This article has been accepted for publication in International Mathematics Research Notices
Published by Oxford University Press.
http://dx.doi.org/10.1093/imrn/rnv021

A characterization of holomorphic Borcherds lifts by symmetries (A running title: Characterization of Borcherds lifts)

Bernhard Heim [*] Atsushi Murase [†]

December 27, 2014

Abstract

In this paper we show that, in the holomorphic automorphic case, the Borcherds lifts on the orthogonal group $O(2, n+2)$ are characterized by certain symmetries and describe the inverse of the Borcherds lifting in terms of Fourier-Jacobi expansion. We also give a characterization of the modular polynomials by certain symmetries.

Keywords: Automorphic forms, Jacobi forms, Borcherds lifts, symmetries, Heegner divisors, modular polynomials

Contents

[*]Department of Mathematics and Science, German University of Technology in Oman (GUtech), P.O. Box: 1816, PC 130, Sultanate of Oman, e-mail:bernhard.heim@gutech.edu.om

[†]Department of Mathematical Science, Kyoto Sangyo University, Motoyama, Kamigamo, Kita-ku, Kyoto 603-8555, Japan, e-mail: murase@cc.kyoto-su.ac.jp (corresponding author)

1 Introduction

1.1 The aim of the paper

Let L be an even lattice of signature $(2, n+2)$ and $V := L \otimes_{\mathbb{Z}} \mathbb{R}$. Let Γ denote the discriminant kernel subgroup of the orthogonal group $O(V)$ of V (for the definition see (5.3)). Borcherds lifts [Bo2, Bo4] are automorphic forms on Γ that have infinite product expansions, and whose zeros and poles are supported on linear combinations of Heegner divisors (special divisors or rational quadratic divisors). They have found various applications in arithmetic, geometry and the theory of Lie algebras. For example they appear as denominator functions of certain generalized Kac-Moody algebras [Bo1, GN1, GN2, Sc1, Sc2], as new infinite product representations of well-known classical functions [Bo1, Bo2, GN1, GN2, Konts], or as the partition function of quarter-BPS dyons [DVV, CD, DMZ]. The lift is also used to construct automorphic forms on

Γ with known vanishing properties to obtain geometric results [Bo3, Bo5, Br1, FS]. We also refer to applications in algebraic and arithmetic geometry [AF, BBK, Kond]. In the study of the graded ring of modular forms, it is also an interesting question to identify Borcherds and Saito-Kurokawa-Oda-Rallis-Schiffmann lifts among the generators (see for example [AI, DK, FS]).

It is a natural and important problem to characterize Borcherds lifts. The first result in this direction was obtained by Bruinier. In [Br1] he characterizes Borcherds lifts as meromorphic automorphic forms whose divisors are linear combinations of Heegner divisors with integral coefficients under the assumption that L splits two hyperbolic planes over $\mathbb{Z}$ (for a precise definition of Heegner divisors, see Section 5.2). A generalization of [Br1] is given in [Br2]. For other proofs of Bruinier's characterization in a special case that L is unimodular, we refer to [BrFr] and [BrFu].

The main aim of the present paper is to give another characterization of Borcherds lifts by certain symmetries and to describe the inverse of the Borcherds lifting in terms of Fourier-Jacobi expansions in the holomorphic automorphic case (we also assume that L splits two hyperbolic planes over $\mathbb{Z}$).

We explain our results more precisely in the $O(2,3)$-case (namely the genus two Siegel modular case). Borcherds lifts are meromorphic continuations of infinite products constructed from weakly holomorphic Jacobi forms ([Bo2, GN1, GN2]) or weakly holomorphic vector valued modular forms ([Bo4, Br1] for example). In this paper we take the former approach using Jacobi forms as input data, since it is more convenient to describe the inverse of the Borcherds lifting.

Let $\mathfrak{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ denote the upper half plane, $\mathfrak{H}_2 := \{Z \in \mathrm{M}_2(\mathbb{C}) \mid {}^tZ = Z, \mathrm{Im}(Z) > 0\}$ the Siegel upper half space of genus two and $\Gamma := \mathrm{Sp}_2(\mathbb{Z})$ the Siegel modular group of genus two acting on $\mathfrak{H}_2$ in a usual way. We write (τ, z, τ') for $\begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathfrak{H}_2$. We put $\mathcal{M}(\Gamma) := \bigcup_{k,\chi} \mathrm{M}_k(\Gamma, \chi)$ (respectively $\mathcal{A}(\Gamma) := \bigcup_{k,\chi} \mathrm{A}_k(\Gamma, \chi)$), k running over the nonnegative integers and χ over the characters of Γ of finite order. Here $\mathrm{M}_k(\Gamma, \chi)$ (respectively $\mathrm{A}_k(\Gamma, \chi)$) is the space of holomorphic (respectively meromorphic) automorphic forms on Γ of weight k and character χ.

Let $\phi(\tau, z) = \sum_{l,r\in\mathbb{Z}} c(l, r)\mathbf{e}(l\tau + rz)$ ($\tau \in \mathfrak{H}, z \in \mathbb{C}$) be a weakly holomorphic Jacobi form of weight 0 and index 1 (for a precise definition of Jacobi forms see Section 3.2). Here $\mathbf{e}(x) := \exp(2\pi ix)$ for $x \in \mathbb{C}$. The Fourier coefficient $c(l, r)$ depends only on $4l - r^2$, for which we write $c(4l - r^2)$. Suppose that

$$c(-m) \in \mathbb{Z} \text{ for } m \in \mathbb{Z}_{>0} \text{ with } m \equiv 0 \text{ or } 1 \pmod 4. \tag{1.1}$$

We also suppose that

$$c(0) \in 2\mathbb{Z} \tag{1.2}$$

for simplicity. Then an infinite product

$$\Psi_\phi(\tau, z, \tau') := \mathbf{e}(\lambda\tau - \rho z + \mu\tau') \prod_{(l,r,m)>0} \left(1 - \mathbf{e}(l\tau + rz + m\tau')\right)^{c(4lm-r^2)}$$

is continued to a meromorphic function on $\mathfrak{H}_2$ and defines an element of $A_{c(0)/2}(\Gamma, \chi)$ with a character χ of Γ of finite order. Here $(\lambda, \rho, \mu) \in \mathbb{Q}^3$ is the Weyl vector determined by ϕ and $(l, r, m) > 0$ means that either "$m > 0$" or "$m = 0, l > 0$" or "$m = l = 0, r > 0$" holds (see Section 5.2). We call Ψ_ϕ the *Borcherds lift* of ϕ. It is known that the divisor of Ψ_ϕ is a linear combination of Heegner divisors with integral coefficients (see Section 5.2; see also Section 2 of [HM1] in $O(2,3)$-case). If furthermore the condition

$$\sum_{j=1}^{\infty} c(-j^2 m) \geq 0 \qquad \text{for any } m \in \mathbb{Z}_{>0} \text{ with } m \equiv 0 \text{ or } 1 \pmod 4 \tag{1.3}$$

is satisfied, then Ψ_ϕ is holomorphic on $\mathfrak{H}_2$ and hence belongs to $\mathcal{M}(\Gamma)$.

In our previous paper [HM2] we showed that $F = \Psi_\phi$ satisfies the following *multiplicative symmetries*; for any $N \geq 2$, we have

$$\prod_{ad=N} \prod_{b=0}^{d-1} F\left(\frac{a\tau + b}{d}, \frac{\sqrt{N}}{d} z, \tau'\right) = \epsilon_N \prod_{ad=N} \prod_{b=0}^{d-1} F\left(\tau, \frac{\sqrt{N}}{d} z, \frac{a\tau' + b}{d}\right) \tag{1.4}$$

with $\epsilon_N \in \mathbb{C}^\times$. Here (a, d) runs over the pairs of positive integers with $ad = N$. It is a natural question to ask whether Borcherds lifts are characterized by multiplicative symmetries. In this paper we give an affirmative answer to this question in the *holomorphic* automorphic case.

Let $F \in \mathcal{M}(\Gamma)$. Then F admits the Fourier-Jacobi expansion

$$F(\tau, z, \tau') = \mathbf{e}(\mu\tau') \sum_{m=0}^{\infty} F_m(\tau, z) \mathbf{e}(m\tau'), \tag{1.5}$$

where $\mu \in \mathbb{Q}$ and $F_0 \neq 0$. We put $f_m(\tau, z) = F_m(\tau, z)/F_0(\tau, z)$ for $m \geq 0$. In general f_m may have poles on $\mathfrak{H} \times \mathbb{C}$ if $m \geq 1$. Nevertheless we can prove the following (for details in the general case see Section 6):

Theorem 1.1. *Suppose that $F \in \mathcal{M}(\Gamma)$ satisfies multiplicative symmetries. Then f_m is holomorphic on $\mathfrak{H} \times \mathbb{C}$ for any $m \geq 1$. Furthermore $\phi_F := -f_1$ is a weakly holomorphic Jacobi form of weight* 0 *and index* 1 *satisfying* (1.1)*,* (1.2) *and* (1.3)*, and F is a constant multiple of the Borcherds lift Ψ_{ϕ_F}.*

One of the keys of the proof of Theorem 1.1 is the following recurrence relations: If F satisfies multiplicative symmetries, the equality

$$\mathcal{G}_N(f_1(\tau, z), \ldots, f_N(\tau, z)) = N^{-1} \sum_{ad=N} \sum_{b=0}^{d-1} f_1\left(\frac{a\tau + b}{d}, az\right) \tag{1.6}$$

holds for any integer $N \geq 2$. Here $\mathcal{G}_N(X_1, \ldots, X_N)$ $(N \geq 1)$ are polynomials defined by

$$1 + X_1 t + X_2 t^2 + \cdots = \exp\left(\sum_{N=1}^{\infty} \mathcal{G}_N(X_1, \ldots, X_N)\, t^N\right) \tag{1.7}$$

(see Section 2).

Remark 1.2. We need to assume the holomorphy of F in order to show that ϕ_F is holomorphic on the whole $\mathfrak{H} \times \mathbb{C}$.

Let $\widetilde{\mathcal{J}}$ be the set of weakly holomorphic Jacobi forms ϕ of weight 0 and index 1 satisfying (1.1) and (1.2), and $\mathcal{J}$ the set of $\phi \in \widetilde{\mathcal{J}}$ satisfying (1.3). Recall that $\Psi_\phi \in \mathcal{A}(\Gamma)$ for $\phi \in \widetilde{\mathcal{J}}$ and that $\Psi_\phi \in \mathcal{M}(\Gamma)$ if and only if $\phi \in \mathcal{J}$. Let

$$\begin{aligned}
\mathcal{M}(\Gamma)_{\mathrm{BL}} &:= \left\{ c\Psi_\phi \mid c \in \mathbb{C}^\times, \phi \in \mathcal{J} \right\}, \\
\mathcal{M}(\Gamma)_{\mathrm{HD}} &:= \{ F \in \mathcal{M}(\Gamma) \setminus \{0\} \mid \text{the divisor of } F \text{ is a linear combination of Heegner divisors} \\
&\qquad\qquad \text{with integral coefficients} \}, \\
\mathcal{M}(\Gamma)_{\mathrm{MS}} &:= \{ F \in \mathcal{M}(\Gamma) \setminus \{0\} \mid F \text{ satisfies multiplicative symmetries} \}.
\end{aligned}$$

Our main results are summarized as follows:

Theorem 1.3. *We have*

$$\mathcal{M}(\Gamma)_{\mathrm{BL}} = \mathcal{M}(\Gamma)_{\mathrm{MS}}. \tag{1.8}$$

The Borcherds lifting $\phi \mapsto (\Psi_\phi \bmod \mathbb{C}^\times)$ *gives rise to a bijection*

$$\mathcal{J} \to \mathcal{M}(\Gamma)_{\mathrm{MS}}/\mathbb{C}^\times$$

and its inverse is given by $F \mapsto \phi_F$.

Remark 1.4. Let

$$\begin{aligned}
\mathcal{A}(\Gamma)_{\mathrm{BL}} &:= \left\{ c\Psi_\phi \mid c \in \mathbb{C}^\times, \phi \in \widetilde{\mathcal{J}} \right\}, \\
\mathcal{A}(\Gamma)_{\mathrm{HD}} &:= \{ F \in \mathcal{A}(\Gamma) \setminus \{0\} \mid \text{the divisor of } F \text{ is a linear combination of Heegner divisors} \\
&\qquad\qquad \text{with integral coefficients} \}, \\
\mathcal{A}(\Gamma)_{\mathrm{MS}} &:= \{ F \in \mathcal{A}(\Gamma) \setminus \{0\} \mid F \text{ satisfies multiplicative symmetries} \}.
\end{aligned}$$

In [Br1] and [Br2] Bruinier showed that

$$\mathcal{A}(\Gamma)_{\mathrm{BL}} = \mathcal{A}(\Gamma)_{\mathrm{HD}}. \tag{1.9}$$

This immediately implies

$$\mathcal{M}(\Gamma)_{\mathrm{BL}} = \mathcal{M}(\Gamma)_{\mathrm{HD}}. \tag{1.10}$$

In Section 7 we give a simple proof of (1.10) based on our characterization of Borcherds lifts by symmetries. One of the referees suggested that we can deduce (1.9) from (1.10) in the $\mathrm{Sp}_2(\mathbb{Z})$-case (for details, see Remark 7.2). It is an interesting open question whether the equality $\mathcal{A}(\Gamma)_{\mathrm{BL}} = \mathcal{A}(\Gamma)_{\mathrm{MS}}$ holds. Note that the inclusion $\mathcal{A}(\Gamma)_{\mathrm{BL}} \subset \mathcal{A}(\Gamma)_{\mathrm{MS}}$ holds ([HM2]).

