L. Barchini. (0000) "Two triangularity results and invariants of $(\mathfrak{sp}((\mathbf{p}+\mathbf{q}),\mathbf{C}),\mathrm{Sp}(\mathbf{p},\mathbf{C})\times\mathrm{Sp}(\mathbf{q},\mathbf{C}))$ and $(\mathfrak{so}(\mathbf{2n},\mathbf{C}),\mathrm{GL}(\mathbf{n},\mathbf{C}))$ modules
International Mathematics Research Notices, Vol. 0000, Article ID rnn000, 21 pages.
doi:10.1093/imrn/rnn000

Two triangularity results and invariants of $(\mathfrak{sp}((\mathbf{p}+\mathbf{q}),\mathbf{C}),\mathbf{Sp}(\mathbf{p},\mathbf{C})\times\mathbf{Sp}(\mathbf{q},\mathbf{C}))$ and $(\mathfrak{so}(\mathbf{2n},\mathbf{C}),\mathbf{GL}(\mathbf{n},\mathbf{C}))$ modules.

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Fix $\mathcal{O}$ a special nilpotent co-adjoint orbit and write $A(\mathcal{O})$ for the group of components of the centralizer of an element f in $\mathcal{O}$. Let $\mathbf{1}$ denotes the trivial representation of $A(\mathcal{O})$. The Springer correspondence attaches to the pair $(\mathcal{O},\mathbf{1})$ an irreducible representation of the Weyl group W which we denote by $Sp(\mathcal{O})$. $Sp(\mathcal{O})$ admits two natural bases consisting of homogeneous polynomials on $\mathfrak{h}^*$. One of the bases is parametrized by a set $\mathrm{Prim}_\rho(\mathcal{O})$, consisting of certain primitive ideals of the universal enveloping algebra. The second basis consists of character polynomials, parametrized by a set of geometric objects. The relation between these bases encodes significant representation theoretical information. We study the change-of-basis matrix and derive consequences relevant to the computation of invariants of Harish-Chandra modules.

1 Introduction

The Springer correspondence attaches to each special complex nilpotent orbit $\mathcal{O}$, an irreducible representation of the Weyl group, $Sp(\mathcal{O})$. This representation admits two natural bases. It was initially suspected that the two bases coincide (up to scaling). However, Tanisaki provided examples showing that this is not the case and conjectured an upper triangular relation between the two, see [22]. The relationship between the bases has been studied by McGovern (using algebraic/combinatorial methods) and by Trapa (using a geometric approach), see ([16], [24]). The aim of this paper is to relate and combine the results in [16] and [24] in order to derive information relevant to the computation of invariants of Harish-Chandra modules.

We focus on the pairs (G,K) of complex groups

$$\begin{array}{l}(\mathrm{Sp}(2n,\mathbf{C}),\mathrm{Sp}(p,\mathbf{C})\times\mathrm{Sp}(q,\mathbf{C})),\ p+q=n\\ (\mathrm{SO}(2n),\mathrm{GL}(n,\mathbf{C}))\end{array}\tag{1.1}$$

and their real forms $G_{\mathbf{R}} = \mathrm{Sp}(p,q)$ and $\mathrm{SO}^*(2n)$.

Before describing our results in detail we introduce some notation. We write $\mathfrak{g}$ for the Lie algebra of G and we let $\mathcal{N}$ denote the nilpotent cone of $\mathfrak{g}$. The variety of Borel subalgebras is denoted by X. We fix a base point $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ and we let $B = HN \subset G$ denote the corresponding Borel subgroup. We write ρ for half the sum of the roots of $\mathfrak{h}$ in $\mathfrak{n}$.

If $\mathcal{O}$ is a special nilpotent orbit, then the set of primitive ideals in the enveloping algebra

$$\begin{aligned} \mathrm{Prim}_\rho(\mathcal{O}) = \{I : \ & \text{primitive 2-sided ideals in } \mathcal{U}(\mathfrak{g}) \\ & I = I(\rho) \text{ contains the augmentation ideal of the center of } \mathcal{U}(\mathfrak{g}) \\ & \text{with respect to the degree filtration, the variety of zeros of } gr(I) = \overline{\mathcal{O}}\} \end{aligned}$$

is non-empty, see [2]. For $\lambda \in \mathfrak{h}^*$ dominant regular, let $I(\rho + \lambda)$ stand for the primitive ideal in $\mathcal{U}(\mathfrak{g})$ obtained from $I(\rho)$ using the translation functor from ρ to $\rho + \lambda$. Jospeh attaches to each $I \in \mathrm{Prim}_\rho(\mathcal{O})$ a polynomial $q_I \in \mathcal{P}(\mathfrak{h}^*)$, the so called Goldie rank polynomial. The value of q_I at $\rho + \lambda$ is the Goldie rank of the primitive quotient $\mathcal{U}(\mathfrak{g}) \ /I(\rho + \lambda)$, see [12]. The set $\{q_I : I \in \mathrm{Prim}_\rho(\mathcal{O})\}$ is linearly independent and its span over $\mathbf{C}$ is the special representation $Sp(\mathcal{O})$.

A second basis of $Sp(\mathcal{O})$ consists of character polynomials, see [12]. Character polynomials are parameterized by a set of geometric objects, i.e.

$$\{\Upsilon : \Upsilon \text{ irreducible component of } \overline{\mathcal{O}} \cap \mathfrak{n}\}. \tag{1.2}$$

These irreducible varieties are H-stable subvarieties of $\mathfrak{n}$. (They are called orbital varieties.) If $I(\Upsilon)$ is the ideal of definition of Υ in the polynomial algebra $\mathcal{P}(\mathfrak{n})$, then the character polynomial p_Υ measures the growth rate of the H-module $\mathcal{P}(\mathfrak{n}) \ /I(\Upsilon)$.

The relation between the bases $\{q_I\}$ and $\{p_\Upsilon\}$ encodes significant representation theoretical information. In [16], the author works within the category $\mathcal{M}_\rho(\mathfrak{g}, B)$ of finitely generated B-finite $(\mathfrak{g}, B)$-modules with infinitesimal character ρ. In this context, the author defines a bijection between the parameter spaces $\mathrm{Prim}_\rho(\mathcal{O})$ and $\{\Upsilon : \Upsilon \text{ irreducible component of } \overline{\mathcal{O}} \cap \mathfrak{n}\}$. He introduces an order on the set $\{\Upsilon\}$ and studies the matrix that relates the bases $\{q_I\}$ and $\{p_\Upsilon\}$.

The results in [24] are presented within the category $\mathcal{M}_\rho(\mathfrak{g}, K)$ of Harish-Chandra modules having trivial infinitesimal character. The author defines a bijection, from $\mathrm{Prim}_\rho(\mathcal{O})$ to $\{\Upsilon\}$ and introduces various orders on the set $\{\Upsilon\}$. In each of such orders, the change-of-basis matrix is upper triangular. This triangularity result imposes restriction on the possible shape of the leading term of the characteristic cycle of irreducible modules in $\mathcal{M}_\rho(\mathfrak{g}, K)$.

For the groups (1.1) we prove that the algebraically defined bijection between $\mathrm{Prim}_\rho(\mathcal{O})$ and $\{\Upsilon\}$ in [16] agrees with the geometrically defined one in [24]. As a consequence, in §5 we derive strong restrictions on the shape of leading term cycles of irreducible Harish-Chandra modules. Low rank computations lead Trapa to ask if leading term cycle of Harish-Chandra modules for the pairs (1.1) were always irreducible. We show that this is not the case, see §5.2. We give an example of one such module with reducible leading term cycle. We outline a general strategy to find examples of $\mathfrak{sp}(n,n)$-modules with reducible leading term cycle. Both Theorem 5.4 and the example just mentioned rely heavily on Trapa's work in [24].

Let $T^*_K(X)$ be the generalized Steinberg variety for the action of K on the flag variety X (the union of conormal bundles to the finitely many K orbits on X). The topological construction of the Springer representation can be adapted to prove that the top Borel-Moore homology $H_{\text{top}}(T_K^*(X))$ is a Weyl group representation. On the other hand, the Grothendieck group of $\mathcal{M}_\rho(\mathfrak{g}, K)$ affords an action of W, via coherent continuation. The characteristic cycle descends to the Grothendieck group $\mathcal{K}(\mathcal{M}_\rho(\mathfrak{g}, K))$ of $\mathcal{M}_\rho(\mathfrak{g}, K)$. The resulting linear map

$$CC : \mathcal{K}(\mathcal{M}_\rho(\mathfrak{g}, K)) \to H_{\text{top}}(T_K^*(X)) \tag{1.3}$$

is W-equivariant, [21]. An important ingredient in the discussion is the notion of Harish-Chandra cells (see, e.g [1]). Cells partition the irreducible modules in $\mathcal{M}_\rho(\mathfrak{g}, K)$. Each Harish-Chandra cell $\mathcal{C}$ determines a cell representation $V_{\mathcal{C}}$, a minimal subquotient of the coherent continuation representation of the Weyl group W on $\mathcal{K}(\mathcal{M}_\rho(\mathfrak{g}, K))$ that is spanned by $\mathcal{C}$. Theorem 4.13 and Theorem 4.17 concern the structure of Harish-Chandra cells as $\mathbb{Q}[W]$-modules. Theorem 4.17 identifies, within each Harish-Chandra cell, a small number of irreducible modules with the property that they generate the cell representation as $\mathbb{Q}[W]$-module. In particular, the computation of invariants of modules in the cell is determined by the invariants of one such module and coherent continuation. A simpler version of Theorem 4.17 is used in [3] to compute the associated cycles of all irreducible representation in cells that contain representations in the discrete series.

2 Background and Notation

2.1 Primitive ideals with trivial infinitesimal character

We summarize in this sub-section some results on primitive ideals that are relevant to our work. We recall the notion of generalized τ-invariant and we include a very brief discussion on the classification of primitive ideals with infinitesimal character ρ in types C and D. The reference cited here is not exhaustive.

