consider the homomorphism $\varphi : \Phi_0[T_Z] \rightarrow A$, $t_i \mapsto D^i(x^2)$, $z_i \mapsto D^i(z)$, $z \in Z$. Clearly, it is a homomorphism of differential algebras which can be extended to a homomorphism of superalgebras

$$\tilde{\varphi} : B_0(\Phi[T_Z], D, t_0) \rightarrow B_0(A^\sharp, D, x^2).$$

Now the composition $\pi \circ \tilde{\varphi}$ maps surjectively $B_0(\Phi[T_Z], D, t_0)$ onto F. By the universal property of F, this map is invertible and hence it is an isomorphism. $\qquad\square$

Corollary 3.7. *The superalgebra $F[Z; x]$ has a base of the form $B \cup Bx$, where B is the set of commutative and associative monomials on the variables*

$$x^2 R_{x,x}^i, \ zR_{x,x}^j, \ z \in Z; \ i, j = 0, 1, \ldots.$$

The proof follows easily from the isomorphism given in the theorem.
$\square$

Consider the superalgebra $F[z; x] \cong B_0(\Phi[T_0 \cup T_z], D, t_0)$. We will be interested in the following homomorphic images of $F[z; x]$:

- $F_t = F[z; x]/((x^2, x, x), ((z, x, x), x, x) \cong B_0(\Phi[t, z, s], s\frac{d}{dz}, t)$,
- $F_1 = F[z; x]/(x^2 - 1, ((z, x, x), x, x)) \cong B(\Phi[z, s], s\frac{d}{dz}, 1)$,
- $F_0 = F[z; x]/(x^2, ((z, x, x), x, x)) \cong B_0(\Phi[z, s], s\frac{d}{dz}, 0)$.

Let $A = A_1 \oplus A_1$ be a (-1,1)-superalgebra, $a \in A_0$, $x \in A_1$. Let, furthermore, $\mathbf{Z}_+$ be the set of non-negative integers and $\mathbf{Z}_+^\infty = \cup_n (\mathbf{Z}_+)^n$ be the set of all ordered finite sequences of elements of $\mathbf{Z}_+$. Denote, for $I = (i_0, i_1, \ldots, i_k) \in \mathbf{Z}_+^\infty$,

$$a^I = a^{i_0}(aR_{x,x})^{i_1}(aR_{x,x}^2)^{i_2} \cdots (aR_{x,x}^k)^{i_k}.$$

Set also $|I| = i_0 + \cdots + i_k$ and $d(I) \leq k+1$ to be the number of nonzero elements in I.

Corollary 3.8. *The superalgebras $F[z; x]$, F_t, F_1, F_0 have following bases in terms of generators z, x:*

$$F[z; x] \ : \ (x^2)^I z^J x^\varepsilon, \ I, J \in \mathbf{Z}_+^\infty, \ \varepsilon \in \{0, 1\}, \ \varepsilon + |I| + |J| > 0;$$
$$F_t \ : \ (x^2)^m z^n (zR_{x,x})^k x^\varepsilon, \ m, n, k \geq 0, \ \varepsilon \in \{0, 1\}, \ \varepsilon + m + n + k > 0;$$
$$F_1 \ : \ z^n (zR_{x,x})^k x^\varepsilon, \ n, k \geq 0, \ \varepsilon \in \{0, 1\};$$
$$F_0 \ : \ z^n (zR_{x,x})^k x^\varepsilon, \ n, k \geq 0, \ \varepsilon \in \{0, 1\}, \ \varepsilon + n + k > 0.$$

Corollary 3.9. *The Grassmann (-1,1)-algebra $G_{(-1,1)}$ has a following base*

$$(x^2)^I x^\varepsilon \otimes e_{j_1} e_{j_2} \cdots e_{j_n}, \ j_1 < j_2 < \cdots < j_n, \ I \in \mathbf{Z}_+^\infty, \ n = \deg((x^2)^I x^\varepsilon).$$

The proof follows from the previous Corollary and Lemma 1.1. $\square$

4. Small varieties of superalgebras

We call a variety $\mathcal{V}$ of strongly (-1,1)-superalgebras to be *small* if it does not contain the superalgebra F_0.

The following result gives a criterion for a unitary closed variety $\mathcal{V}$ to be small.

Proposition 4.1. *A variety $\mathcal{V}$ of strongly (-1,1)-superalgebras over a field Φ of characteristic 0 is small if and only if it satisfies the identity*

(18) $$(z, x, x)^k = 0$$

for any central z and odd x and some $k > 0$.

Proof. Note that for any n the element $(z, x, x)^n$ belongs to the base of F_0 given in Corollary 3.8, hence it is non-zero. Therefore, the condition above is sufficient for a variety $\mathcal{V}$ to be small.

Assume now that $F_0 \notin \mathcal{V}$. Consider the free $\mathcal{V}$-superalgebra $F_\mathcal{V}[a; x]$ on an even generator a and odd generator x. Then the quotient superalgebra $S = F_\mathcal{V}[a; x]/(x^2, ((a, x, x), x, x), [a, F_\mathcal{V}[a; x]])$ is a proper homomorphic image of F_0. Therefore, the images of basic elements of F_0 are linearly dependent in S. In other words, we have in $F_\mathcal{V}[a; x]$ a relation of the form

$$\sum_{n,k} \alpha_{n,k} a^n (a, x, x)^k x^\varepsilon = x^2 U + ((a, x, x), x, x)V + \sum_i [a, f_i]W_i$$

for some $f_i = f_i(a, x) \in F_\mathcal{V}[a; x]$ and $U = U(a, x), V = V(a.x), W_i = W_i(a, x)$ from the multiplication algebra of $F_\mathcal{V}[a; x]$. Since Φ is infinite, we may assume that the relation above is homogeneous in a, x, that is, has a form

$$a^n (a, x, x)^k x^\varepsilon = x^2 U + ((a, x, x), x, x)V + \sum_i [a, f_i]W_i.$$

The application $D = R_{x,x}$ in view of (3) is a derivation of the superalgebra S such that $D(x) = D((a, x, x)) = 0$, hence we have in S

$$D^n(a^n (a, x, x)^k x^\varepsilon) = (a, x, x)^{k+n} x^\varepsilon.$$

Returning to $F_\mathcal{V}[a; x]$, we will get a relation of the form

$$(a, x, x)^k x^\varepsilon = x^2 U + ((a, x, x), x, x)V + \sum_i [a, f_i]W_i.$$

If $\varepsilon = 1$, multiplying the relation above by a and x we get in the superalgebra S

$$\begin{aligned}
(((a, x, x)^k x)a)x &= ((a, x, x)^k a)x \cdot x = ((a, x, x)^k a, x, x) \\
&= ((a, x, x)^k, x, x)a + (a, x, x)^k(a, x, x) \\
&= k(a, x, x)^{k-1}((a, x, x), x, x)a + (a, x, x)^{k+1} \\
&= (a, x, x)^{k+1}.
\end{aligned}$$

Therefore, without loss of generality, we may assume that we have a homogeneous relation in $F_{\mathcal{V}}[a; x]$ of the form

$$(a, x, x)^k = x^2 U + ((a, x, x), x, x)V + \sum_i [a, f_i]W_i.$$

Denote $F_{\mathcal{V}}[z; x] = F_{\mathcal{V}}[a; x]/([a, F_{\mathcal{V}}[a; x]])$, then we have in $F_{\mathcal{V}}[z; x]$:

$$(z, x, x)^k = x^2 U(z, x) + ((z, x, x), x, x)V(z, x).$$

Consider in the free central extension $F[s, z; x]$ the element $u = sU(z, x)$. By Corollary 3.7, u may be written as a linear combination of monomials on the variables $x^2 R^i_{x,x}$, $s R^j_{x,x}$, $z R^l_{x,x}$. By homogeneity, since $\deg_z(u) = k$, every monomial contains k variables of type $z R^l_{x,x}$. Since $\deg_x u = 2(k-1)$, at least one of these variables should be just z, henceforth $u = z u_1$. Since x^2 is a central element in $F_{\mathcal{V}}[z; x]$, we have a homomorphism $\varphi : F[s, z; x] \to F_{\mathcal{V}}[z; x]$, $x \mapsto x$, $z \mapsto z$, $s \mapsto x^2$. Then $x^2 U(z, x) = \varphi(u) = z\varphi(u_1)$.