1.2 Organization of the paper

The paper is organized as follows.

Section 2 is of preliminary nature. We recall several properties of polynomials $\mathcal{G}_N(X_1, \ldots, X_N)$ defined by (1.7). Note that such polynomials appear in other areas of mathematics (for example see [KT]).

To prove our main results on a characterization of Borcherds lifts by symmetries, we need to show similar results for weakly holomorphic Jacobi forms, which are of independent interest. In Section 3, after recalling the definition of weakly holomorphic Jacobi forms, we define a Jacobi form ϕ_υ attached to an integral vector system υ as an infinite product after [Bo2] and [Bo4]. In Section 4 we state and prove the main results for Jacobi forms; a characterization of weakly holomorphic Jacobi forms attached to integral vector systems by certain symmetries. To be more precise, let φ be a weakly holomorphic Jacobi form and let

$$\varphi(\tau, z) = \mathbf{e}(\lambda\tau) \sum_{N=0}^{\infty} A_N(z)\mathbf{e}(N\tau) \qquad (\tau \in \mathfrak{H}, z \in \mathbb{C}^n)$$

be its Fourier expansion in τ with $\lambda \in \mathbb{Q}$ and $A_0(z) \neq 0$. Put $B_N(z) := A_N(z)/A_0(z)$ for $N \geq 0$. In general $B_N(z)$ may have poles on $\mathbb{C}^n$ for $N \geq 1$. We say that a function ϕ on $\mathfrak{H} \times \mathbb{C}^n$ satisfies multiplicative symmetries of Jacobi type if, for any integer $N \geq 2$, we have

$$\prod_{ad=N} \prod_{b=0}^{d-1} \phi\left(\frac{a\tau + b}{d}, az\right) = \epsilon_N \prod_{ad=N} \phi(\tau, az)^d$$

with a nonzero constant ϵ_N. Note that, for any integral vector system υ, ϕ_υ satisfies multiplicative symmetries of Jacobi type. Suppose that a weakly holomorphic Jacobi form φ satisfies multiplicative symmetries of Jacobi type. Then we show the following results (Theorem 4.2):

(a) For any $N \geq 1$, $B_N(z)$ is holomorphic.

(b) The Fourier coefficients of $-B_1(z)$ define an integral vector system υ.

(c) φ is a constant multiple of ϕ_υ.

The key of the proofs is the recurrence relations satisfied by $B_N(z)$ similar to (1.6):

$$\mathcal{G}_N(B_1(z), \ldots, B_N(z)) = \sum_{d|N} d^{-1} B_1(dz) \qquad (N \geq 1) \tag{1.11}$$

(Proposition 4.4). We use this recurrence relations twice; in the proofs of (a) and (c). The plan of the proofs is explained at the end of Section 4.2.

In Sections 5 and 6 we study Borcherds lifts on $\Gamma \subset O(2, n+2)$ and their characterization by symmetries, assuming that $n > 0$ and that L splits two hyperbolic planes over $\mathbb{Z}$. The case $n = 0$ is treated separately in Section 8. We need to assume that L splits two hyperbolic planes over $\mathbb{Z}$, since we do not have suitable symmetries otherwise. Note that, in the Hilbert modular case, Borcherds lifts satisfy certain multiplicative symmetries ([HM3]). In Section 5 we recall the definitions of holomorphic automorphic forms on Γ and Borcherds lifts of weakly holomorphic Jacobi forms of weight 0. In Section 6 we state and prove the main results of the present paper (Theorem 6.3 and Corollary 6.4); a characterization of holomorphic Borcherds lifts by multiplicative symmetries and an explicit form of the inverse map of Borcherds lifting in the holomorphic automorphic case. The proofs are similar to those of the main results for Jacobi forms and we explain the plan of the proofs at the end of Section 6.2.

Sections 7 and 8 are independent of the previous sections. In Section 7 we show additive symmetries for Heegner divisors (Proposition 7.1), which might be known to experts. As an application of this fact and Corollary 6.4, we give a simple proof of the equality $\mathcal{M}(\Gamma)_{\mathrm{BL}} = \mathcal{M}(\Gamma)_{\mathrm{HD}}$. As explained before, this equality also follows from much more general results due to Bruinier ([Br1]; see also [Br2]), though the proof of [Br1] is different from ours.

In Section 8 we show that the modular polynomials are characterized by certain symmetries as an application of characterization results in the case of automorphic forms on $\mathrm{SL}_2(\mathbb{Z})\times\mathrm{SL}_2(\mathbb{Z})$ (the $O(2,2)$-case), in which case we need a separate treatment since the Koecher principle does not hold.

1.3 Notation

For $x \in \mathbb{C}$, we put $\mathbf{e}(x) := \exp(2\pi i x)$. For a symmetric matrix T of degree m and $x, y \in \mathbb{C}^m$, we put $T[x] := {}^t x T x$ and $T(x,y) := {}^t x T y$. We denote the set of nonnegative (respectively positive) integers by $\mathbb{Z}_{\geq 0}$ (respectively by $\mathbb{Z}_{>0}$). Until the end of Section 6 we fix a positive definite even integral symmetric matrix S of size $n \geq 1$. Let $L_0 := \mathbb{Z}^n$, $L_0^* := S^{-1}L_0$ and identify $V_0 := L_0 \otimes_{\mathbb{Z}} \mathbb{R}$ with $\mathbb{R}^n$. We put $S' := 2^{-1}S$.

2 The polynomials $\mathcal{G}_m$

In this section we recall the definition and properties of certain polynomials used later.

Let $X_1, X_2, \ldots$ be indeterminates. We define polynomials $\mathcal{G}_m(X_1, \ldots, X_m)$ $(m \geq 1)$ by

$$1 + \sum_{l=1}^{\infty} X_l t^l = \exp\left(\sum_{m=1}^{\infty} \mathcal{G}_m(X_1, \ldots, X_m) t^m\right). \tag{2.1}$$

For example we have $\mathcal{G}_1(X_1) = X_1$, $\mathcal{G}_2(X_1, X_2) = X_2 - 2^{-1}X_1^2$ and $\mathcal{G}_3(X_1, X_2, X_3) = X_3 - X_1X_2 + 3^{-1}X_1^3$. It is easily seen that

$$\mathcal{G}_m(X_1, \ldots, X_m) = X_m + (\text{a polynomial in } X_1, \ldots, X_{m-1}). \tag{2.2}$$

The following elementary facts will be used later.

Lemma 2.1. *(1) We have*

$$\mathcal{G}_m\left(\frac{1}{1!}, \frac{1}{2!}, \ldots, \frac{1}{m!}\right) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m \geq 2. \end{cases}$$

(2) For $m \geq 1$ put

$$R_m(t; X_1, X_2, \ldots) := \prod_{j=0}^{m-1} \left(1 + \sum_{l=1}^{\infty} X_l \mathbf{e}(jl/m) t^l\right). \tag{2.3}$$

We then have

$$R_m(t; X_1, X_2, \ldots) = 1 + m\mathcal{G}_m(X_1, \ldots, X_m) t^m + O(t^{2m}).$$

(3) Let ρ be a positive real number. Assume that a sequence $\{a_m\}_{m\geq 1}$ of complex numbers satisfies the following two conditions:

(i) *The power series $f(z) := \sum_{m=1}^{\infty} a_m z^m$ is absolutely and uniformly convergent in $\mathbf{D}_\rho := \{z \in \mathbb{C} \mid |z| \leq \rho\}$.*

(ii) *We have $\sum_{m=1}^{\infty} |a_m z^m| < 1$ for $z \in \mathbf{D}_\rho$.*

Then the series

$$\sum_{m=1}^{\infty} \mathcal{G}_m(a_1, \ldots, a_m) z^m \tag{2.4}$$

is absolutely and uniformly convergent in $\mathbf{D}_\rho$, and the equality

$$1 + f(z) = \exp\left(\sum_{m=1}^{\infty} \mathcal{G}_m(a_1, \ldots, a_m) z^m\right) \tag{2.5}$$

holds for $z \in \mathbf{D}_\rho$.

Proof. The first and second assertions are easily verified. We give a proof of the third for completeness. First observe that

$$\mathcal{G}_m(X_1, \ldots, X_m) = -\sum_{k=1}^{m} \frac{(-1)^k}{k} \mathcal{G}_{m,k}(X_1, \ldots, X_m),$$

where

$$\mathcal{G}_{m,k}(X_1, \ldots, X_m) := \sum_{j_1, \ldots, j_k \geq 1,\, j_1 + \cdots + j_k = m} X_{j_1} \cdots X_{j_k}.$$

Fix $z \in \mathbf{D}_\rho$ and put $r := |z|$. By condition (ii), we have $|f(z)| < 1$ and

$$\log(1 + f(z)) = -\sum_{k=1}^{\infty} \frac{(-1)^k}{k} f(z)^k = -\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sum_{m=1}^{\infty} \mathcal{G}_{m,k}(a_1, \ldots, a_m) z^m. \tag{2.6}$$

We now show that the double series

$$\sum_{k,\, m=1}^{\infty} \left| \frac{(-1)^k}{k} \mathcal{G}_{m,k}(a_1, \ldots, a_m) z^m \right| \tag{2.7}$$

is convergent. Since $\mathcal{G}_{m,k}(X_1, \ldots, X_m)$ is a polynomial in $X_1, \ldots, X_m$ with nonnegative coefficients, we have $|\mathcal{G}_{m,k}(a_1, \ldots, a_m)| \leq \mathcal{G}_{m,k}(\alpha_1, \ldots, \alpha_m)$ with $\alpha_i := |a_i|$. We also note that the series $A := \sum_{m=1}^{\infty} \alpha_m r^m$ is convergent and $0 \le A < 1$ by conditions (i) and (ii). For any positive integers K and M, we have

$$\sum_{k=1}^{K} \sum_{m=1}^{M} \left| \frac{(-1)^k}{k} \mathcal{G}_{m,k}(a_1, \dots, a_m) z^m \right|$$
$$\le \sum_{k=1}^{K} \frac{1}{k} \sum_{m=1}^{\infty} \mathcal{G}_{m,k}(\alpha_1, \dots, \alpha_m) r^m = \sum_{k=1}^{K} \frac{1}{k} \left(\sum_{m=1}^{\infty} \alpha_m r^m \right)^k \le \sum_{k=1}^{\infty} \frac{1}{k} A^k = -\log(1-A),$$

which proves the convergence of the series (2.7). Thus we can change the order of summation in the right-hand side of (2.6) and get

$$\log(1+f(z)) = -\sum_{m=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \mathcal{G}_{m,k}(a_1, \dots, a_m) \right) z^m = \sum_{m=1}^{\infty} \mathcal{G}_m(a_1, \dots, a_m) z^m.$$

We here note that $\mathcal{G}_{m,k} = 0$ if $k > m$. Exponentiating the above equality, we obtain (2.5). It is easily seen that the series (2.4) is absolutely and uniformly convergent in $\mathbf{D}_\rho$. □

3 Jacobi forms attached to vector systems

In this section we recall the definition of Jacobi forms attached to vector systems introduced by Borcherds [Bo2], [Bo4].

3.1 Jacobi groups

Let H be an algebraic group defined over $\mathbb{Q}$ with $H(\mathbb{Q}) = \mathbb{Q}^n \times \mathbb{Q}^n \times \mathbb{Q}$ and multiplication law

$$(u, v, t)(u', v', t') = (u + u', v + v', t + t' + S(u, v')).$$

The group SL_2 acts on the *Heisenberg group* H on the right via

$$(u, v, t) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} := (u_1, v_1, t + S'(u_1, v_1) - S'(u, v)) \qquad (u_1 = au + cv,\ v_1 = bu + dv).$$

We call the semidirect product $G^J := \mathrm{SL}_2 \ltimes H$ the *Jacobi group* and $\Gamma^J := G^J(\mathbb{Z})$ the *Jacobi modular group*.

