RECOGNIZING PRODUCTS OF SURFACES AND SIMPLY CONNECTED 4-MANIFOLDS

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Abstract. We give necessary and sufficient conditions for a closed smooth 6-manifold N to be diffeomorphic to a product of a surface F and a simply connected 4-manifold M in terms of basic invariants like the fundamental group and cohomological data. Any isometry of the intersection form of M is realized by a self-diffeomorphism of $M \times F$.

1. Introduction

Simply-connected closed 6-manifolds were classified by Wall [13], Jupp [6], and Žubr [14]. However, if the fundamental group is non-trivial, such complete information is not within reach of current techniques except in special cases.

In this paper we consider the following problem: given a closed, oriented 6-manifold N, can we identify a closed, oriented surface F and a simply-connected, closed 4-manifold M such that N is diffeomorphic to $M \times F$? Since simply connected 6-manifolds are already classified, we assume from now on that $F \neq S^2$ has genus ≥ 1, but the results remain true in the simply connected case. First we discuss some of the necessary conditions.

Condition 1. *The fundamental group $\pi_1(N)$ is isomorphic to the fundamental group of a closed, oriented surface F.*

We choose a base-point preserving classifying map $u\colon N \to F$ for the universal covering. Up to homotopy and choice of base points this is equivalent to choosing an isomorphism $\alpha\colon \pi_1(N) \to \pi_1(F)$, where $u_\# = \alpha$.

The next condition concerns the second homology group of the universal covering $H_2(\widetilde{N})$, which for the product of F with a simply connected 4-manifold M is a trivial module over $\pi_1(N)$ and so we require this:

Condition 2. *$H_2(\widetilde{N})$ is a trivial $\pi_1(N)$-module.*

Under this condition, the Serre spectral sequence for the fibration over $F = K(\pi_1(N), 1)$ with fibre $\widetilde{N}$, implies that we have an exact sequence

$$0 \to H^2(F) \xrightarrow{u^*} H^2(N) \xrightarrow{p^*} H^2(\widetilde{N}) \to 0,$$

where p is the universal covering projection. It follows that $H_2(\widetilde{N})$ is a finitely-generated free abelian group.

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The key to our recognition result is the observation that the cohomology algebra of N provides a candidate for the intersection form of a closed, simply-connected topological 4-manifold M. To identify this candidate, suppose that Condition 1 holds. Then the trilinear cup product form on $H^2(N)$ induces a well-defined symmetric bilinear form

$$I(N) : H^2(N)/u^*H^2(F) \times H^2(N)/u^*H^2(F) \to \mathbb{Z}$$

by mapping x and y to $\langle u^*([F]) \cup x \cup y, [N]\rangle$, where $[F] \in H^2(F)$ is the cohomology fundamental class. If Condition 2 holds, then $V := H^2(N)/u^*H^2(F) \cong H^2(\widetilde{N})$ is a finitely-generated free abelian group.

Under the assumption that $N \approx M \times F$, this form $I(N)$ is *unimodular* and the finitely-generated free abelian group V is isomorphic to $H^2(M)$. Moreover, the form $I(N)$ and the intersection form

$$s_M \colon H^2(M) \times H^2(M) \to \mathbb{Z}$$

are isometric. We recall that vanishing of the Kirby-Siebenmann invariant of M, denoted $KS(M)$ is a necessary and sufficient condition for $M \times F$ to be smoothable (see [7, Theorem 5.14, p. 318]). If M is a spin manifold, then this condition is assured by requiring $\operatorname{sign}(M) \equiv 0 \pmod{16}$. Thus we have:

Condition 3. *The symmetric bilinear form $I(N)$ is unimodular, and* $\operatorname{sign} I(N) \equiv 0 \pmod{16}$ *if N is a spin manifold.*

If $I(N)$ is unimodular there exists a closed, simply-connected topological 4-manifold M with this intersection form, by the foundational results of Freedman [4, Theorem 1.5]. If M is non-spin, then Freedman shows that we may assume $KS(M) = 0$. In either case, if $KS(M) = 0$ the manifold M is *uniquely* determined (up to homeomorphism) by its intersection form s_M. Moreover, the smooth structures on $M \times F$ are determined by lifts of its stable topological tangent bundle $\tau_{M\times F}$ (see [7, Theorem 10.1, p. 194] for the precise statement).