We can assume that $\mathfrak{g}$ is any complex reductive Lie algebra except when indicated otherwise. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. The choice of Borel subalgebra determines a positive system $\Delta^+(\mathfrak{g}, \mathfrak{h})$. Let $\rho \in \mathfrak{h}^*$ stand for half the sum of the roots of $\mathfrak{h}$ in $\mathfrak{n}$. Let w_o denote the long element in the Weyl group $W(\mathfrak{g}, \mathfrak{h})$. Write $\text{Prim}_\rho(\mathcal{U}(\mathfrak{g}))$ for the set of primitive ideals in $\mathcal{U}(\mathfrak{g})$ with infinitesimal character ρ. For $w \in W(\mathfrak{g}, \mathfrak{h})$ put

- $M_w = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbf{C}_{ww_o\rho-\rho}$
- $L_w =$ irreducible quotient of M_w
- $I_w =$ annihilator of L_w in $\mathcal{U}(\mathfrak{g})$.

Duflo proved in [7] that the map from $W(\mathfrak{g}, \mathfrak{h})$ to $\text{Prim}_\rho(\mathcal{U}(\mathfrak{g}))$ given by $w \to I_w$ is surjective. Hence the classification of $\text{Prim}_\rho(\mathcal{U}(\mathfrak{g}))$ is reduced to the determination of the fiber of Duflo's map. Relevant to such classification are the notions of τ-invariant and generalized τ-invariant. For any $w \in W$ the τ-invariant is defined as

$$\tau(w) = \{\text{simple roots in } \Delta(\mathfrak{g}, \mathfrak{h}) : w\alpha \notin \Delta^+(\mathfrak{g}, \mathfrak{h})\}.$$

Following the notation in [16] we write $\tau^r(w)$, the right τ-invariant of w, for

$$\tau^r(w) = \{\text{simple roots in } \Delta(\mathfrak{g}, \mathfrak{h}) : w^{-1}\alpha \notin \Delta^+(\mathfrak{g}, \mathfrak{h})\}.$$

Even when $\tau(w)$ depends only on the primitive ideal I_w, it is not a complete invariant of I_w. Vogan introduced in [25] the notion of generalized τ-invariant; denoted by τ_∞. He proved, in type A_n, that the generalized τ-invariant (together with infinitesimal character) determines primitive ideals. Analogous results hold when $\mathfrak{g}$ is of classical type (with an appropriate definition of τ_∞ in type D), see for example [2], [8], [9].

The notion of generalized τ-invariant is closely related to that of coherent continuation action of the Weyl group $W(\mathfrak{g},\mathfrak{h})$ (implemented by composition of translation functors) on the Grothendieck group of highest weight modules. If α,β are adjacent simple roots, Vogan defined sets $D_{\alpha,\beta}=\{w\in W(\mathfrak{g},\mathfrak{h}):\beta\in\tau(w) \text{ and } \alpha\notin\tau(w)\}$ and operators

$$\begin{aligned} T_{\alpha,\beta} :&D_{\alpha,\beta}\to D_{\beta,\alpha}\\ T_{\alpha,\beta}(w)&=\{s_\alpha w, s_\beta w\}\cap D_{\beta,\alpha}. \end{aligned}$$

When α,β are roots of the same length $T_{\alpha,\beta}(w)$ is single-valued. When α,β are roots of different length $T_{\alpha,\beta}(w)$ is a one or two-elements set, see [25]. When $\mathfrak{g}$ is of type D, the operators $T_{\alpha,\beta}$ are supplemented by an operator T_D, see [9]. Two Weyl group elements w,w' have the same generalized τ-invariant if, for every (possibly empty) composition C of operators $T_{\alpha,\beta}$ and T_D, Cw is defined if and only if Cw' is and the elements in Cw and Cw' match up in such a way that matching elements have the same τ-invariant.

A second action of $W(\mathfrak{g},\mathfrak{h})$ on the Grothendieck group of highest weight modules is implemented, for example by compositions of Enright completion functors. This action is called left coherent continuation action in [16], (unfortunately the literature is not consistent on the usage of right versus left.) Then one can analogously define the notion of generalized right τ-invariant, τ^r_∞. A careful explanation of these notions is given for example in [[8], II, page 4].

When $\mathfrak{g}$ is of type A_n, Joseph determined when two irreducible highest weight modules have the same annihilator using the Robinson-Schensted algorithm. For other classical Lie algebras, an algorithmic characterization of primitive ideals is given in [2]. Garfinkle proved, for $\mathfrak{g}$ classical, that the generalized τ-invariant is a complete invariant of primitive ideals with trivial infinitesimal character, see [8]. Garfinkle extends the Robinson-Schensted algorithm to classical Weyl groups other than type A_n. To each $w\in W(\mathfrak{g},\mathfrak{h})$ she attaches a pair of standard domino tableaux $(L(w),R(w))$ of the same shape that satisfy $R(w)=L(w^{-1})$. Such tableaux consists, in types C_n and D_n of n dominos labeled by integers from one to n occupying the shape of a Young diagram, so that rows are left justified and the length decreases weakly as one moves down. The labels on the dominos increase as one moves down or to the right. As in type A_n, two Weyl group elements determine the same primitive ideal if they have the same left domino tableaux. The converse is not true. To remedy this, Garfinkle introduced an equivalence relation on domino tableaux. Within each equivalence class there is exactly one tableau of "special" shape. She proved that $I_w=I_{w'}$ if and only if $L(w)$ and $L(w')$ are equivalent to the same standard domino tableau of special shape, see [8]. Furthermore, she interpreted the operators $T_{\alpha,\beta}$ and T_D as operators at the level of domino tableaux and characterized the set of pairs of domino tableau that can be connected via compositions of such operators. As a result the τ_∞ of a primitive ideal can be read of the corresponding left domino tableau , [8].

2.2 Cells of the Weyl group

In the study of primitive ideals of complex semisimple Lie algebras, Joseph introduced the notion of cells of the Weyl group W. For each $w \in W$ and $I_w = \mathrm{Ann}(L_w)$ Joseph associates a left and a right cell. These cells are equivalence classes for the relations $w \sim_L w'$ if $I(w) = I(w')$ and $w \sim_R w'$ if $I(w^{-1}) = I(w'^{-1})$, respectively. Then

- $\mathcal{C}^L_w = \{w' \in W : I(w') = I(w)\}$
- $\mathcal{C}^R_w = \{w' \in W : I(w'^{-1}) = I(w^{-1})\}$
- $\mathcal{D}_w = \{$ smallest set generated by $\sim_L, \sim_R$ from $w\}$

$\mathcal{D}_w$ is called the double cell of w. At times we write $\mathcal{C}_I$ for $\{w' \in W : I(w') = I\}$.

The triangularity results studied in this paper are closely related to the notion of associated variety of L_w, see §2.3. It is important for us to recall that the associated variety is an invariant of the right cells, see [[1], Prop. 2.9].

Each cell can be regarded in a natural manner as a basis of a (non-necessarily irreducible) representation of W, see for example [1]. We denote the respective representation by V^L_w, V^R_w and $V_{\mathcal{D}_w}$.

2.3 Geometric background

Through the next sections (G, K) is one of the pairs (1.1). The subgroup K is the fixed point group of an involution Θ of G. The (complexified) Cartan decomposition of the Lie algebra of G (for θ be the differential of Θ) is written as $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. We fix $\mathfrak{h}$ a Cartan subalgebra contained in $\mathfrak{k}$ and a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \subset \mathfrak{g}$. Relevant to our discussion are various geometric objects. Some are the K-orbits on the flag variety X, the conormal bundle to the K-orbit $\mathcal{Q}$, $T^*_{\mathcal{Q}}X$. Also relevant are the nilpotent cone $\mathcal{N}$, $\mathcal{N}_\theta = \mathcal{N} \cap \mathfrak{p}$, the moment map $\mu : T^*X \to \mathcal{N}$. We describe some of these objects more explicitly.

We identify the cotangent bundle of X with the homogeneous bundle $G \underset{B}{\times} \mathfrak{n}$. Under this identification, the moment map for the natural action of G on T^*X is

$$\mu : G \underset{B}{\times} \mathfrak{n} \to \mathcal{N} \tag{2.1}$$

$$\mu(g, \xi) = g \cdot \xi \ (:= \mathrm{Ad}(g)\xi). \tag{2.2}$$

The fibers to μ are referred to as Springer fibers; we often use the common notation X^f for $\mu^{-1}(f)$.

The group K acts on X with finitely many orbits. The moment map image of the conormal bundle to a K-orbit $\mathcal{Q}$ is a subvariety of $\mathcal{N}_\theta$. Since μ is proper and $T^*_{\mathcal{Q}}X$ is irreducible, $\mu(T^*_{\mathcal{Q}}X)$ is irreducible. Indeed,

$$\mu(\overline{T^*_{\mathcal{Q}}X}) = \overline{K \cdot f} \tag{2.3}$$

is the closure of a single nilpotent K-orbit.

For $f \in \mathcal{N}_\theta$ we let $\mathcal{O} = G \cdot f$ and write $A(\mathcal{O})$ for the component group of $Z_G(f)$, the centralizer of f in G. Similarly $A_K(f)$ is the component group of $Z_K(f)$. For the groups under consideration $A(\mathcal{O})$ is a product of copies of $\mathbf{Z}/\mathbf{Z}_2$. The finite group $A(\mathcal{O})$ acts on the set of irreducible components of the Springer fiber $\mu^{-1}(f)$. It is known that conormal bundles to K-orbits in X partition the fiber of μ into $A_K(f)$-orbits. For the groups under consideration, one can show that those $A_K(f)$-orbits are singletons.

There is a closed relation between $A(\mathcal{O})$-obits in the set of irreducible components of the Springer fiber $\mu^{-1}(f)$ and the set of orbital varieties for $\mathcal{O}$, [19]. Indeed, Spaltenstein defined a bijection

$$\{\Upsilon: \text{ irreducible comp. } \overline{\mathcal{O}} \cap \mathfrak{n}\} \leftrightarrow \{A(\mathcal{O})\text{-orbits in } \mathrm{Irr}(\mu^{-1}(f))\}. \tag{2.4}$$

On the other hand, as $T^*_{\mathcal{Q}}(X) \cap \mu^{-1}(f)$ is dense in a unique irreducible component $C_{\mathcal{Q}}$, the assingment

$$\Upsilon \equiv A(\mathcal{O}) \cdot C \leftrightarrow S_{\Upsilon} = \{\mathcal{Q}_i \in K\ /X : \mu(\overline{T^*_{\mathcal{Q}_i} X}) = \overline{K \cdot f} \text{ and } C_{\mathcal{Q}_i} \in A(\mathcal{O}) \cdot C\} \tag{2.5}$$

defines a bijection between $\{\Upsilon: \text{ irreducible comp. } \overline{\mathcal{O}} \cap \mathfrak{n}\}$ and sets consisting of K-orbits in X. The map (2.5) will play an important role in our discussion on invariants of Harish-Chandra modules.