Similarly, the element $v = sV(z, x) \in F[s, z; x]$ may be written as $v = z v_1$ and therefore $((z, x, x), x, x)V(z, x) = z\varphi(v_1)$ where $\varphi : F[s, z; x] \to F_{\mathcal{V}}[z; x]$, $x \mapsto x$, $z \mapsto z$, $s \mapsto ((z, x, x)x, x)$.

Resuming, we have in $F_{\mathcal{V}}[z; x]$ a relation

$$(z, x, x)^k = zw(z, x).$$

Linearizing this relation on z, we get

$$(z_1, x, x) \cdots (z_k, x, x) = \sum_{i=1}^k z_i w_i.$$

Substitute z_1^2 instead of z_1:

$$(z_1^2, x, x) \prod_{i \geq 2}(z_i, x, x) = z_1^2 w_1 + \sum_{i \geq 2} z_i w_i'.$$

But $(z_1^2, x, x) = 2z_1(z_1, x, x)$, hence

$$(z_1^2, x, x) \prod_{i \geq 2}(z_i, x, x) = 2z_1(z_1, x, x) \prod_{i \geq 2}(z_i, x, x) = 2z_1^2 w_1 + \sum_{i \geq 2} z_i w_i''.$$

Comparing the two expressions, we get $z_1^2 w_1 = \sum_{i \geq 2} z_i(w_i' - w_i'')$, which implies

$$(z_1^2, x, x) \prod_{i \geq 2}(z_i, x, x) = \sum_{i \geq 2} z_i w_i^{(1)}.$$

Repeating the same arguments, we get after $k - 1$ steps

$$\left(\prod_{i=1}^{k-1}(z_i^2, x, x)\right)(z_k, x, x) = z_k w_k^{(k-1)},$$

and eventually

$$\prod_{i=1}^{k}(z_i^2, x, x) = 0.$$

Due to the inclusion $D(A)^3 \subseteq D(A^2)A^\sharp$ which holds for any algebra A with a derivation D, we have for $D = R_{x,x}$

$$(z, x, x)^3 = \sum_i (z_i^2, x, x)z_i' \quad \text{for some } z_i, z_i' \in K(F_{\mathcal{V}}[z; x]),$$

and finally $(z, x, x)^{3k} = 0.$ □

5. THE SUPERALGEBRAS F_0, F_1, $F_{(-1,1)}[\emptyset; x]$ HAVE THE SAME IDENTITIES

We will start with the following auxiliary result.

Lemma 5.1. *Let A be a superalgebra, $Z = Z(A)$ be the (full) cener of A. Assume that for any $a \in A$ there exists $z \in Z$ such that $az \neq 0$. Then the variety $Var A$ is unitary closed.*

Proof. Recall that a variety V is unitary closed if for any element f from the T-ideal $T(V)$ of identities of V all the partial derivatives $f\Delta_i$ belong to $T(V)$ as well (see [33]). Since $char \Phi = 0$, we may assume that f is multilinear. Assume that $Var A$ is not unitary closed, then there exists a multilinear $f = f(x_1, \ldots, x_n) \in T(A)$ such that $f\Delta_i \notin T(A)$ for some i. Let, for example, $f\Delta_n \notin T(A)$, then there exist $a_1, \ldots, a_{n-1} \in A$ such that $(f\Delta_n)(a_1, \ldots, a_{n-1}) = b \neq 0$. Choose $z \in Z$ with $bz \neq 0$, then we have

$$f(a_1, \ldots, a_{n-1}, z) = (f\Delta_n)(a_1, \ldots, a_{n-1})z = bz \neq 0.$$

The contradiction proves the Lemma. □

Lemma 5.2. *Let $F_0[Z; x] = F[Z; x]/(x^2)$. Then the superalgebras $F_0[Z; x]$ and F_0 generate the same variety.*

Proof. Since F_0 is a homomorphic image of $F_0[Z; x]$, it suffices to prove that $F_0[Z; x]$ belongs to the variety $\mathcal{V}$ generated by F_0. Consider first the case when Z is a singleton: $Z = \{z\}$. Assume that $F_0[z; x] \notin \mathcal{V}$, then some nontrivial linear combination $f = f(z, x)$ of basic elements of $F_0[z; x]$ vanishes in F_0 for any substitution $z = a \in K(F_0)$, $x = y \in (F_0)_1$. More exactly, we may assume that $f = \sum_I \alpha_I z^I x$, where $I = (i_0, \ldots, i_k) \in \mathbf{Z}_+^\infty$, $\alpha_I \in \Phi$. Since $char \Phi = 0$, we may assume that f is homogeneous, that is, the numbers $|I|$ and $i_1 + \cdots + ki_k = i$ are fixed. Let m be a maximal value of the index i_0 for I in f. The algebra F_0 contains in its center the polynomial ring $\Phi[s]$ which acts

without torsion on F_0. By Lemma 5.1, the variety $Var\, F_0$ is unitary closed. Therefore, F_0 satisfies the partial linearization $f\Delta_z^m$ which has the form $\sum_J a_J z^J x$, where all J are of the form $J = (0, j_1, \ldots, j_t)$. Without loss of generality we may assume that all I in f have $i_0 = 0$.

Choose a lexicographically minimal m-tuple $I_0 = (0, i_1, \ldots, i_m)$ in f, and consider a partial linearization $f\Delta$ of f for $\Delta = \Delta_z^{i_1}(z) \cdots \Delta_z^{i_m}(z^m)$, then $f\Delta$ vanishes in F_0 as well (see [33]). Observe that in F_0 we have

$$(z^n)R_{x,x}^m = \begin{cases} 0, & m > n, \\ \frac{n!}{(n-m)!} z^{n-m}(z,x,x)^m, & m \le n. \end{cases}$$

Therefore, calculating $f\Delta$ in F_0, we will have $(z^I x)\Delta = 0$ for $I \ne I_0$ and $(z^{I_0}x)\Delta = (1!)^{i_1} \cdots (m!)^{i_m}(z, x, x)^i x$. Thus

$$0 = f\Delta = \alpha_{I_0}(1!)^{i_1} \cdots (m!)^{i_m}(z, x, x)^i x,$$

implying $\alpha_{I_0} = 0$ and $f = 0$.