Definition 1.1. The *standard smooth structure* on $M \times F$ is the one determined by product of the unique lift of $\tau_M \colon M \to BTOP$ to BO, together with $\tau_F \colon F \to BO$. The lift of τ_M is unique because $TOP/O \simeq TOP/PL = K(\mathbb{Z}/2, 3)$ in this range of dimensions.

We then fix the standard smooth structure on $M \times F$ and take the product orientation with respect to given orientations on M and F. This is our candidate for recognizing N as the product $M \times F$.

Finally, we need some more information about the *oriented* integral cohomology ring of N and the Pontrjagin class $p_1(N) \in H^4(N)$. Let $q_1 \colon M \times F \to M$ and $q_2 \colon M \times F \to F$ denote the first and second factor projection maps. Note that the integral cohomology of $M \times F$ is $\mathbb{Z}$-torsion free, so any map $H^*(M \times F) \to H^*(N)$ of integral cohomology rings reduced mod 2 induces a map on $\mathbb{Z}/2$-cohomology.

Condition 4. *Let M be a closed, oriented, simply-connected topological 4-manifold with $s_M \cong I(N)$ and $KS(M) = 0$. There exists an isomorphism*

$$\phi \colon H^*(M \times F) \to H^*(N)$$

of oriented integral cohomology rings. We assume that

(i) $\phi([M] \times [F]) = [N] \in H^6(N)$,

(ii) $\phi \circ q_2^* = u^* \colon H^*(F) \to H^*(N)$, *and*

(iii) ϕ *preserves the second Stiefel-Whitney class:*

$$\phi(w_2(M \times F)) = w_2(N) \in H^2(N; \mathbb{Z}/2).$$

(iv) *Moreover, the relation*

$$\langle \phi(x) \cup p_1(N), [N] \rangle = \begin{cases} 3\,\mathrm{sign}(M) & \text{if } x = q_2^*([F]) \in H^2(M \times F), \\ 0 & \text{if } x = q_1^*(y) \end{cases}$$

holds for all $y \in H^2(M)$.

Example 1.2. Unless $M = S^4$, the cohomology cohomology ring determines the Steenrod operations, and so ϕ preserves the second Stiefel-Whitney class. On the other hand, consider an oriented 4-sphere bundle N over F with $w_2(N) \neq 0$. Then N has the same cohomology ring as $S^4 \times F$ but is not diffeomorphic to the product.

Now we are ready to formulate our main result.

Theorem A. *Let N be a closed, oriented smooth 6-manifold, and $\alpha \colon \pi_1(N) \cong \pi_1(F)$ for some closed, oriented surface F, such that Conditions 1-3 hold. Suppose that*

(i) *M is the closed, simply-connected topological 4-manifold M, such that $s_M \cong I(N)$, with $KS(M) = 0$, and*

(ii) *$\phi \colon H^*(M \times F) \xrightarrow{\approx} H^*(N)$ is a ring isomorphism satisfying Condition 4.*

Then, there is an orientation and base-point preserving diffeomorphism $f \colon N \to M \times F$ such that $f_\# = \alpha$ and $f^* = \phi$.

We can also ask which automorphisms of the second cohomology of $M \times F$ are induced by self-diffeomorphisms. In particular, we consider automorphisms of $H^2(M)$, and extend them by the identity on $H^2(F)$ via the identification:

$$(q_1^*, q_2^*) \colon H^2(M) \oplus H^2(F) \cong H^2(M \times F).$$

From the ring structure in cohomology, a necessary condition is that the automorphism on $H^2(M)$ is an isometry of the intersection form.