Remark 2.6. When $\mathcal{O}$ is special, $\#\mathrm{Prim}(\mathcal{O})_\rho = \#\{A(\mathcal{O})\backslash\mathrm{Irr}(\mu^{-1}(f))\}$. □

As in the case of primitive ideals, orbital varieties are parametrized by appropriate standard domino tableaux. Given $\mathcal{O}$ a special nilpotent orbit attached to a partition $\mathbf{p}$, McGovern parametrized orbital varieties for $\mathcal{O}$ by standard tableaux of shape $\mathbf{p}$ (with Roman numerical I or II in type D), see [[14], Thms. 1 and 3]. His parametrization relies in the generalized Robinson-Schensted algorithm as Garfinkle's parametrization of primitive ideals does. Garfinkle's equivalence relation at the level of domino tableaux is too strong to describe orbital varieties. McGovern defines a new equivalence relation and shows that within each equivalence class there is a unique domino tableau of shape $\mathbf{p}$. Once again the notion of τ-invariant is defined at the level of domino tableaux. In particular if $\Upsilon_w = \overline{B(\mathfrak{n} \cap \mathfrak{n}^w)}$ corresponds to a domino tableau T, then $\tau(T) = \tau^r(w) = \tau(w^{-1})$. It is worth mentioning that a different parametrization of orbital varieties in terms of signed domino tableux is given in [17]. Both McGovern and Pietraho's parametrization yield tableux with the same generalized τ-invariant. (This is verified by inspecting the action of $T_{\alpha,\beta}$ at the level of tableaux.)

3 Characteristic Cycles and Associated Varieties

3.1 Category $\mathcal{M}_\rho(\mathfrak{g}, B)$

Consider the category of finitely generated B-finite $(\mathfrak{g}, B)$-modules, $\mathcal{M}_\rho(\mathfrak{g}, B)$. The study of invariants of such modules gave rise to a rich and deep theory involving objects such as primitive ideals, nilpotent orbits, the flag variety and representations of the Weyl group W. Crucial to this is the equivalence of the category $\mathcal{M}_\rho(\mathfrak{g}, B)$ and the category $\mathcal{M}_{\mathrm{coh}}(\mathcal{D}_X, B)$ of B-equivariant coherent $\mathcal{D}_X$-modules on X, [4]. The characteristic cycle of a module in $\mathcal{M}_\rho(\mathfrak{g}, B)$, or rather its localization in the flag variety X, is defined as usual in the theory of $\mathcal{D}_X$-module, see for example [5]. The characteristic cycle of $\mathcal{L}_w$, the localization of L_w, is a formal integer combination of closures of conormal bundles to B-orbits on X. These B-orbits are parameterized by W; we write them as $X(y)$. Thus, the characteristic cycle of $L_w \in \mathcal{M}_\rho(\mathfrak{g}, B)$ may be written as

$$CC(L_w) = CC(\mathcal{L}_w) = \sum_{y \in W} m_{w,y} \overline{T^*_{X(y)}X}. \tag{3.1}$$

Theorem 3.2 ([5], §6.9 and 6.11)**.** (1) If $CC(\mathcal{L}_w) = \sum_{y \in W} m_{w,y} \overline{T^*_{X(y)}X}$, then $CC(\mathcal{L}_{w^{-1}}) = \sum_{y \in W} m_{w,y} \overline{T^*_{X(y^{-1})}X}$.

(2) If $m_{w,y} \neq 0$, then $\tau(w) \subset \tau(y)$ and $\tau^r(w) \subset \tau^r(y)$.

(3) $AV(L_w) = \mu(CC(\mathcal{L}_w)) = \cup \Upsilon_i$ where Υ_i is orbital for $\mathcal{O}$ with $\overline{\mathcal{O}} = AV(Ann(L_w))$.

□

As mentioned in the introduction, Joseph associated to each Υ a homogeneous polynomial $p_\Upsilon \in \mathcal{P}(\mathfrak{h}^*)$, see [12]. The set $\{p_\Upsilon : \Upsilon \text{ irreducible component of } \overline{\mathcal{O}} \cap \mathfrak{n}\}$ is linearly independent.

$$Sp(\mathcal{O}) = \text{span}_{\mathbf{C}}\{p_\Upsilon : \Upsilon \text{ irreducible component of } \overline{\mathcal{O}} \cap \mathfrak{n}\} \tag{3.3}$$

is an irreducible representation of W; it is called the special representation attached to $\mathcal{O}$. $Sp(\mathcal{O})$ admits a second basis that arises in the study of primitive ideals in the enveloping algebra. The second basis, also defined by Joseph, associates the Goldie rank polynomial, q_I, to each primitive ideal in $\text{Prim}_\rho(\mathcal{O})$. A very strong connection between the two sets of polynomials and the associated variety is the following.

Theorem 3.4. [10] Suppose $L_{w^{-1}}$ is an irreducible highest weight module and write the Goldie rank polynomial of $\text{Ann}(L_{w^{-1}})$ as $q_{\text{Ann}(L_{w^{-1}})} = \sum m_i \, p_{\Upsilon_i}$. Then,

$$m_i \neq 0 \text{ if and only if } \Upsilon_i \text{ is an irreducible component of} AV(L_w).$$

□

When $\mathfrak{g}$ is a classical Lie algebra, McGovern studied the change-of-basis matrix between $\{q_I\}$ and $\{p_\Upsilon\}$ in connection to the structure of associated varieties of highest weight modules.

Theorem 3.5. [[14], Theorem 2] Assume $\mathfrak{g}$ is a classical Lie algebra. For each L_w, there exists $w' \in W$ with $AV(L_w) = AV(L_{w'})$ such that the orbital variety $\Upsilon_{w'} = \overline{B(\mathfrak{n} \cap \mathfrak{n}^{w'})}$ is an irreducible component of $AV(L_w)$. Moreover, $\Upsilon_{w'}$ is parametrized by domino tableau of special shape T such that, in the notation of §2.1, $\tau_\infty(T) = \tau_\infty(w^{-1})$. □

Theorem 3.6. [[14] and [16]] Keep the notation of Theorem 3.5. The map

$$\begin{aligned} \text{Prim}_\rho(\mathcal{O}) &\to \{\Upsilon : \Upsilon \text{ irreducible component of } \overline{\mathcal{O}} \cap \mathfrak{n}\} \\ I_{w^{-1}} &\to \Upsilon_{w'} = \Upsilon_T \end{aligned}$$

is a bijection of sets. □

Using the parametrization of orbital varieties via standard domino tableaux, as mentioned in §2.3, and Theorems (3.5) (3.6) McGovern introduced a combinatorial order on the set $\{\Upsilon : \Upsilon \text{ irreducible component of } \overline{\mathcal{O}} \cap \mathfrak{n}\}$, see [16]. For Υ_1, Υ_2 parametrized by T_1, T_2, he defined $\Upsilon_1 < \Upsilon_2$ if $\tau_\infty(T_1) \subset \tau_\infty(T_2)$ and showed that in that order the relation between Goldie rank polynomials and characteristic polynomials is upper triangular.

3.2 Category $\mathcal{M}_\rho(\mathfrak{g}, K)$

Let $\mathcal{D}_X$ denote the sheaf of algebraic differential operators on the flag variety X of $\mathfrak{g}$. Let M be a Harish-Chandra module with infinitesimal character ρ. Since $\mathcal{U}(\mathfrak{g})$ acts by global differential operators on X, the localization $\mathcal{M} = \mathcal{D}_X \otimes_{\mathcal{U}(\mathfrak{g})} M$ is a well defined $(\mathcal{D}_X, K)$-module. The support of a Harish-Chandra module M with infinitesimal character ρ is defined to be the support of its localization $\mathcal{M}$. When M is irreducible, the support of $\mathcal{M}$ is the closure of a unique K-orbit in X. Keeping the notation used by Trapa in [24], we write for M irreducible

$$\mathrm{supp}(M) = \overline{\mathrm{supp}_o(M)}$$

.

The $(\mathcal{D}_X, K)$-module $\mathcal{M}$ admits good K-equivariant filtrations compatible with the degree filtration of $\mathcal{D}_X$. Then $\mathrm{gr}(\mathcal{M})$ becomes a $(\mathcal{O}_{T^*X}, K)$-module, where $\mathcal{O}_{T^*X}$ denotes the ring of regular function on T^*X. The characteristic cycle of $\mathcal{M}$ is then the support (counting multiplicities) of $gr(\mathcal{M})$. In particular, for each irreducible $M \in \mathcal{M}_\rho(\mathfrak{g}, K)$ there are positive integers n_i and K-orbits $\mathcal{Q}_i$ so that

$$CC(M) = CC(\mathcal{M}) = \sum_i n_i [\, \overline{T^*_{\mathcal{Q}_i} X} \,]. \tag{3.7}$$

The characteristic variety of $\mathcal{M}$ is the support of $gr(\mathcal{M})$, i.e

$$CV(M) = CV(\mathcal{M}) = \cup_{\mathcal{Q}_i : n_i \neq 0} \overline{T^*_{\mathcal{Q}_i} X}.$$

The associated cycle (see, [26]) is related to the characteristic cycle through the moment map $\mu : T^*X \to \mathcal{N}$. In fact,

$$AV(M) = \mu(CV(M)) \subset \mathcal{N}_\theta$$

By (2.3) $AV(M)$ is a union of nilpotent K-orbits closure. If $\mathcal{O}_K$ is open in $AV(M)$, then $\overline{\mathcal{O}} = \overline{G \cdot \mathcal{O}_K} = AV(\mathrm{Ann}(M))$.

