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by

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$\mathcal{D}$-modules on the representation of $Sp(2n, \mathbb{C}) \times GL(2, \mathbb{C})$

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Abstract

We give an algebraic classification of regular holonomic $\mathcal{D}$-modules on $(\mathbb{C}^{2n})^2$ related to the action of the group $Sp(2n, \mathbb{C}) \times GL(2, \mathbb{C})$ product of the symplectic linear transformations group by the general linear group.

Keywords: $\mathcal{D}$-modules, holonomic $\mathcal{D}$-modules, invariant differential operators, Capelli identity, invariant sections, symplectic group, general linear group, multplicity free spaces.

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1 Introduction

Let V be a finite dimensional linear representation of a complex (connected) reductive algebraic group G. We denote by $\mathbb{C}[V]$ the algebra of polynomials on V and by $G' = [G, G]$ the derived subgroup of G. We assume that G acts on V with an open orbit then the representation (G, V) is called a prehomogeneous vector space (see M. Sato [35], [37] or T. Kimura [19, chap. 2]). We assume furthermore that the representation (G, V) is multiplicity free that is the associated representation of G on $\mathbb{C}[V]$ decomposes without multiplicities (i.e. each irreducible representation of G occurs at most once in $\mathbb{C}[V]$). Then V. G. Kac [13] asserts that there are finitely many orbits $(V_k)_{k \in K}$. Let us denote by $\Lambda := \bigcup_{k \in K} \overline{T^*_{V_k} V}$ the lagrangian variety which is the union of the closure of conormal bundles to the G- orbits (see [32]).
As usual $\mathcal{D}_V$ is the sheaf of rings of differential operators on V with holomorphic coefficients.
Denote by $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$ the full category whose objects are regular holonomic $\mathcal{D}_V$-modules with characteristic variety contained in Λ.

Let us now point out that the action of G on $\mathbb{C}[V]$ extends to $\Gamma(V, \mathcal{D}_V)^{\mathrm{pol}}$ the $\mathbb{C}$-algebra of differential operators on V with polynomial coefficients in $\mathbb{C}[V]$. Then there is a natural algebra arising in this situation: the quotient algebra $\mathcal{A}$ of G'-invariant differential operators $\bar{\mathcal{A}} := \Gamma\left(V, \mathcal{D}_V\right)^{G'}$ by a suitable ideal $\bar{\mathcal{J}}$ of $\bar{\mathcal{A}}$ described in section 3 i.e. $\mathcal{A} := \bar{\mathcal{A}}/\bar{\mathcal{J}}$.
Let $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ be the category whose objects are graded $\mathcal{A}$-modules of finite type for θ the Euler vector field on V.

One says (see [22]) that the multiplicity free representation (G, V) has a one-dimensional quotient if there exists a polynomial f on V such that the algebra $\mathbb{C}[V]^{G'}$ of G'-invariant polynomials on V is the algebra of polynomials in f (i.e. $\mathbb{C}[V]^{G'} = \mathbb{C}[f]$) and $f \notin \mathbb{C}[V]^G$.

T. Levasseur [22, Conjecture 5. 17, p. 508] gave the following conjecture:

Conjecture: Suppose (G, V) is an irreducible multiplicity free representation with a one-dimensional quotient satisfying some Capelli condition (see [11, p. 581, (10.3): the abstract Capelli Problem)], [38], [22])) then the categories $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$ and $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ are equivalent.

We should note that this conjecture has been proved by the author in the following cases: $(G = GL(n,\mathbb{C}) \times SL(n,\mathbb{C}),\ V = M_n(\mathbb{C}))$; $(G = SO(n,\mathbb{C}) \times \mathbb{C}^*,\ V = \mathbb{C}^n)$; $(G = GL(n,\mathbb{C})$, $V = \Lambda^2\mathbb{C}^n$, n even), $(G = GL(n,\mathbb{C})$, $V = S^2\mathbb{C}^n)$ (see [25], [26], [27], [28], [29], [30]) .

Our aim in this paper is to prove Levasseur's conjecture for the linear action of the group $G = Sp(2n,\mathbb{C}) \times GL(2,\mathbb{C})$ on the vector space $V = (\mathbb{C}^{2n})^2$ which we will think of as the space of $2n$ by 2 complex matrices. It would be interesting to prove this conjecture for the representation of the more general group $Sp(2n,\mathbb{C}) \times GL(n,\mathbb{C})$ on the space of $2n$ by n matrices. Unfortunately the Capelli property (see [11, p. 581, (10.3)]) fails in some cases like $(Sp(2n,\mathbb{C}) \times GL(3,\mathbb{C}),\ M_{2n,3}(\mathbb{C}))$ (see Howe-Umeda [11, p. 598, Lemma 11.7.9] or Umeda [38]).
Actually, the equivalence between the categories $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$ and $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ leads to an algebraic description of regular holonomic $\mathcal{D}_V$-modules in $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$. More precisely the objects in the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ are encoded in terms of finite diagrams of linear maps.

By the way we should note that the problem of classifying regular holonomic $\mathcal{D}$-modules or equivalently perverse sheaves on a complex manifold (thanks to the Riemann-Hilbert correspondence due to M. Kashiwara [15] and Z. Mebkhout [24] independently) has been treated by several authors. In [2] L. Boutet de Monvel gives a description of regular holonomic $\mathcal{D}$-modules in one variable by using pairs of finite dimensional $\mathbb{C}$-vector spaces and certain linear maps. P. Deligne [4] gives a combinatorial description of the category of perverse sheaves on $\mathbb{C}$ with respect to the stratification $\{0\}$, $\mathbb{C} - \{0\}$. It uses a characterization of constructible sheaves given in [5], [6]. A. Galligo, M. Granger and P. Maisonobe [7] obtained using the Riemann-Hilbert correspondence, a classification of regular holonomic $\mathcal{D}_{\mathbb{C}^n}$-modules with singularities along the hypersurface $x_1 \cdots x_n = 0$ by 2^n-tuples of $\mathbb{C}$-vector spaces with a set of linear maps. L. Narváez-Macarro [31] treated the case $y^2 = x^p$ using the method of Beilinson and Verdier and generalized this study to the case of reducible plane curves. R. MacPherson and K. Vilonen [23] treated the case with singularities along the curve $y^n = x^m$ etc.

This paper is organized as follows:

In section 2, we review some useful results. In particular the one's saying that: any coherent $\mathcal{D}_V$-module equipped with a good filtration, invariant under the action of the Euler vector field θ, is generated by finitely many global sections of finite type for θ.

Section 3 deals with the description of $\overline{\mathcal{A}}$ the algebra of G'-invariant differential operators following a method by T. Levasseur [22].

We arrive, in section 4, at a deep result: Theorem 14 saying that any $\mathcal{D}_V$-module in the category $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$ is generated by its invariant global sections under the action of G'. The proof uses $\mathcal{D}_V$-modules with support in the closure of the G'-orbits.

This result leads, in section 5, to the main Theorem 18: there is an equivalence of categories between the category $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$ and the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$. Note that here the image by this equivalence of a regular holonomic $\mathcal{D}_V$-module is its set of global sections which are both of finite type for the Euler vector field θ and invariant under the action of G'.

This study ends in section 6 by the description of objects in the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ in terms of finite diagrams of linear maps on finite dimensional vector spaces.

We refer the reader to [3], [9] [16], [17], [18] for notions on $\mathcal{D}$-modules theory.