Corollary B. *Let M be a closed topological 4-manifold with $KS(M) = 0$ and F a closed oriented surface. Then each isometry of the intersection form of M is induced by a self-diffeomorphism of $M \times F$.*

Proof. There is an automorphism ϕ of $H^*(M \times F)$, which on $H^2(M \times F)$ is the given isometry on $H^2(M)$ extended by the identity on $H^2(F)$. By Theorem A there is a self-diffeomorphism of $M \times F$ inducing ϕ, and therefore the given isometry on $H^2(M)$. □

Remark 1.3. In the case where M is itself smooth, Donaldson theory (see [5, Theorem 6]) provides examples of isometries of $H^2(M)$ which cannot be realized by self-diffeomorphisms of M. We also remark that an alternate argument can be given for Corollary B by using further results of Freedman and Kirby-Siebenmann. By [4, p. 371] there is a homeomorphism $h\colon M \to M$ realizing the given isometry. Consider the s-cobordism

$$W^5 := (M \times I) \cup_h (M \times I)$$

obtained by gluing two cylinders $M \times I$ via h. Since $H^4(W, \partial W; \mathbb{Z}/2) = H^3(M; \mathbb{Z}/2) = 0$, we can pick a lift of $\tau_W\colon W \to BO$ extending the standard lift of τ_M on both boundary components. Taking the product of the s-cobordism with F, we obtain an s-cobordism $W \times F$ with the product lift of $\tau_{W\times F}$ over BO. By Kirby-Siebenmann [7, Theorem 10.1, p. 194] there is a smooth structure on $W \times F$ which restricts to the standard smooth structure on both ends. The s-cobordism theorem then gives a self-diffeomorphism of the standard smooth structure on $M \times F$, realizing the given isometry.

Finally, we note that the smooth structure on $M \times F$ is actually *unique* up to diffeomorphism.

Corollary C. *Let M be a closed, simply-connected topological 4-manifold with $KS(M) = 0$, and let F be a closed oriented surface. Then $M \times F$ has a unique smooth structure.*

Proof. We can apply Theorem A to the topological manifold $M \times F$ equipped with two different smooth structures. By Novikov [10, Theorem 1], we have Condition 4 with $\phi = id$. □

Remark 1.4. The results of Kirby and Siebenmann [7, Theorem 5.4, p. 318] show that the set of distinct smoothings of $M \times F$ is in bijection with

$$[M \times F, TOP/O] = [M \times F, TOP/PL] = H^3(M \times F; \mathbb{Z}/2),$$

since in this dimension every PL manifold admits a unique smooth structure. Theorem A shows that Homeo$(M \times F)$ acts transitively on the set of smoothings. It would be interesting to construct a corresponding homeomorphism for each $\alpha \in H^3(M \times F; \mathbb{Z}/2)$.

Here a *smoothing* is a pair (N, h), where N^6 is a smooth 6-manifold and $h\colon N \to M \times F$ is a homeomorphism; two smoothings (N, h) and (N', h') are equivalent if there exists a diffeomorphism $\varphi\colon N \to N'$ such that h and $h' \circ \varphi$ are topologically isotopic.

Remark 1.5. The effectiveness of our recognition result in practice will depend on the difficulty of verifying Conditions 3 and 4, but most of this is linear algebra. After obtaining Conditions 1 and 2, one might proceed by showing that $H^*(\widetilde{N})$ is isomorphic to a 4-dimensional algebra Λ^*, with $\Lambda^0 = \Lambda^4 = \mathbb{Z}$, $\Lambda^1 = \Lambda^3 = 0$, carrying the symmetric bilinear form $I(N)\colon \Lambda^2 \otimes \Lambda^2 \to \mathbb{Z}$ on a free abelian group $\Lambda^2 \cong \mathbb{Z}^r$. This gives the Euler characteristic equation $\chi(N) = \chi(F) \cdot (r + 2)$, and shows that $H^3(N) \cong H^1(F) \otimes H^2(\widetilde{N})$ is torsion-free. Now Poincaré duality for $H^3(N)$ shows that $I(N)$ is unimodular. After that, it will be necessary to check that $H^*(N) \cong \Lambda^* \otimes H^*(F)$ as graded algebras, and proceed to construct a cohomology ring isomorphism $\phi\colon H^*(M \times F) \to H^*(N)$ with the required conditions on $w_2(N)$ and $p_1(N)$.

However complicated the process, at least the conditions depend only on the primary algebraic topology of N and do not involve determining the full homotopy type of N. For example, we do not assume anything about $\pi_3(N)$.