Theorem 3.8. ([24], Theorem 5.2) Assume that $G_{\mathbf{R}} = Sp(p, q)$ or $G_{\mathbf{R}} = SO^*(2n)$. Let M be an irreducible $(\mathfrak{g}, K)$ module with support $\overline{\mathcal{Q}}$. Then,

$$AV(M) = \mu(\overline{T^*_{\mathcal{Q}}(X)}).$$

In particular, $AV(M)$ is irreducible. □

Definition 3.9. Assume that $G_{\mathbf{R}} = Sp(p, q)$ or $G_{\mathbf{R}} = SO^*(2n)$. Let M be an irreducible $(\mathfrak{g}, K)$ module with $AV(M) = \overline{\mathcal{O}_K}$. Define, the leading term of M as

$$LTC(M) = \sum_{\mathcal{Q}_i \in \mu^{-1}(\mathcal{O}_K)} n_i [\, \overline{T^*_{\mathcal{Q}_i} X} \,]$$

where $\mu^{-1}(\mathcal{O}_K) = \{\mathcal{Q} : \mu(\overline{T^*_{\mathcal{Q}}(X)}) = \mathcal{O}_K\}$. □

Remark 3.10. The set $\mu^{-1}(\mathcal{O}_K)$ is partially ordered by the closure order on $K \backslash X$. □

Keep the assumption $G_{\mathbf{R}} = Sp(p,q)$ or $G_{\mathbf{R}} = SO^*(2n)$ and let M be an irreducible $(\mathfrak{g}, K)$ module. Write $Ann(M) = I_{w^{-1}} = Ann(L_{w^{-1}})$, $\overline{\mathcal{O}} = AV(Ann(M))$, $AV(M) = \overline{\mathcal{O}_K} = \overline{K \cdot f}$ and in the notation of (1.2) $\overline{\mathcal{O}} \cap \mathfrak{n} = \cup\, \Upsilon_i$. Recall the polynomials q_{I_w}, p_{Υ_i} of §1. In [24], the author uses the geometry of $LTC(M)$ to define various orders on $\{\Upsilon_i\}$ and to deduce that in such orders the matrix that relates $\{q_{I_w}\}$ to $\{p_{\Upsilon_i}\}$ is upper triangular. Key to this result is bijection (2.5) that assigns to each orbital variety Υ a set $S_\Upsilon \subset \mu^{-1}(\mathcal{O}_K)$. Within each S_Υ, Trapa choses an orbit $\mathfrak{Q}_\Upsilon \in S_\Upsilon$ such that $\mathfrak{Q}_\Upsilon$ is minimal in the order closure restricted to S_Υ and defines

$$\mu^{-1}(K \cdot f)' = \{\mathfrak{Q}_{\Upsilon_i}\}.$$

Theorem 3.11. ([24], Theorem 3.20) Choose any total order order on $\mu^{-1}(K \cdot f)' \subset K\backslash X$ compatible with closure order inclusion in $K\backslash X$. Enumerate $\mu^{-1}(K \cdot f)'$ as $\mathfrak{Q}_0, \dots \mathfrak{Q}_r$ in this total order. Write $M(\mathfrak{Q}_i)$ for the $(\mathfrak{g}, K)$ module with support $\overline{\mathfrak{Q}_i}$ and write $I_j = Ann(M(\mathfrak{Q}_j))$. Use (2.5) to compatibly order the set $\{\Upsilon_i\}$. The matrix $(m_{i,j})$ such that

$$q_{I_j} = \sum m_{i,j}\; p_{\Upsilon_i}.$$

is upper triangular. □

A very practical tool in relating invariants attached to irreducible modules in $\mathcal{M}_\rho(\mathfrak{g}, B)$ to those of irreducible modules in $\mathcal{M}_\rho(\mathfrak{g}, K)$ is the geometric interpretation of Theorem 3.4 given below.

Theorem 3.12. ([24], Corollary 4.2 and Theorem 5.2) Assume that $G_{\mathbf{R}} = Sp(p,q)$ or $G_{\mathbf{R}} = SO^*(2n)$. For M irreducible write $AV(M) = \overline{K \cdot f}$ and $\mathrm{Ann}(M) = \mathrm{Ann}(L_{w^{-1}})$.

(1) If the conormal bundle $\overline{T^*_{\mathfrak{Q}}(X)}$ contributes to the leading term cycle of M then the orbital variety Υ that corresponds to $A(\mathcal{O}) \cdot (\mu^{-1}(f) \cap \overline{T^*_{\mathfrak{Q}}(X)}) = A(\mathcal{O}) \cdot C$ occurs and it is open in $AV(L_w)$.
(2) If Υ is open in $AV(L_w)$ then there exists $T^*_{\mathfrak{Q}}(X)$ contributing to the leading term cycle of M so that Υ is the orbital variety attached to $A(\mathcal{O}) \cdot (\mu^{-1}(f) \cap \overline{T^*_{\mathfrak{Q}}(X)})$.

□

4 Harish-Chandra Cells and Harish-Chandra Cells Representations

The set of Harish-Chandra modules with infinitesimal character ρ partitions into cells. We call these cells Harih-Chandra cells. Cells are designed to capture information of tensoring Harish-Chandra modules with finite dimensional representations. Indeed, all the representations in a Harish-Chandra cell have the same associated variety. In §4.1 we summarize relevant information on Harish-Chandra cells when $G_{\mathbf{R}} = \mathrm{Sp}(p,q)$ or $\mathrm{SO}^*(2n)$. Proposition 4.2 will play a key role in our discussion on leading term cycles.

The Weyl group W acts on the Grothendieck group of Harish-Chandra modules with infinitesimal character ρ via coherent continuation. Each Harish-Chandra cell defines a W-representation $V_{\mathcal{C}}$, a minimal subquotient of the coherent continuation representation which is spanned by the irreducible modules in $\mathcal{C}$. See for example [1]. Sub-section 4.2 concerns the W-structure of Harish-Chandra cell representations for the pairs (1.1).

4.1 **Harish-Chandra cells**

When $G_{\mathbf{R}} = Sp(p,q)$ or $G_{\mathbf{R}} = SO^*(2n)$, cells of Harish-Chandra modules have special properties that make the study of leading term cycles possible. We summarize such properties in the following Theorem.

Theorem 4.1. Assume that $G_{\mathbf{R}} = Sp(p,q)$ or $G_{\mathbf{R}} = SO^*(2n)$ and let $\mathcal{C}$ be a cell of Harish-Chandra modules with infinitesimal character ρ.

(1) Each irreducible module in $\mathcal{C}$ has irreducible associated variety.
(2) If $M \in \mathcal{C}$ has $\operatorname{supp}(M) = \overline{\mathcal{Q}}$, then $\mu(\overline{T^*_{\mathcal{Q}}(X)}) = AV(M)$.
(3) Let $\mathcal{O}$ be a special complex nilpotent orbit and $\mathcal{O} \cap (\mathfrak{g}/\mathfrak{k})^* = \cup_j \mathcal{O}^j_K$ with $\mathcal{O}^j_k$ nilpotent K-orbits. Then for each j there exists exactly one Harish-Chandra cell $\mathcal{C}_j$ so that irreducible modules in $\mathcal{C}_j$ have associated variety $\overline{\mathcal{O}^j_K}$.
(4) $\#\mathcal{C} = \#\{\mathcal{Q} \in K\backslash\mathcal{B} : \mu(\overline{T^*_{\mathcal{Q}}(X)}) = \overline{K \cdot f}\} = \#\operatorname{Irr}(\mu^{-1}(f))$.

□

McGovern proved that Harish-Chandra cells are parametrized by nilpotent K-orbits, see [[15], Theorem 6]. Parts (3) follows from [[15], Theorem 6] and [[24], Theorem 5.2]. Parts (1) and (4) are due to Peter Trapa, see [[24], Theorem 5.2].

Proposition 4.2. Let $\mathcal{C}$ be a cell of Harish-Chandra modules with infinitesimal character ρ. Denote by $\overline{K \cdot f}$ the associated variety of modules in $\mathcal{C}$. Assume that $M, M' \in \mathcal{C}$ are distinct modules and write their supports as $\overline{\mathcal{Q}}$ and $\overline{\mathcal{Q}'}$, respectively. Let $C = \mu^{-1}(f) \cap \overline{T^*_{\mathcal{Q}}(X)}$ and $C' = \mu^{-1}(f) \cap \overline{T^*_{\mathcal{Q}'}(X)}$. If $C' \notin A(\mathcal{O}) \cdot C$, then $\operatorname{Ann}(M) \neq \operatorname{Ann}(M')$. □

Proof. We proceed by contradiction. Assume that $\operatorname{Ann}(M) = \operatorname{Ann}(M')$. Choose an order on $\mu^{-1}(K \cdot f)'$ compatible with orbit closure inclusion on $K\backslash\mathfrak{B}$. Interpret such order as an order on $A(\mathcal{O})\backslash\operatorname{Irr}(\mu^{-1}(f))$.

By Theorem 4.1 and Theorem 3.11,

$$LTC(M) = \overline{T^*_{\mathcal{Q}}(X)} + \sum m_k \overline{T^*_{\mathcal{Q}_k}(X)}$$

with $m_k \neq 0$ only if $A(\mathcal{O}) \cdot C_k < A(\mathcal{O}) \cdot C$ in the chosen order. Similarly

$$LTC(M') = \overline{T^*_{\mathcal{Q}'}(X)} + \sum m_\ell \overline{T^*_{\mathcal{Q}_\ell}(X)}$$

with $m_\ell \neq 0$ only if $A(\mathcal{O}) \cdot C_\ell < A(\mathcal{O}) \cdot C'$. Without lost of generality assume that $A(\mathcal{O}) \cdot C < A(\mathcal{O}) \cdot C'$. Write Υ_i for the orbital variety that corresponds, via Spaltenstein bijection, to $A(\mathcal{O}) \cdot C_i$. Similarly write Υ (Υ') the orbital variety dual to $A(\mathcal{O}) \cdot C$ ($A(\mathcal{O}) \cdot C'$, repectively). By Theorem 3.11, there are constants β_i, $i = 1, 2$ so that

$$\beta_1\, q_{\operatorname{Ann}(M)} = p_\Upsilon + \sum m_k\, p_{\Upsilon_k} \tag{4.3}$$

$$\beta_2\, q_{\operatorname{Ann}(M')} = p_{\Upsilon'} + \sum m_\ell\, p_{\Upsilon_\ell}. \tag{4.4}$$

Since we are assuming that $\operatorname{Ann}(M) = \operatorname{Ann}(M')$, the right hand sides of (4.3) and (4.4) are proportional. Observe that $p_{\Upsilon'}$ does not contribute to the right hand side of (4.3). Hence, $p_{\Upsilon'}$ is a linear combination of polynomials $p_\Upsilon, p_{\Upsilon_i}$. This is a contradiction as such polynomials are linearly independent. ■