2 Preliminaries

Let G be an algebraic group and $G' = [G, G]$ its derived subgroup. Given a symplectic form σ on $\mathbb{C}^{2n}$ the symplectic group is the subgroup of $GL(2n, \mathbb{C})$ that preserves the form σ namely $Sp(2n, \mathbb{C}) = \{s \in GL(2n, \mathbb{C}) \ / \ \sigma(sx, sy) = \sigma(x, y) \ \ \forall \ (x, y) \in (\mathbb{C}^{2n})^2\} = \{s \in GL(2n, \mathbb{C}) \ / \ sJs^T = J\}$ where s^T is the transpose matrix of s and $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ is the alternating matrix with I_n the n by n identity matrix, and 0 the n by n matrix with entries that are zero.

Definition 1 *Given a vector space V endowed with a symplectic form σ,*
(i) a subspace $W \subset V$ is isotropic if $\sigma(x, y) = 0 \ \ \forall \ (x, y) \in W^2$,
(ii) an isotropic subspace W determines a corresponding maximal parabolic subgroup of $Sp(V)$ that is the stabilizer of W in $Sp(V)$:

$$P := \{p \in Sp(V) \ \ / \ pW = W\} \quad W \subset V \ \ \textit{isotropic}, \tag{1}$$

Remark 2 *If $v \neq 0$ is any non zero vector which spans a G- stable line in V, then it is clear that*

$$g.v = \chi(g)v \qquad \textit{for } g \in G \tag{2}$$

defines a character of G.

Let us denote by V the complex vector space $(\mathbb{C}^{2n})^2 \simeq \mathbb{C}^{2n} \otimes \mathbb{C}^2 \simeq \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^{2n})$ which we will think of as the space of $2n \times 2$-matrices $V \simeq M_{2n,2}(\mathbb{C})$. Denote by $G = Sp(2n, \mathbb{C}) \times GL(2, \mathbb{C})$ the product of the symplectic linear transformations group by the general linear group and $G' = Sp(2n, \mathbb{C}) \times SL(2, \mathbb{C})$ its derived subgroup. The group G acts linearly on V as follows: $(s, g) \cdot v = svg^T$ where (s, g) is in G (g^T is the transpose matrix of g and v is a $2n$ by 2 complex matrix. Then we can determine the orbits for G by two data: (a) the rank and (b) the isometry type (see Witt theorem [12, Chap V]). These data are described as (a) $\ r = \text{rank of } v$ and (b) $\ q = \text{rank of } vJv^T$ where J is the alternating matrix defining the symplectic group $Sp(2n, \mathbb{C})$. We have obvious limitations like $r \leq 2$ and $q \leq 2$. Also the integer q must be even and $q \geq 2r - 2n$. Let us denote by $V_{r,q} := \{v \in V, \text{rank}(v) = r \ \text{ and } \ \text{rank}(vJv^T) = q\}$ $(0 \leq q \leq r \leq 2, \text{ with } q \neq 1)$ the orbits with the two data. These are:

$$\begin{cases} V_{0,0}, V_{1,0}, V_{2,2}, & \text{if } \ n = 1, \\ \\ V_{0,0}, V_{1,0}, V_{2,0}, V_{2,2} & \text{if } \ n \geq 2. \end{cases}$$

Note that if $n \geq 2$, the G'-orbits are $V_{0,0}, V_{1,0}, V_{2,0}$. We have the following proposition:

Proposition 3 *The orbits $V_{0,0}, V_{1,0}, V_{2,0}$ of the linear action of the group $G' = Sp(2n, \mathbb{C}) \times SL(2, \mathbb{C})$ on the space V are simply connected i.e. $\pi_1(V_{r,q}) = \{1\}$.*

Proof. Consider the action of the derived subgroup $G' = Sp(2n,\mathbb{C}) \times SL(2,\mathbb{C})$ on $V = \mathbb{C}^{2n} \otimes \mathbb{C}^2$: $(s,g) \cdot v = svg^T$ where (s,g) is in G' and $v \in V$. We start off by noting that there is a natural relationship between the orbits of a group action and the isotropy group (or the stabilizer) $H = H_v := \{g \in G', g.v = v\}$ of the point $v \in V$. Indeed the G'-orbits V_{rq} can be identified with the space of cosets G'/H under the correspondence $g.v \longrightarrow gH$.
We have the exact homotopy sequence of the fibration $H \longrightarrow G' \longrightarrow G'/H$ namely

$$\cdots \longrightarrow \pi_1(H) \longrightarrow \pi_1(G') \longrightarrow \pi_1(G'/H) \longrightarrow \pi_0(H) \longrightarrow \pi_0(G') \cdots \tag{3}$$

Since the group $G' = Sp(2n,\mathbb{C}) \times SL(2,\mathbb{C})$ is simply connected and connected (i.e. $\pi_1(G') = \pi_0(G') = \{1\}$) the exact homotopy sequence shows that

$$\pi_1(G'/H) \simeq \pi_0(H). \tag{4}$$

So we have to compute all the possible stabilizer groups H of points in V and see that their are connected i.e. $\pi_0(H) \simeq \{1\}$.
The orbit $V_{0,0} = \{0\}$ is obviously simply connected. Let us check that points in the set $V_{1,0} = \{v \in V/ \operatorname{rank}(v) = 1 \text{ and } \operatorname{rank}(v^T J v) = 0\}$, of rank exactly one matrices on which the symplectic form σ vanishes, have connected stabilizer groups.
Let (e_1, e_2) be the standard basis vectors of $V = (\mathbb{C}^{2n})^2$ and σ be the symplectic form on $\mathbb{C}^{2n}$. Using the right action of $SL(2,\mathbb{C})$ on $V_{1,0}$ the rank-one $2n \times 2$-matrices, we can reduce a $2n \times 2$-matrix v to a matrix $(w, 0)$ where w is a $2n \times 1$-matrix which is an isotropic vector (i.e. $\sigma(w,w) = 0$).
Now using the left $Sp(2n,\mathbb{C})$- action, we can make $w = e_1$ (the first standard basis vector) since every isotropic vector can be embedded in an isotropic basis ($\sigma(e_1,e_1) = \sigma(e_2,e_2) = 0$).
Denote by W_{e_1} the subspace spanned by e_1. Let

$$P := \{p \in Sp(2n,\mathbb{C}) \text{ such that } pW_{e_1} = W_{e_1}\}$$

be the subgroup of $Sp(2n,\mathbb{C})$ that fixes the one-dimensional space W_{e_1}. The group P is a maximal parabolic subgroup of $Sp(2n,\mathbb{C})$. It turns out that the action of P on e_1 is by a character χ the first fundamental weight of $Sp(2n,\mathbb{C})$ extended to be 1 on the unipotent radical of P (see remark 2, (2)). Then the stabilizer $H_{(e_1,0)}$ of the $2n \times 2$-matrix $(e_1, 0)$ is the subgroup of element (p,q) in $G' = Sp(2n,\mathbb{C}) \times SL(2,\mathbb{C})$ where $p \in P$ and q is a lower-triangular matrix in $SL(2,\mathbb{C})$ with diagonal elements $\chi(p)^{-1}$ and $\chi(p)$ namely

$$H_{(e_1,0)} = \{(p,q) \in G' \text{ such that } p \in P \text{ and } q = \begin{pmatrix} \chi(p)^{-1} & 0 \\ \star & \chi(p) \end{pmatrix} \in SL(2,\mathbb{C})\ \}. \tag{5}$$

So as an algebraic group the stabilizer $H_{(e_1,0)}$ is isomorphic to $P \times U$:

$$H_{(e_1,0)} \simeq P \times U \tag{6}$$

where U is the subgroup of unipotent lower triangular matrices in $SL(2,\mathbb{C})$.
Note that as a subgroup of G', the stabilizer $H_{(e_1,0)}$ is a fibered product $P \times_\chi B$ where B is the lower-triangular Borel subgroup of $SL(2,\mathbb{C})$. Since P and B are connected then $H_{(e_1,0)}$ is also connected i.e. $\pi_0(H_{(e_1,0)}) = \{0\}$. Hence $V_{1,0}$ is simply connected.
Next, in the case of V_{20} the result is also true since the stabilizer of a point in V_{20} is isomorphic to the kernel of a character of a parabolic subgroup which stabilizes an isotropic plane in V. ■

As in introduction we shall denote by $\mathcal{D}_V$ the sheaf of rings of differential operators on V with holomorphic coefficients. Let $\theta := \sum_{1 \le i \le 2n; 1 \le j \le 2} x_{ij} \frac{\partial}{\partial x_{ij}} \in \mathcal{D}_V$ be the Euler vector field on V.