3.2 Weakly holomorphic Jacobi forms

Let $k, m \in \mathbb{Z}$. For a function φ on $\mathfrak{H} \times \mathbb{C}^n$ and $g = (u, v, t) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G^J(\mathbb{R})$, we set

$$\left(\varphi|_{k,mS'} g\right)(\tau, z) = (c\tau + d)^{-k} \mathbf{e}_m \left(-\frac{c}{c\tau + d} S'[z] + \frac{a\tau + b}{c\tau + d} S'[u] + \frac{2}{c\tau + d} S'(u, z) + t \right)$$
$$\times \varphi \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} + \frac{a\tau + b}{c\tau + d} u + v \right),$$

where $\mathbf{e}_m(x) := \exp(2\pi i m x)$ for $x \in \mathbb{C}$. Let χ be a character of Γ^J of finite order. Let φ be a holomorphic function on $\mathfrak{H} \times \mathbb{C}^n$ satisfying

$$\varphi|_{k,mS'}\gamma = \chi(\gamma)\varphi \qquad (\gamma \in \Gamma^J). \tag{3.1}$$

Then φ admits the Fourier expansion

$$\varphi(\tau, z) = \sum_{l \in l_0 + \mathbb{Z},\, \alpha \in \alpha_0 + L_0^*} a_\varphi(l, \alpha)\mathbf{e}(l\tau + S(\alpha, z)) \tag{3.2}$$

with some $l_0 \in \mathbb{Q}$ and $\alpha_0 \in \mathbb{Q}^n$. Here we recall $L_0^* = S^{-1}\mathbb{Z}^n$. Let $J^!_{k,mS',\chi}$ be the space of holomorphic functions φ on $\mathfrak{H} \times \mathbb{C}^n$ satisfying (3.1) and

$$a_\varphi(l, \alpha) = 0 \quad \text{if } l - mS'[\alpha] \text{ is sufficiently small.} \tag{3.3}$$

We call $J^!_{k,mS',\chi}$ the space of *weakly holomorphic Jacobi forms* of weight k, index mS' and character χ.

3.3 Vector systems

A $\mathbb{C}$-valued function υ on L_0^* is called a *vector system* on L_0^* if the following three conditions are satisfied (see [Bo2] and [Bo4]):

(V1) $R := \{\alpha \in L_0^* \mid \upsilon(\alpha) \neq 0\}$ is a finite set.

(V2) We have $\upsilon(-\alpha) = \upsilon(\alpha)$ for $\alpha \in L_0^*$.

(V3) There exists a constant μ_υ such that

$$\sum_{\alpha \in R} \upsilon(\alpha)S(\alpha, u)\alpha = 2\mu_\upsilon u$$

holds for any $u \in V_0 = \mathbb{R}^n$.

We call μ_υ the *index* of υ. It is known that

$$\mu_\upsilon = \frac{1}{2n}\sum_{\alpha \in R} \upsilon(\alpha)S[\alpha].$$

A vector system υ is said to be *integral* if $\upsilon(\alpha) \in \mathbb{Z}$ for any $\alpha \in R$.

A connected component of $V_0 \setminus \left(\bigcup_{\alpha \in R\setminus\{0\}} \{u \in V_0 \mid S(\alpha, u) = 0\}\right)$ is called a *Weyl chamber* with respect to υ. We fix a Weyl chamber W and put $R^+ := \{\alpha \in R \mid \alpha > 0\}$, where $\alpha > 0$ means $S(\alpha, u_0) > 0$ for some $u_0 \in W$. Define the *Weyl vector* $\Lambda_\upsilon := (\lambda_\upsilon, \rho_\upsilon, \mu_\upsilon) \in \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}$ with respect to (υ, W) by

$$\lambda_\upsilon := \frac{1}{24}\sum_{\alpha \in R} \upsilon(\alpha),\ \rho_\upsilon := \frac{1}{2}\sum_{\alpha \in R^+} \upsilon(\alpha)\alpha. \tag{3.4}$$

We also put

$$k_\upsilon := \frac{1}{2}\upsilon(0). \tag{3.5}$$

3.4 Jacobi forms attached to vector systems

Let υ be an integral vector system on L_0^*. We define a function ϕ_υ on $\mathfrak{H} \times \mathbb{C}^n$ by

$$\phi_\upsilon(\tau, z) := \mathbf{e}(\lambda_\upsilon \tau - S(\rho_\upsilon, z)) \\ \times \prod_{l=1}^{\infty} \prod_{\alpha \in R} (1 - \mathbf{e}(l\tau + S(\alpha, z)))^{\upsilon(\alpha)} \prod_{\alpha \in R^+} (1 - \mathbf{e}(S(\alpha, z)))^{\upsilon(\alpha)} \quad (\tau \in \mathfrak{H}, z \in \mathbb{C}^n).$$

Suppose that

$$\upsilon(0) \in 2\mathbb{Z} \quad \text{and} \quad \mu_\upsilon \in \mathbb{Z}. \tag{3.6}$$

Then ϕ_υ defines a meromorphic function on $\mathfrak{H} \times \mathbb{C}^n$ satsifying (3.1) with $k = k_\upsilon, m = \mu_\upsilon$ and some character $\chi = \chi_\upsilon$ of Γ^J of finite order. We call ϕ_υ the *Jacobi form attached to* υ. It is easy to see that ϕ_υ is holomorphic on $\mathfrak{H} \times \mathbb{C}^n$ if and only if υ satisfies

$$\sum_{j=1}^{\infty} \upsilon(j\alpha) \geq 0 \quad \text{for any} \;\; \alpha \in R \setminus \{0\}. \tag{3.7}$$

Thus ϕ_υ belongs to $J^!_{k_\upsilon, \mu_\upsilon S', \chi_\upsilon}$ for an integral vector system υ satisfying (3.6) and (3.7).

4 Multiplicative symmetries for Jacobi forms

In this section we give a characterization of Jacobi forms attached to vector systems by certain symmetries in the weakly holomorphic case (Theorem 4.2 and Corollary 4.3).

4.1 Multiplicative symmetries of Jacobi type

A function φ on $\mathfrak{H} \times \mathbb{C}^n$ is said to satisfy *multiplicative symmetries of Jacobi type* if

$$\prod_{ad=N} \prod_{b=0}^{d-1} \varphi\left(\frac{a\tau + b}{d}, az\right) = \epsilon_N \prod_{ad=N} \varphi(\tau, az)^d \tag{4.1}$$

holds for any integer $N \geq 2$ with a nonzero constant ϵ_N depending only on φ and N. It is straightforward to see the following:

PROPOSITION 4.1. *For any integral vector system υ, ϕ_υ satisfies multiplicative symmetries of Jacobi type.*

4.2 The main results for Jacobi forms: A characterization of the weakly holomorphic Jacobi forms attached to vector systems by symmetries

The object of this section is to show the inverse of Proposition 4.1 in the weakly holomorphic case. Namely we prove that, if a nonzero weakly holomorphic Jacobi form φ satisfies multiplicative symmetries of Jacobi type, then there exists an integral vector system υ satisfying (3.6) and (3.7) such that φ is a constant multiple of ϕ_υ. We also show that υ is explicitly expressed in terms of the Fourier expansion of φ.

To be more precise, let $J^! = \bigcup_{k,m,\chi} J^!_{k,mS',\chi}$, where k and m run over $\mathbb{Z}$ and χ over the characters of Γ^J of finite order. A nonzero element φ of $J^!$ admits the Fourier expansion in τ:

$$\varphi(\tau, z) = q^\lambda \sum_{N=0}^{\infty} A_N(z) q^N, \tag{4.2}$$

where λ is a rational number, $q := \mathbf{e}(\tau)$, $A_N(z)$ is a holomorphic function on $\mathbb{C}^n$ for $N \geq 0$ and $A_0(z) \neq 0$. We set

$$B_N(z) := \frac{A_N(z)}{A_0(z)} \qquad (N \in \mathbb{Z}_{\geq 0}). \tag{4.3}$$

Note that $B_0(z) = 1$ and that $B_N(z)$ is invariant under translation by elements of L_0. In general $B_N(z)$ may have poles on $\mathbb{C}^n$ if $N \geq 1$.

Let $J^!_{\mathrm{MS}}$ be the set of nonzero elements of $J^!$ satisfying multiplicative symmetries of Jacobi type and $\mathcal{V}$ the set of integral vector systems υ on L_0^* satisfying (3.6) and (3.7). Recall that ϕ_υ is in $J^!_{\mathrm{MS}}$ if $\upsilon \in \mathcal{V}$. The main results of this section are now stated as follows:

Theorem 4.2. *Let $\varphi \in J^!_{\mathrm{MS}}$.*

(1) For any $N \geq 1$, $B_N(z)$ is holomorphic on $\mathbb{C}^n$.

(2) Let

$$-B_1(z) = \sum_{\alpha \in L_0^*} \upsilon_\varphi(\alpha) \mathbf{e}(S(\alpha, z)) \tag{4.4}$$

be the Fourier expansion of $-B_1(z)$. Then $\upsilon_\varphi \in \mathcal{V}$ and φ is a constant multiple of ϕ_{υ_φ}.

Corollary 4.3. *(1) We have $J^!_{\mathrm{MS}} = \{c\phi_\upsilon \mid c \in \mathbb{C}^\times, \upsilon \in \mathcal{V}\}$.*

(2) The mapping $\upsilon \mapsto (\phi_\upsilon \bmod \mathbb{C}^\times)$ defines a bijection from $\mathcal{V}$ to $J^!_{\mathrm{MS}}/\mathbb{C}^\times$. Its inverse is given by $\varphi \mapsto \upsilon_\varphi$.

We explain the plan of the proof of Theorem 4.2. In Section 4.3, we derive the recurrence relations (4.5) satisfied by $B_N(z)$ from multiplicative symmetries of Jacobi type for φ. In Section 4.4 we show the holomorphy of $B_N(z)$ by using (4.5) and Lemma 2.1 (1). In Section 4.5, again using (4.5) together with Lemma 2.1 (3), we show that φ is a constant multiple of an infinite product attached to υ_φ. In Section 4.5 we show that υ is a vector system by using automorphy of φ. The integrality of υ follows from the facts that φ is single-valued on $\mathfrak{H} \times \mathbb{C}^n$ and that the weight of φ is an integer.

4.3 The recurrence relations satisfied by quotients of Fourier coefficients

From now on until the end of this section we let φ be a nonzero element of $J^!_{k,mS',\chi}$ with $k, m \in \mathbb{Z}$ and a character χ of Γ^J of finite order, and suppose that φ satisfies multiplicative symmetries of Jacobi type. Let $A_N(z)$ and $B_N(z)$ be as in the previous subsection. The following recurrence relations satisfied by $B_N(z)$ ($N \geq 1$) play a crucial role in the proof of Theorem 4.2.

Proposition 4.4. *We have*

$$\mathcal{G}_N\left(B_1(z),\ldots,B_N(z)\right)=\sum_{d\mid N} d^{-1}B_1(dz)\qquad (N\geq 1). \tag{4.5}$$

Proof. The equality (4.5) is trivial for $N=1$. Let $N\geq 2$. Observe that $(\tau,z)\mapsto q^{-\lambda}A_0(z)^{-1}\varphi(\tau,z)=1+\sum_{l=1}^{\infty}B_l(z)q^l$ satisfies multiplicative symmetries of Jacobi type and hence that

$$\prod_{ad=N}\prod_{0\leq j\leq d-1}\left\{1+\sum_{l=1}^{\infty}B_l(az)\mathbf{e}\left(d^{-1}jl\right)q^{al/d}\right\}=\epsilon'_N\prod_{ad=N}\left\{1+\sum_{l=1}^{\infty}B_l(az)q^l\right\}^d \tag{4.6}$$

holds with $\epsilon'_N\in\mathbb{C}^\times$. We consider the Fourier expansions of both sides of (4.6) in τ. Comparing the constant terms, we get $\epsilon'_N=1$. The left-hand side of (4.6) is equal to

$$\begin{aligned}\prod_{ad=N}R_d(q^{a/d};B_1(az),B_2(az),\ldots)&=\prod_{ad=N}\left(1+d\mathcal{G}_d(B_1(az),\ldots,B_d(az))q^a+O(q^{2a})\right)\\&=1+N\mathcal{G}_N(B_1(z),\ldots,B_N(z))q+O(q^2)\end{aligned}$$

in view of Lemma 2.1 (2) (for the definition of R_d see (2.3)). On the other hand the right-hand side of (4.6) is equal to

$$1+\left(\sum_{ad=N}dB_1(az)\right)q+O(q^2).$$

Comparing the coefficients of q of both sides of (4.6), we obtain

$$\mathcal{G}_N(B_1(z),\ldots,B_N(z))=\sum_{ad=N}a^{-1}B_1(az),$$

which proves the proposition. □

4.4 The holomorphy of quotients of Fourier coefficients

In this subsection we prove the first assertion of Theorem 4.2 that $B_N(z)$ is holomorphic on $\mathbb{C}^n$ for any $N\geq 1$. Let $t_1,\ldots,t_n$ be indeterminates. The polynomial ring $\mathcal{R}:=\mathbb{C}[t_1,t_1^{-1},\ldots,t_n,t_n^{-1}]$ of $t_1,t_1^{-1},\ldots,t_n,t_n^{-1}$ over $\mathbb{C}$ is a unique factorization domain. For $P\in\mathcal{R}$, let f_P be the holomorphic function on $\mathbb{C}^n$ defined by $f_P(z):=P(\mathbf{e}(z_1),\mathbf{e}(-z_1),\ldots,\mathbf{e}(z_n),\mathbf{e}(-z_n))$ for $z=(z_1,\ldots,z_n)\in\mathbb{C}^n$. Let $\mathcal{L}$ be the integral domain consisting of f_P ($P\in\mathcal{R}$) and $\mathcal{K}$ the quotient field of $\mathcal{L}$. Note that an element $f\in\mathcal{K}$ is in $\mathcal{L}$ if and only if f is holomorphic on $\mathbb{C}^n$. Denote by $\mathcal{P}$ the set of irreducible elements in $\mathcal{R}$. For $P\in\mathcal{P}$, define a divisor on $\mathbb{C}^n$ by $D_P:=\{z\in\mathbb{C}^n\mid f_P(z)=0\}$. The divisor of $f\in\mathcal{K}$ is of the form

$$\operatorname{div}(f)=\sum_{P\in\mathcal{P}}\nu_P(f)D_P,$$

where $\nu_P(f)\in\mathbb{Z}$ and $\nu_P(f)=0$ except for a finite number of P. For $f\in\mathcal{K}$, we put $\mathcal{M}(f):=\max\{-\nu_P(f)\mid P\in\mathcal{P}\}$. Note that f is in $\mathcal{L}$ if and only if $\mathcal{M}(f)\leq 0$.