Corollary 4.5. Let $\mathcal{C}$ be a cell of Harish-Chandra modules with infinitesimal character ρ. Denote by $\overline{K \cdot f}$ the associated variety of modules in $\mathcal{C}$. For $M \in \mathcal{C}$ write $\overline{\mathcal{Q}} = \mathrm{supp}(M)$ and let $C = \mu^{-1}(f) \cap \overline{T^*_{\mathcal{Q}}(X)}$. Then,

$$\#\{Y \in \mathcal{C} : \mathrm{Ann}(Y) = \mathrm{Ann}(M)\} = \#\{C' \in \mathrm{Irr}(\mu^{-1}(f)) : C' \in A(\mathcal{O}) \cdot C\}.$$

□

Proof. On the one hand, $\#\mathcal{C} = \sum_{I \in \mathrm{Prim}(\mathcal{O})_\rho} \#\{Y \in \mathcal{C}; \mathrm{Ann}(Y) = I\}$. On the other hand, by Theorem 4.1, $\#\mathcal{C} = \#\ \mathrm{Irr}(\mu^{-1}(f))$. Hence,

$$\begin{aligned} \#\mathcal{C} = \#\ \mathrm{Irr}(\mu^{-1}(f)) &= \sum_{\{A(\mathcal{O}) \cdot C_i : i=1\ldots d\}} \#\{A(\mathcal{O})) \cdot C_i\} \\ &= \sum_{I \in \mathrm{Prim}(\mathcal{O})_\rho} \#\{Y \in \mathcal{C}; \mathrm{Ann}(Y) = I\}. \end{aligned} \tag{4.6}$$

By Proposition 4.2 we know that $\#\{Y \in \mathcal{C} : \mathrm{Ann}(Y) = \mathrm{Ann}(M)\} \leq \#\{C' \in \mathrm{Irr}(\mu^{-1}(f)) : C' \in A(\mathcal{O}) \cdot C\}$. Since both sums in equation (4.6) have the same number of summands, see Remark 2.6, the Corollary follows. ■

Corollary 4.7. $M, M' \in \mathcal{C}$ have the same annihilator if and only if the component $C, C' \in \mathrm{Irr}(\mu^{-1}(f))$, attached to the support of M and M', belong to the same $A(\mathcal{O})$-orbit. □

4.2 **Harish-Chandra cell representations**

We keep the notation $\mathcal{C}$ for a Harish-Chandra cell and $\mathcal{C}^L$ for a Kazhdan-Lusztig left cell. Similarly, $V_{\mathcal{C}}$ denotes the Harish-Chandra cell representation generated by the irreducible modules in $\mathcal{C}$ and $V_{\mathcal{C}^L}$ stands for the Kazhdan-Lusztig left cell representation attached to $\mathcal{C}^L$. An important result by McGovern states, for the pairs (1.1), that Harish-Chandra cell representations are isomorphic as W-modules to certain Kazhdan-Lusztig left cell representation, see Theorem 4.9. The aim of this sub-section is to study $V_{\mathcal{C}}$ in more detail. Relevant to our discussion is the following definition.

Definition 4.8. Fix $\mathcal{O} = G \cdot f$ a special nilpotent orbit. Write $\mathcal{O}_K = K \cdot f$ and let $\mathcal{C}$ be a Harish-Chandra cell with associated variety $\overline{\mathcal{O}_K}$. Let I be an ideal in $\mathrm{Prim}_\rho(\mathcal{O})$. Define $\mathcal{C}_I = \{Y \in \mathcal{C} : \mathrm{Ann}(Y) = I\}$ and $n_I = \#\mathcal{C}_I$. □

Theorem 4.9. ([15], Theorem 6) Assume that $G_{\mathbf{R}} = Sp(p,q)$ or $G_{\mathbf{R}} = SO^*(2n)$. Let $\mathcal{C}$ be a cell of Harish-Chandra modules with infinitesimal character ρ and denote by $V_{\mathcal{C}}$ the corresponding Harish-Chandra cell representation. There exists a left cell $\mathcal{C}^L_{I_o}$ so that $V_{\mathcal{C}}$ is isomorphic as W-module to $V_{\mathcal{C}^L_{I_o}}$. Moreover, if $\overline{K \cdot f}$ is the associated variety of irreducible modules in $\mathcal{C}$ then

$$V_{\mathcal{C}} \simeq V_{\mathcal{C}^L_{I_o}} \simeq H_{\mathrm{top}}(X^f) \text{ as } W\text{-module}.$$

□

Theorem 4.9 states that as W-module $V_{\mathcal{C}}$ is isomorphic to a W-left cell representation. The notation $\mathcal{C}^L_{I_o}$ is used to emphasize that the left cell in Theorem 4.9 determines a primitive ideal denoted by I_o. It follows from [[13], Theorem 12.13] that Harish-Chandra cell representations for the groups under consideration are multiplicity free W-modules. Moreover, the number of irreducible constituents in $V_{\mathcal{C}}$ is known; see [[13], Chapter 12]. It is useful to recall that $H_{\text{top}}(X^f)$ affords an action of the finite group $A(\mathcal{O})$. The actions of $A(\mathcal{O})$ and W commute. For the pairs (1.1), the group $A(\mathcal{O})$ is a product of copies of $\mathbf{Z}/\mathbf{Z}_2$. The set $\hat{A(\mathcal{O})}$ of irreducible representations consists of characters. As W-module

$$H_{\text{top}}(X^f) = \bigoplus_{\sigma \in S} H(\sigma), \tag{4.10}$$

where S is a subset of the irreducible representations of $A(\mathcal{O})$ and $H(\sigma)$ is the σ-isotypic subspace of $H_{\text{top}}(X^f)$. If $\overline{A(\mathcal{O})} = A(\mathcal{O}) \setminus \cap_{\sigma \in S} \ker(\sigma)$, then $\# S = \# \overline{A(\mathcal{O})}$.

Theorem 4.11. ([15], Proof of Theorem 1 and Corollary 3) Assume that $G_{\mathbf{R}} = Sp(p,q)$ or $G_{\mathbf{R}} = SO^*(2n)$. Let $\mathcal{C}$ be a cell of Harish-Chandra modules with infinitesimal character ρ.

(1) Fix $I \in \text{Prim}(\mathcal{O})_\rho$ and assume that $M \in \mathcal{C}_I$. Then, $\mathbf{Q}[W] \cdot M \subset V_{\mathcal{C}}$ decomposes into the sum of n_I irreducible W-modules. Each such submodule is isomorphic to a submodule of $V_{\mathcal{C}^L_I}$.
(2) The number of irreducible constituents in the Harish-Chandra cell representation $V_{\mathcal{C}} \simeq V_{\mathcal{C}^L_{I_o}}$ is n_{I_o}.

□

Corollary 4.12. There exists $C \in \text{Irr}(\mu^{-1}(f))$ such that the number of irreducible constituents in $H_{\text{top}}(X^f)$ equals $\#\{C' \in \text{Irr}(\mu^{-1}(f)) : C' \in A_G(f) \cdot C\}$. □

Proof. By Theorem 4.9, there exists a left cell $\mathcal{C}^L_{I_o}$ such that $V_{\mathcal{C}} \simeq H_{\text{top}}(X^f) \simeq V_{\mathcal{C}^L_{I_o}}$ as W-modules. By Theorem 4.11 the number of irreducible constituents in $V_{\mathcal{C}^L_{I_o}}$ is n_{I_o}. On the other hand, Theorem 4.11 implies that $\mathbf{Q}[W] \cdot M$ has exactly n_{I_o} distinct W-irreducible constituents whenever $M \in \mathcal{C}_{I_o}$. Thus, $\mathbf{Q}[W] \cdot M \simeq V_{\mathcal{C}^L_{I_o}}$ and the number of irreducible constituents in $H_{\text{top}}(X^f)$ equals $\#\{Y \in \mathcal{C} : \text{Ann}(Y) = I_o\}$. Now the Corollary follows from Corollary 4.5. ■

Theorem 4.13. Assume that $G_{\mathbf{R}} = Sp(p,q)$ or $G_{\mathbf{R}} = SO^*(2n)$. Let $\mathcal{C}$ be a cell of Harish-Chandra modules with infinitesimal character ρ. For $M \in \mathcal{C}$, write $AV(M) = \overline{K \cdot f}$.

(1) There exists $C \in \text{Irr}(\mu^{-1}(f))$ so that $\mathbf{Q}[W] \cdot [C] \equiv H_{\text{top}}(X^f)$ as W-module.
(2) If C' is an arbitrary component of $\mu^{-1}(f)$ and $A_G(f, C') = \{z \in A(\mathcal{O}) : z \cdot C' = C'\}$, then $\mathbf{Q}[W] \cdot [C'] \simeq H_{\text{top}}(X^f)^{A_G(f,C')}$ as W-module.

□

Proof. Theorem 4.9 tells us that the top-homology of the Springer fiber X^f is isomorphic as W-module to the Harish-Chandra cell representation $V_{\mathcal{C}}$. On the other hand, by Theorem

4.11, there exists an irreducible module $M \in \mathcal{C}$ with annihilator $\mathrm{Ann}(M) = I_o$ such that $\mathbf{Q}[W] \cdot M \equiv V_{\mathcal{C}}$ as W-module. The number of irreducible W-constituents in $\mathbf{Q}[W] \cdot M$ is n_{I_o}. Write $\overline{\mathcal{Q}}$ for the support of M. The intersection $\mu^{-1}(f) \cap \overline{T^*_{\mathcal{Q}}(X)}$ is dense in a component C of the Springer fiber. We claim that $\mathbf{Q}[W] \cdot [C] \equiv H_{\mathrm{top}}(X^f)$.

Write, as in equation(4.10),

$$H_{\mathrm{top}}(X^f) = \bigoplus_{\sigma \in S} H(\sigma). \tag{4.14}$$

In order to prove the claim it is enough to show that C has non-zero projection to each $H(\sigma)$ with $\sigma \in S$. We use an argument similar to that in ([3], Proposition 2.2).

By Corollary 4.5

$$n_{I_o} = \#\{Y \in \mathcal{C} : \mathrm{Ann}(Y) = I_o)\} = \#A(\mathcal{O})/A_G(f, C). \tag{4.15}$$

Thus, $\#S = \#\overline{A(\mathcal{O})} = \#A(\mathcal{O})/A_G(f, C)$.