Definition 4 *Let $\mathcal{M}$ be a $\mathcal{D}_V$-module.*
A section u in $\mathcal{M}$ is said to be homogeneous if $\dim_{\mathbb{C}} \mathbb{C}[\theta] u < \infty$. The section u is said to be homogeneous of degree $\lambda \in \mathbb{C}$, if there exists $j \in \mathbb{N}$ such that $(\theta - \lambda)^j u = 0$.

Let us recall the following result which will be used later (see [27, Theorem 1.3.]):

Theorem 5 *Let $\mathcal{M}$ be a coherent $\mathcal{D}_V$-module equipped with a good filtration $(\mathcal{M}_k)_{k\in\mathbb{Z}}$ stable under the action of θ:*

$$\theta \mathcal{M}_k \subset \mathcal{M}_k.$$

Then

i) $\mathcal{M}$ is generated over $\mathcal{D}_V$ by finitely many homogeneous global sections,

ii) For any $k \in \mathbb{N}$, $\lambda \in \mathbb{C}$, the vector space $\Gamma(V, \mathcal{M}_k) \bigcap \left[\bigcup_{p\in\mathbb{N}} \ker(\theta - \lambda)^p \right]$ of homogeneous global sections in $\mathcal{M}_k$ of degree λ is finite dimensional:

$$\dim_{\mathbb{C}} \left\{ \Gamma(V, \mathcal{M}_k) \bigcap \left[\bigcup_{p\in\mathbb{N}} \ker(\theta - \lambda)^p \right] \right\} < +\infty. \tag{7}$$

Remark 6 *We will describe a holomorphic classification of regular holonomic $\mathcal{D}_V$-modules in* $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$ *but Theorem 5 permits to reduce these objects to algebraic (homogeneous) $\mathcal{D}_V$-modules.*

3 Algebra of $Sp(2n, \mathbb{C}) \times SL(2, \mathbb{C})$-invariant differential operators

As in the introduction, we denote by G a connected reductive algebraic group over $\mathbb{C}$ and G' its derived subgroup. We consider (G, V) a linear representation of G on a finite dimensional complex vector space V. The action of G on V extends to various algebras namely: $\mathbb{C}[V]$ the algebra of polynomial functions on V and $\Gamma(V, \mathcal{D}_V)^{\mathrm{pol}}$ the algebra of differential operators with polynomial coefficients in $\mathbb{C}[V]$. We will denote by $\mathbb{C}[V]^G$ (resp. $\mathbb{C}[V]^{G'}$) and $\Gamma(V, \mathcal{D}_V)^G$ (resp. $\Gamma(V, \mathcal{D}_V)^{G'}$) the algebras of G (resp. G')-invariant polynomial functions and invariant differential operators on V respectively.

In this section we describe the algebra $\overline{\mathcal{A}} := \Gamma(V, \mathcal{D}_V)^{G'}$ in the case $G := Sp(2n, \mathbb{C}) \times GL(2, \mathbb{C})$, $G' := Sp(2n, \mathbb{C}) \times SL(2, \mathbb{C})$ and $V := (\mathbb{C}^{2n})^2 \simeq \mathbb{C}^{2n} \otimes \mathbb{C}^2$. For this purpose, we use T. Levasseur's results [22, Section 4].

3.1 A review on invariant differential operators on a class of multiplicity free spaces with one dimensional quotient

We recall that a finite dimensional linear representation (G, V) is called prehomogeneous vector space if G has a Zariski open (hence dense) orbit (see Kimura [19], Sato [35] or Sato-Kimura [37]).

In this case it is known that there are some relative invariants $f_j \in \mathbb{C}[V]$, $0 \le j \le m$, i.e. there exist rational characters $\chi_j \in \mathcal{X}(G)$ such that

$$g \cdot f_j = \chi_j(g) f_j \qquad \text{for all } g \in G \tag{8}$$

(see [19, Theorem 2.9, p.26]). Moreover there exist relative invariants $f_j^*(\partial) \in \mathbb{C}[V^*]$ with weight χ_j^{-1}, $0 \le j \le m$ (see [22, Section 3.1]). We set $\Delta_j := f_j^*(\partial)$ for $0 \le j \le m$. In this situation, T. Levasseur [22, lemma 4.2, (d) and formula (4.3) p. 487] proved that the algebra of G-invariant polynomials is generated by the relative invariants $f_0, \cdots, f_m$:

$$\mathbb{C}[V]^G = \mathbb{C}[f_0, \cdots, f_m], \tag{9}$$

and also the algebra of G-invariant differential operators with constant coefficients is generated by $\Delta_0, \cdots, \Delta_m$:

$$\mathbb{C}[V^*]^G = \mathbb{C}[\Delta_0, \cdots, \Delta_m]. \tag{10}$$

Now the prehomogeneous vector space (G, V) is said to be multiplicity free if the associated representation of G on $\mathbb{C}[V]$ decomposes without multiplicities i.e. each irreducible representation of G occurs at most once in $\mathbb{C}[V]$ (see Levasseur [22, p. 484, Definition 4.1] or Howe - Umeda [11] for details). Note that multiplicity free spaces have been classified by some authors notably Benson and Ratcliff [1], V. G. Kac [13] and A. Leahy [21].
If (G, V) is multiplicity free, it turns out that, following Benson and Ratcliff [1], Howe - Umeda [11], Knopp [20] and Levasseur [22], the algebra of G-invariant differential operators decomposes as a direct sum of one dimensional irreducible G-modules $\mathbb{C}E_\gamma$:

$$\Gamma(V, \mathcal{D}_V)^G := \bigoplus_{\gamma_j \equiv \chi_j} \mathbb{C}E_{\gamma_j}. \tag{11}$$

Actually to each G-module $\mathbb{C}E_\gamma$ one can associate a G-invariant differential operator

$$E_j := E_{\gamma_j}(x, \partial_x), \quad 0 \le j \le m.$$

These invariant differential operators are called Capelli operators. We arrive now at a deep result proved by Howe-Umeda [11]

Theorem 7 *(Howe - Umeda) For a fix multiplicity free representation (G, V), the algebra*

$$\Gamma(V, \mathcal{D}_V)^G = \mathbb{C}\left[E_0, \cdots, E_m\right] \tag{12}$$

is a commutative polynomial ring.