Proposition 4.5. *For any $N \geq 1$, $B_N(z)$ belongs to $\mathcal{L}$ and hence is holomorphic on $\mathbb{C}^n$.*

Proof. Put $A'_N(z) := \mathbf{e}(-S(\alpha_0, z))A_N(z) = \sum_{\alpha \in L_0^*} a_\varphi(\lambda+N, \alpha_0+\alpha)\mathbf{e}(S(\alpha, z))$ (see (3.2)). Then we have $A'_N \in \mathcal{L}$, $B_N = A'_N/A'_0 \in \mathcal{K}$ and

$$\sup_{N \geq 0} \{\mathcal{M}(B_N)\} < \infty. \tag{4.7}$$

To prove the proposition it suffices to show that $B_1 \in \mathcal{L}$ in view of (2.2) and the recurrence relation (4.5). Suppose the contrary. Then $r := \mathcal{M}(B_1) > 0$. Take an element P of $\mathcal{P}$ with $\nu_P(B_1) = -r$. For $N \geq 1$, we put $B_N^*(z) = f_P(z)^{rN} B_N(z)$. Then $\nu_P(B_1^*) = 0$. We now show that

$$B_N^* = \frac{(B_1^*)^N}{N!} \qquad \text{on } D_P \tag{4.8}$$

by induction on N. The assertion for $N = 1$ is trivial. Let $N \geq 2$ and put $\mathcal{G}'_N(X_1, \ldots, X_{N-1}) := -\mathcal{G}_N(X_1, \ldots, X_N) + X_N$ (see (2.2)). Then

$$\mathcal{G}'_N(X_1, \ldots, X_{N-1}) = \sum_{i_1, \ldots, i_{N-1} \geq 0,\, i_1+2i_2+\cdots+(N-1)i_{N-1}=N} c(i_1, \ldots, i_{N-1}) X_1^{i_1} \cdots X_{N-1}^{i_{N-1}} \tag{4.9}$$

with $c(i_1, \ldots, i_{N-1}) \in \mathbb{Q}$ and

$$\mathcal{G}'_N\left(\frac{1}{1!}, \ldots, \frac{1}{(N-1)!}\right) = \frac{1}{N!}$$

by Lemma 2.1 (1). In view of the recurrence relations (4.5) and (4.9), we have

$$B_N^*(z) = \sum_{d|N} d^{-1} f_P(z)^{rN} B_1(dz) + \mathcal{G}'_N(B_1^*(z), \ldots, B_{N-1}^*(z)).$$

Observe that $f_P(z)^{rN} B_1(dz)$ vanishes on D_P for $d \geq 1$. By induction, for $z \in D_P$, we have

$$\begin{aligned} B_N^*(z) &= \mathcal{G}'_N\left(\frac{B_1^*(z)}{1!}, \ldots, \frac{B_1^*(z)^{N-1}}{(N-1)!}\right) \\ &= \mathcal{G}'_N\left(\frac{1}{1!}, \ldots, \frac{1}{(N-1)!}\right) B_1^*(z)^N \\ &= \frac{B_1^*(z)^N}{N!} \end{aligned}$$

and hence (4.8) has been proved. The equality (4.8) implies that $\nu_P(B_N) = -rN$ $(N \geq 1)$, which contradicts (4.7). We have completed the proof of the proposition. □

4.5 Proof of the main results for Jacobi forms

In this subsection we prove the remaining parts of Theorem 4.2.

To simplify the notation we write υ for υ_φ. Since $B_1 \in \mathcal{L}$ and $B_1(-z) = B_1(z)$, we see that $R := \{\alpha \in L_0^* \mid \upsilon(\alpha) \neq 0\}$ is a finite set and that $\upsilon(-\alpha) = \upsilon(\alpha)$ for $\alpha \in R$.

We next show that $A_0(z)^{-1}\varphi(\tau,z)$ has an infinite product expansion. Recall that

$$\mathbf{e}(-\lambda\tau)A_0(z)^{-1}\varphi(\tau,z) = 1 + \sum_{N=1}^{\infty} B_N(z)q^N. \tag{4.10}$$

Since the series of the right-hand side of (4.10) is absolutely and locally uniformly convergent on $\mathfrak{H}\times\mathbb{C}^n$, there exist a positive real number T and an open neighborhood U of $(0,\ldots,0)$ in $\mathbb{C}^n$ such that

$$\sum_{N=1}^{\infty} \left|B_N(z)q^N\right| < 1 \quad \text{for } (\tau,z)\in\mathcal{U}(T,U),$$

where $\mathcal{U}(T,U) := \{(\tau,z)\in\mathfrak{H}\times\mathbb{C}^n \mid \mathrm{Im}(\tau) > T, z\in U\}$. For $(\tau,z)\in\mathcal{U}(T,U)$, we set

$$\Phi(\tau,z) := \sum_{N=1}^{\infty} \mathcal{G}_N(B_1(z),\ldots,B_N(z))q^N. \tag{4.11}$$

It follows from Lemma 2.1 (3) that the series (4.11) is absolutely convergent and

$$1 + \sum_{N=1}^{\infty} B_N(z)q^N = \exp\left(\Phi(\tau,z)\right) \tag{4.12}$$

for $(\tau,z)\in\mathcal{U}(T,U)$. By the recurrence relations (4.5), we obtain

$$\begin{aligned}\Phi(\tau,z) &= \sum_{N=1}^{\infty}\sum_{d|N} d^{-1}B_1(dz)\,q^N = \sum_{d=1}^{\infty}\sum_{l=1}^{\infty} d^{-1}B_1(dz)q^{dl}\\ &= -\sum_{\alpha\in R} \upsilon(\alpha)\sum_{l=1}^{\infty}\sum_{d=1}^{\infty} d^{-1}\mathbf{e}(l\tau+S(\alpha,z))^d = \sum_{\alpha\in R}\upsilon(\alpha)\log\left(\prod_{l=1}^{\infty}(1-\mathbf{e}(l\tau+S(\alpha,z)))\right)\end{aligned}$$

and hence the equality

$$\varphi(\tau,z) = \mathbf{e}(\lambda\tau)A_0(z)\prod_{l=1}^{\infty}\prod_{\alpha\in R}(1-\mathbf{e}(l\tau+S(\alpha,z)))^{\upsilon(\alpha)} \tag{4.13}$$

holds for $(\tau,z)\in\mathcal{U}(T,U)$. Since the infinite product of the right-hand side of (4.13) is continued to a meromorphic function on $\mathfrak{H}\times\mathbb{C}^n$, the equality (4.13) holds for $(\tau,z)\in\mathfrak{H}\times\mathbb{C}^n$ by analytic continuation.

Observe that $\varphi(\tau,z)$ is a single valued function on $\mathfrak{H}\times\mathbb{C}^n$ and that $1-\mathbf{e}(l\tau+S(\alpha,z))$ has zeros on $\mathfrak{H}\times\mathbb{C}^n$ if $l\geq 1$ and $\alpha\neq 0$. In view of (4.13), we have $\sum_{j=1}^{\infty}\upsilon(j\alpha)\in\mathbb{Z}$ for any $\alpha\in L_0^*\setminus\{0\}$. This implies that $\upsilon(\alpha)\in\mathbb{Z}$ for any $\alpha\in L_0^*\setminus\{0\}$.

We next show that υ is a vector system on L_0^*. The idea of the proof is based on the proof of Theorem 6.5 in [Bo2]. Observe that

$$\varphi(\tau,z) = \mathbf{e}(\lambda\tau)A_0(z)\prod_{l=1}^{\infty}\left(1-q^l\right)^{\upsilon(0)}\prod_{l=1}^{\infty}\prod_{\alpha\in R^+}\left(\left(1-q^l\zeta^{\alpha}\right)\left(1-q^l\zeta^{-\alpha}\right)\right)^{\upsilon(\alpha)} \tag{4.14}$$

for $\tau \in \mathfrak{H}, z \in \mathbb{C}^n$. Here we write ζ^α for $\mathbf{e}(S(\alpha, z))$. Let $u \in L_0 \cap W$. Since $\varphi(\tau, z - u\tau) = \chi((u,0,0))^{-1} q^{-mS'[u]} \zeta^{mu} \varphi(\tau, z)$, we have

$$\frac{A_0(z-u\tau)}{A_0(z)} = \chi((u,0,0))^{-1} q^{-mS'[u]} \zeta^{mu} \times \prod_{\alpha \in R^+} \left(\prod_{l=1}^{\infty} \frac{\left(1-q^l\zeta^\alpha\right)\left(1-q^l\zeta^{-\alpha}\right)}{\left(1-q^{l-\sigma_\alpha}\zeta^\alpha\right)\left(1-q^{l+\sigma_\alpha}\zeta^{-\alpha}\right)} \right)^{\upsilon(\alpha)},$$

where we put $\sigma_\alpha := S(\alpha, u)$. Since $\sigma_\alpha \in \mathbb{Z}_{>0}$ for $\alpha \in R^+$, we have

$$\begin{aligned} \prod_{l=1}^{\infty} \frac{\left(1-q^l\zeta^\alpha\right)\left(1-q^l\zeta^{-\alpha}\right)}{\left(1-q^{l-\sigma_\alpha}\zeta^\alpha\right)\left(1-q^{l+\sigma_\alpha}\zeta^{-\alpha}\right)} &= \prod_{l=1}^{\sigma_\alpha} \frac{1-q^l\zeta^{-\alpha}}{1-q^{l-\sigma_\alpha}\zeta^\alpha} \\ &= (-1)^{\sigma_\alpha} q^{2^{-1}\sigma_\alpha(\sigma_\alpha+1)} \zeta^{-\sigma_\alpha \alpha} \frac{1-q^{-\sigma_\alpha}\zeta^\alpha}{1-\zeta^\alpha}. \end{aligned}$$

It follows that

$$\frac{A_0(z-u\tau)}{A_0(z)} = \chi((u,0,0))^{-1} (-1)^{2S(\rho_\upsilon, u)} q^{B(u)} \zeta^{C(u)} \prod_{\alpha \in R^+} \left(\frac{1-q^{-S(\alpha,u)}\zeta^\alpha}{1-\zeta^\alpha} \right)^{\upsilon(\alpha)}, \tag{4.15}$$

where

$$\begin{aligned} B(u) &= -mS'[u] + \frac{1}{2} \sum_{\alpha \in R^+} \upsilon(\alpha) S(\alpha, u)^2 + S(\rho_\upsilon, u), \\ C(u) &= mu - \sum_{\alpha \in R^+} \upsilon(\alpha) S(\alpha, u) \alpha \end{aligned}$$

(for the definition of ρ_υ, see (3.4)). Taking the limit $\tau \to 0$ (and hence $q \to 1$) in (4.15), we obtain $1 = \chi((u,0,0))^{-1} (-1)^{2S(\rho_\upsilon,u)} \zeta^{C(u)}$ and hence

$$\chi((u,0,0)) = (-1)^{2S(\rho_\upsilon, u)}, \quad C(u) = 0 \tag{4.16}$$

for any $u \in L_0 \cap W$. Since $L_0 \cap W$ generates L_0, (4.16) holds for any $u \in L_0$. We thus have $\sum_{\alpha \in R} \upsilon(\alpha) S(\alpha, u) \alpha = 2mu \ \ (u \in V_0)$, which implies that υ is a vector system on L_0^* of index m. Since $2^{-1} \sum_{\alpha \in R^+} \upsilon(\alpha) S(\alpha, u)^2 = mS'[u] \ \ (u \in V_0)$, we have $B(u) = S(\rho_\upsilon, u)$ for any $u \in L_0$. It follows that

$$\frac{A_0(z-u\tau)}{A_0(z)} = q^{S(\rho_\upsilon, u)} \left(\frac{1-q^{-S(\alpha,u)}\zeta^\alpha}{1-\zeta^\alpha} \right)^{\upsilon(\alpha)} \qquad (u \in L_0, \tau \in \mathfrak{H}).$$

Put

$$\widetilde{A}_0(z) := \frac{A_0(z)}{\zeta^{-\rho_\upsilon} \prod_{\alpha \in R^+} (1-\zeta^\alpha)^{\upsilon(\alpha)}}.$$

Then $\widetilde{A}_0(z - u\tau) = \widetilde{A}_0(z)$ holds for any $u \in L_0$ and $\tau \in \mathfrak{H}$. This implies that $\widetilde{A}_0(z)$ is a constant function on $\mathbb{C}^n$ and hence that $A_0(z)$ is a constant multiple of

$$\mathbf{e}(-S(\rho_\upsilon, z)) \prod_{\alpha \in R^+} (1 - \mathbf{e}(S(\alpha, z)))^{\upsilon(\alpha)}.$$

Combining this with (4.13) we see that φ is a constant multiple of ϕ_υ.