Let now $f = f(z_1, \ldots, z_k, x) \in F_0[Z; x]$ be a nontrivial linear combination of basic elements of $F_0[Z; x]$ which vanishes in F_0 for any substitution $z_i = a_i \in K(F_0)$, $x = y \in (F_0)_1$. We prove that $f = 0$ by induction on k. Write f in the form

$$f = \sum_{I \in \mathbf{Z}_+^\infty} f_I z_1^I,$$

where $f_I = f_I(z_2, \ldots, z_k, x) \ne 0$. Arguing as above, we may choose I_0 and the operator Δ of partial linearization in z_1 such that $(z_1^I)\Delta = 0$ for $I \ne I_0$ and $(z_1^{I_0})\Delta = n(z_1, x, x)^i$ for some natural numbers i, n. Then

$$f\Delta = f_{I_0} n(z_1, x, x)^i$$

is an identity in F_0, which implies easily that f_{I_0} is an identity in F_0. Then by induction $f_{I_0} = 0$, a contradiction. $\square$

Lemma 5.3. $Var\,(F_0) = Var\,(F_{(-1,1)}[\emptyset; x])$.

Proof. Observe first that the superalgebra $F_{(-1,1)}[\emptyset; x]$ does not satisfy identity (18). In fact, it has a base formed by the elements $(x^2)^I x^\varepsilon$, $I \in \mathbf{Z}_+^\infty$, $\varepsilon \in \{0, 1\}$. In particular, $(x^2, x, x)^k \ne 0$ in $F_{(-1,1)}[\emptyset; x]$ for any k. Therefore, by Proposition 4.1, the variety $Var\,(F_{(-1,1)}[\emptyset; x])$ is not small, that is, contains the superalgebra F_0.

To prove the converse inclusion, it suffices to show that no nontrivial linear combination of basic elements of the superalgebra $F_{(-1,1)}[\emptyset; x]$ is an identity in F_0. Assume it is not the case, that is, F_0 satisfies the

identity

$$(19) \qquad \sum_{I \in \mathbf{Z}_+^\infty} \alpha_I (x^2)^I x^\varepsilon = 0, \ \alpha_I \in \Phi.$$

By Lemma 5.2, the superalgebra $F_0[z; x]$ satisfies identity (19) as well for any odd x. Let $q = \min\{d(I)\}$ for I in (19), then we have in $F_0[z; x]$ the identity $G\Delta_x^q(zx) = 0$, where G stands for the right part of (19) and $\Delta_x^i(y)$ is the operator of partial linearization in x of degree i (see [33, 1.4]). It is clear that $((x^2)^I x^\varepsilon)\Delta_x^q(zx) = 0$ if $d(I) > q$ (since $x^2 = 0$ in $F_0[z; x]$), hence we have in $F_0[z; x]$ the equality

$$(20) \qquad \sum_{d(I)=q} \alpha_I ((x^2)^I x^\varepsilon)\Delta_x^q(zx) = 0.$$

Furthermore, for every I with non-zero components $i_{k_1}, \ldots, i_{k_q}$ we have

$$\begin{aligned} ((x^2)^I x^\varepsilon)\Delta_x^q(zx) &= ((x^2)\Delta_x^1(zx))^I x^\varepsilon = (x \cdot zx + zx \cdot x)^I x^\varepsilon \\ &= (\tfrac{3}{2}(z,x,x))^I x^\varepsilon = (\tfrac{3}{2})^q (zR_{x,x}^{k_1+1})^{i_{k_1}} \ldots, (zR_{x,x}^{k_q+1})^{i_{k_q}} x^\varepsilon. \end{aligned}$$

Evidently, the obtained elements are linearly independent in $F_0[z; x]$ for different I, hence $\alpha_I = 0$ for all I with $d(I) = q$, and identity (19) is trivial. $\qquad \square$

Lemma 5.4. $F[Z; x] \in Var\,(F_0)$.

Proof. Again, it suffices to prove that no nontrivial linear combination f of basic elements of $F[Z; x]$ is an identity in F_0. Write f in the form

$$f = \sum_{I \in \mathbf{Z}_+^\infty} (x^2)^I f_I$$

where

$$f_I = \sum_{I_1, \ldots, I_k} a_{I_1, \ldots, I_k} z_1^{I_1} \cdots z_k^{I_k} x^\varepsilon \neq 0,$$

$z_i \in Z$, $I_s \in \mathbf{Z}_+^\infty$, $a_{I_1, \ldots, I_k} \in \Phi$.

Let $z \in Z \setminus \{z_1, \ldots, z_k\}$ and $q = \min\{d(I)\}$ in the expression for f. Arguing as in the proof of Lemma 5.3, we have in F_0 the identity

$$0 = f\Delta_x^q(zx) = (\tfrac{3}{2})^q \sum_{I, I_1, \ldots, I_k, \, d(I)=q} a_{I_1, \ldots, I_k}(z, x, x)^I z_1^{I_1} \cdots z_k^{I_k} x^\varepsilon.$$

By Lemma 5.2, this implies that all $a_{I_1, \ldots, I_k} = 0$ and $f_I = 0$, a contradiction. $\qquad \square$

Lemma 5.5. $Var\, F_0 = Var\, F_1$.

Proof. Since the superalgebra F_1 is a homomorphic image of $F_t = F[z; x]$, by Lemma 5.4 we have $F_1 \in Var\, F_0$. On the other hand, F_1 evidently does not satisfy identity (18), hence by Proposition 4.1 it is not small and $F_0 \in Var\, F_1$. $\square$

Corollary 5.6. $Var\, A_0 = Var\, \overline{A}_0 = Var\, G(F_0)$.

Proof. For a variety $\mathcal{V}$ of algebras denote by $\widetilde{\mathcal{V}}$ the variety of $\mathcal{V}$-superalgebras. Clearly, it suffices to prove that $\widetilde{Var\, \overline{A}_0} = \widetilde{Var\, A_0} = Var\, F_0$.

Recall that $\overline{A}_0$ is a homomorphic image of A_0, and $A_0 \subseteq G(F_1)$. Therefore,

$$\widetilde{Var\, \overline{A}_0} \subseteq \widetilde{Var\, A_0} \subseteq Var\, F_1 = Var\, F_0,$$

and it suffices to prove that the variety $\widetilde{Var\, \overline{A}_0}$ is not small. Consider the superalgebra $G \otimes \overline{A}_0 = G_0 \otimes \overline{A}_0 + G_1 \otimes \overline{A}_0$ which evidently belongs to $\widetilde{Var\, \overline{A}_0}$. Take in $G \otimes \overline{A}_0$ the elements $z = 1 \otimes z$, $x_k = g_1 \otimes e_1 + \cdots + g_{2k} \otimes e_{2k}$, where by g_i we denote the canonical generators of the Grassmann algebra G, in order not confuse them with the generators e_i of $\overline{A}_0$. We have $(z, x_k, x_k) = \sum_{i \neq j} g_i g_j \otimes (z, e_i, e_j)$ and furthermore, $(z, x_k, x_k)^k = (2k)! g_1 g_2 \cdots g_{2k-1} g_{2k} \otimes (z, e_1, e_2) \cdots (z, e_{2k-1}, e_{2k}) \neq 0$. Therefore, the superalgebra $G \otimes \overline{A}_0$ does not satisfy identity (18) for any k and hence the variety $\widetilde{Var\, \overline{A}_0}$ is not small. $\square$

6. Proof of Theorems A and B

6.1. **Proof of Theorem A.** We will present the proof in three steps.

Lemma 6.1. $G(B) \in Var\,(\overline{A}_0)$, *where* $B = B(\Gamma, D, \gamma)$, $\Gamma = \Gamma_0$.