We recall the following definition:

Definition 8 *(see [22]) A multiplicity free space (G, V) is said to have a one-dimensional quotient if there exists a polynomial f such that $\mathbb{C}[V]^{G'} = \mathbb{C}[f]$ and $f \notin \mathbb{C}[V]^G$.*

A list of multiplicity free representations with one dimensional quotient is given in [21], [22, p. 508]. In this situation, we know from Sato-Bernstein-Kashiwara (see [19, Proposition 2.22] and [14]) that there exists a polynomial $b(s) \in \mathbb{R}[s]$ of degree $n = \deg f$ such that

$$\begin{array}{llll} i) & b(s) & = & c\prod_{j=0}^{n-1}(s + \lambda_j + 1), \qquad c > 0; \\ ii) & \Delta f^{s+1} & = & b(s) f^s; \\ iii) & \lambda_{j+1} \in \mathbb{Q}^{*+}, & & 0 \le j \le n-1, \qquad \lambda_0 = 0 \end{array} \tag{13}$$

where $\Delta := \Delta_0$, $f := f_0$.

In the sequel, we are particularly interested in the algebras of invariant differential operators on the so called multiplicity free representations with one dimensional quotient. Recall that $\overline{\mathcal{A}}$ denotes the algebra of G'-invariant differential operators and θ is the Euler vector field on V: $\overline{\mathcal{A}} \supset \Gamma(V, \mathcal{D}_V)^G$, $\theta \in \Gamma(V, \mathcal{D}_V)^G$. When (G, V) is a multiplicity free space of one dimensional quotient, T. Levasseur [22, Theorem 4.11, (i)] gives the following result:

Theorem 9 *For a fix multplicity free representation (G, V) of one dimensional quotient, we have*

$$\overline{\mathcal{A}} = \mathbb{C}\langle f, \Delta, \theta, E_2, \cdots, E_m\rangle . \tag{14}$$

Here $E_1 = \theta$ is the Euler vector field. This last result generalizes the one's of H. Rubenthanler (see [33, Proposition 3.1] or [34, Theorem 5.3.3.]) obtained when $(\widetilde{G} : V)$ is an irreducible regular prehomogeneous representation of "commutative parabolic type".

3.2 Application: $(G = Sp(2n, \mathbb{C}) \times GL(2, \mathbb{C}), V = \mathbb{C}^{2n} \otimes \mathbb{C}^2)$

As in the introduction we set $G = Sp(2n, \mathbb{C}) \times GL(2, \mathbb{C})$, $G' = Sp(2n, \mathbb{C}) \times SL(2, \mathbb{C})$. The action of G' on $V = \mathbb{C}^{2n} \otimes \mathbb{C}^2$ and on $\mathcal{D}_V$ are defined as follows: for $(s, g) \in G'$ and $(v, \frac{\partial}{\partial v}) \in V \times \mathcal{D}_V$ we have

$$(s, g) \cdot v = svg^T, \qquad (s, g) \cdot \frac{\partial}{\partial v} = (s^T)^{-1} \frac{\partial}{\partial v} g^{-1}. \tag{15}$$

This is a multiplicity free action with one dimensional quotient (see [22, p. 508 Appendix, (5)], [11, p. 612 table (15.1)] or [37, p. 145, (13)]). So we can apply the preceding results.
Here the G'-invariant polynomial function is $f = f_0 := \sigma$ the symplectic form on V of degree 2 (see [19, p. 265]) and $\Delta := \sigma(\partial)$ is the dual of σ:

$$f = \sigma \quad \text{and} \quad \Delta := \sigma(\partial).$$

Moreover in this case $m = 1$, $E_m = E_1 = \theta$ and by theorem 9 the algebra $\overline{\mathcal{A}}$ is generated by σ, θ, Δ:

$$\overline{\mathcal{A}} = \mathbb{C}\langle \sigma, \Delta, \theta\rangle . \tag{16}$$

Consider $\mathcal{J} := \mathrm{ann}\mathbb{C}[\sigma]$ the ideal annihilator of G'-invariant polynomials $\mathbb{C}[V]^{G'} = \mathbb{C}[\sigma]$ that is $\mathcal{J} := \left\{P \in \overline{\mathcal{A}}, \text{ such that } Pf = 0 \text{ for } f \in \mathbb{C}[\sigma]\right\}$

Proposition 10 *The following relations hold in the quotient algebra $\overline{\mathcal{A}}/\mathcal{J}$:*

$$[\theta, \sigma] = 2\sigma, \qquad [\theta, \Delta] = -2\Delta, \tag{17}$$

$$\sigma\Delta = \frac{1}{4}\theta(\theta + 4n - 2), \tag{18}$$

$$\Delta\sigma = \frac{1}{4}(\theta + 2)(\theta + 4n), \tag{19}$$

$$[\Delta, \sigma] = \theta + 2n. \tag{20}$$

To prove relations (17)-(20) of Proposition 10, we need to recall the existence of a b-function of the symplectic form σ that satisfies the following equation:

$$\Delta\sigma^{k+1} = (k+1)(k+2n)\sigma^k, \qquad k \in \mathbb{Z} \tag{21}$$

(see T. Kimura [19, p. 265] or T. Levasseur [22, p. 508 Appendix: (5)])

Proof. Since the symplectic form σ is an homogeneous polynomial of degree 2, and its dual Δ is homogeneous of degree -2 then we have relations (17). We recall that the algebra $\overline{\mathcal{A}}$ acts on the ring $\mathbb{C}[\sigma]$ of polynomials of the symplectic form σ. In particular the differential operators $\sigma\Delta$ and $\Delta\sigma$, which are homogeneous of degree 0 (i.e. $[\theta, \Delta\sigma] = 0$ and $[\theta, \sigma\Delta] = 0$), act also on the polynomials of the symplectic form $\mathbb{C}[\sigma]$. This means that these differential operators are polynomials of σ and $\frac{\partial}{\partial\sigma}$ namely $\Delta\sigma, \sigma\Delta \in \mathbb{C}[\sigma, \frac{\partial}{\partial\sigma}]$ such that $\sigma\frac{\partial}{\partial\sigma} = \frac{1}{2}\theta$. Thus $\sigma\Delta$ (resp. $\Delta\sigma$) is a polynomial in θ. In order to prove (18), (19) we write $\sigma\Delta$ and $\Delta\sigma$ as

$$\sigma\Delta = a_2\theta^2 + a_1\theta^1 + a_0 \quad \textit{mod } \mathcal{J} \tag{22}$$

$$\Delta\sigma = b_2\theta^2 + b_1\theta^1 + b_0 \qquad \text{mod } \mathcal{J} \tag{23}$$

differential operators of degree 2 with constant coefficients $a_k, b_k \in \mathbb{C}$ $(0 \leq k \leq 2)$. Then we determine successively the coefficients a_k (resp. b_k) by applying the operator $\sigma\Delta$ (resp. $\Delta\sigma$) on the polynomials σ^k 's $(0 \leq k \leq 2)$ and by making use of the above Bernstein-Sato formula (21). Finally the relation (20) follows immediately. ■

Let $\overline{\mathcal{J}} \subset \overline{\mathcal{A}}$ be the preimage in $\overline{\mathcal{A}}$ of the ideal in $\overline{\mathcal{A}}/\mathcal{J}$ generated by the relations (17), (18), (19). We denote by $\mathcal{A} := \overline{\mathcal{A}}/\overline{\mathcal{J}}$ the quotient algebra of $\overline{\mathcal{A}}$ by $\overline{\mathcal{J}}$ which will be used in the sequel.

4 Invariant sections in regular holonomic $\mathcal{D}_V$-modules

As in the introduction we denote by $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$ the full category consisting of regular holonomic $\mathcal{D}_V$-modules whose characteristic variety Λ is contained in the union of the closure of conormal bundles to the G -orbits (see Panyushev [32]).