Write $[C] = \sum h_\sigma$ according to the decomposition (4.10). Apply $z \in \overline{A(\mathcal{O})}$ to get

$$[z \cdot C] = \sum_\sigma z \cdot h_\sigma. \tag{4.16}$$

Let σ' be an irreducible representation of $\overline{A(\mathcal{O})}$ and write $\chi_{\sigma'}$ for its character. Multiple both sides of (4.16) by $\frac{\dim(\sigma')}{\#(\overline{A(\mathcal{O})}}\chi_{\sigma'}(z)$ and sum over $z \in \overline{A(\mathcal{O})}$:

$$\frac{\dim(\sigma')}{\#(\overline{A(\mathcal{O})})} \sum_z \chi_{\sigma'}(z)\,[z \cdot C] = \sum_\sigma \Big(\frac{\dim(\sigma')}{\#(\overline{A(\mathcal{O})}} \sum_z \chi_{\sigma'}(z)\, z \cdot h_\sigma \Big) = h_{\sigma'}.$$

The last equality holds because $P_{\sigma'} = \frac{\dim(\sigma')}{\#(\overline{A(\mathcal{O})})} \sum_z \chi_{\sigma'}(z)\, z$ is the projection onto the σ'-isotypic subspace. The left-hand side is nonzero since $\{[z \cdot C] : z \in \overline{A(\mathcal{O})}\}$ is independent and $\chi_{\sigma'}(z) \neq 0$ for some $z \in \overline{A(\mathcal{O})}$. We conclude that in the expression for $[C]$, $h_\sigma \neq 0$ for *each* $\sigma \in S$. Since $H(\sigma)$ is an irreducible W-representation we have

$$\mathbf{Q}[W] \cdot [C] \supset \mathbf{Q}[W] \cdot P_\sigma([C]) = H(\sigma).$$

Therefore, $\mathbf{Q}[W] \cdot [C]$ contains all isotypic subspaces. This proves the first assertion of the Theorem.

Let $C' \in \mathrm{Irr}(\mu^{-1}(f))$ be arbitrary. By Part (1), we have $[C'] \in \mathbf{Q}[W] \cdot [C]$. In particular, $\mathbf{Q}[W] \cdot [C'] \subset H_{\mathrm{top}}(X^f)^{A_G(f,C')}$. Now, an argument analogous to that used to settle part (1) proves that $P_\sigma(C') \neq 0$ for each σ so that $\sigma|_{A_G(f,C')} = 1$. ■

The following theorem is a generalization of Proposition 12 and Corollary 13 in [3]. These results have shown instrumental in the computation of associated cycles of Harish-Chandra modules.

Theorem 4.17. Assume that $G_{\mathbf{R}} = Sp(p,q)$ or $G_{\mathbf{R}} = SO^*(2n)$. Let $\mathcal{C}$ be a cell of Harish-Chandra modules with infinitesimal character ρ. Write the associated variety of irreducible modules in $\mathcal{C}$ as $\overline{K \cdot f}$. Let $M \in \mathcal{C}$ be an irreducible module with $\mathrm{Ann}(M) = I$ and $\mathrm{supp}(M) = \overline{\mathcal{Q}}$. Write $C_{\mathcal{Q}} = \mu^{-1}(f) \cap \overline{T^*_{\mathcal{Q}}(X)}$. Then,

$$\mathbf{Q}[W] \cdot M \equiv \mathbf{Q}[W] \cdot [C] \simeq H_{\mathrm{top}}(X^f)^{A_G(f,C)}.$$

□

Proof. By Theorem 4.11, if M and M' are irreducible $(\mathfrak{g}, K)$-modules in $\mathcal{C}$ with the same annihilator, then $\mathbf{Q}[W] \cdot M \equiv \mathbf{Q}[W] \cdot M'$. Moreover, if the closures of the K-orbits $\mathcal{Q}$ and $\mathcal{Q}'$ are the supports of M and M' respectively, then the components $C = \mu^{-1}(f) \cap \overline{T^*_{\mathcal{Q}}(X)}$ and $C' = \mu^{-1}(f) \cap \overline{T^*_{\mathcal{Q}'}(X)}$ belong to the same $A(\mathcal{O})$-orbit; see Theorem 4.2 and Corollary 4.5. Consider the set $\{\mathrm{supp}(Y) : Y \in \mathcal{C}_{H,\mathrm{Ann}(M)}\}$ and impose on the set an order compatible with orbit closure inclusion of K-orbits in X. Let $\mathcal{Q}$ be the smallest orbit in such set with respect to this order. Let $M(\mathcal{Q})$ be the irreducible module in $\mathcal{C}_{H,\mathrm{Ann}(M)}$ with $\mathrm{supp}(M(\mathcal{Q})) = \overline{\mathcal{Q}}$. It follows that the leading term cycle of M is of the form $LTC(M(\mathcal{Q})) = \overline{T^*_{\mathcal{Q}}(X)} + \sum_i m_i\, \overline{T^*_{\mathcal{Q}_i}(X)}$ where $C_i = \mu^{-1}(f) \cap \overline{T^*_{\mathcal{Q}_i}(X)}$ do not belong to $A(\mathcal{O}) \cdot C$. The assingment

$$M_o \to [C] + \sum_i m_i\, [C_i] \in H_{\mathrm{top}}(X^f),$$

extends to a W-equivariant map $\psi : \mathbf{Q}[W] \cdot M \to H_{\mathrm{top}}(X^f)$, see ([18]). By composing ψ with the projection from $H_{\mathrm{top}}(X^f)$ to $H_{\mathrm{top}}(X^f)^{A_G(f,C)}$ we obtain a W-map

$$\psi' : \mathbf{Q}[W] \cdot M \to H_{\mathrm{top}}(X^f)^{A_G(f,C)}$$
$$\psi'(M) = [C] + \sum_i m_i \big(\sum_{z \in A_G(f,C)} z \cdot [C_i] \big).$$

Since $\{\sum_{z \in A_G(f,C)} z \cdot [C_j] : C_j \in \mathrm{Irr}(\mu^{-1}(f))\}$ is a basis of $H_{\mathrm{top}}(X^f)^{A_G(f,C)}$, ψ' is a non-zero W-equivariant map. We claim that ψ' is an isomorphism. Arguing as in Theorem 4.13 we show that ψ' is onto. By part (2) in Theorem 4.13, the number of irreducible constituents in $H_{\mathrm{top}}(X^f)^{A_G(f,C)}$ is equal to $\#A(\mathcal{O}) \cdot C$, which in turn equals $\#\{Y \in \mathcal{C} : \mathrm{Ann}(Y) = I\}$, according to Corollary 4.5. On the other hand, by Theorem 4.9, we know that the number of irreducible constituent in $\mathbf{Q}[W] \cdot X$ is $\#\{Y \in \mathcal{C} : \mathrm{Ann(Y)} = I\}$. It follows that ψ' is also injective. ■

Example 4.1. Assume that $G_{\mathbf{R}} = Sp(1,1)$ and let $\mathcal{C}$ be the Harish-Chandra cell of modules with infinitesimal character ρ that contains the holomorphic discrete series. The associated variety of the annihilator of modules in this cell is the nilpotent G-orbit $\mathcal{O}$ parametrized by the partition $[2,2]$.

The cell consists of three representations; π_1, π_2 in the discrete series and a third representation π_3 that is cohomologically induced. One can verify that $\mathrm{Ann}(\pi_1) = \mathrm{Ann}(\pi_2)$ while π_3 has a different annihilator. (These computations can be verified by using the software ATLAS, for example.)

It follows that

$$\mathbb{Q}[W]\pi_i \simeq V_{\mathcal{C}},\ \text{for } i = 1, 2$$
$$\mathbb{Q}[W]\pi_3 \subsetneq V_{\mathcal{C}}$$

On the other hand, as each of the irreducible modules π_i is cohomologically induced, we have

$$CC(\pi_i) = T^*_{\mathfrak{Q}_i}(X) \text{ with } \overline{\mathfrak{Q}_i} = \text{supp}(\pi_i).$$

Write C_i for the component of the Springer fiber corresponding to $T^*_{\mathfrak{Q}_i}(X) \cap \mu^{-1}(f)$. A basis for $H_{top}(\mu^{-1}(f))^{A_G(f,C_3)}$ is given by $\{[C_1+C_2],[C_3]\}$. As $Q[W]\pi_3 \simeq H_{top}(\mu^{-1}(f))^{A_G(f,C_3)}$ and the characteristic cycle is W-equivariant, we can explicitly describe the action of W on this basis,i.e.

$$\begin{aligned} s_{\alpha_1}[C_3] &= -[C_3] & s_{\alpha_1}[C_1+C_2] &= [C_1+C_2] + 2[C_3] \\ s_{\alpha_2}[C_3] &= [C_3] + [C_1+C_2] & s_{\alpha_2}[C_1+C_2] &= -[C_1+C_2]. \end{aligned}$$

□

5 Leading term cycle

5.1 Two triangularity results and the leading term cycles

Throughout this section $(\mathfrak{g}, K)$ stands for $(\mathfrak{sp}(2n), \mathrm{GL}(n, \mathbf{C}))$ or $(\mathfrak{sp}(2n), \mathrm{Sp}(p, \mathbf{C}) \times \mathrm{Sp}(q, \mathbf{C}))$ with $p+q=n$. We fix $\mathcal{O}$ a nilpotent orbit and we assume that $\mathcal{O}$ occurs as the associated variety of the annihilator of some $(\mathfrak{g}, K)$-module. In [24], a geometric argument is used to define a bijection between $\mathrm{Prim}_\rho(\mathcal{O})$ and the set of orbital varieties $\{\Upsilon : \Upsilon \text{ irreducible component of } \overline{\mathcal{O}} \cap \mathfrak{n}\}$. On the other hand, in [14] a bijection between these sets is defined by using an algebraic/combinatoric argument. In this section we prove that the algebraically defined and the geometrically defined bijections coincide. We use results from [24] and [14] to derive information on leading term cycles of irreducible Harish-Chandra modules.