It remains to show that $\upsilon \in \mathcal{V}$. Comparing the weights of φ and ϕ_υ, we have $\upsilon(0) = 2k \in 2\mathbb{Z}$ and hence υ is an integral vector system satisfying (3.6). The condition (3.7) is also satisfied for υ since ϕ_υ is holomorphic on $\mathfrak{H} \times \mathbb{C}^n$. We thus have proved $\upsilon \in \mathcal{V}$ and the proof of Theorem 4.2 has been completed.

5 Borcherds lifts

In this section we recall the definitions of automorphic forms and Borcherds lifts on $O(2, n+2)$ introduced and studied in [Bo2] and [Bo4] (see also [GN1], [GN2] and [Br1]).

5.1 Automorphic forms on $O(2, n+2)$

We set

$$Q_0 := -S,\ Q_1 := \begin{pmatrix} & & 1 \\ & -S & \\ 1 & & \end{pmatrix},\ Q := \begin{pmatrix} & & 1 \\ & Q_1 & \\ 1 & & \end{pmatrix} = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & -S & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix},$$

$$L_0 := \mathbb{Z}^n,\ L_1 := \mathbb{Z}^{n+2},\ L := \mathbb{Z}^{n+4},$$
$$L_0^* := Q_0^{-1}L_0,\ L_1^* := Q_1^{-1}L_1,\ L^* := Q^{-1}L,$$

and

$$q_0(x) := 2^{-1}Q_0[x] = -S'[x],\ q_1(y) := 2^{-1}Q_1[y],\ q(z) := 2^{-1}Q[z]$$

for $x \in \mathbb{C}^n, y \in \mathbb{C}^{n+2}, z \in \mathbb{C}^{n+4}$. Let G be the orthogonal group of Q viewed as an algebraic group over $\mathbb{Q}$. Then $G(\mathbb{R})$, the group of real points of G, acts on $\mathcal{D}' := \{Z \in \mathbb{C}^{n+2} \mid q_1(\mathrm{Im}(Z)) > 0\}$ in the following way: For $g \in G(\mathbb{R})$ and $Z \in \mathcal{D}'$, there exist $g\langle Z\rangle \in \mathcal{D}'$ and $J(g, Z) \in \mathbb{C}^\times$ such that $J(g, Z)i(g\langle Z\rangle) = g\, i(Z)$, where

$$i(Z) := \begin{pmatrix} -q_1(Z) \\ Z \\ 1 \end{pmatrix} \in \mathbb{C}^{n+4}.$$

Let $\mathcal{D}^\pm$ be the connected component of $\mathcal{D}'$ containing $\pm(i, 0, i)$. Then $\mathcal{D}' = \mathcal{D}^+ \cup \mathcal{D}^-$ and $\mathcal{D}^\pm = \{Z = (\tau, z, \tau') \in \mathcal{D}' \mid \pm\tau, \pm\tau' \in \mathfrak{H}\}$. We write $\mathcal{D}$ for $\mathcal{D}^+$ and denote by $G(\mathbb{R})^+$ the subgroup of elements in $G(\mathbb{R})$ preserving $\mathcal{D}$. Let

$$\gamma_0 := \begin{pmatrix} 1 & & & & \\ & 0 & & 1 & \\ & & 1_n & & \\ & 1 & & 0 & \\ & & & & 1 \end{pmatrix}. \tag{5.1}$$

Then $\gamma_0 \in G(\mathbb{R})^+$ and $\gamma_0\langle(\tau, z, \tau')\rangle = (\tau', z, \tau)$. For a function F on $\mathcal{D}$, $k \in \mathbb{Z}$ and $g \in G(\mathbb{R})^+$, we put $(F|_k g)(Z) := J(g, Z)^{-k}F(g\langle Z\rangle)$ $(Z \in \mathcal{D})$. We regard $G^J(\mathbb{R})$ as a subgroup of $G(\mathbb{R})^+$ via an embedding

$$(u,v,t)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1 & S(u,v)-t & {}^t vS & 0 & \frac{1}{2}S[v] \\ 0 & 1 & 0 & 0 & 0 \\ 0 & u & 1_n & 0 & v \\ 0 & \frac{1}{2}S[u] & {}^t uS & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} a & & & -b & \\ & a & & & b \\ & & 1_n & & \\ -c & & & d & \\ & c & & & d \end{pmatrix}. \tag{5.2}$$

Let

$$\Gamma := \{\gamma \in G(\mathbb{R})^+ \mid \gamma L = L, \gamma x \equiv x \pmod{L} \text{ for any } x \in L^*\} \tag{5.3}$$

be the *discriminant kernel subgroup* of $G(\mathbb{R})^+$. We have $\Gamma^J \subset \Gamma$. For an integer k and a character χ of Γ, we denote by $\mathrm{A}_k(\Gamma,\chi)$ (respectively $\mathrm{M}_k(\Gamma,\chi)$) the space of meromorphic (respectively holomorphic) functions F on $\mathcal{D}$ satisfying $F|_k\gamma = \chi(\gamma)F$ for $\gamma \in \Gamma$. Let

$$\mathcal{A}(\Gamma) := \bigcup_{k,\chi} \mathrm{A}_k(\Gamma,\chi),$$
$$\mathcal{M}(\Gamma) := \bigcup_{k,\chi} \mathrm{M}_k(\Gamma,\chi),$$

where k runs over the nonnegative integers and χ over the characters of Γ of finite order. Since $\gamma_0 \in \Gamma$, we have

$$F(\tau',z,\tau) = \pm F(\tau,z,\tau') \tag{5.4}$$

for $F \in \mathcal{A}(\Gamma)$.

5.2 Borcherds lifts

To simplify the notation we write $J^!_{0,S'}$ for $J^!_{0,S',\mathbf{1}}$, where $\mathbf{1}$ denotes the trivial character of Γ. Let $\phi \in J^!_{0,S'}$ and

$$\sum_{l\in\mathbb{Z},\,\alpha\in L_0^*} c_\phi(l,\alpha)\mathbf{e}(l\tau + S(\alpha,z))$$

be its Fourier expansion. We say that ϕ has *integral principal part* if $c_\phi(l,\alpha)$ is an integer for any $(l,\alpha) \in \mathbb{Z}\times L_0^*$ with $l - S'[\alpha] < 0$.

Let ϕ be an element of $J^!_{0,S'}$ with integral principal part. Define a function υ on L_0^* by $\upsilon(\alpha) := c_\phi(0,\alpha)$ $(\alpha \in L_0^*)$. Then υ is a vector system with $\upsilon(\alpha) \in \mathbb{Z}$ for $\alpha \neq 0$ (see Theorem 10.5 in [Bo4]). Choose and fix a Weyl chamber $W \subset V_0 = \mathbb{R}^n$ with respect to υ and let (λ,ρ,μ) be the Weyl vector with respect to (υ,W) (see Section 3.3). Define the *Borcherds lift* of ϕ by

$$\Psi_\phi(\tau,z,\tau') := \mathbf{e}(\lambda\tau - S(\rho,z) + \mu\tau') \prod_{\substack{l,m\in\mathbb{Z},\,\alpha\in L_0^* \\ (l,\alpha,m)>0}} \left(1 - \mathbf{e}(l\tau + S(\alpha,z) + m\tau')\right)^{c_\phi(lm,\alpha)} \tag{5.5}$$

for $(\tau,z,\tau') \in \mathcal{D}$. Here $(l,\alpha,m) > 0$ means that either "$m > 0$" or "$m = 0, l > 0$" or "$m = l = 0, \alpha > 0$" holds. The fundamental properties of Ψ_ϕ are stated as follows (see [Bo2], [Bo4]):

(1) The infinite product (5.5) is absolutely convergent if $q_1(\mathrm{Im}(Z)) = \mathrm{Im}(\tau)\mathrm{Im}(\tau') - S'[\mathrm{Im}(z)]$ is sufficiently large.

(2) The function Ψ_ϕ is continued to a meromorphic function on $\mathcal{D}$ satisfying

$$\Psi_\phi|_{k_\phi}\gamma = \chi(\gamma)\Psi_\phi(Z) \qquad (\gamma \in \Gamma)$$

with $k_\phi := 2^{-1}c_\phi(0,0)$ and a multiplier χ of Γ of finite order.

(3) The divisor of Ψ_ϕ is given by

$$\frac{1}{2}\sum_{\alpha\in L^*/L}\ \sum_{m\in q(\alpha)+\mathbb{Z},\, m<0} c_\phi(m-q(\alpha),\alpha)H(\alpha,m). \tag{5.6}$$

Here

$$H(\alpha,m) := \sum_{\xi\in\alpha+L,\, q(\xi)=m} \{Z\in\mathcal{D} \mid Q(\xi, i(Z)) = 0\} \tag{5.7}$$

is the *Heegner divisor* of discriminant $(\alpha,m) \in (L^*/L)\times\mathbb{Q}$ with $m - q(\alpha) \in \mathbb{Z}$ and $m < 0$. We identify L^*/L with L_0^*/L_0 in a natural manner and make a convention that the multiplicities of $H(\alpha,m)$ are 2 if $2\alpha = 0$ in L^*/L and 1 if $2\alpha \neq 0$ in L^*/L.

If $c_\phi(0,0)$ is an even integer, then χ is a character of Γ.

Let

$$\widetilde{\mathcal{J}} := \{\phi \in J^!_{0,S'} \mid \phi \text{ satisfies (J1)}\},$$
$$\mathcal{J} := \{\phi \in J^!_{0,S'} \mid \phi \text{ satisfies (J1) and (J2)}\}.$$

Here

(J1) ϕ has integral principal part and $c_\phi(0,0)$ is an even integer.

(J2) For any $(l,\alpha) \in \mathbb{Z}\times L_0^*$ with $l - S'[\alpha] < 0$, we have

$$\sum_{j=1}^{\infty} c_\phi(j^2 l, j\alpha) \geq 0.$$

Note that $\Psi_\phi \in \mathcal{M}(\Gamma)$ for $\phi \in \mathcal{J}$. We set

$$\mathcal{A}(\Gamma)_{\mathrm{BL}} := \left\{c\Psi_\phi \mid c\in\mathbb{C}^\times,\, \phi\in\widetilde{\mathcal{J}}\right\} \subset \mathcal{A}(\Gamma), \tag{5.8}$$

$$\mathcal{M}(\Gamma)_{\mathrm{BL}} := \left\{c\Psi_\phi \mid c\in\mathbb{C}^\times,\, \phi\in\mathcal{J}\right\} \subset \mathcal{M}(\Gamma). \tag{5.9}$$

6 Main results

In this section we state and prove the main results of the paper on a characterization of holomorphic Borcherds lifts by multiplicative symmetries. We keep the notation of the previous section.