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2. The normal 2-type and normal 2-smoothings

For the proof we use the methods from [8] and assume that the reader is familiar with the basic concepts and theorems although we repeat the relevant definitions briefly.

We abbreviate $V := H^2(N)/u^*H^2(F) \cong H^2(\widetilde{N})$, and let $H = \pi_2(N) = H_2(\widetilde{N})$. We have $H^2(\widetilde{N}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_2(\widetilde{N}), \mathbb{Z})$, so that $H \cong \mathrm{Hom}_{\mathbb{Z}}(V, \mathbb{Z}) = V^*$. The following remark is immediate from the definitions.

Lemma 2.1. *If N satisfies Condition 4 with respect to $M \times F$, then $H^2(M) \cong V$.*

We start by determining the normal 2-type of N. By definition, this is a fibration B over BSO where the homotopy groups of the fibre vanish in degree ≥ 3 and such there is a lift of the normal Gauss map of N over B, which is a 3-equivalence. We have to distinguish two cases, where the symmetric bilinear form $I(N) \colon V \times V \to \mathbb{Z}$ is a even or odd. In the first case, the normal 2-type is

$$p_{even} \colon B = K(H, 2) \times F \times BSpin \to BSO,$$

where the map is the composition of the projection to $BSpin$ and the canonical projection to BSO. If the form $I(N)$ is odd, one chooses a primitive characteristic element $v \in V$, and a complex line bundle L_v over $K(H, 2)$ with first Chern class v. Then the normal 2-type is

$$p_{odd} \colon B = K(H, 2) \times F \times BSpin \to BSO,$$

where p_{odd} is the map given by the projection to $K(H, 2) \times BSpin$ composed by the map given by the Whitney sum of line bundle L_v and the canonical map to BSO (of course, we have to replace this map by a fibration).

Lemma 2.2. *The normal 2-types of $M \times F$ and N are given by (B, p_{even}), if M is spin, or (B, p_{odd}) if M is non-spin.*

Proof. We first look at the second stage of the Postnikov tower of N, this is a fibration over $K(\pi_1(N), 1)$ with fibre $K(\pi_2(N), 2)$, where in our situation $\pi_2(N) = H$. These fibrations are classified by the action of $\pi_1(N)$ on $\pi_2(N)$ and the k-invariant $k \in H^3(\pi_1(N); \pi_2(N))$. This group is zero, and so the action of $\pi_1(N)$ on $\pi_2(N)$ determines the Postnikov tower. If the $\pi_1(N)$-action is trivial, then we have the trivial fibration. Next, we use our data to construct a 3-equivalence

$$c_{M \times F} := g_{M \times F} \times h_{M \times F} \colon M \times F \to K(H, 2) \times F,$$

and a 3-equivalence

$$c_N := g_N \times h_N \colon N \to K(H, 2) \times F,$$

which is compatible with our data α and ϕ. For this we consider the map

$$g_{M \times F} \colon M \times F \to K(H, 2)$$

such that $(g_{M \times F})^* \colon V \to H^2(M \times F) = H^2(M) \oplus H^2(F) = V \oplus H^2(F)$ is the inclusion onto the first summand (see Lemma 2.1), and choose a base point preserving map $g_N \colon N \to$

$K(H,2)$ such that $(g_N)^* = \phi \circ (g_{M\times F})^*$. Then we consider the projection $h_{M\times F} = q_2\colon M\times F \to F$ and $h_N = u\colon N \to F$. From Conditions 1 - 4 it is clear that the maps $c_{M\times F}$ and c_N are 3-equivalences, with $(c_N)^* = \phi\circ (c_{M\times F})^*$.

If N is Spin-manifold, then by assumption $M\times F$ is a Spin-manifold and we equip both manifolds with an arbitrary Spin structure ω_N and $\omega_{M\times F}$. If N and so $M\times F$ are not Spin-manifolds, then we choose a primitive class $v\in H^2(M\times F;\mathbb{Z})$, such that its component in $H^2(F;\mathbb{Z})$ is zero, which reduces to $w_2(M\times F)$ and a spin structure $\omega_{M\times F}$ on $\nu(M\times F)\oplus L_v$, where L_v is the complex line bundle classified by v. Similarly, we choose a Spin structure ω_N on $\nu(M)\oplus L_{\phi(v)}$. The maps $c_{M\times F}$ and c_N together with the (twisted) Spin-structures are normal 2-smoothings in $(B, p_{odd/even})$. □

3. The bordism groups

The next step in the proof of Theorem A is to show that, under the given conditions, the normal 2-smoothings constructed in Section 2 are bordant in $\Omega_6(B,\xi)$, where ξ is the bundle classified by p_{odd} or p_{even} depending on the normal 2-type.