For the pairs $(\mathfrak{g}, K)$ under consideration, in view of Proposition 4.2, the geometric bijection

$$\phi : \mathrm{Prim}_\rho(\mathcal{O}) \to \{\Upsilon : \Upsilon \text{ irreducible component of } \overline{\mathcal{O}} \cap \mathfrak{n}\} \tag{5.1}$$

can be described as follows. Let $\mathcal{C}$ be a Harish-Chandra cell of modules with trivial infinitesimal character such that $AV(\mathrm{Ann}(M)) = \overline{\mathcal{O}}$ for $M \in \mathcal{C}$. Write AV of irreducible modules in $\mathcal{C}$ as $\overline{K \cdot f}$. Given $I \in \mathrm{Prim}_\rho(\mathcal{O})$, choose $M \in \mathcal{C}$ with $\mathrm{Ann}(M) = I$. Let $\mathrm{supp}_o(M) \in K\backslash X$ be the K-orbit dense in the support of M and write $C_{\mathrm{supp}_o(M)} \in \mathrm{Irr}(\mu^{-1}(f))$ for the component attached to $T^*_{\mathrm{supp}_o(M)}X$. Then,

$$\phi(I) = \Upsilon_{\mathrm{supp}_o(M)}$$

where $\Upsilon_{\mathrm{supp}_o(M)}$ is the orbital variety that corresponds to $A(\mathcal{O}) \cdot C_{\mathrm{supp}_o(M)}$ under the Spaltenstein bijection. It is worth observing that:

(1) the geometric bijection attaches to a primitive ideal an orbit of $A(\mathcal{O})$ acting on $\mu^{-1}(f)$. By (2.5), such $A(\mathcal{O})$-orbit corresponds to a set of K-orbit contained in $\mu^{-1}(K \cdot f)$.
(2) $\Upsilon_{\mathrm{supp}_o(M)} = \Upsilon_{\mathrm{supp}_o(M')}$ implies that $\mathrm{Ann}(M) = \mathrm{Ann}(M')$, see Proposition 4.2.

In the algebraic setting, $I \in \mathrm{Prim}_\rho(\mathcal{O})$ is identified as $I = \mathrm{Ann}(L_{w^{-1}})$ for some $w \in W$. Then, [[14], Theorem 2] gives a Weyl group element $w' \in W$ with two key properties (a) $AV(L_w) = AV(L_{w'})$ and (b) $\Upsilon_{w'}$ is open in $AV(L_w)$. Note that w, w' belong to the same right Khazdan-Lusztig cell. In particular, $\tau_\infty(w^{-1}) = \tau_\infty(w'^{-1})$. The algebraic bijection assigns to I the orbital variety $\Upsilon_{w'}$. In the combinatorial language, $\Upsilon_{w'}$ is indexed by a standard domino tableau as explained in §3.1 . We write $\Upsilon_{w'} = \Upsilon_{T_I}$.

Theorem 5.2. The algebraically defined and geometrically defined bijections between $\mathrm{Prim}_\rho(\mathcal{O})$ and $\{\Upsilon : \Upsilon$ irreducible component of $\overline{\mathcal{O}} \cap \mathfrak{n}\}$ coincide. □

Proof. Fix $\mathcal{O}_K = K \cdot f \subset \overline{\mathcal{O}} = \overline{G \cdot f}$. Let $\mathcal{C}$ be the Harish-Chandra cell with associated variety $\overline{\mathcal{O}_K}$. For $\mathcal{Q} \in K\backslash X$ denote by $M(\mathcal{Q})$ the irreducible Harish-Chandra module with support equal to $\overline{\mathcal{Q}}$.

Identify each ideal $I \in \mathrm{Prim}_\rho(\mathcal{O})$, via the geometric bijection, with the set $\{\mathcal{Q} \in K\backslash X : \mu(\overline{T^*_{\mathcal{Q}}X}) = \overline{\mathcal{O}_K}$ and $\mathrm{Ann}(M(\mathcal{Q})) = I\}$. Choose from each such set of K-orbits in the flag variety X one orbit as prescribed in [[24], §3.3] to form the set $\mu^{-1}(\mathcal{O}_K)'$, see (3.2). Order the set $\mu^{-1}(\mathcal{O}_K)'$ as in [[24], §3.3]. List the orbits in $\mu^{-1}(\mathcal{O}_K)' = \{\mathcal{Q}_0, \mathcal{Q}_1, \ldots, \mathcal{Q}_r\}$ in the chosen order.

Let $I_0 = \mathrm{Ann}(M(\mathcal{Q}_0)) = \mathrm{Ann}(L_{w_0^{-1}})$. By Trapa's upper triangularity result we know that $AV(L_{w_0}) = \Upsilon_{\mathrm{supp}_o(M(\mathcal{Q}_0))}$. On the other hand, [[14], Theorem 2] says the $\Upsilon_{T_{I_0}}$ is open in $AV(L_{w_0})$. Hence, $\Upsilon_{\mathrm{supp}_o(M(\mathcal{Q}_0))} = \Upsilon_{T_{I_0}}$. Similarly, for $\mathcal{Q}_1 \in \mu^{-1}(\mathcal{O}_K)'$ write $I_1 = \mathrm{Ann}(M(\mathcal{Q}_1)) = \mathrm{Ann}(L_{w_1^{-1}})$. By the triangularity result in [24], $AV(L_{w_1})$ contains $\Upsilon_{\mathrm{supp}_o(M(\mathcal{Q}_1))}$ and no other orbital variety but possibly $\Upsilon_{\mathrm{supp}_o(M(\mathcal{Q}_0))}$. On the other hand, [Theorem 2, [14]] guarantees that $\Upsilon_{T_{I_1}}$ is open in $AV(L_{w_1})$. Since $\Upsilon_{T_{I_0}} \neq \Upsilon_{T_{I_1}}$, we conclude that $\Upsilon_{\mathrm{supp}_o(M(\mathcal{Q}_1))} = \Upsilon_{T_{I_1}}$. An inductive argument settles the theorem. ■

There is no known algorithm to compute characteristic cycles or leading term cycles of Harish-Chandra modules. For the pairs (1.1) information concerning leading term cycles is implicit in the material presented so far. For example, we know

(1) $T^*_{\mathrm{supp}_o(M)}X$ contributes to $LTC(M)$,
(2) if $T^*_{\mathcal{Q}}X$ contributes to $LTC(M)$, then $\mathcal{Q}$ should be smaller than $\mathrm{supp}_o(M)$ in the orders defined in [24],
(3) if $M \in \mathcal{C}$ with $AV(M) = \overline{K \cdot f}$ and $T^*_{\mathcal{Q}}X$ contributes to $LTC(M)$, then $\overline{\mathcal{Q}}$ is the support of an irreducible module $M' \in \mathcal{C}$.

Our observation (3) follows from three facts:

(1) each irreducible module has its support contributing to the leading term cycle, [24];
(2) there exists exactly one Harish-Chandra cell with associated variety $\overline{\mathcal{O}_K}$, [14];
(3) $\#\mathcal{C} = \#\{\mathcal{Q} \in K\backslash X : \mu(\overline{T^*_{\mathcal{Q}}X}) = \overline{\mathcal{O}_K}\}$.

The following Proposition relies on the triangularity result given in [24].

Proposition 5.3. Assume that (G, K) is one of the pairs (1.1). Fix $\mathcal{O}$ a special nilpotent orbit and let $\mathcal{O}_K = K \cdot f \subset \mathcal{O} \cap \mathfrak{p}$. Let $\mathcal{C}$ be the Harish-Chandra cell attached to $\mathcal{O}_K$. If $M \in \mathcal{C}$ and $LTC(M) = T^*_{\mathrm{supp}_o(M)}X + \sum_i m_i\, T^*_{\mathcal{Q}_i}X$ with $m_i \neq 0$, then $\tau(\mathrm{Ann}(M)) \subset \tau(\mathrm{Ann}\ (M(\mathcal{Q}_i)))$. □

Proof. Write, with the notation already introduced, $\mu^{-1}(\mathcal{O}_K)' = \{\mathcal{Q}_0, \mathcal{Q}_1, \ldots, \mathcal{Q}_r\}$ and $I_i = \mathrm{Ann}(M(\mathcal{Q}_i)) = \mathrm{Ann}(L_{w_i^{-1}})$. For each $M(\mathcal{Q})$ write $C_{\mathcal{Q}} \in \mu^{-1}(f)$ for the irreducible component of the Springer fiber that is dense in $\mu^{-1}(f) \cap T^*_{\mathcal{Q}}X$.

Let M be an irreducible module in $\mathcal{C}$. Assume first that $\mathrm{Ann}(M) = I_0$. The argument in the proof of Theorem 5.2 gives $AV(L_{w_0}) = \Upsilon_{\mathrm{supp}_o(M)}$. The triangularity result in [24] implies that any conormal bundle $T^*_{\mathcal{Q}}X$ contributing to $LTC(M)$ has $A(\mathcal{O}) \cdot C_{\mathcal{Q}} = A(\mathcal{O}) \cdot C_{\mathrm{supp}_o(M)}$. By Proposition 4.2, $\mathrm{Ann}(M) = \mathrm{Ann}(M(\mathcal{Q}))$. Hence, the proposition follows in this case. Next, assume that $\mathrm{Ann}(M) = \mathrm{Ann}(L_{w_1^{-1}}) = I_1$. By the geometric triangularity result, $AV(L_{w_1})$ contains $\Upsilon_{\mathrm{supp}_o(M)}$ and possibly $\Upsilon_{\mathrm{supp}_o(M(\mathcal{Q}_0))}$. As before, if $T^*_{\mathcal{Q}}X$ contributes to $LTC(M)$, then $\mathrm{Ann}(M)$ is either $I_0 = \mathrm{Ann}(L_{w_0^{-1}})$ or $I_1 = \mathrm{Ann}(L_{w_1^{-1}})$. Moreover, if $\Upsilon_{\mathrm{supp}_o(M(\mathcal{Q}_0))}$ is open $AV(L_{w_1})$ then $CC(L_{w_1})$ contains a conormal bundle $T^*_{X(w')}X$ such that $\mu(\overline{T^*_{X(w')}X}) = \Upsilon_{w'} = \Upsilon_{T_{I_0}}$. By (3.2), $\tau(w_1^{-1}) \subset \tau(w'^{-1})$. Since $\tau(w_1^{-1}) = \tau(I_1)$ and $\tau(w'^{-1}) = \tau(I_0)$ we conclude that $\tau(\mathrm{Ann}(M)) \subset \tau(\mathrm{Ann}\ (M(\mathcal{Q}_0))$. The general case is settled in a similar manner. ■

Proposition 5.3 can be strengthen if we use the triangularity result in [16]. The argument is analogue to the one used in the proof of the proposition.