6.1 Multiplicative symmetries

We define two commuting embeddings $\iota^{\uparrow}$ and $\iota^{\downarrow}$ of SL_2 into G by

$$\iota^{\uparrow}(h) := \begin{pmatrix} a & & & -b & \\ & a & & & b \\ & & 1_n & & \\ -c & & & d & \\ & c & & & d \end{pmatrix}, \quad \iota^{\downarrow}(h) := \begin{pmatrix} a & -b & & & \\ -c & d & & & \\ & & 1_n & & \\ & & & a & b \\ & & & c & d \end{pmatrix} \tag{6.1}$$

for $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2$. For a function F on $\mathcal{D}$ and an integer $N \geq 2$, we define

$$\begin{aligned} F|T_{\Pi}^{\uparrow}(N)(Z) &:= \prod_{ad=N,\, 0\leq b<d} F\left(\iota^{\uparrow}\left(\frac{1}{\sqrt{N}}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right)\langle Z\rangle\right) \\ &= \prod_{ad=N,\, 0\leq b<d} F\left(\frac{a\tau+b}{d}, \frac{\sqrt{N}}{d}z, \tau'\right), \\ F|T_{\Pi}^{\downarrow}(N)(Z) &:= \prod_{ad=N,\, 0\leq b<d} F\left(\iota^{\downarrow}\left(\frac{1}{\sqrt{N}}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right)\langle Z\rangle\right) \\ &= \prod_{ad=N,\, 0\leq b<d} F\left(\tau, \frac{\sqrt{N}}{d}z, \frac{a\tau'+b}{d}\right) \end{aligned}$$

for $Z = (\tau, z, \tau') \in \mathcal{D}$.

Remark 6.1. These operators are induced from the classical Hecke operators via the embeddings $\iota^{\pm}$.

Remark 6.2. The operators $T_{\Pi}^{\uparrow}(N)$ for prime numbers N in the special case $O(2,3)$ are similar to the operators defined in [GN2] (Definition 3.1). Gritsenko and Nikulin proved some interesting functorial properties (see Theorem 3.3 [GN2]), which are not used in the present paper. Similar results had been obtained by Guerzhoy [G] in the $O(2,1)$-case.

We say that F satisfies *multiplicative symmetries* if

$$F|T_{\Pi}^{\uparrow}(N)(Z) = \epsilon_N\, F|T_{\Pi}^{\downarrow}(N)(Z) \tag{6.2}$$

holds for any $N \geq 2$ with a nonzero constant ϵ_N depending only on F and N. Denote by $\mathcal{A}(\Gamma)_{\mathrm{MS}}$ (respectively $\mathcal{M}(\Gamma)_{\mathrm{MS}}$) the set of nonzero elements of $\mathcal{A}(\Gamma)$ (respectively $\mathcal{M}(\Gamma)$) satisfying multiplicative symmetries. In our previous paper [HM2] we have proved the inclusion $\mathcal{A}(\Gamma)_{\mathrm{BL}} \subset \mathcal{A}(\Gamma)_{\mathrm{MS}}$ holds. This immediately implies $\mathcal{M}(\Gamma)_{\mathrm{BL}} \subset \mathcal{M}(\Gamma)_{\mathrm{MS}}$. One of the main aim of the present paper is to show the equality $\mathcal{M}(\Gamma)_{\mathrm{BL}} = \mathcal{M}(\Gamma)_{\mathrm{MS}}$.

6.2 The main results: A characterization of holomorphic Borcherds lifts by symmetries and the inverse of the Borcherds lifting

To state our main results more explicitly, we first observe that a nonzero element F of $\mathcal{M}(\Gamma)$ admits the Fourier-Jacobi expansion

$$F(\tau, z, \tau') = e(\mu\tau') \sum_{m=0}^{\infty} F_m(\tau, z)\mathbf{e}(m\tau'),$$

where $\mu \in \mathbb{Q}$, F_m is holomorphic on $\mathfrak{H} \times \mathbb{C}^n$ for $m \geq 0$ and $F_0 \neq 0$. Set

$$f_m(\tau, z) := \frac{F_m(\tau, z)}{F_0(\tau, z)} \qquad (m \in \mathbb{Z}_{\geq 0}, (\tau, z) \in \mathfrak{H} \times \mathbb{C}^n). \tag{6.3}$$

We see that $f_m|_{0,mS'}\gamma = f_m$ for any $\gamma \in \Gamma^J$. In general f_m may have poles on $\mathfrak{H} \times \mathbb{C}^n$ if $m \geq 1$. We now state the main results of the paper:

THEOREM 6.3. *Let F be an element of $\mathcal{M}(\Gamma)_{\mathrm{MS}}$. Then we have the following:*

(1) For $m \geq 1$, f_m is holomorphic on $\mathfrak{H} \times \mathbb{C}^n$ and belongs to $J^!_{0,mS'}$.

(2) Put $\phi_F := -f_1$. Then ϕ_F belongs to $\mathcal{J}$ and F is a constant multiple of the Borcherds lift Ψ_{ϕ_F} of ϕ_F.

COROLLARY 6.4. *(1) We have $\mathcal{M}(\Gamma)_{\mathrm{BL}} = \mathcal{M}(\Gamma)_{\mathrm{MS}}$.*

(2) The Borcherds lifting defined by $\phi \mapsto (\Psi_\phi \bmod \mathbb{C}^\times)$ is a bijection from $\mathcal{J}$ to $\mathcal{M}(\Gamma)_{\mathrm{MS}}/\mathbb{C}^\times$ and its inverse is given by $F \mapsto \phi_F$.

REMARK 6.5. It is an open question whether the equality $\mathcal{A}(\Gamma)_{\mathrm{BL}} = \mathcal{A}(\Gamma)_{\mathrm{MS}}$ holds.

We now explain the plan of the proof of Theorem 6.3, which is quite similar to that of Theorem 4.2. In Section 6.3, we deduce the following two results from multiplicative symmetries for F and Theorem 4.2:

(a) F_0 is a constant multiple of ϕ_υ (for the definition see Section 3.4) with an integral vector system $\upsilon \in \mathcal{J}$.

(b) f_m satisfies recurrence relations (6.5).

The first result (a) implies that the orders of poles of f_m ($m \geq 1$) are bounded from above. Using this and (b), we show the holomorphy of f_m in Section 6.3. In particular ϕ_F belongs to $J^!_{0,S'}$. In Section 6.4, using again (b) and Lemma 2.1 (3), we show that F has an infinite product expansion attached to ϕ_F. We show that ϕ_F has integral principal part by using the fact that F is single-valued on $\mathcal{D}$. We then deduce that ϕ_F is in $\mathcal{J}$ and that F is a constant multiple of the Borcherds lift Ψ_{ϕ_F}.

6.3 The holomorphy of quotients of Fourier-Jacobi coefficients

In this subsection we let $F \in \mathcal{M}(\Gamma)_{\mathrm{MS}}$ and let F_m and f_m be as in the previous subsection. To prove holomorphy of f_m, we first show certain recurrence relations satisfied by f_m ($m \geq 1$). We recall the definition of Hecke-like operators V_N introduced in [EZ]; for a function ϕ on $\mathfrak{H} \times \mathbb{C}^n$ and an integer $N \geq 1$ we put

$$(\phi|V_N)(\tau, z) = N^{-1} \sum_{ad=N} \sum_{b=0}^{d-1} \phi\left(\frac{a\tau+b}{d}, az\right). \tag{6.4}$$

Note that $\phi|V_N \in J^!_{0,NS'}$ if $\phi \in J^!_{0,S'}$.

Proposition 6.6. *(1) The first nonvanishing Fourier-Jacobi coefficient F_0 is a constant multiple of ϕ_v with an integral vector system $v \in \mathcal{J}$.*

(2) For any integer $N \geq 2$, we have

$$\mathcal{G}_N(f_1(\tau, z), \ldots, f_N(\tau, z)) = (f_1|V_N)(\tau, z). \tag{6.5}$$

Proof. Since

$$(\tau, z, \tau') \mapsto \mathbf{e}(-\mu\tau')F(\tau, z, \tau') = \sum_{m=0}^{\infty} F_m(\tau, z)\mathbf{e}(m\tau')$$

satisfies multiplicative symmetries, the equality

$$\begin{aligned} &\prod_{ad=N} \prod_{b=0}^{d-1} \left\{ \sum_{m=0}^{\infty} F_m\left(\frac{a\tau+b}{d}, d^{-1}\sqrt{N}z\right) \mathbf{e}(m\tau') \right\} \\ &= \epsilon_N \prod_{ad=N} \prod_{b=0}^{d-1} \left\{ \sum_{m=0}^{\infty} F_m\left(\tau, d^{-1}\sqrt{N}z\right) \mathbf{e}\left(m\,\frac{a\tau'+b}{d}\right) \right\} \end{aligned} \tag{6.6}$$

holds for $N \geq 2$ with $\epsilon_N \in \mathbb{C}^\times$. We now compare the coefficients of the Fourier expansions in τ' of both sides of (6.6). Comparing the constant terms, we obtain

$$\prod_{ad=N} \prod_{b=0}^{d-1} F_0\left(\frac{a\tau+b}{d}, d^{-1}\sqrt{N}z\right) = \epsilon_N \prod_{ad=N} F_0\left(\tau, d^{-1}\sqrt{N}z\right)^d \tag{6.7}$$

and hence F_0 satisfies multiplicative symmetries of Jacobi type. The first assertion of the proposition now follows from Theorem 4.2.

Dividing (6.6) by (6.7) and letting $z \mapsto \sqrt{N}z$, we obtain

$$\begin{aligned} &\prod_{ad=N} \prod_{b=0}^{d-1} \left\{ 1 + \sum_{m=1}^{\infty} f_m\left(\frac{a\tau+b}{d}, az\right) \mathbf{e}(m\tau') \right\} \\ &= \prod_{ad=N} \prod_{b=0}^{d-1} \left\{ 1 + \sum_{m=1}^{\infty} f_m(\tau, az)\, \mathbf{e}\left(m\,\frac{a\tau'+b}{d}\right) \right\} \end{aligned} \tag{6.8}$$

The coefficient of $q_1 := \mathbf{e}(\tau')$ of the left-hand side of (6.8) is equal to

$$\sum_{ad=N} \sum_{b=0}^{d-1} f_1\left(\frac{a\tau+b}{d}, az\right).$$

On the other hand the right-hand side of (6.8) is equal to

$$\begin{aligned}\prod_{ad=N} R_d\left(q_1^{a/d}; f_1(\tau, az), f_2(\tau, az), \ldots\right) &= \prod_{ad=N} \left(1 + d\mathcal{G}_d(f_1(\tau, az), \ldots, f_d(\tau, az))q_1^a + O(q_1^{2a})\right) \\ &= 1 + N\mathcal{G}_N(f_1(\tau, z), \ldots, f_N(\tau, z))q_1 + O(q_1^2)\end{aligned}$$

by Proposition 2.1. It follows that

$$\mathcal{G}_N(f_1(\tau, z), \ldots, f_N(\tau, z)) = \frac{1}{N} \sum_{ad=N} \sum_{b=0}^{d-1} f_1\left(\frac{a\tau+b}{d}, az\right) = (f_1|V_N)(\tau, z).$$

□

PROPOSITION 6.7. *For any integer $m \geq 1$, f_m is holomorphic on $\mathfrak{H} \times \mathbb{C}^n$.*

Proof. The proof of the proposition is similar to that of Proposition 4.5. For a meromorphic function f on $\mathfrak{H} \times \mathbb{C}^n$, let $\mathrm{div}(f) = \sum_D c_D(f)D$ be the divisor of f, where D runs over the irreducible divisors on $\mathfrak{H} \times \mathbb{C}^n$ and $c_D(f) \in \mathbb{Z}$. Set $\mathcal{N}(f) := \sup_D\{-c_D(f)\}$. In view of Proposition 6.6 (1), we have $\sup_D\{c_D(F_0)\} < \infty$. Since $F_0(\tau, z)f_m(\tau, z)$ is holomorphic on $\mathfrak{H} \times \mathbb{C}^n$, we have

$$\sup_{m\geq 0}\{\mathcal{N}(f_m)\} < \infty. \tag{6.9}$$

In view of (2.2) and (6.5) it suffices to show that f_1 is holomorphic on $\mathfrak{H} \times \mathbb{C}^n$. Suppose the contrary. Then $r := \mathcal{N}(f_1)$ is a positive integer. Take an irreducible divisor D_0 such that $c_{D_0}(f_1) = -r$ and let φ_0 be the defining equation of D_0. An argument similar to the proof of Proposition 4.5 shows the equality

$$\varphi_0^{rm} f_m = \frac{(\varphi_0^r f_1)^m}{m!} \qquad \text{on } D_0,$$

which implies that $\mathcal{N}(f_m) \geq rm$ for $m \geq 1$. This contradicts (6.9) and we are done. □

6.4 Proof of the main results

In this subsection we complete the proof of Theorem 6.3. As stated before, the idea of the proof is similar to that of the proof of Theorem 4.2. Let F, F_m, f_m be as in Theorem 6.3. In view of Proposition 6.7, f_m belongs to $J^!_{0,mS'}$ for any $m \geq 1$.