The method of proof is based on detecting elements in the bordism group by explicit invariants. We have $H\cong \mathbb{Z}^r$ so that $K := K(H,2) = (\mathbb{CP}^\infty)^r$. Let $Dp_1(N)\in H_2(N)$ denote the Poincaré dual of the first Pontrjagin class.

Proposition 3.1. *There is an injection $\Omega_6(B,\xi)\to \mathbb{Z}\oplus H_6(K)\oplus H_4(K)\oplus H_2(K)$, given by* $\operatorname{sign} I(N)$, *and the images of $[N]$, $[N]\cap u^*([F])$, $Dp_1(N)$ under the reference maps $c_N\colon N\to B$ for the normal 2-types.*

To compute the bordism groups we consider the functor associating to a space X the bordism group of $p_{odd/even}\colon X\times K(H,2)\times BSpin\to BSO$, where the maps are defined as above in the case $X = F$. This is a homology theory denoted by $h_k(X)$ and so we can use the Mayer-Vietoris sequence to compute it, by writing a surface of genus g as $D_2\cup Y$, where Y is a wedge of $2g$ circles. Then we obtain an exact sequence

$$\tilde h_7(S^2)\to \tilde h_6(Y)\to \tilde h_6(F)\to \tilde h_6(S^2)\to \tilde h_5(Y),$$

or, if we apply the suspension isomorphism, the exact sequence:

$$h_5(pt)\to \sum_{2g} h_5(pt)\to \tilde h_6(F)\to h_4(pt)\to . \tag{3.2}$$

The map from $h_6(F)$ to $h_4(pt)$ is defined by sending $[N,c_N]\mapsto [Q,c_Q]$, where $c_N\colon N\to B$ is a lift of the normal Gauss map, and $Q\subset N$ is the pre-image of a regular value of the composition of the map to B with the projection to F. The reference map $c_Q\colon Q\to B$ is given by the restriction of c_N to $K := K(H,2)$, together with the induced bundle and (twisted) Spin-structure.

To proceed further we need information about $h_k(pt) = \Omega_k^{\mathrm{Spin}}((\mathbb{CP}^\infty)^r)$, for p_{even}, and $h_k(pt) = \Omega_k^{\mathrm{Spin}}((\mathbb{CP}^\infty)^r, L)$, for p_{odd}. We begin with the case $r = 1$.

Lemma 3.3. *Let L denote the Hopf bundle over $\mathbb{CP}^\infty$.*

(i) *The map $\Omega_4^{\mathrm{Spin}}(\mathbb{CP}^\infty)\to \mathbb{Z}\oplus\mathbb{Z}$ given by the signature and the image of the fundamental class is injective.*

(ii) $\Omega_6^{\text{Spin}}(\mathbb{CP}^\infty) = \mathbb{Z} \oplus \mathbb{Z}$, *detected by the image of the fundamental class and the image* $Dp_1(N)$,

(iii) $\Omega_4^{\text{Spin}}(\mathbb{CP}^\infty; L) \cong \mathbb{Z} \oplus \mathbb{Z}$, *detected by the image of the fundamental class and the image of* $Dp_1(N)$.

Proof. The E^2-term of the Atiyah-Hirzebruch spectral sequence computing $\Omega_4^{\text{Spin}}(\mathbb{CP}^\infty)$ gives $\mathbb{Z}$ in position $(0,4)$ and $(4,0)$, and $\mathbb{Z}/2$ in position $(2,2)$. The differential

$$d\colon H_4(\mathbb{CP}^\infty;\mathbb{Z}) \to H_2(\mathbb{CP}^\infty;\mathbb{Z}/2)$$

is the reduction mod 2 composed by the dual of Sq^2 [12, Proposition 1, p. 750] and so is nontrivial. This implies that

$$\Omega_4^{\text{Spin}}(\mathbb{CP}^\infty) \to \mathbb{Z} \oplus \mathbb{Z}$$

given by the signature and the image of the fundamental class is injective.