Definition 11 *One says that a Lie group G acts on a $\mathcal{D}_V$-module $\mathcal{M}$ if it preserves the good filtration on $\mathcal{M}$ and there exists an isomorphism of $\mathcal{D}_{G\times V}$-modules $u : p_1^+(\mathcal{M}) \xrightarrow{\sim} p_2^+(\mathcal{M})$ where $p_1 : G \times V \longrightarrow V, (g, v) \longmapsto v$ is the projection on V and $p_2 : G \times V \longrightarrow V, (g, v) \longmapsto g \cdot v$ is the action of G on V (satisfying the associativity conditions).*

Let us denote by $\mathfrak{g}$ the Lie algebra of G and $U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. We recall the following definition:

Definition 12 *(see [22, Definition 5.1, p. 497]) A representation (G, V) is said to be of Capelli type if it is an irreducible multiplicity free representation such that $\Gamma(V, \mathcal{D}_V)^G$ is the image of the center of $U(\mathfrak{g})$ under the differential $\tau : \mathfrak{g} \longrightarrow \Gamma(V, \mathcal{D}_V)^{\mathrm{pol}}$ of the G-action i.e.*

$$\tau(Z(U(\mathfrak{g})) = \Gamma(V, \mathcal{D}_V)^G. \tag{24}$$

Note that the multiplicity free representations of Capelli type have been studied by Howe-Umeda [11], [38]. They fall into eight cases (see [22, p. 508, Appendix]). The case we are interested in $(G = Sp(2n, \mathbb{C}) \times GL(2, \mathbb{C}), V = \mathbb{C}^{2n} \otimes \mathbb{C}^2)$ is multiplicity free of Capelli type with one dimensional quotient. In this case if G_1 denotes the simply connected cover of $G' = Sp(2n,\mathbb{C}) \times SL(2,\mathbb{C})$, T. Levasseur [22, Lemma 5.15] showed that the category of G_1-equivariant $\mathcal{D}_V$-modules is equivalent to the category $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$. In particular we have the following remark:

Remark 13 *The action of $G = Sp(2n,\mathbb{C}) \times GL(2,\mathbb{C})$ on V extends to an action of its universal covering G_1 on $\mathcal{D}_V$-modules $\mathcal{M}$ in $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$. In particular $G' = Sp(2n,\mathbb{C}) \times SL(2,\mathbb{C})$ acts on $\mathcal{M}$.*

This section consists in the proof of the main general argument of the paper. We show that any $\mathcal{D}_V$-module $\mathcal{M}$ in the category $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$ is generated by its invariant global sections under the action of G'. For the proof we need to study $\mathcal{D}_V$-modules with support in the closure of the G'- orbits.

Theorem 14 *A $\mathcal{D}_V$-module $\mathcal{M}$ in $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$ is generated by its G'-invariant global sections.*

Let us prepare some basic results which will be used in the proof of this central theorem.

4.1 $\mathcal{D}$-modules with support on the closure of $Sp(2n,\mathbb{C}) \times SL(2,\mathbb{C})$-orbits

As in the preliminaries, we denote the G- orbits by $V_{r,q} := \left\{v \in V,\ \ \mathrm{rank}(v) = r \ \text{ and } \ \mathrm{rank}(vJv^T) = q\right\}$ $(0 \le r, q \le 2$ with $q \ne 1)$. Let us consider $\overline{V}_{r,q}$ the closure of the G- orbit $V_{r,q}$, that is, $\overline{V}_{r,q} := \left\{v \in V, \mathrm{rank}(v) \le r \ \text{ and } \ \mathrm{rank}(vJv^T) \le q\right\}$ the set of $2n{\times}2$-matrices v such that $(\mathrm{rank}(v),\ \mathrm{rank}(vJv^T))$ is at most (r, q) for $0 \le q \le r \le 2,\ q \ne 1$.
Denote again by $\sigma : V = (\mathbb{C}^{2n})^2 \longrightarrow \mathbb{C}, v = (x, y) \longmapsto \sigma(x, y)$ the mapping associated to the symplectic form. Then $\overline{V}_{2,0}$ is the hypersurface defined by the symplectic form σ i.e. $\overline{V}_{2,0} := \{v = (x, y) \in V$ such that $\sigma(x, y) = 0\}$.
In this section we are interested in $\mathcal{D}_V$-modules with support on $\overline{V}_{0,0}, \overline{V}_{1,0}, \overline{V}_{2,0}$ the G'-orbits. These modules will be used to prove the central theorem 14.

4.1.1 Study of sub-modules and quotients of $\mathcal{O}_V(\frac{1}{\sigma})$

Recall that we have denoted $\sigma := \sigma(v)$ and its dual $\Delta := \sigma(\frac{\partial}{\partial v})$ for $v \in V$ and $\frac{\partial}{\partial v} := \left(\frac{\partial}{\partial x_{ij}}\right) \in \mathcal{D}_V$ respectively. Here we describe the sub-quotient modules of $P := \mathcal{O}_V\left(\frac{1}{\sigma}\right)$. Actually P is generated by its G'-invariant sections $e_{-k} := \sigma^{-k}$ $(k \ge 0)$ which are homogeneous of degree $-2k$ $(k \ge 0)$ and that satisfy the following relations:

$$\sigma e_{-k} = e_{-k+1} \tag{25}$$

$$\theta e_{-k} = -2k e_{-k} \tag{26}$$

$$\Delta e_{-k} = -k(-k + 2n - 1)e_{-k-1} \tag{27}$$

where the relation (27) comes from (21).

4.1.2 Relations

We consider the sub-modules $P_k := \mathcal{D}_V e_{-k}$ of P generated respectively by e_{-k} $(0 \le k \le 2)$:

$$P_0 := \mathcal{O}_V \subset \quad P_1 := \mathcal{D}_V e_{-1} \subset \quad P_2 := \mathcal{D}_V e_{-2}. \tag{28}$$

Note that the relation (27) shows that the $\mathcal{D}_V$-module P_1 generated by e_{-1} contains all the e_{-k} for $k = 1, 2, \cdots, 2n-1$ so that $P_2 = P_1$.

Denote by $P^k := P_k/P_{k-1}$ the quotient module associated with P_k $(0 \leq k \leq 2)$. We have $P^2 := P_2/P_1 = 0$. The quotient modules $P^0 = P_0$, $P^1 := P_1/P_0$ are irreducible holonomic $\mathcal{D}_V$ -modules of multiplicity 1 supported respectively by $\overline{V}_{2,2}$, $\overline{V}_{2,0}$: Indeed

$$P^0 := \mathcal{O}_V, \;\; P^1 := \mathcal{D}_V \sigma^{-1}/\mathcal{O}_V. \tag{29}$$

They are described by the following generators and relations: $P^0 = P_0 = \mathcal{O}_V$ such that

$$P^0 = \begin{cases} \text{generator} \quad e_0 := 1_V, \\ \qquad \theta e_0 = 0, \\ \qquad \Delta e_0 = 0 \end{cases}$$

is supported by $\overline{V}_{2,2}$,

$$P^1 := P_1/P_0 = \begin{cases} \text{generator } \; \overline{e}_{-1} := e_{-1} \mod P_0, \\ \qquad \theta \overline{e}_{-1} = -2\overline{e}_{-1}, \\ \qquad \sigma \overline{e}_{-1} = 0 \mod P_0 \end{cases}$$

is supported by $\overline{V}_{2,0}$.

4.1.3 Extension

We show that any section u of the $\mathcal{D}_V$-module P_1 in the complement of $\overline{V}_{1,0}$ extends to the whole V.