We first show that $F(\tau, z, \tau')/F_0(\tau, z)$ has an infinite product expansion. Observe that the series

$$1 + \sum_{m=1}^{\infty} f_m(\tau, z)\mathbf{e}(m\tau') = \mathbf{e}(-\mu\tau')F_0(\tau, z)^{-1} F(\tau, z, \tau').$$

is absolutely and locally uniformly convergent on $\mathcal{D}$. It follows that there exist a positive real number T and an open neighbourhood U of $(0,\ldots,0)$ in $\mathbb{C}^n$ such that

$$\sum_{m=1}^{\infty}\left|f_m(\tau,z)\mathbf{e}(m\tau')\right|<1 \text{ for } (\tau,z,\tau')\in\mathcal{D}(T,U),$$

where $\mathcal{D}(T,U):=\{(\tau,z,\tau')\in\mathcal{D}\mid \mathrm{Im}(\tau),\mathrm{Im}(\tau')>T, z\in U\}$. For $(\tau,z,\tau')\in\mathcal{D}(T,U)$, we set

$$\Phi(\tau,z,\tau')=\sum_{m=1}^{\infty}\mathcal{G}_m(f_1(\tau,z)\ldots,f_m(\tau,z))\mathbf{e}(m\tau'). \tag{6.10}$$

In view of Lemma 2.1 (3), the series (6.10) is absolutely convergent on $\mathcal{D}(T,U)$ and

$$\mathbf{e}(-\mu\tau')F_0(\tau,z)^{-1}F(\tau,z,\tau')=\exp\left(\Phi(\tau,z,\tau')\right)\qquad((\tau,z,\tau')\in\mathcal{D}(T,U)). \tag{6.11}$$

To simplify the notation we write ϕ and $c(l,\alpha)$ for $\phi_F=-f_1$ and $c_{\phi_F}(l,\alpha)$ respectively. In view of Proposition 6.6 (2), we obtain

$$\begin{aligned}
&\Phi(\tau,z,\tau')\\
&=\sum_{m=1}^{\infty}\left(f_1|_{0,S'}V_m\right)(\tau,z)\mathbf{e}(m\tau')\\
&=-\sum_{m=1}^{\infty}m^{-1}\sum_{d|m}\sum_{b=0}^{d-1}\phi\left(\frac{d^{-1}m\tau+b}{d},d^{-1}mz\right)e(m\tau')\\
&=-\sum_{m=1}^{\infty}m^{-1}\sum_{d|m}\sum_{b=0}^{d-1}\sum_{l\in\mathbb{Z},\,\alpha\in L_0^*}c(l,\alpha)\mathbf{e}\left(l\frac{d^{-1}m\tau+b}{d}+S(\alpha,d^{-1}mz)+m\tau'\right)\\
&=-\sum_{m=1}^{\infty}\sum_{d|m}\sum_{l\in\mathbb{Z},\,\alpha\in L_0^*}dm^{-1}c(dl,\alpha)\mathbf{e}\left(d^{-1}m(l\tau+S(\alpha,z)+d\tau')\right)\\
&=\sum_{d\geq1,\,l\in\mathbb{Z},\,\alpha\in L_0^*}c(dl,\alpha)\log\left(1-\mathbf{e}(l\tau+S(\alpha,z)+d\tau')\right).
\end{aligned}$$

We thus have

$$F(\tau,z,\tau')=\mathbf{e}(\mu\tau')F_0(\tau,z)\prod_{m\geq1,\,l\in\mathbb{Z},\,\alpha\in L_0^*}\left(1-\mathbf{e}(l\tau+S(\alpha,z)+m\tau')\right)^{c(lm,\alpha)} \tag{6.12}$$

for $(\tau,z,\tau')\in\mathcal{D}(T,U)$.

If $l-S'[\alpha]<0$, the divisor $\{(\tau,z,\tau')\in\mathcal{D}\mid l\tau+S(\alpha,z)+\tau'=0\}$ intersects $\{Z\in\mathcal{D}\mid q_1(\mathrm{Im}(Z))>N\}$ for $N\gg0$. Since F is single-valued on $\mathcal{D}$, we have $\sum_{j=1}^{\infty}c(j^2l,j\alpha)\in\mathbb{Z}$ for any (l,α) with $l-S'[\alpha]<0$. This implies that $c(l,\alpha)\in\mathbb{Z}$ for any (l,α) with $l-S'[\alpha]<0$, proving that ϕ has integral principal part.

We next show that F is a constant multiple of the Borcherds lift Ψ_ϕ. By (6.12) and analytic continuation, we have

$$F(\tau,z,\tau')=\mathbf{e}(A\tau')\psi(\tau,z)\Psi_\phi(\tau,z,\tau') \tag{6.13}$$

with $A \in \mathbb{Q}$ and a meromorphic function ψ on $\mathfrak{H} \times \mathbb{C}^n$. By (5.4), we have

$$\mathbf{e}(-A\tau)\psi(\tau, z) = \pm\,\mathbf{e}(-A\tau')\psi(\tau', z) \qquad (\tau, \tau' \in \mathfrak{H}, z \in \mathbb{C}^n),$$

which implies that there exists a meromorphic function $\varphi(z)$ on $\mathbb{C}^n$ such that $\psi(\tau, z) = \mathbf{e}(A\tau)\varphi(z)$. On the other hand, by (6.13) and automorphy of F and Ψ_ϕ with respect to $\Gamma^J \subset \Gamma$ (see (5.2)), we have $\psi|_{k-k_\phi, AS'}\gamma = \chi'(\gamma)\psi$ for any $\gamma \in \Gamma^J$, where χ' is a multiplier of Γ^J of finite order. It follows that, for any $u \in \mathbb{Z}^n, \tau \in \mathfrak{H}, z \in \mathbb{C}^n$, we have

$$\psi(\tau, z + u\tau) = \kappa(u)\mathbf{e}(-A(S'[u]\tau + S(u, z)))\psi(\tau, z)$$

and hence

$$\varphi(z + u\tau) = \kappa(u)\mathbf{e}(-A(S'[u]\tau + S(u, z)))\varphi(z),$$

where κ is a character of $\mathbb{Z}^n$ of finite order. Letting $\tau \to 0$, we get $\varphi(z) = \kappa(u)\mathbf{e}(-AS(u, z))\varphi(z)$ and hence $A = 0$ and $\kappa(u) = 1$ for any $u \in \mathbb{Z}^n$. We thus have $\varphi(z + u\tau) = \varphi(z)$ for any $u \in \mathbb{Z}^n, \tau \in \mathfrak{H}$ and hence φ is a constant function on $\mathbb{C}^n$. This concludes that ψ is a constant function on $\mathfrak{H} \times \mathbb{C}^n$ and hence that F is a constant multiple of Ψ_ϕ.

Since Ψ_ϕ is holomorphic on $\mathcal{D}$ and $c(0, 0) = 2k \in 2\mathbb{Z}$, we see that $\phi \in \mathcal{J}$, which completes the proof of Theorem 6.3.

7 Symmetries for Heegner divisors

Recall that the Heegner divisor $H(\alpha, m)$ of discriminant (α, m) is defined by (5.7). We denote by $\mathcal{A}(\Gamma)_{\mathrm{HD}}$ the set of nonzero elements of $\mathcal{A}(\Gamma)$ whose divisors are of the form

$$\frac{1}{2} \sum_{\alpha \in L^*/L,\, m \in q(\alpha)+\mathbb{Z},\, m<0} c(\alpha, m)H(\alpha, m) \qquad \text{(a finite sum)}$$

with $c(\alpha, m) \in \mathbb{Z}$ for any (α, m) (see Section 5.2). We set $\mathcal{M}(\Gamma)_{\mathrm{HD}} = \mathcal{A}(\Gamma)_{\mathrm{HD}} \cap \mathcal{M}(\Gamma)$. In view of [Bo2] and [Bo4], we have $\mathcal{A}(\Gamma)_{\mathrm{BL}} \subset \mathcal{A}(\Gamma)_{\mathrm{HD}}$. Bruinier ([Br1], [Br2]) showed the equality

$$\mathcal{A}(\Gamma)_{\mathrm{BL}} = \mathcal{A}(\Gamma)_{\mathrm{HD}}. \tag{7.1}$$

This immediately implies

$$\mathcal{M}(\Gamma)_{\mathrm{BL}} = \mathcal{M}(\Gamma)_{\mathrm{HD}}. \tag{7.2}$$

In this section we give a simple proof of (7.2) by showing *additive symmetries* for $H(\alpha, m)$ (Proposition 7.1) and using Corollary 6.4.

7.1 Additive symmetries for Heegner divisors

For a divisor X of $\mathcal{D}$ and an integer $N \geq 2$, we put

$$X|T_\Sigma^\uparrow(N) = \sum_{ad=N,\, 0\leq b<d} X * \iota^\uparrow\left(\frac{1}{\sqrt{N}}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right),$$

$$X|T_\Sigma^\downarrow(N) = \sum_{ad=N,\, 0\leq b<d} X * \iota^\downarrow\left(\frac{1}{\sqrt{N}}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right),$$

where we put $X * g := \{g^{-1}\langle Z\rangle \mid Z \in X\}$ for $g \in G(\mathbb{R})^+$ and $\iota^\uparrow$ and $\iota^\downarrow$ are defined by (6.1). The following fact might be known to experts, though we give a proof for completeness.

PROPOSITION 7.1. *Let $\alpha \in L^*/L$ and $m \in q(\alpha) + \mathbb{Z}$, $m < 0$. Then*

$$H(\alpha, m)|T_\Sigma^\uparrow(N) = H(\alpha, m)|T_\Sigma^\downarrow(N) \tag{7.3}$$

holds for any integer $N \geq 2$.

Proof. To prove the proposition we may and do assume that $\alpha = (0, 0, \alpha_0, 0, 0)$ with $\alpha_0 \in L_0^*/L_0$. For $x = (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4$ and $\lambda_0 \in L_0^*$, let

$$X(x, \lambda_0) := \left\{ Z \in \mathcal{D} \mid Q\left(\begin{pmatrix} x_1 \\ x_2 \\ \lambda_0 \\ x_3 \\ x_4 \end{pmatrix}, i(Z)\right) = 0 \right\}$$

be a divisor of $\mathcal{D}$. We then have

$$H(\alpha, m) = \sum_{\lambda_0 \in \alpha_0 + L_0} \sum_{\substack{x \in \mathbb{Z}^4 \\ \ell(x) = m - q_0(\lambda_0)}} X(x, \lambda_0),$$

where $\ell(x) := x_1x_4 + x_2x_3$.

For $a, b, d \in \mathbb{Z}_{\geq 0}$ with $ad = N$ we define maps $f_{a,b,d}^\uparrow$ and $f_{a,b,d}^\downarrow$ from $\mathbb{Z}^4$ to itself by

$$f_{a,b,d}^\uparrow(x) := (dx_1 - bx_3, dx_2 + bx_4, ax_3, ax_4),$$
$$f_{a,b,d}^\downarrow(x) := (dx_1 - bx_2, ax_2, dx_3 + bx_4, ax_4)$$

for $x = (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4$. It is easily verified that

$$f_{a,b,d}^\downarrow \circ i = i \circ f_{a,b,d}^\uparrow\,, \tag{7.4}$$

where $i(x_1, x_2, x_3, x_4) := (x_1, x_3, x_2, x_4)$. A straightforward calculation shows the equalities

$$\begin{aligned} H(\alpha, m)|T_\Sigma^\uparrow(N) &= \sum_{\lambda_0 \in \alpha_0 + L_0} \sum_{\substack{x \in \mathbb{Z}^4 \\ \ell(x) = m - q_0(\lambda_0)}} \sum_{ad = N,\, 0 \leq b < d} X(f_{a,b,d}^\uparrow(x), \sqrt{N}\lambda_0) \\ &= \sum_{\lambda_0 \in \alpha_0 + L_0} \sum_{\substack{x \in \mathbb{Z}^4 \\ \ell(x) = N(m - q_0(\lambda_0))}} \nu_N^\uparrow(x) X(x, \sqrt{N}\lambda_0) \end{aligned}$$

and

$$\begin{aligned} H(\alpha, m)|T_\Sigma^\downarrow(N) &= \sum_{\lambda_0 \in \alpha_0 + L_0} \sum_{\substack{x \in \mathbb{Z}^4 \\ \ell(x) = m - q_0(\lambda_0)}} \sum_{ad = N,\, 0 \leq b < d} X(f_{a,b,d}^\downarrow(x), \sqrt{N}\lambda_0) \\ &= \sum_{\lambda_0 \in \alpha_0 + L_0} \sum_{\substack{x \in \mathbb{Z}^4 \\ \ell(x) = N(m - q_0(\lambda_0))}} \nu_N^\downarrow(x) X(x, \sqrt{N}\lambda_0). \end{aligned}$$