Analyzing the Atiyah-Hirzebruch spectral sequence for $\Omega_6^{\text{Spin}}(\mathbb{CP}^\infty)$ gives an entry $\mathbb{Z}$ at position $(2,4)$ and $(6,0)$ and $\mathbb{Z}/2$ at position $(4,2)$. This time the differential vanishes and so the bordism group is either $\mathbb{Z}\oplus\mathbb{Z}$ or $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/2$. It was proven in [9, p. 258] that

$$\Omega_6^{\text{Spin}}(\mathbb{CP}^\infty) = \mathbb{Z} \oplus \mathbb{Z},$$

detected by the image of the fundamental class and the image of $Dp_1(N)$.

Now we consider the bordism groups twisted by the line bundle L. We reduce the 4-th bordism group to the untwisted case by using the isomorphism given by taking the transversal preimage of $\mathbb{CP}^{N-1}$, where we replace $\mathbb{CP}^\infty$ by $\mathbb{CP}^N$ for a large N:

$$\Omega_6^{\text{Spin}}(\mathbb{CP}^\infty) \cong \Omega_4^{\text{Spin}}(\mathbb{CP}^\infty; L)$$

(here we use that $\Omega_6^{\text{Spin}} = \Omega_5^{\text{Spin}} = 0$) implying that $\Omega_4^{\text{Spin}}(\mathbb{CP}^\infty; L) \cong \mathbb{Z}\oplus\mathbb{Z}$ again detected by the image of the fundamental class and the signature.

Finally the computation of $\Omega_4^{\text{Spin}}(\mathbb{CP}^\infty; L) \cong \mathbb{Z} \oplus \mathbb{Z}$, again detected by the image of the fundamental class and the image of $Dp_1(N)$, follows from the Atiyah-Hirzebruch spectral sequence. This time the E^∞-term is torsion free in the 6-line, since the differential $d\colon H_6(\mathbb{CP}^\infty;\mathbb{Z}) \to H_4(\mathbb{CP}^\infty;\mathbb{Z}/2)$ is the reduction mod 2 composed by the dual of Sq^2 plus $c_1(L) \cup ...$) (see again [12, Proposition 1, p. 750]) and so is trivial. □

Lemma 3.4. $h_5(pt)$ *is zero. The map* $h_6(pt) \to H_6(K) \oplus H_2(K)$ *given by the image of the fundamental class and the image of* $Dp_1(N)$ *is injective. The map given by the signature and the image of the fundamental class is an injection* $h_4(pt) \to \mathbb{Z} \oplus H_4(K)$.

Proof. Now we consider the general case. If we show that the bordism groups are again torsion free, then the statements follow from the Atiyah-Hirzebruch spectral sequence. We first note that by applying an appropriate isomorphism of $H \cong \mathbb{Z}^r$ we can assume in the twisted case that $c_1(L) = (0, ..., 0, 1)$. With this we write $(\mathbb{CP}^\infty)^r = X \times \mathbb{CP}^\infty$ and compute $\Omega_k^{\text{Spin}}(X \times \mathbb{CP}^\infty)$ and $\Omega_k^{\text{Spin}}(X \times \mathbb{CP}^\infty; L)$ for $k = 4$ and 6, where $X = (\mathbb{CP}^\infty)^{r-1}$ and L is the Hopf bundle over the last factor. We assume inductively that $\Omega_k(X)$ is torsion free for $k = 4$ and $k = 6$. Using again the transversal preimage of $\mathbb{CP}^{N-1}$, where we replace $\mathbb{CP}^\infty$ by $\mathbb{CP}^N$ for a large N, we have an exact Gysin sequence (see [2, Section I.6, p. 315], [11]):

$$\Omega_5^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty; L) \to \Omega_6^{\mathrm{Spin}}(X) \to \Omega_6^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty) \to \Omega_4^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty; L).$$