Proof. The inclusion in lemma is equivalent to the inclusion $B \in \widetilde{Var\, \overline{A}_0} = Var\,(F_0) = Var\,(F[Z; x])$. Let Z be such a set that Γ is isomorphic to a homomorphic image of the polynomial ring without constant terms $\Phi_0[Z]$ under a homomorphism φ. Consider the polynomial ring $\Phi_0[T_Z]$, where $T_Z = T_0 \cup (\cup_{z \in Z} T_z)$, $T_0 = \{t_0, t_1, \ldots\}$, $T_z = \{z = z_0, z_1, z_2, \ldots\}$, $z \in Z$, and let D be the derivation of $\Phi[T_Z]$ defined by $D(t_i) = t_{i+1}$, $D(z_i) = z_{i+1}$. By Proposition 3.6, $F[Z; x] \cong B(\Phi_0[T_Z], D, t_0)$. Extend the homomorphism φ to a homomorphism $\tilde{\varphi} : \Phi_0[T_Z] \to \Gamma$ by setting $\tilde{\varphi}(t_i) = D^i(\gamma)$, $\tilde{\varphi}(z_i) = D^i(\varphi(z))$, $z \in Z$; then evidently $\tilde{\varphi}$ is a homomorphism of differential algebras $\tilde{\varphi} : (\Phi[T_Z], D) \to (\Gamma, D)$. Clearly, $\tilde{\varphi}$ induces a surjective homomorphism of superalgebras $\tilde{\tilde{\varphi}} : B(\Phi_0[T_Z], D, t_0) \to B(\Gamma, D, \gamma)$. Therefore, $B(\Gamma, D, \gamma) \in Var\,(F[Z; x])$. $\square$

Lemma 6.2. $Var\,(G(B)) = Var\,(\overline{A}_0)$, *where* B *is a simple non-associative (-1,1)-superalgebra.*

Proof. In view of [23], a simple (-1,1)-superalgebra $B = A+M$, $(A = B_0,\ M = B_1)$ over a field of characteristic 0 has the following properties:

a) A is a unital differentially simple associative and commutative algebra without zero divisors and M is an associative commutative A-bimodule;

b) If $0 \neq x \in M$, then $R_{x,x}$ is a nonzero derivation of A.

It follows from a) and b) that B does not satisfy identity (18) for any k, hence $F_0 \in Var\,B$ and $\overline{A}_0 \subseteq G(F_0) \in Var\,(G(B))$.

On the other hand, the arguments from [32] given for simple Jordan superalgebras with associative even part are applied to simple (-1.1)-superalgebras as well and show that the superalgebra $B = A+M$ can be embedded into a twisted superalgebra $B(\Gamma, D, \gamma)$ of vector type, where Γ is the field of fractions of A. Therefore, by Lemma 6.1, $Var\,(G(B)) \subseteq Var\,(G(B(\Gamma, D, \gamma))) \subseteq Var\,\overline{A}_0$. $\qquad\square$

Lemma 6.3. *Let* $A_{VF} = A(V, \Phi, \tau, \lambda)$ *be a (-1,1)-superalgebra of vector fields over a field* Φ *of characteristic* 0. *Suppose that there exists* $u \in V$ *with* $\tau(u) \neq 0$. *Then* $Var\,(G(A_{VF})) = Var\,(\overline{A}_0)$.

Proof. By [21, Theorem 1], the superalgebra A_{VF} is of even vector type, hence $G(A_{VF}) \in Var\,(\overline{A}_0)$ by Lemma 6.1. On the other hand, we have in A_{VF} the equality $(a_u, x_v, x_v) = 4\tau(u)a_{\lambda(u+2v)}$ which implies that $(a_u, x_v, x_v)^k = 4^k(\tau(u))^k a_{k\lambda(u+2v)} \neq 0$ for any k. Therefore, by Proposition 4.1 $F_0 \in Var\,(A_{VF})$ and $\overline{A}_0 \in Var\,(G(F_0)) \subseteq Var\,(G(A_{VF}))$. $\square$

Theorem A now follows from Lemmas 6.1 - 6.3, Lemma 5.3, and Corollary 5.6.

It follows from Lemma 6.3 that the prime (-1,1)-monsters of characteristic 0, constructed in [13] and [21] are isomorphic.

Corollary 6.4. *The free algebra of countable rank in the variety generated by the Grassmann envelope of the (-1,1)-superalgebra of vector type* $B(\Phi[t], \frac{d}{dt}, 0)$ *over a field* Φ *of characteristic* 0 *is prime.*

6.2. **Proof of Theorem B.** The proof of Theorem B we also give in three steps.

Lemma 6.5. *The variety* $\mathcal{V}_0 := Var\,(\overline{A}_0)$ *over a field* Φ *of characteristic 0 is the smallest variety of (-1,1)-algebras that contains a prime non-associative algebra.*

Proof. Let $\mathcal{V}$ be a variety containing a prime nonassociative algebra A. Since A is a strongly (-1,1)-algebra [3], we may assume, without loss of generality, that $\mathcal{V}$ is strongly (-1,1). Assume that $\mathcal{V}$ does not contain $\overline{A}_0$, then the corresponding variety $\tilde{\mathcal{V}}$ of $\mathcal{V}$-superalgebras does not contain F_0. By Proposition 4.1, the identity $(z, x, x)^k = 0$ holds in $\tilde{\mathcal{V}}$ for any central z and odd x.

Consider the $\mathcal{V}$-superalgebra $G \otimes A = G_0 \otimes A + G_1 \otimes A$. Let $z_1, \ldots, z_k \in K(A)$ and $a_1, \ldots, a_{2k} \in A$; consider in $G \otimes A$ the elements $z = e_1 e_2 \otimes z_1 + \cdots + e_{2k-1} e_{2k} \otimes z_k$ and $x = e_{2k+1} \otimes a_1 + \cdots + e_{4k} \otimes a_{2k}$, where $e_1, \ldots, e_{4k}$ are the elements of the canonical base of G. Then $z \in K(G \otimes A)$, $x \in (G \otimes A)_1$, hence we have

$$
\begin{aligned}
0 &= (z, x, x)^k = \prod_{i,s,t} e_{2i-1} e_{2i} e_s e_t \otimes (z_i, a_s, a_t) \\
&= \pm e_1 \cdots e_{4k} \otimes \prod_{i,s,t} (z_i, a_s, a_t),
\end{aligned}
$$

where $i \in \{1, \cdots, k\}$, $s, t \in \{1, \ldots, 2k\}$. In view of identities (9), (10), the product $(z, a, b)(z', c, d)$ for $z, z' \in K(A)$ is skewsymmetric in a, b, c, d. Moreover, the associators (z_i, a_s, a_t) lie in the associative and commutative algebra $K(A)$. Therefore, we have

$$
0 = (4k)! \, e_1 e_2 \cdots e_{4k} \otimes (z_1, a_1, a_2) \cdots (z_k, a_{2k-1}, a_{2k}).
$$

Returning to the algebra A, we have in it the equality $(K_1)^k = 0$, where $K_1 = (K(A), A, A)$. Since the center $K(A)$ is closed under the operators $R_{x,y}$, the set $K_1 A^\sharp$ is an ideal of A, and by induction it is easy to see that $(K_1 A^\sharp)^N \subseteq K_1^N A^\sharp$. Then the ideal $K_1 A^\sharp$ is nilpotent, hence $K_1 = 0$ and $K(A) \subseteq Z(A)$. In particular, the center $Z(A)$ contains nilpotent elements, such as commutators [6], a contradiction. $\quad\square$

Lemma 6.6. *Let F be the free algebra of countable rank in the variety $\mathcal{V}_0 = Var\,(A_0)$. Then every metaideal I of finite index of the algebra F generates the same variety $\mathcal{V}_0$.*

Proof. Let I be a metaideal of the algebra F. Since an ideal of a prime (-1,1)-algebra is a prime algebra [18], I is a prime algebra. In addition, an ideal of a non-associative prime (-1,1)-algebra can not be an associative algebra [17]. Hence by Lemma 6.5 and Corollary 5.6 $\mathcal{V}_0 \subseteq Var\,(I)$. The converse inclusion is evident. $\quad\square$

Recall that the ideal of an algebra A generated by all commutators $[a, b]$, $a, b \in A$, is called the *commutant* of the algebra A. We will denote the commutant of an algebra A by A'.