Proposition 15 *A section $u \in \Gamma\left(V \backslash \overline{V}_{1,0}, \; P_1\right)$ of the $\mathcal{D}_V$-module P_1 in the complement of $\overline{V}_{1,0}$ extends to the whole V.*

Proof. Note that $\overline{V}_{2,0}$ is smooth out of $\overline{V}_{1,0}$ and it is a normal variety along $V_{1,0}$. Actually P_1 is an holonomic $\mathcal{D}_V$-module such that the associated graded modules $\mathrm{gr}(P_1)$ is the sum of modules $\mathcal{O}_{T^*_{V_{1,0}}V}\overline{e}_1$ and $\mathcal{O}_{T^*_{V_{1,0}}V}\overline{e}_0$. In this case the property of extension here is true for functions because $\overline{V}_{1,0}$ is normal along $V_{0,0}$. ∎

4.2 Proof of theorem 14

Recall that we have denoted $P_0 = \mathcal{O}_V \;\; \subset \;\; P_1 = \mathcal{D}_V(\frac{1}{\sigma}) \;\; \subset \;\; P_2 = \mathcal{D}_V(\frac{1}{\sigma^2})$.
Let $\mathcal{M}$ be an object in the category $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$. We denote by $\mathcal{M}^{G'}$ the submodule of $\mathcal{M}$ generated over $\mathcal{D}_V$ by G'-invariant homogeneous global sections i.e.

$$\mathcal{M}^{G'} := \mathcal{D}_V\{u \;\in\; \Gamma(V, \mathcal{M})^{G'} \text{ such that } \dim_{\mathbb{C}}\mathbb{C}[\theta]u < \infty\}$$

We are going to see that the quotient $\mathcal{D}_V$-module $\mathcal{M}/\mathcal{M}^{G'}$ is supported successively by the closure of the G'- orbits $V_{2,0}$, $V_{1,0}$, $V_{0,0}$ and the monodromy is trivial since the orbits are simply connected (see Proposition 3).
First the $\mathcal{D}_V$-module $\mathcal{M}/\mathcal{M}^{G'}$ is supported by $\overline{V}_{2,0}$: To see it, we will use an algebraic point of view.

Denote by $U := V \backslash \overline{V}_{2,0}$ (the complement of the symplectic hypersurface) the algebraic variety, and $U(\mathbb{C})$ the set of its complex points with its usual topology. On the Zariski open set U, the given of $\mathcal{M}$ is equivalent to that of a local system $\mathcal{F}$ on U (i.e. on $U(\mathbb{C})$). Since G' is simply connected, such a local system $\mathcal{F}$ is an inverse image by the symplectic map σ of a local system $\mathcal{L}$ on $\mathbb{G}_m := \mathbb{P}^1 \backslash \{0, \infty\}$ (i.e. on $\mathbb{C}^*$) (one has a fiber bundle with connected and simply connected fibers):

$$\mathcal{F} = \sigma^{-1}\mathcal{L} \quad \text{with} \quad \sigma : U \longrightarrow \mathbb{G}_m. \tag{30}$$

The corresponding D-module $\mathcal{N}$ on $\mathbb{G}_m$ is generated by its sections $u_1, \cdots, u_p \in \Gamma(\mathbb{G}_m, \mathcal{N})$ ($\mathbb{G}_m$ is affine):

$$\mathcal{N} = D_{\mathbb{G}_m} \langle u_1, \cdots, u_p \rangle . \tag{31}$$

The inverse images on U of these sections are G'-invariant and on U they generate $\mathcal{M}$:

$$\sigma^{-1}(u_1), \cdots, \sigma^{-1}(u_p) \in \Gamma(U, \mathcal{M})^{G'} \tag{32}$$

and

$$\mathcal{M}_{|U} = D_U \left\langle \sigma^{-1}(u_1), \cdots, \sigma^{-1}(u_p) \right\rangle \tag{33}$$

(because the action of G' on the inverse image comes from the action of G' on U, which is compatible with the projection $\sigma : U \longrightarrow \mathbb{G}_m$ (i.e. $\sigma : U \longrightarrow \mathbb{C}^*$) and compatible with the trivial action on $\mathbb{G}_m$).

Note that each of these invariant sections $\sigma^{-1}(u_1), \cdots, \sigma^{-1}(u_p)$ extends from U to the whole space V after a multiplication by a large enough power of the symplectic σ (see [8], [36]):

$$\sigma^{-1}(u_1), \cdots, \sigma^{-1}(u_p) \in \Gamma(V, \mathcal{M}) . \tag{34}$$

Moreover, after a multiplication by another power of σ, the extension is G'-invariant (see [8], [36]):

$$\sigma^{-1}(u_1), \cdots, \sigma^{-1}(u_p) \in \Gamma(V, \mathcal{M})^{G'} . \tag{35}$$

Now taking the quotient of $\mathcal{M}$ by $\mathcal{M}^{G'}$ the module generated by G'-invariant sections, we then deduce from (33) and (35) that

$$\mathcal{M}/\mathcal{M}^{G'} = 0 \quad \text{on} \quad U. \tag{36}$$

Therefore the quotient modules $\mathcal{M}/\mathcal{M}^{G'}$ is supported by $\overline{V}_{2,0}$.

Now if $\mathcal{M}$ is supported by $\overline{V}_{2,0}$, it is isomorphic out of $\overline{V}_{1,0}$ to the sum of a finite number of copies of P_1/P_0 (the Dirac module supported by $\overline{V}_{2,0}$). Then there is a morphism $\mathcal{M} \longrightarrow (P_1/P_0)^N$ whose sections extend (see Proposition 15). The image of this morphism is a sub-module of P_1/P_0 so it is generated by its invariant sections. Therefore the quotient $\mathcal{M}/\mathcal{M}^{G'}$ is supported by $\overline{V}_{1,0}$.

Next if $\mathcal{M}$ is supported by $\overline{V}_{1,0}$ it is isomorphic out of $\overline{V}_{0,0}$ to the sum of a finite number of copies of $\delta_{\overline{V}_{1,0}}$ the Dirac module supported by $\overline{V}_{1,0}$ then there is a morphism $\mathcal{M} \longrightarrow (\delta_{\overline{V}_{1,0}})^N$ whose sections extend (see Proposition 15) so that the quotient $\mathcal{M}/\mathcal{M}^{G'}$ is supported by $V_{0,0}$ (corresponding to the Dirac modules supported by $V_{0,0} = \{0\}$) because $\delta_{\overline{V}_{1,0}}$ is generated by its global invariant sections. Finally, if $\mathcal{M}$ is supported by $V_{0,0}$ the result is clear.

5 Equivalence of categories

In this section we establish the main result of this paper: Theorem 18.