Here

$$\nu_N^{\uparrow}(x) := \#\left\{(a,b,d)\in\mathbb{Z}^3 \mid ad=N,\ a,d>0,\ 0\le b<d,\ x\in f^{\uparrow}_{a,b,d}(\mathbb{Z}^4)\right\},$$
$$\nu_N^{\downarrow}(x) := \#\left\{(a,b,d)\in\mathbb{Z}^3 \mid ad=N,\ a,d>0,\ 0\le b<d,\ x\in f^{\downarrow}_{a,b,d}(\mathbb{Z}^4)\right\}$$

for $x\in\mathbb{Z}^4$. By (7.4) we have $\nu_N^{\uparrow}(x)=\nu_N^{\downarrow}(x)$ for any $x\in\mathbb{Z}^4$, which implies (7.3). □

7.2 Proof of (7.2)

It is sufficient to show that $\mathcal{M}(\Gamma)_{\mathrm{HD}}\subset\mathcal{M}(\Gamma)_{\mathrm{BL}}$. Let $F\in\mathcal{M}(\Gamma)_{\mathrm{HD}}$ and $N\ge 2$ be an integer. Let

$$Q^{(N)} := \begin{pmatrix} &&&&1\\ &&&1&\\ &&-NS&&\\ &1&&&\\ 1&&&& \end{pmatrix}$$

and let $\mathcal{D}^{(N)}$ be the symmetric domain corresponding to $Q^{(N)}$. Observe that both $F^{\uparrow}(\tau,z,\tau') := F|T^{\uparrow}_{\Pi}(N)(\tau,\sqrt{N}z,\tau')$ and $F^{\downarrow}(\tau,z,\tau') := F|T^{\downarrow}_{\Pi}(N)(\tau,\sqrt{N}z,\tau')$ are holomorphic automorphic forms on an arithmetic subgroup Γ' of $O(Q^{(N)})$ of the same weight and the same divisor by Proposition 7.1. Then the quotient $F^{\uparrow}(Z)/F^{\downarrow}(Z)$ is a holomorphic function on $\mathcal{D}^{(N)}$ invariant under Γ' and hence a constant function on $\mathcal{D}^{(N)}$ by the Koecher principle (see [BB]). This implies that $F\in\mathcal{M}(\Gamma)_{\mathrm{MS}}$. We now conclude that $F\in\mathcal{M}(\Gamma)_{\mathrm{BL}}$ in view of Corollary 6.4.

Remark 7.2. One of the referees suggested the following argument.

Assume that, for any $(\alpha,m)\in L^*\times\mathbb{Q}$ with $m\in q(\alpha)+\mathbb{Z}$ and $m<0$, there exists a holomorphic Borcherds lift vanishing on $H(m,\alpha)$. Then we can deduce (7.1) from (7.2) in the following way: Let $F\in\mathcal{A}(\Gamma)_{\mathrm{HD}}$. The assumption implies that there exists $F_1\in\mathcal{M}(\Gamma)_{\mathrm{BL}}$ such that $F_2 := FF_1\in\mathcal{M}(\Gamma)$. Since both F_1 and F_2 belong to $\mathcal{M}(\Gamma)_{\mathrm{HD}}$, we have $F_1,F_2\in\mathcal{M}(\Gamma)_{\mathrm{BL}}$ by (7.2) and hence $F=F_2/F_1\in\mathcal{A}(\Gamma)_{\mathrm{BL}}$.

Note that the above assumption holds in the case of $\mathrm{Sp}_2(\mathbb{Z})$ (for example see [HM1], Theorem 4.1).

8 Modular polynomials and symmetries

In this section we characterize the modular polynomials by certain symmetries. In this section we put $\Gamma_1 := \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma := \mathrm{SL}_2(\mathbb{Z})\times\mathrm{SL}_2(\mathbb{Z})$.

8.1 Weakly holomorphic modular forms on $\mathrm{SL}_2(\mathbb{Z})$

Let $\mathrm{M}^!_k(\Gamma_1)$ be the space of weakly holomorphic modular forms on Γ_1 of weight k. By definition $\mathrm{M}^!_k(\Gamma_1)$ is the space of holomorphic functions ϕ on $\mathfrak{H}$ satisfying the following two conditions:

(i) For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$, we have $\phi\left(\dfrac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \phi(\tau)$.

(ii) Let $\phi(\tau) = \sum_{l\in\mathbb{Z}} c_\phi(l) q^l$ be the Fourier expansion of ϕ ($q := \mathbf{e}(\tau)$). We have $c_\phi(l) = 0$ if l is sufficiently small.

Let $j(\tau)$ be the modular invariant (the unique element of $\mathrm{M}_0^!(\Gamma_1)$ with $j(\tau) = q^{-1} + 744 + O(q)$). Then $\mathrm{M}_0^!(\Gamma_1) = \mathbb{C}[j]$.

8.2 A characterization of modular polynomials by symmetries

For any positive integer m, there uniquely exists a polynomial $\Phi_m(X, Y) \in \mathbb{Z}[X, Y]$ satisfying

$$\Phi_m(X, j(\tau)) = \prod_{\substack{ad=m,\, 0\le b<d \\ \gcd(a,b,d)=1}} \left(X - j\left(\frac{a\tau+b}{d}\right)\right),$$

where $\gcd(a, b, d)$ stands for the greatest common divisor of a, b, d (for example see [Co], page 230). It is known that $\Phi_m(X, Y)$ is irreducible in $\mathbb{C}[X, Y]$ and that

$$\Phi_m(Y, X) = \pm\, \Phi_m(X, Y). \tag{8.1}$$

The polynomial $\Phi_m(X, Y)$ is called the *primitive modular polynomial* of order m. The main object of this section is to show the following result:

Theorem 8.1. *Let $P(X, Y)$ be an irreducible polynomial in $\mathbb{C}[X, Y]$ with $P(Y, X) = \pm P(X, Y)$. Then the following two conditions are equivalent:*

(1) *For any integer $N \ge 2$, the equality*

$$\prod_{ad=N}\prod_{b=0}^{d-1} P\left(j\left(\frac{a\tau+b}{d}\right), j(\tau')\right) = \epsilon_N \prod_{ad=N}\prod_{b=0}^{d-1} P\left(j(\tau), j\left(\frac{a\tau'+b}{d}\right)\right) \quad (\tau, \tau' \in \mathfrak{H})$$

holds with a constant $\epsilon_N \in \mathbb{C}^\times$.

(2) *The polynomial P is a constant multiple of Φ_m with some positive integer m.*

To prove the theorem we need a characterization of holomorphic Borcherds lifts on $\Gamma = \Gamma_1 \times \Gamma_1$ by multiplicative symmetries. We need a separate treatment in this case since the Koecher principle does not holds for Γ.

8.3 Automorphic forms and Borcherds lifts on $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$

For $k \in \mathbb{Z}$, let $\mathrm{M}_k^!(\Gamma)$ be the space of holomorphic functions F on $\mathfrak{H} \times \mathfrak{H}$ satisfying the following three conditions:

(i) We have

$$F\left(\frac{a_1\tau_1+b_1}{c_1\tau_1+d_1}, \frac{a_2\tau_2+b_2}{c_2\tau_2+d_2}\right) = (c_1\tau_1+d_1)^k(c_2\tau_2+d_2)^k F(\tau_1,\tau_2) \qquad \left(\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \Gamma_1, i=1,2\right).$$

(ii) We have $F(\tau_2,\tau_1) = \pm F(\tau_1,\tau_2)$.

(iii) Let $F(\tau_1,\tau_2) = \sum_{m_1,m_2\in\mathbb{Z}} c_F(m_1,m_2)\mathbf{e}(m_1\tau_1+m_2\tau_2)$ be the Fourier expansion of F. Then there exists a positive integer n_0 such that $c_F(m_1,m_2) = 0$ if $m_1 < -n_0$ or $m_2 < -n_0$.

For $P(X,Y) \in \mathbb{C}[X,Y]$, set

$$F_P(\tau_1,\tau_2) := P(j(\tau_1), j(\tau_2)) \qquad (\tau_1,\tau_2 \in \mathfrak{H}). \tag{8.2}$$

Then $\mathrm{M}_0^!(\Gamma) = \{F_P \mid P \in \mathbb{C}[X,Y], P(Y,X) = \pm P(X,Y)\}$.

Let $\phi \in \mathrm{M}_0^!(\Gamma_1)$ and suppose that

$$c_\phi(l) \in \mathbb{Z} \ \text{ if } \ l < 0. \tag{8.3}$$

For simplicity we further assume that

$$c_\phi(0) \text{ is an integer divisible by } 24. \tag{8.4}$$

We put

$$\mu_\phi := \frac{1}{24}c_\phi(0) - \sum_{m=1}^{\infty}\left(\sum_{d|m} d\right) c_\phi(-m),$$
$$k_\phi := \frac{c_\phi(0)}{2}.$$

Define

$$\Psi_\phi(\tau_1,\tau_2) := q_1^{\mu_\phi} q_2^{c_\phi(0)/24} \prod_{m_1,m_2} (1-q_1^{m_1}q_2^{m_2})^{c_\phi(m_1m_2)} \qquad (\tau_i \in \mathfrak{H}, q_i := \mathbf{e}(\tau_i), i=1,2), \tag{8.5}$$

where (m_1,m_2) runs over the pairs of integers such that "$m_1 > 0$" or "$m_1 = 0$ and $m_2 > 0$". The infinite product (8.5) is absolutely convergent if $\mathrm{Im}(\tau_1)\mathrm{Im}(\tau_2)$ is sufficiently large, and continued to a meromorphic function on $\mathfrak{H}\times\mathfrak{H}$ satisfying conditions (i)–(ii) of Section 8.3 with $k = k_\phi$. We call Ψ_ϕ the *Borcherds lift* of ϕ. Note that the Borcherds lift of $j(\tau) - 744 = q^{-1} + 196884q + \cdots$ is $j(\tau_1) - j(\tau_2)$. The following is easily verified.

Proposition 8.2. *Let ϕ be an element of $\mathrm{M}_0^!(\Gamma_1)$ satisfying (8.3) and (8.4). Then*

$$\Psi_\phi(\tau_1,\tau_2) = (\Delta(\tau_1)\Delta(\tau_2))^{c_\phi(0)/24} \prod_{m\geq 1} \Phi_m\left(j(\tau_1), j(\tau_2)\right)^{\sum_{f=1}^{\infty} c_\phi(-f^2m)}.$$

Here $\Delta(\tau) := q\prod_{l=1}^{\infty}(1-q^l)^{24}$ is the modular discriminant.

8.4 A characterizations of holomorphic Borcherds lifts on $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$

A nonzero element F of $\mathrm{M}_k^!(\Gamma)$ admits the Fourier expansion in τ_2:

$$F(\tau_1, \tau_2) = \mathbf{e}(\mu\tau_2) \sum_{m=0}^{\infty} F_m(\tau_1)\mathbf{e}(m\tau_2),$$

where $\mu \in \mathbb{Z}$, F_m is holomorphic on $\mathfrak{H}$ and $F_0 \neq 0$. Then $F_m \in \mathrm{M}_k^!(\Gamma_1)$ for any $m \in \mathbb{Z}_{\geq 0}$. Put

$$\phi_F(\tau) := -\frac{F_1(\tau)}{F_0(\tau)}.$$

Note that ϕ_F may have poles on $\mathfrak{H}$ in general.

We say that a function F on $\mathfrak{H} \times \mathfrak{H}$ satisfies multiplicative symmetries if the equality

$$\prod_{ad=N} \prod_{b=0}^{d-1} F\left(\frac{a\tau + b}{d}, \tau'\right) = \epsilon_N \prod_{ad=N} \prod_{b=0}^{d-1} F\left(\tau, \frac{a\tau' + b}{d}\right) \quad (\tau, \tau' \in \mathfrak{H})$$

holds for any integer $N \geq 2$ with a constant $\epsilon_N \in \mathbb{C}^\times$. Theorem 8.1 is a direct consequence of Proposition 8.2 and the following result, whose proof is similar to that of Theorem 6.3 and we omit.

THEOREM 8.3. *Suppose that $F \in \mathrm{M}_k^!(\Gamma)$ satisfies multiplicative symmetries.*

(1) *The function ϕ_F is an element of $\mathrm{M}_0^!(\Gamma_1)$ satisfying* (8.3) *and* (8.4)*.*

(2) *F is a constant multiple of the Borcherds lift of ϕ_F.*

Fundings

This work was supported by Grants-in-Aids from Japan Society for the Promotion of Science [20540031, 23540033 to A.M.]; and Kyoto Sangyo University Research Grants [E1105 to A.M.].

Acknowledgements

We would like to thank the referees for their very helpful comments and suggestions. We are grateful to N. Scheithauer for stimulating discussions with him, and to Max-Planck-Institut für Mathematik in Bonn for their support during our stay. The first author is grateful to German University of Technology in Oman for their support.

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