Lemma 6.7. *A metaideal of the algebra F containing in its commutant F' can not be a free algebra of any variety.*

Proof. Let I be a metaideal of F containing in F'. Assume that it is a free algebra with a set of free generators $f_1, f_2, \ldots$. Since the commutant F' is a nil-algebra [3], there exists a number n such that $f_1^n = 0$. Hence I is a nil-algebra of bounded index, and by [19] it is solvable. This is a contradiction, since a prime (-1,1)-algebra can not be solvable. $\square$

Theorem B now follows from Lemmas 6.5 - 6.7.

7. Jordan algebras of type $A^{(+)}$ for (-1,1)-algebra A

7.1. **The operator f^+.** In a Jordan algebra J, we denote by $D_{a,b}$ the inner derivation operator $D_{a,b} = [R_a, R_b] : x \mapsto (a, x, b)$. If $J = A^{(+)}$, then for $a \in A$ we denote by R_a^+ the operator of right multiplication in the algebra $A^{(+)}$.

More generally, let $f = f(x_1, ..., x_n)$ be a nonassociative polynomial, i.e., an element of the free nonassociative algebra $\Phi\{X\}$. Consider the subalgebra $\Phi^+\{X\}$ of the algebra $\Phi\{X\}^{(+)}$ generated by the set X, and the homomorphism $\varphi : \Phi\{X\} \to \Phi^+\{X\}$ extending the identity mapping X onto itself. We set $f^+ = \varphi(f)$. In other words, f^+ denotes the polynomial obtained from f by replacement the multiplication $\cdot$ by its symmetrization "$\odot$". Similar notation is used for operators. Thus, if ρ is an element of the multiplication algebra of the algebra $\Phi\{X\}$, the operator ρ^+ is defined by $x\rho^+ = (x\rho)^+$, where $x \in X$. Observe that this agrees with the previous notation R_a^+.

Since a right alternative algebra A satisfies identity [33, p. 69]

$$4(x, y, z)^+ = 2(y, x, z) + [y, [x, z]],$$

every strongly (-1,1)-algebra satisfies the identity

(21) $$2(x, y, z)^+ = (y, x, z),$$

or $R_{a,b} = 2D_{a,b}^+$, where $D_{a,b}^+ = [R_a^+, R_b^+]$.

7.2. **The alternative center of a Jordan superalgebra.** Following E. Zelmanov [30], define *the alternative center $Z_{alt}(J)$ of a Jordan superalgebra J* via

$$Z_{alt}(J) = \{z \in J \mid (z, x, y) + (-1)^{xy}(z, y, x) = 0\}$$

for any homogeneous $x, y \in J$. For the sake of brevity, we will call the elements of $Z_{alt}(J)$ *central*. It is easy to see that the center $Z_{alt}(J)$ has properties similar to those of $K(A)$.

Lemma 7.1. *The center $Z_{alt} = Z_{alt}(J)$ satisfies the following properties:*

    a) $(Z_{alt}, Z_{alt}, J) = (Z_{alt}, J, Z_{alt}) = 0$;
    b) *for any $z \in Z_{alt}$, $a, b \in J$ holds $(a, z, b) = 2(-1)^{az}(z, a, b)$;*
    c) $(Z_{alt}, J, J) + (J, Z_{alt}, J) \subseteq Z_{alt}$;
    d) $Z_{alt}(J)$ *is a subsuperalgebra of J;*
    e) *if A is a (-1,1)-superalgebra then $K(A) \subseteq Z_{alt}(A^{(+)})$.*
    f) *if $J = J(\Gamma, D)$ is a superalgebra of vector type then $Z_{alt}(J) = \Gamma$.*

Proof. Making a standart passage to Grassmann envelopes, we reduce the proof to the case of algebras. One can directly check that any commutative algebra satisfies the associator Jacoby identity (1) and also the flexibility identity

$$(22) \qquad\qquad\qquad (x, y, x) \;\; = \;\; 0.$$

Let $z \in Z_{alt}$, $a, b \in J$, then by (1) we have

$$(z, a, b) + (a, b, z) + (b, z, a) = 0.$$

By the definition of Z_{alt}, $(z, a, b) = -(z, b, a) = (a, b, z)$, hence $2(z, a, b) + (b, z, a) = 0$ and by linearized (22) $2(z, a, b) = -(b, z, a) = (a, z, b)$, which proves b).

Now, for $z_1, z_2 \in Z_{alt}$ we have by b)

$$(z_1, z_2, a) = 2(z_2, z_1, a) = 4(z_1, z_2, a),$$

hence $(z_1, z_2, a) = 0$, which implies a).

Furthermore, one can easily check that Z_{alt} is invariant under derivations of J. In particular, $(a, Z_{alt}, b) = Z_{alt}D_{a,b} \subseteq Z_{alt}$, which in view of b) implies c). Moreover, the fact that the applications $R_{a,b}$ are derivations of Z_{alt} implies easily d).

Finally, e) follows from relations (21),(7), and f) is proved directly.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Let us call a subspace $V \subseteq Z_{alt}(J)$ to be *D-invariant* if $(V, J, J) \subseteq V$.

Corollary 7.2. *The following properties are true.*

    a) *If U, V are D-invariant then so is UV;*
    b) *If U is D-invariant then $UJ^{\sharp}$ is an ideal of J;*
    c) *If U, V are D-invariant then $(UJ)(VJ) \subseteq (UV)J^{\sharp}$.*

Proof. Let us prove c) which is the only non-evident. We have

$$
\begin{aligned}
(UJ)(VJ) \;\subseteq\;& (UJ \cdot V)J + (UJ, V, J) = (UV \cdot J)J + (V, UJ, J) \\
\subseteq\;& (UV)J + (UV, J, J) + U(V, J, J) + (V, U, J)J \\
\subseteq\;& (UV)J + UV + (UV \cdot J)J + (V \cdot UJ)J \\
\subseteq\;& (UV)J^{\sharp} + (UV, J, J) + (UV \cdot J)J \subseteq (UV)J^{\sharp}.
\end{aligned}
$$

□

Proposition 7.3. *In a prime nonassociative (-1,1)-algebra A the equality $K(A) = Z_{alt}\left(A^{(+)}\right)$ holds.*

Proof. It is known [3] that a prime nonassociative (-1,1)-algebra is strongly (-1,1). If $z \in Z_{alt}\left(A^{(+)}\right)$, then, $(\forall x \in A)\,(z, x, x)^+ = 0$ and hence $z \in V(A)$ by (21). The same identity gives the opposite inclusion, which yields $V(A) = Z_{alt}(A^{(+)})$. Finally, in a prime algebra A we have $V(A) = K(A)$ [17, Lemma 21]. □

7.3. **Functions $k(x, y; z, t)$ and $h_x(y, z)$.** Consider the following function in a Jordan algebra J:

$$k(x, y; z, t) := (xy, z, t) - (x, z, t)y - x(y, z, t).$$

We will also use the notation:

$$k(x; y, z) := k(x, x; y, z), \ \ k(x; y) := k(x, x; y, y).$$

This function was used in the theory of Jordan algebras by E. Zelmanov [30] and V. Skosyrskii [27]. It is easy to see that if $J = A^{(+)}$ for an associative algebra A then $k(x; y) = \frac{1}{2}[x, y]^2$ in A.