Since the odd dimensional groups are by the Atiyah-Hirzebruch spectral sequence torsion, we see that $\Omega_6^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty)$ is torsion free, if $\Omega_4^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty; L)$ is torsion free. For this we consider the corresponding exact Gysin sequence (again, see [2, Section I.6]):

$$\Omega_3^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty; L \oplus L) \to \Omega_4^{\mathrm{Spin}}(X) \to \Omega_4^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty; L) \to \Omega_2^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty; L \oplus L).$$

The Atiyah-Hirzebruch spectral sequence implies that

$$\Omega_2^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty; L \oplus L) \cong H_2(X \times \mathbb{CP}^\infty) \oplus \mathbb{Z}/2.$$

Now we compare this exact sequence with that for X a point:

$$\Omega_3^{\mathrm{Spin}}(\mathbb{CP}^\infty; L \oplus L) \to \Omega_4^{\mathrm{Spin}} \to \Omega_4^{\mathrm{Spin}}(\mathbb{CP}^\infty; L) \to \Omega_2^{\mathrm{Spin}}(\mathbb{CP}^\infty; L \oplus L).$$

We have maps from the first to the second exact sequence given by the projection from X to a point. Now suppose that $\Omega_4^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty; L)$ contains a torsion element. Then, since by assumption $\Omega_4^{\mathrm{Spin}}(X)$ is torsion free, this maps to the non-trivial torsion element in $\Omega_2^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty; L \oplus L)$. But then the image in $\Omega_4^{\mathrm{Spin}}(\mathbb{CP}^\infty; L)$ is again a non-trivial torsion element, since in $\Omega_2^{\mathrm{Spin}}(\mathbb{CP}^\infty; L \oplus L)$ it maps to the non-trivial element. But this is a contradiction to what we have shown above that $\Omega_4^{\mathrm{Spin}}(\mathbb{CP}^\infty; L)$ is torsion free.

Now we have shown half of our statements, namely that $\Omega_6^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty)$ is torsion free as well as $\Omega_4^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty; L)$. We prove the other cases by a similar argument using this time the exact Gysin sequences:

$$\Omega_5^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty; L^{\oplus 2}) \to \Omega_6^{\mathrm{Spin}}(X) \to \Omega_6^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty; L) \to \Omega_4^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty; L^{\oplus 2})$$

and

$$\Omega_3^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty; L^{\oplus 3}) \to \Omega_4^{\mathrm{Spin}}(X) \to \Omega_4^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty; L^{\oplus 2}) \to \Omega_2^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty; L^{\oplus 3}).$$

This case is easier since $\Omega_2^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty; L^{\oplus 3})$ is torsion free, the torsion in the E^2 term is killed by the d_2-differential.

Finally we show that $\Omega_4^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty)$ is torsion free using the exact sequence:

$$\Omega_3^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty; L) \to \Omega_4^{\mathrm{Spin}}(X) \to \Omega_4^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty) \to \Omega_2^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty; L).$$

By the same argument as above the group $\Omega_2^{\mathrm{Spin}}(X \times \mathbb{CP}^\infty; L)$ is torsion free finishing the argument.

Now we show that $h_5(pt) = 0$. On the line corresponding to $h_5(pt)$ the only non-trivial entry in the E_2-term is $H_4(K; \mathbb{Z}/2)$. If $I(N)$ is even, the differentials are even given by the dual of Sq^2. If $I(N)$ is odd, where we had to use twisted Spin-structures, the differentials are given by the dual of Sq^2 plus $x \mapsto Sq^2 x + w_2 \cup x$, where w_2 is the reduction of c mod 2. It is an easy exercise to show that the E^3-term is zero in both cases. □

With this information we show that the bordism classes of N and $M \times F$, equipped with the normal 2-smoothings constructed in Section 2, agree when identified via the maps α and ϕ. By the exact sequence (3.2) and Lemma 3.4, this amounts to showing (i) the bordism classes in $h_6(pt)$ agree, and (ii) that the classes in $h_4(pt)$ agree, which we obtain as transversal preimages of a regular value of the map to F given by composing our normal 2-smoothings with the projection to F.