Recall that $\overline{\mathcal{A}}$ is the algebra of G'-invariant differential operators, and $\mathcal{J} := \operatorname{ann}\mathbb{C}[\sigma]$ is the ideal

annihilator of G'-invariant polynomials $\mathbb{C}[V]^{G'} = \mathbb{C}[\sigma]$. We have denoted $\overline{\mathcal{J}} \subset \overline{\mathcal{A}}$ the preimage in $\overline{\mathcal{A}}$ of the ideal in $\overline{\mathcal{A}}/\mathcal{J}$ generated by the relations (17), (18), (19) of Proposition 10. Then we have put $\mathcal{A} := \overline{\mathcal{A}}/\overline{\mathcal{J}}$ the quotient algebra of $\overline{\mathcal{A}}$ by $\overline{\mathcal{J}}$.
As in the introduction we denote by $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ the category consisting of graded $\mathcal{A}$-modules T of finite type for the Euler vector field θ i.e. such that $\dim_{\mathbb{C}}\mathbb{C}[\theta]u < \infty$ for any u in T. In other words, $T = \bigoplus_{\lambda\in\mathbb{C}} T_\lambda$ is a direct sum of finite dimensional $\mathbb{C}$- vector spaces

$$T_\lambda := \bigcup_{p\in\mathbb{N}} \ker{(\theta-\lambda)^p} \ \text{(with } \dim_{\mathbb{C}} T_\lambda < \infty) \tag{37}$$

equipped with the endomorphisms σ, θ, Δ of degree 2, 0, -2 respectively and satisfying the relations (17), (18), (19) of Proposition 10 with $(\theta-\lambda)$ being a nilpotent operator on each vector space T_λ.
Recall that $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$ stands for the category consisting of regular holonomic $\mathcal{D}_V$- modules whose characteristic variety is contained in Λ the union of the closure of conormal bundles to the orbits for the action of G on V.

Let $\mathcal{M}$ be an object in the category $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$, denote by $\Psi(\mathcal{M})$ the sub-module of $\Gamma(V,\mathcal{M})$ consisting of G'-invariant homogeneous global sections $u \in \Gamma(V,\mathcal{M})^{G'}$ such that $\dim_{\mathbb{C}} \mathbb{C}[\theta]\, u < \infty$:

$$\Psi(\mathcal{M}) := \left\{u \in \Gamma(V,\mathcal{M})^{G'},\ \dim_{\mathbb{C}} \mathbb{C}[\theta]\, u < \infty\right\}. \tag{38}$$

Let us recall (Theorem 5) that for $\lambda \in \mathbb{C}$,

$$\Psi(\mathcal{M})_\lambda := [\Psi(\mathcal{M})] \bigcap \left[\bigcup_{p\in\mathbb{N}} \ker(\theta-\lambda)^p\right] \tag{39}$$

is the finite dimensional $\mathbb{C}$-vector space of homogeneous global sections of degree λ in $\Psi(\mathcal{M})$ and

$$\Psi(\mathcal{M}) = \bigoplus_{\lambda\in\mathbb{C}} \Psi(\mathcal{M})_\lambda\,. \tag{40}$$

Then we can see that $\Psi(\mathcal{M})$ is an object in $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$:

Indeed, let $(\gamma_1,\cdots,\gamma_p) \in \Gamma(V,\mathcal{M})^{G'}$ be finitely many invariant (homogeneous) global sections generating the $\mathcal{D}_V$-module $\mathcal{M}$ (see Theorem 14), we are going to see that the family $(\gamma_1,\ \cdots,\ \gamma_p)$ generates also $\Psi(\mathcal{M})$ as an $\mathcal{A}$-module:
Actually, if $\gamma = \sum_{j=1}^{p} q_j\left(v, \frac{\partial}{\partial v}\right)\gamma_j \in \Gamma(V,\mathcal{M})^{G'}$ is an invariant section of $\mathcal{M}$, denote by $\widetilde{q_j}$ the average of q_j over $Sp(n)\times SU(2)$ (compact maximal subgroup of G'), then $\widetilde{q_j} \in \overline{\mathcal{A}}$. Let f_j be the class of $\widetilde{q_j}$ modulo $\overline{\mathcal{J}}$, that is, $f_j \in \mathcal{A}$, then we also have $\gamma = \sum_{j=1}^{p} \widetilde{q_j}\gamma_j = \sum_{j=1}^{p} f_j\gamma_j$. Therefore $\Psi(\mathcal{M})$ is an object in $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$.

Conversely, let T be an object in the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$, one associates to it the $\mathcal{D}_V$-module

$$\Phi(T) := \mathcal{D}_V \bigotimes_{\overline{\mathcal{A}}} T. \tag{41}$$

Then $\Phi(T)$ is an object in the category $\mathrm{Mod}_\Lambda^{\mathrm{rh}}(\mathcal{D}_V)$.

Thus, we have defined two functors

$$\Psi : \mathrm{Mod}_\Lambda^{\mathrm{rh}}(\mathcal{D}_V) \longrightarrow \mathrm{Mod}^{\mathrm{gr}}(\mathcal{A}),\ \Phi : \mathrm{Mod}^{\mathrm{gr}}(\mathcal{A}) \longrightarrow \mathrm{Mod}_\Lambda^{\mathrm{rh}}(\mathcal{D}_V). \tag{42}$$

We need the two following lemmas:

Lemma 16 *The canonical morphism*

$$T \longrightarrow \Psi(\Phi(T)),\ t \longmapsto 1 \otimes t \tag{43}$$

is an isomorphism, and defines an isomorphism of functors $\mathrm{Id}_{\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})} \longrightarrow \Psi \circ \Phi$.

Proof. Let us consider $\mathcal{M}_0 := \mathcal{D}_V/\overline{\mathcal{J}}$. Denote by ε (the class of $1_{\mathcal{D}}$ modulo $\overline{\mathcal{J}}$) the canonical generator of $\mathcal{M}_0$. Let $h \in \mathcal{D}_V$, denote by $\widetilde{h} \in \overline{\mathcal{A}}$ its average on $Sp(n) \times SU(2)$ and by φ the class of $\widetilde{h}$ modulo $\overline{\mathcal{J}}$, that is, $\varphi \in \mathcal{A}$.
Since ε is G'-invariant, we get $\widetilde{h\varepsilon} = \widetilde{h}\varepsilon = \varepsilon\varphi$. Moreover, we have $\widetilde{h}\varphi = 0$ if and only if $\widetilde{h} \in \overline{\mathcal{J}}$. In other words $\varphi = 0$. Therefore the average operator (over $Sp(n) \times SU(2)$) $\mathcal{D}_V \longrightarrow \overline{\mathcal{A}}$, $h \longmapsto \widetilde{h}$ induces a surjective morphism of $\mathcal{A}$-modules $v : \mathcal{M}_0 \longrightarrow \mathcal{A}$. More generally, for any $\mathcal{A}$-module T in the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ the morphism $v \otimes 1_T$ is surjective

$$v_T : \mathcal{M}_0 \bigotimes_{\mathcal{A}} T \longrightarrow \mathcal{A} \bigotimes_{\mathcal{A}} T = T. \tag{44}$$

Note that v_T is the left inverse of the morphism

$$u_T : T \longrightarrow \mathcal{M}_0 \bigotimes_{\mathcal{A}} T,\ t \longmapsto \varepsilon \otimes t, \tag{45}$$

that is, $(v \otimes 1_T) \circ (\varepsilon \otimes 1_T) = v(\varepsilon) = 1_T$. This means that the morphism u_T is injective. Next, the image of u_T is exactly the set of invariant sections of $\mathcal{M}_0 \bigotimes_{\mathcal{A}} T = \Phi(T)$ namely $\Psi(\Phi(T))$: indeed if $s = \sum_{i=1}^{p} h_i \otimes t_i$ is an invariant section in $\mathcal{M}_0 \bigotimes_{\mathcal{A}} T$, we may replace each h_i by its average $\widetilde{h_i} \in \mathcal{A}$, then we get

$$s = \sum_{i=1}^{p} \widetilde{h_i} \otimes t_i = \varepsilon \otimes \sum_{i=1}^{p} \widetilde{h_i} t_i \ \in \ \varepsilon \otimes T \tag{46}$$

hence $\sum_{i=1}^{p} \widetilde{h_i} t_i \in T$. Therefore the morphism u_T is an isomorphism from T to $\Psi(\Phi(T))$ and defines an isomorphism of functors. ∎

Next we have the following lemma

Lemma 17 *The canonical morphism*

$$w : \Phi(\Psi(\mathcal{M})) \longrightarrow \mathcal{M} \tag{47}$$

is an isomorphism and defines an isomorphism of functors $\Phi \circ \Psi \longrightarrow \mathrm{Id}_{\mathrm{Mod}_\Lambda^{\mathrm{rh}}(\mathcal{D}_V)}$.