Lemma 7.4. *The function $k(x, y; z, t)$ is symmetric in x, y and in z, t. Moreover, $k(x, y; z, t) = k(z, t; x, y)$.*

Proof. It is clear that the function k is symmetric in x, y. Furthermore, the following identity holds in every Jordan algebra [33]:

$$(23) \qquad (x, yz, t) = y(x, z, t) + (x, y, z)t.$$

Therefore,

$$
\begin{aligned}
k(x, y; z, t) \quad &= \quad (xy, z, t) - x(y, z, t) - (x, z, t)y \\
&\overset{(5)}{=} \quad -(x, y, zt) + (x, yz, t) + (x, y, z)t - y(x, z, t) \\
&\overset{(22),(23)}{=} \quad (zt, y, x) + y(x, z, t) + (x, y, t)z - t(z, y, x) - y(x, z, t) \\
&= \quad (zt, y, x) - z(t, y, x) - (z, y, x)t = k(z, t; y, x).
\end{aligned}
$$

The obtained identitiy yields the other statements of the lemma. □

Lemma 7.5. *If A is a strongly (-1,1)-algebra, then $k(x, y; z, t)^+ \in K(A)$.*

Proof. Since the function $k(x, y; z, t)^+$ is symmetric in x, y and in z, t, it sufficies to verify that $k(x; y)^+ \in K(A)$. Identity (4) implies

$$(24) \qquad (y, x^2, z) = 2(y, x, z)x + (y, x, [z, x]).$$

Therefore, in view of (21) and (24) we have:

$$
\begin{aligned}
2k(x;y)^+ &= (y, x^2, y) - 2x \odot (y, x, y) \\
&= 2(y, x, y)x + (y, x, [y, x]) - 2x \odot (y, x, y) \\
&= (y, x, [y, x]) + [(y, x, y), x].
\end{aligned}
$$

Clearly, $[(y, x, y), x] \in [A, A] \subseteq K(A)$, and by (7)

$$
(y, x, [y, x]) \in (A, A, K(A)) \subseteq (K(A), A, A) \subseteq K(A).
$$

$\square$

Define now the functions

$$
\begin{aligned}
Q_a &: \quad x \mapsto (a, a, x), \\
h_x(a, b) &:= x[Q_a, Q_b] = (b, b, (a, a, x)) - (a, a, (b, b, x)).
\end{aligned}
$$

Lemma 7.6. *If A is a strongly (-1,1)-algebra, then $h_x^+(a, b) \in K(A)$.*

Proof. It was proved by I.Hentzel and H.Smith [5, identity (19)] that the function $h_x(a, b)$ is central in the variety of binary (-1,1)-algebras. If A is a strongly (-1,1)-algebra, then by (21) we have

$$
4h_x^+(a, b) = h_x(a, b) \in K(A).
$$

$\square$

7.4. Some identities of $G(J(\Gamma, \delta))$.

Proposition 7.7. *The Grassmann envelope $G(J(\Gamma, \delta))$ of a Jordan superalgebra $J(\Gamma, \delta)$ of vector type satisfies the identities:*

$$
k(x; y)Q_c = h_x(a, b)Q_c = k(x, y)^2 = h_x(a, b)^2 = (a, a, b)^2 = 0.
$$

Proof. Consider the twisted superalgebra $B := B(\Gamma, D, 0)$ of vector type, where $D = 2\delta$. Observe that $B^{(+)} = J(\Gamma, \delta)$. The superalgebra B is a strongly (-1,1), hence its Grassmann envelope $A := G(B)$ satisfies Lemmas 7.5 and 7.6, and we have the inclusions $k(x, y)^+, \; h_x^+(a, b) \in K(A) \subseteq Z_{alt}(A^{(+)})$. This proves the first two equalities of the lemma. In addition, it follows from [17, Lemma 23] and the Kleinfeld identity $[x, y]^3 = 0$ which is valid in any strongly (-1,1)-algebra that $k(x; y)^2 = 0$ in the algebra $A^{(+)}$. Furthermore, $4((a, a, b)^+)^2 = (a, a, b)^2 = 0$ [3]. Finally, by [17, Lemma 6] we have $h_x(a, b)^2 = 0$ in $A^{(+)}$. Therefore, the algebra $G(J) = G(B^{(+)}) = (G(B))^{(+)}$ satisfies the identities of the lemma. $\square$

Lemma 7.8. *If a Jordan algebra J satisfies the identities $k(x, y; z, t) = 0$ and $h_x(y, z) = 0$, then it satisfies the identity $aQ_bQ_c = 0$.*

Proof. Let $(a, b, c)_\sigma$ denotes any of the six associators obtained from the associator (a, b, c) by an arbitrary permutation of variables. The identity $k(x, y; z, t) = 0$ implies that the associator (a, b, c) is a derivation on all the variables. Therefore,

$$
\begin{aligned}
4(a, (a, b, c)_\sigma, b) &= 4\, a(a, b, c)_\sigma \cdot b - 4\, a \cdot (a, b, c)_\sigma b \\
&= 2\, (a^2, b, c)_\sigma b - 2\, (a, b^2, c)_\sigma a \\
&= (a^2, b^2, c)_\sigma - (a^2, b^2, c)_\sigma = 0.
\end{aligned}
$$

Furthermore,

$$(a, aQ_b, c) = -(a, (a, b, b), c) = -(a, (a, b, c), c)\Delta_c^1(b) + (a, (a, b, c), b) = 0,$$

where $\Delta_c^1(b)$ is the operator of partial linearization with respect to c [33, 1.4].

The application $a \mapsto aQ_b$ is a derivation. Therefore, by the previous identity,

$$
\begin{aligned}
(aQ_b, a, c) &= -(a, aQ_b, c) - (a, a, cQ_b) + (a, a, c)Q_b \\
&= -cQ_bQ_a + cQ_aQ_b = h_c(a, b) = 0.
\end{aligned}
$$

From the Associator Jacobi identity (1) we have

$$(aQ_b, c, a) = -(c, a, aQ_b) - (a, aQ_b, c) = (aQ_b, a, c) - (a, aQ_b, c) = 0.$$

Finally,

$$aQ_bQ_c = ((a, b, b), c, c) = -(aQ_b, a, c)\Delta_a^1(c) + (cQ_b, a, c) = 0.$$

$\square$

7.5. The Jordan central extension $J[Z; x]$.

Denote by $J[Z; x]$ the quotient superalgebra of the free Jordan superalgebra $Jord\,[Z; x]$ generated by a nonempty set Z of even elements and an odd element x, by the ideal generated by the elements of the form

$$(z, a, b) + (-1)^{ab}(z, b, a), \text{ where } z \in Z;\ a, b \in Jord[Z; x].$$

Clearly, $J[Z; x]$ satisfies the following universal property:
for any Jordan superalgebra J, any odd $y \in J_1$, and for any mapping $\varphi : Z \to Z_{alt}(J)$ there exists a unique homomorphism $\tilde{\varphi} : J[Z; x] \to J$ such that $\tilde{\varphi}|_Z = \varphi$, $\tilde{\varphi}(x) = y$.

The superalgebra $J[Z; x]$ plays for Jordan algebras the same role as the superalgebra $F[Z; x]$ does in the variety $\mathcal{St}$ of strongly (-1,1)-algebras.