By Lemma 3.4, the first invariant is given by two invariants, the image of the fundamental class in $H_6(K(H,2))$ and the image of $Dp_1(N)$ in $H_2(K(H,2))$. The image of the fundamental class in $H_6(K(H,2))$ is (by the cohomological structure of $K(H,2)$) equivalent to the triple product $g^*(x) \cup g^*(y) \cup g^*(z)$ for classes x, y, z in $H^2(K(H,2))$. But these products vanish for $M \times F$ with $g = g_{M\times F}$, and for N with $g = g_N$, since ϕ is an isometry of the cohomology rings and $(g_N)^* = \phi \circ (g_{M\times F})^*$. The image of $Dp_1(N)$ in $H_2(K(H,2))$ is determined by the products $g^*(x) \cup p_1$ for all $x \in H^2(K(H,2))$ and vanishes for $M \times F$ and for N by Condition 4.

Thus we are left with the invariant in $h_4(pt)$. Let $Q \subset N$ be the transversal preimage of a regular value of the map $u\colon N \to F$. By Lemma 3.4, bordism classes in $h_4(pt)$ are determined by the signature of the underlying 4-manifold, and the image of the fundamental class $[Q]$ in $H_4(K(H,2)$. For a class $\beta \in H^4(N)$ we have the adjunction formula

$$\langle u^*([F]) \cup \beta, [N]\rangle = \langle i^*(\beta), [Q]\rangle,$$

where $i\colon Q \to N$ is the inclusion. Applying this to $\beta = p_1(N)$ we obtain:

$$\langle p_1(N) \cup u^*([F]), [N]\rangle = \langle p_1(Q), [Q]\rangle,$$

since the normal bundle of Q is trivial. The signature theorem for Q and Condition 4 (iv) imply that

$$\langle p_1(N) \cup u^*([F]), [N]\rangle = 3\,\mathrm{sign}(Q) = 3\,\mathrm{sign}(M),$$

proving the equality for the first invariant in $h_4(pt)$.

For the second invariant we note that the image of the fundamental class of Q in $H_4(K(H,2)$ is determined by the numbers

$$\langle i^*g^*(x) \cup i^*g^*(y), [Q]\rangle.$$

We apply again the adjunction formula for $\beta = g^*(x) \cup g^*(y)$, and get

$$\langle g^*(x) \cup g^*(y) \cup u^*([F]), [N]\rangle = \langle i^*g^*(x) \cup i^*g^*(y), [Q]\rangle,$$

where $g = g_N$. A similar formula holds for $M \times F$ and $g = g_{M\times F}$. The left side agrees for N and $M \times F$, since ϕ is an isomorphism of the cohomology ring. Thus also the second invariant for the element in $h_4(pt)$ agrees. Summarizing, we have shown:

Proposition 3.5. *If the conditions of Theorem A are fulfilled, then the bordism classes*

$$[N, c_N] = [M \times F, c_{M\times F}] \in \Omega_6(B, \xi),$$

for the normal 2*-smoothings on* N *and* $M \times F$ *constructed in Lemma* 2.2.

4. The proof of Theorem A

We consider N and $M \times F$ equipped with normal 2-smoothings compatible with α and ϕ. By Proposition 3.5, the corresponding bordism classes are equal. Choose a B-bordism W between these two normal 2-smoothings. Since the Euler characteristics of N and $M \times F$ agree, there is an obstruction $\theta(W) \in l_7(\pi_1(F))$ which is elementary if and only if W is B-bordant to an s-cobordism. We first note that the Whitehead group for $\pi_1(F)$ vanishes by a result of Farrell-Hsiang [3], so that we can ignore decorations in the l-monoids and L-groups. Next we note that the intersection form on $\pi_3(M \times F) \cong \pi_3(M)$ with values in the group ring vanishes identically (since $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}G) = 0$ for G an infinite group). By [8, Proposition 8, p. 739], this implies that $\theta(W)$ sits in the ordinary L-group $L_7(\pi_1(F))$. But by Cappell [1, Theorem 18], there is a closed 7-manifold with B-structure so that after taking the disjoint union of W with this manifold the obstruction in $L_7(\pi_1(F))$ vanishes. This completes the proof.

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