Proof. Following Theorem 14 the $\mathcal{D}_V$-module $\mathcal{M}$ is generated by finitely many invariant sections $(\zeta_i)_{i=1,\cdots,p} \in \Psi(\mathcal{M})$ so that the morphism w is surjective. Now consider $\mathcal{K}$ the kernel of the morphism $w : \Phi(\Psi(\mathcal{M})) \longrightarrow \mathcal{M}$. It is also generated over $\mathcal{D}_V$ by its invariant sections namely by $\Psi(\mathcal{K})$. Then we get

$$\Psi(\mathcal{K}) \subset \Psi[\Phi(\Psi(\mathcal{M}))] = \Psi(\mathcal{M}) \tag{48}$$

where we used $\Psi \circ \Phi = Id_{\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})}$ (see the previous Lemma 16). Since the morphism $\Psi(\mathcal{M}) \longrightarrow \mathcal{M}$ is injective ($\Psi(\mathcal{M}) \subset \Gamma(V, \mathcal{M})$) we obtain $\Psi(\mathcal{K}) = 0$. Therefore $\mathcal{K} = 0$ (because $\Psi(\mathcal{K})$ generates $\mathcal{K}$). ∎

This section ends by Theorem 18 established by means of the previous lemmas.

Theorem 18 *The functors Φ and Ψ induce equivalence of categories*

$$\mathrm{Mod}_{\Lambda}^{\mathrm{rh}}(\mathcal{D}_V) \overset{\sim}{\longrightarrow} \mathrm{Mod}^{\mathrm{gr}}(\mathcal{A}). \tag{49}$$

6 Some quivers associated with $\mathcal{A}$-modules

This section consists in the description of objects in the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$. Actually a graded $\mathcal{A}$-module T in $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ defines an infinite diagram consisting of finite dimensional vector spaces T_λ (with $(\theta - \lambda)$ being a nilpotent operator on each T_λ, $\lambda \in \mathbb{C}$) and linear maps between them deduced from σ, Δ, θ:

$$\cdots \rightleftarrows T_\lambda \underset{\Delta}{\overset{\sigma}{\rightleftarrows}} T_{\lambda+2} \rightleftarrows \cdots \tag{50}$$

satisfying the following relations $(\theta - \lambda) T_\lambda \subset T_\lambda$,

$$\sigma\Delta = \frac{1}{4}\theta(\frac{\theta}{2} + 4n - 2) \ \text{ on } \ T_\lambda, \tag{51}$$

$$\Delta\sigma = \frac{1}{4}(\theta + 2)(\theta + 4n) \ \text{ on } \ T_\lambda. \tag{52}$$

These diagrams are determined by finite subsets of objects and arrows:

a) For $\eta \in \mathbb{C}/2\mathbb{Z}$, denote by $T^\eta \subset T$ the sub-module $T^\eta = \bigoplus_{\lambda = \eta \mod 2\mathbb{Z}} T_\lambda$. Then T is generated by the finite direct sum of T^η's

$$T = \bigoplus_{\eta \in \mathbb{C}/2\mathbf{Z}} T^\eta = \bigoplus_{\eta \in \mathbb{C}/2\mathbf{Z}} \left(\bigoplus_{\lambda = \eta \mod 2\mathbb{Z}} T_\lambda \right). \tag{53}$$

b) If $\eta \neq 0 \mod 2\mathbb{Z}$ then the linear maps σ and Δ are bijective. Therefore T^η is determined by one T_λ with the nilpotent action of $(\theta - \lambda)$.

c) If $\eta = 0 \mod 2\mathbb{Z}$ then T^η is determined by one diagram of 3 elements:

$$T_{-4n} \underset{B}{\overset{A}{\rightleftarrows}} T_{-2} \underset{\Delta}{\overset{\sigma}{\rightleftarrows}} T_0 \tag{54}$$

where $A=\sigma^{2n-1}$ and $B=\Delta^{2n-1}$. In the other degrees σ or Δ are bijective. Indeed, we have

$$\sigma^k T_0 \simeq T_{2k} \simeq T_0 \tag{55}$$

and

$$\Delta^k T_{-4n} \simeq T_{-4n-2k} \simeq T_{-4n} \ (k\in\mathbb{N}) \tag{56}$$

thanks to relations (51), (52). The operator $\sigma\Delta$ (resp. $\Delta\sigma$) on T_λ has only one eigenvalue $\frac{\lambda}{4}(\lambda+4n-2)$ (resp. $\frac{1}{4}(\lambda+2)(\lambda+4n)$) so that the equations (51), (52) has each of them a unique solution θ of eigenvalue λ if λ is not a critical value. Here $\lambda=0,\ -2,\ -4n+2,-4n$ thus it is always the case.

6.1 Examples of diagrams

Example 19 *We consider the $\mathcal{D}_V$-module of holomorphic functions $\mathcal{O}_V$. It is generated by $g_0 := 1_V$ an homogeneous section of degree 0 such that*

$$\theta g_0 = 0 \ \text{ and } \ \Delta g_0 = 0.$$

This yields a graded $\mathcal{A}$-module of finite type in $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ with a basis (g_r) where $r=2k$ ($k\in\mathbb{N}$) such that $\Delta g_0=0$ and satisfying the following system:

$$E_0 = \begin{cases} \theta g_r = r g_r \ \text{ for } r=2k,\ \ k\in\mathbb{N} \\ \\ \sigma g_r = g_{r+2}, \\ \\ \Delta g_r = \frac{1}{4}r(r+4n-2)e_{r-2}. \end{cases} \tag{57}$$

Since $\Delta e_0 = 0$ (i.e. $\Delta T_0=0$) the arrows on the left of T_0 in the diagram vanish, that is,

$$0 \leftarrow T_0 \ \underset{\Delta}{\overset{\sigma}{\rightleftarrows}} \ T_2 \ \rightleftarrows \cdots \tag{58}$$

Example 20 *Here we look at the Dirac module supported by $V_{0,0}=\{0\}$ that is $\mathcal{B}_{\{0\}|V}$. It is the Fourier transform of the module of holomorphic functions $\mathcal{O}_V$ and it is generated by g_{-4n} an homogeneous section of degree $-4n$ satisfying the equations:*

$$\theta g_{-4n} = -4n g_{4n} \ \text{ and } \ \sigma g_{-4n} = 0.$$

This yields a graded $\mathcal{A}$-module whose basis is (g_p) where $p=-4n-2k$ ($k\in\mathbb{N}$) such that $\sigma g_{-4n}=0$ satisfying the system:

$$E_1 = \begin{cases} \theta g_p = p g_p \ \text{ for } p=-4n-2k,\ k\in\mathbb{N} \\ \\ \sigma g_p = \frac{1}{4}(p+2)(p+4n)g_{p+2}. \end{cases} \tag{59}$$

Since $\sigma g_{-4n}=0$ (i.e. $\sigma T_{-4n}=0$) the arrows on the right of T_{-4n} in the diagram vanish, that is,

$$\cdots \rightleftarrows \ T_{-4n-2} \ \underset{\Delta}{\overset{\sigma}{\rightleftarrows}} \ T_{-4n} \rightarrow 0. \tag{60}$$

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