Consider again the polynomial ring $\Phi[T_Z]$ on the set of variables $T_Z = \{z_0 = z, z_1, \ldots, \mid z \in Z\}$ with the derivation $D : z_i \mapsto z_{i+1}$, $i = 0, 1, \ldots$.

Proposition 7.9. *The superalgebra $J[Z; x]$ is isomorphic to the subsuperalgebra $\Phi_0[T_Z] \oplus \overline{\Phi[T_Z]}$ of the superalgebra of vector type $J(\Phi[T_Z], D)$.*

Proof. Denote $J = J[Z; x]$. Consider the subalgebra A of J_0 generated by the elements $zR_{x,x}^i$, $z \in Z$, $i \geq 0$. Denote $D = R_{x,x}$, then by Lemma 7.1 D is a derivation of $Z_{alt}(J)$ and $A \subseteq Z_{alt}(J)$. Moreover, $D(A) \subseteq A$ hence D is a derivation of A. Let us prove that $J = A + A^\sharp x$. Since $Z \cup \{x\} \subseteq A$, it suffices to prove that $A + A^\sharp x$ is a subsuperalgebra of J. By Lemma 7.1 $A^\sharp x$ is an associative bimodule over A, hence we need only to consider the product $(A^\sharp x)(A^\sharp x)$. Let $a, b \in A^\sharp$, then by Lemma 7.1 we have

$$
\begin{aligned}
(ax)(bx) &= (ax \cdot b)x - (ax, b, x) = ((ab)x)x - 2(b, ax, x) \\
&= (ab, x, x) - 2a(b, x, x) = (ab)R_{x,x} - 2a(bR_{x,x}) \\
&= D(a)b - aD(b) \in A.
\end{aligned}
$$

Therefore, $A + A^\sharp x$ is a subsuperalgebra of J and $J = A + A^\sharp x$. Moreover, the obtained equality shows that $J^\sharp$ is a homomorphic image of the superalgebra of vector type $J(A^\sharp, D)$ under the homomorphism $a + \bar{b} \mapsto a + ax$.

Consider the homomorphism $\varphi : \Phi[T_Z] \to A^\sharp$, $z_i \mapsto D^i(z)$, $z \in Z$. Clearly, it is a homomorphism of differential algebras which can be extended to a homomorphism of superalgebras

$$
\tilde{\varphi} : J(\Phi[T_Z], D) \to J(A^\sharp, D).
$$

Now the composition $\pi \circ \tilde{\varphi}$ maps surjectively $\Phi_0[T_Z] \oplus \overline{\Phi[T_Z]}$ onto J. By the universal property of J, in view of Lemma 7.1.f), this map is invertible and hence it is an isomorphism. $\qquad\square$

Corollary 7.10. *The superalgebra $J[Z; x]$ has a base of the form $B \cup Bx$ where B consists of associative and commutative monomials on the elements $zR_{x,x}^i$, $z \in Z$, $i \geq 0$.*

Recall that for $I = (i_0, i_1, \ldots, i_k) \in \mathbf{Z}_+^\infty$ we denote

$$
a^I = a^{i_0}(aR_{x,x})^{i_1}(aR_{x,x}^2)^{i_2} \cdots (aR_{x,x}^k)^{i_k}.
$$

Corollary 7.11. *The superalgebra $J[Z \cup \{s\}; x]$ is isomorphic to the superalgebra $F[Z; x]^{(+)}$ under the isomorphism*

$$
s^{I_0} z_1^{I_1} \ldots z_k^{I_k} x^\varepsilon \mapsto (x^2)^{I_0} z_1^{I_1} \ldots z_k^{I_k} x^\varepsilon.
$$

In particular, the superalgebra $J[z; x]$ is isomorphic to the superalgebra $F[\emptyset; x]^{(+)}$.

Proof. By Propositions 7.9 and 3.6, we have the isomorphisms

$$J[Z\cup\{s\};x]^\sharp \cong J(\Phi[T_{Z\cup\{s\}}],D) \cong B(\Phi[T_{Z\cup\{s\}}],D,s)^{(+)} \cong (F[Z;x]^\sharp)^{(+)}.$$

The restriction of these isomorphisms on "nonunital parts" gives the isomorphism of the corollary.   □

8. Proof of Theorem C

8.1. **Superalgebra JF_0 and variety $J\mathcal{V}_0$.** Following the case of $(-1,1)$-algebras, consider the superalgebra

$$JF_0 := J[z;x]/(((z,x,x),x,x)).$$

It is easy to see that $JF_0 \cong J_0(\Phi[z,s],s\frac{d}{dt}) \cong F_0^{(+)}$ and has a base $z^n(z,x,x)^k x^\varepsilon,\ n,k \geq 0,\ \varepsilon+n+k > 0$.

Similarly to Proposition 4.1, we have the following characterization of varieties of Jordan superalgebras that do not contain JF_0.

Proposition 8.1. *A variety $\mathcal{V}$ of Jordan superalgebras does not contain JF_0 if and only if it satisfies the identity*

$$(z,x,x)^k = 0$$

for any central z and odd x and some $k > 0$.

The proof repeats that of Proposition 4.1 with evident modifications.
                                                                      □

Denote by $J\mathcal{V}_0$ the variety of Jordan algebras generated by the Grassmann envelope $G(JF_0)$ of the superalgebra JF_0.

Proposition 8.2. *Variety $J\mathcal{V}_0$ is generated by any one of the following algebras*

$$\bar{J}_0,\ A_0^{(+)},\ G_{(-1,1)}^{(+)},$$
$$G(J[Z;x])\ \text{with}\ Z \neq \emptyset,$$
$$G(J(\Gamma,\delta)),\ \text{where}\ \Gamma = \Gamma_0\ \text{and}\ \Gamma^\delta\ \text{is not nilpotent.}$$

Proof. Observe first that if $Var\,A = Var\,B$ then $Var\,A^{(+)} = Var\,B^{(+)}$. Furthermore, it is easy to check that $G(A^{(+)}) \cong (G(A))^{(+)}$. By the results of Sections 5 and 6, all the algebras

$$\bar{A}_0,\ A_0,\ G_{(-1,1)},\ G(F_{(-1,1)}[\emptyset;x]),\ G(F[Z;x]),\ G(B(\Gamma,D,\gamma))$$

generate the same variety $\mathcal{V}_0 = Var\,A_0$. Therefore, the algebras $\bar{J}_0 = (\bar{A}_0)^{(+)},\ A_0^{(+)},\ G_{(-1,1)}^{(+)},\ G(J[Z;x]) \cong (G(F[Z\setminus\{s\};x]))^{(+)}$ for some element $s \in Z,\ G(J(\Gamma,\delta)) \cong (G(B(\Gamma,D,0))^{(+)}$ generate the same variety $J\mathcal{V}_0$.   □

Corollary 8.3. *The free algebra of countable rank in the variety $J\mathcal{V}_0$ over a field Φ of characteristic 0 is prime and degenerate.*

This was proved in [13], [10], and in [21] for the varieties $Var\,(A_0)^{(+)}$, $Var\,J(\Phi[t], \frac{d}{dt})$, and $Var\,J_{VT}$, respectively, which in fact are all equal to $J\mathcal{V}_0$.

