Addendum to the paper: "Artin Prime Producing Quadratics" [Abh. Math. Sem. Univ. Hamburg 77 (2007), 109–127; MR2379332 (2008m:11194)] by P. Moree

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Abstract

A record mentioned in the paper was recently improved on by Akbary and Scholten. However, the record mentioned was not the then record. The then record, due to Gallot (2004), actually slightly improves on that obtained recently by Akbary and Scholten.

Given an integer g and a polynomial $f(X) \in \mathbb{Z}[X]$, let $p_1(g,f), p_2(g,f), \ldots$ be the sequence of primes that is obtained on going through the sequence $f(0), f(1), \ldots$ and writing down the primes not dividing g as they appear (called Artin primes). We let r be the largest integer r (if this exists) such that g is a primitive root mod p for all primes $p_j(g,f)$ with $1 \leq j \leq r$. We let $c_g(f)$ be the number of distinct primes amongst $p_j(g,f)$ with $1 \leq j \leq r$.

In [6] the problem was addressed of finding an integer g and a quadratic polynomial f such that $c_g(f)$ is as large as possible and it was stated that

$$c_g(f) = 31082$$

was the current record (obtained by Yves Gallot). On preparing the paper for publication (fall 2006) the author failed to recall an e-mail by Gallot from June 2004. That e-mail actually stated what in 2006 still would be the true current record (due to Gallot), namely

$$c_g(f) = 38639.$$

It is obtained on taking $f(X) = 32X^2 + 39721664X + 1822153811472858449$ and $g = 593856338459898$. Perhaps a more elegant reformulation is: for those 38639 integers n in $[620651, 1749283]$ for which

$$h(n) := 32n^2 + 1822153688206406068 17$$

is prime, the number 593856338459898 is a primitive root modulo $h(n)$.

In a recent paper by Akbary and Scholten [1] the authors find a g and a quadratic f such that $c_g(f) = 37951$. This improves on the record indicated in [6], but falls slightly below the 'hidden record' indicated above.

Akbary and Scholten go beyond Moree in that they in addition consider the case where f linear and f cubic and obtain here record values for consecutive Artin primes for certain integers g of 6355, respectively 10011.

Finally, let us mention some highly interesting work by Pollack [7]. He merges the method of proof of Hooley [3] of Artin's conjecture (under GRH) with the method of Maynard-Tao [4, 5] in order to produce bounded gaps between primes: On GRH for every nonsquare $g \neq -1$ and every m, there are infinitely many runs of m consecutive primes all possessing g as a primitive root and lying in an interval of order $O_m(1)$. For related work see Baker and Pollack [2].

References

[1] A. Akbary and K. Scholten, Artin prime producing polynomials, arXiv:1310.5198, to appear in Mathematics of Computation. (Electronically published in December 2014.)

[2] R.C. Baker and P. Pollack, Bounded gaps between primes with a given primitive root, II, arXiv:1407.7186.

[3] C. Hooley, On Artin's conjecture, *J. Reine Angew. Math.* **225** (1967), 209–220.

[4] J. Maynard, Small gaps between primes, *Ann. Math.*, to appear.

[5] D.H.J. Polymath, Variants of the Selberg sieve, and bounded intervals containing many primes, arXiv:1407.4897.

[6] P. Moree, Artin prime producing quadratics, arXiv:1407.4897. *Abh. Math. Sem. Univ. Hamburg* **77** (2007), 109–127.

[7] P. Pollack, Bounded gaps between primes with a given primitive root, arXiv:1404.4007v3.