8.2. **Minimality of variety $J\mathcal{V}_0$.**

Proposition 8.4. *Let $\mathcal{V}$ be a subvariety of $J\mathcal{V}_0$ which contains a prime Jordan algebra J which is not associative. Then $\mathcal{V} = J\mathcal{V}_0$.*

Proof. Assume that $\mathcal{V}$ does not contain the algebra $\overline{J}_0$. Then the arguments from the proof of Lemma 6.5 in view of Proposition 8.1 show that the prime algebra J satisfies the equality $Z_1^m = 0$ for some m, where $Z_1 = (Z_{alt}(J), J, J)$. Since the center $Z_{alt}(J)$ is invariant under the operators $R_{x,y}$, by Corollary 7.2 the subspace $Z_1 J^\sharp$ is an ideal of J. By induction, using Corollary 7.2 again, it is easy to see that $(Z_1 J^\sharp)^k \subseteq Z_1^k J^\sharp$. Then the ideal $Z_1 J^\sharp$ is nilpotent. Let $Z_1^k = 0$, $Z_1^{k-1} \neq 0$, then $(Z_1)^{k-1} J^\sharp$ is a non-zero trivial ideal in J which should be zero. Therefore, $Z_1 = 0$ and $Z_{alt}(J)$ is contained in the center $Z(J)$.

Since $J \in \mathcal{V} \subset J\mathcal{V}_0$, it satisfies the identities of $G(J(\Gamma, D))$. By Proposition 7.7, the functions $k(x; y)$ and $h_x(a, b)$ produce nilpotent central elements in J and therefore are zero in J. Then by Lemma 7.8 $(a, b, b) \in Z_{alt}(J) = Z(J)$ for any $a, b \in J$. By Proposition 7.7 again, $(a, b, b)^2 = 0$, hence $(a, b, b) = 0$ in J and $J = Z_{alt}(J)$, a contradiction. $\square$

Theorem C follows from Propositions 8.2 and 8.4.

Observe that, contrary to the case of variety $\mathcal{V}_0$, we do not know whether $J\mathcal{V}_0$ lies in every variety that contains a prime degenerate algebra. Below we show that it is true for the prime degenerate algebra related with the Jordan superalgebra of Poisson brackets [10] and for the algebra constructed by V. G. Skosyrskii [28].

Proposition 8.5. *Let $\mathcal{V}_{PB}$ be the variety generated by the Grassmann envelope of the Jordan algebra of free Poisson bracket and $\mathcal{V}_{Skos}$ be the variety generated by the algebra of Skosyrskii [28]. Then we have*

$$J\mathcal{V}_0 \subseteq \mathcal{V}_{Skos} \subset \mathcal{V}_{PB}.$$

Proof. It was proved in [9] that every Jordan superalgebra of brackets (or, in other terms, every *Kantor double*) can be embedded into a Jordan superalgebra of Poisson brackets. In particular, this is true for the superalgebra of even vector type $J(\Gamma, \delta)$. Therefore,

$$J\mathcal{V}_0 = Var\,G(J(\Gamma, D)) \subset \mathcal{V}_{PB}.$$

The inclusion is strict since the Grassmann envelope of the superalgebra of Poisson brackets does not satisfies Proposition 7.7.

The algebra of Skosyrskii has the form $J(G, D)_0$, where G is the Grassmann algebra on infinite number of generators $e_1, \ldots, e_n, \ldots$ and D is the derivation of G defined by the condition $D(e_i) = e_{i+1}$. It follows from [22] that $J(G, D)_0 \cong B(G, D, 0)_0^{(+)}$, where $B(G, D, 0)$ is a $(-1, 1)$-superalgebra of vector type. It is clear that $B(G, D, 0)_0$ is a nonassociative prime (-1,1)-algebra, hence $\mathcal{V}_0 \subseteq Var\, B(G, D, 0)_0$. On the other hand, as it was mentioned above, $Var\, J(G, D) \subseteq \mathcal{V}_{PB}$. Therefore, $J\mathcal{V}_0 \subseteq \mathcal{V}_{Skos} \subset \mathcal{V}_{PB}$. $\qquad\square$

It remains an open question whether $J\mathcal{V}_0 = \mathcal{V}_{Skos}$?

9. Enveloping $\mathcal{V}_0$-algebra for $\mathcal{F}_{J\mathcal{V}_0}[X]$

In this section we relate the free algebra $\mathcal{F}_{J\mathcal{V}_0}[X]$ and the strongly $(-1, 1)$-algebra $\mathcal{F}_{\mathcal{V}_0}[X]$.

Proposition 9.1. *Let $\mathcal{F}_{\mathcal{V}_0}[X]$ be the free algebra of countable rank in the variety $\mathcal{V}_0$, and let $J[X]$ be the subalgebra generated by set X in the algebra $(\mathcal{F}_{\mathcal{V}_0}[X])^+$. Then $J[X]$ is isomorphic to the free algebra $\mathcal{F}_{J\mathcal{V}_0}[X]$.*

Proof. Notice first that $J[X]$ is a relatively free algebra over the set X, that is, any relation $f(x_1, \ldots, x_n) = 0$ in the algebra $J[X]$ is its identity. In fact, since this relation is an identity of $\mathcal{F}_{\mathcal{V}_0}[X]$, then $f(a_1, \ldots, a_n) = 0$ for any $a_1, \ldots, a_n \in \mathcal{F}_{\mathcal{V}_0}[X]$, which obviously implies that f is an identity in $J[X]$.

We now show that $J[X]$ and $\mathcal{F}_{J\mathcal{V}_0}[X]$ have the same identities. Assume that $f(x_1, \ldots, x_n) \neq 0$ in $J[X]$. Since $\mathcal{V}_0 = Var\, A_0$, there exist $a_1, \ldots, a_n \in A_0$ such that $f(a_1, \ldots, a_n) \neq 0$. But then f is not an identity in $A_0^{(+)}$ and hence $f(x_1, \ldots, x_n) \neq 0$ in $\mathcal{F}_{J\mathcal{V}_0}[X]$.

Conversely, if $f(x_1, \ldots, x_n) \neq 0$ in the algebra $\mathcal{F}_{J\mathcal{V}_0}[X]$ then there exist $a_1, \ldots, a_n \in A_0^{(+)}$ such that $f(a_1, \ldots, a_n) \neq 0$. But then $f(a_1, \ldots, a_n) \neq 0$ in the algebra A_0 as well, which means that $f(x_1, \ldots, x_n) \neq 0$ in $\mathcal{F}_{\mathcal{V}_0}[X]$ and in $J[X]$.

Thus, a relatively free algebra $J[X]$ and $\mathcal{F}_{J\mathcal{V}_0}$ have the same identities, hence they are isomorphic. $\qquad\square$

It remains an open question on relation between the varieties $J\mathcal{V}_0$ and $\mathcal{V}_0$: *is it true that every $J\mathcal{V}_0$-algebra has an $\mathcal{V}_0$-enveloping algebra?* One can show that the algebra $J[X]$ is not isomorphic to the algebra $A^{(+)}$ for any strongly $(-1, 1)$-algebra A.

We recall also the following open question which was first formulated in [12]: *Is it true that $\mathcal{V}_0 = \mathcal{St}$?*

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FINANCE UNIVERSITY UNDER THE GOVERNMENT OF THE RUSSIAN FEDERATION, AND MOSCOW CITY PEDAGOGICAL UNIVERSITY
  E-mail address: `pchelinzev@mail.ru`

INSTITUTO DE MATEMÁTICA E ESTATÍSTICA, UNIVERSIDADE DE SÃO PAULO, SÃO PAULO, BRAZIL, AND SOBOLEV INSTITUTE OF MATHEMATICS, NOVOSIBIRSK, RUSSIA
  E-mail address: `shestak@ime.usp.br`