Theorem 5.4. Assume that (G, K) is one of the pairs (1.1). Fix $\mathcal{O}$ a special nilpotent orbit and let $\mathcal{O}_K = K \cdot f \subset \mathcal{O} \cap \mathfrak{p}$. Let $\mathcal{C}$ be the Harish-Chandra cell attached to $\mathcal{O}_K$. If $M \in \mathcal{C}$ and $LTC(M) = T^*_{\mathrm{supp}_o(M)}X + \sum_i m_i\, T^*_{\mathcal{Q}_i}X$ with $m_i \neq 0$, then $\tau_\infty(\mathrm{Ann}(M)) \subset \tau_\infty(\mathrm{Ann}(M(\mathcal{Q}_i))$. □

5.2 A module with reducible leading term cycle

In this sub-section we give an example of a $(\mathfrak{sp}(8), \mathrm{Sp}(4, \mathbf{C}) \times \mathrm{Sp}(4, \mathbf{C}))$ Harish-Chandra module with reducible leading term cycle. We outline a method to find families of $(\mathfrak{sp}(2n), \mathrm{Sp}(n, \mathbf{C}) \times \mathrm{Sp}(n, \mathbf{C}))$ Harish-Chandra modules with reducible leading term cycle. A detailed and systematic construction of such families will appear in a sequel to this paper. Trapa asked if $(\mathfrak{sp}(2n), \mathrm{Sp}(n, \mathbf{C}) \times \mathrm{Sp}(n, \mathbf{C}))$ Harish-Chandra modules always have irreducible leading term cycle. This is not the case.

The results presented in this sub-section rely heavily on

(1) the relationship between leading term cycles of irreducible Harish-Chandra modules and associated varieties of highest weight modules, see [[24], Cor. 4.2]
(2) the fact that there exist $\mathrm{Sp}(2n, \mathbf{R})$ and $\mathrm{Sp}(n, n)$ modules that share the same annihilator.

The strategy to construct $(\mathfrak{sp}(2n), \mathrm{Sp}(n) \times \mathrm{Sp}(n))$ Harish-Chandra modules with reducible leading term cycle is as follows:

(1) Construct $(\mathfrak{sp}(2n), \mathrm{GL}(n, \mathbf{C}))$ Harish-Chandra modules which have reducible leading term cycle. This can be done by using cohomological induction. The key idea is an observation made by Peter Trapa that the computation of leading terms is not closed under induction.

(2) For a $(\mathfrak{sp}(2n), \mathrm{GL}(n, \mathbf{C}))$ module M as in (1), write $\mathrm{Ann}(M) = \mathrm{Ann}(L_{w^{-1}})$ and $\overline{\mathcal{O}} = \overline{G \cdot f} = AV(\mathrm{Ann}(L_{w^{-1}}))$. Use Corollary 4.2 in [24] to conclude that $AV(L_w)$ is reducible. Here is a reason. The finite groups $A(\mathcal{O})$ and $A_K(f)$ act on $\mu^{-1}(f)$. When $G_{\mathbf{R}} = \mathrm{Sp}(2n, \mathbf{R})$ the $A(\mathcal{O})$ and $A_K(f)$ orbits on $\mu^{-1}(f)$ coincide. Hence, distinct conormal bundles contribution to the leading term cycle of M yield distinct irreducible components of $AV(L_w)$.

(3) Identify $\mathrm{Sp}(n, n)$ modules with annihilator equal to $\mathrm{Ann}(L_{w^{-1}})$. Apply Corollary 4.2 in [24] to conclude that the leading term cycles of such modules are reducible.

A concrete example when $G_{\mathbf{R}} = \mathbf{Sp}(8, \mathbf{R})$.

We work in the category $\mathcal{M}_\rho(\mathfrak{sp}(8), \mathrm{GL}(4, \mathbf{C}))$ and construct a cohomologicaly induced highest weight module with reducible leading term cycle. Write $\mathfrak{g} = \mathfrak{sp}(8) = \mathfrak{k} \oplus \mathfrak{p}$. As $GL(4, \mathbf{C})$-representation $\mathfrak{p}$ decomposes into the direct sum $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$. Our construction will yield a module M with $AV(M) = \mathfrak{p}^+$. (Observe that $\mathfrak{p}^+$ is the closure of a $K = GL(4, \mathbf{C})$-nilpotent orbit, i.e. $\mathfrak{p}^+ = \overline{K \cdot f}$ for an appropriate nilpotent element f.)

Fix $\mathfrak{h} \subset \mathfrak{k}$ a Cartan subalgebra. Fix a positive system $\Delta^+(\mathfrak{g}, \mathfrak{h}) = \Delta^+(\mathfrak{k}, \mathfrak{h}) \cup \Delta(\mathfrak{p}^+)$. Let $\mathfrak{b}$ be the Borel subalgebra determined by $\Delta^+(\mathfrak{g}, \mathfrak{h})$. Define $\mathfrak{s} = \mathfrak{l} \oplus \mathfrak{u}$ the θ-stable parabolic subalgebra with $\mathfrak{l} \simeq \mathfrak{sp}(4, \mathbf{C}) \oplus \mathbf{C}^2$ and $\mathfrak{u} \cap \mathfrak{p} = \mathfrak{u} \cap \mathfrak{p}^-$. In what follows we identify a $(\mathfrak{l}, L \cap K)$ Harish-Chandra module Y such that $M = R_{\mathfrak{s}}(Y)$ has reducible leading term cycle and $AV(M) = \mathfrak{p}^+$.

Consider the Harish-Chandra cell of $(\mathfrak{l}, L \cap K)$-modules that contains the holomorphic discrete series. This cell consists of three highest weight Harish-Chandra modules. Moreover, the associated variety of the annihilators of these modules is the closure of the nilpotent orbit $\mathcal{O}$ attached to the partition $[2, 2]$, see [6]. It is useful to recall that the dimension of the special representation associated to the orbit with partition $[2, 2]$ is 2, see [6]. We argue that exactly one of the irreducible modules in this cell has reducible characteristic cycle. This is the module Y that is relevant to our construction.

The Harish-Chandra cell under consideration contains (π_1, Y_1) the holomorphic discrete series, (π_2, Y_2) a module that is cohomologicaly induced from a character and a third module (π, Y). As (π_j, Y_j) with $j = 1, 2$ are cohomologically induced from a character, it is known that $CC(Y_j) = \overline{T^*_{\mathcal{Q}_j}(X_L)}$ where $\overline{\mathcal{Q}_j} = \mathrm{supp}(Y_j)$ and X_L is the flag variety for L. The bijections (2.4) and (2.5) associate to $\mathcal{Q}_j, j = 1, 2$ two distinct orbital varieties $\Upsilon_j, j = 1, 2$. By Theorem 3.4, the characteristic polynomials $p_{\Upsilon_j}, j = 1, 2$ are linearly independent. As $\dim Sp(\mathcal{O}) = 2$, the set $\{p_{\Upsilon_1}, p_{\Upsilon_2}\}$ is a basis of $Sp(\mathcal{O})$. It follows form this discussion that for $\overline{\mathcal{Q}} = \mathrm{supp}(Y)$, $\mu(\overline{T^*_{\mathcal{Q}}(X_L)}) \subsetneq \mathfrak{p}^+$. On the other hand, $AV(Y) = AV(Y_i)$ with $i = 1, 2$. Hence, $CC(Y)$ must contain $\overline{T^*_{\mathcal{Q}'}(X_L)}$ with $\mathcal{Q}' \subset \overline{\mathcal{Q}}$ and $\mu(\overline{T^*_{\mathcal{Q}}(X_L)}) = \mathfrak{p}^+$. It is well-known how to compute moment map images of conormal bundles when $\mathfrak{g}$ is a classical Lie algebra , see for example [28], [23]. Using the algorithm in [23] we conclude that $CC(Y) = \overline{T^*_{\mathcal{Q}}(X_L)} + \overline{T^*_{\mathcal{Q}_1}(X_L)}$. In particular, $LTC(Y)$ is irreducible but $CC(Y)$ is not.

Next, we consider the characteristic cycle of $M = R_{\mathfrak{s}}(Y)$. It is known that $CC(R_{\mathfrak{s}}(Y))$ can be written in terms of $CC(Y)$. Let $\mathcal{F}$ be the generalized flag variety of all parabolic subalgebra conjugate to $\mathfrak{s}$. Denote by $p : X \to \mathcal{F}$ the canonical projection. Then

$$CC(R_{\mathfrak{s}}(Y)) = \overline{T^*_{\widetilde{\mathcal{Q}}}(X)} + \overline{T^*_{\widetilde{\mathcal{Q}_1}}(X)}$$

where $\widetilde{\mathcal{Q}}$ ($\widetilde{\mathcal{Q}_1}$) is the $GL(4, \mathbf{C})$-orbit that fibers over its projection to $\mathcal{F}$ with fiber isomorphic to $\mathcal{Q}$ ($\mathcal{Q}_1$ respectively). We use once more the algorithm in [23] to show that both $\mu(T^*_{\widetilde{\mathcal{Q}}}(X))$

and $\mu(T^*_{\tilde{\mathcal{Q}}}(X))$ are open in $AV(M)$. In other words, M has reducible leading term cycle. The associated variety of the annihilator of M is the orbit with parametrized by $[2,2,2,2]$. As this orbit also occurs as associated variety of annihilators of $(\mathfrak{sp}(8), \mathrm{Sp}(2,C) \times \mathrm{Sp}(2,\mathbf{C}))$ modules, we conclude that there exists $\widetilde{M}$ a $(\mathfrak{sp}(8), \mathrm{Sp}(2,C) \times \mathrm{Sp}(2,\mathbf{C}))$ module with reducible leading term cycle.

The modules M and $\widetilde{M}$ can be identified using the ATLAS software. The Langlands data of M and $\widetilde{M}$ can be computed using the software REALX. Here we use the Beilinson-Berstein classification of irreducible modules to identify M and $\widetilde{M}$ via their support $\overline{\mathcal{Q}}$ and $\overline{\tilde{\mathcal{Q}}}$. Write $\mathcal{Q} = GL(4,\mathbf{C}) \cdot \mathfrak{b}_{\mathcal{Q}}$ and $\tilde{\mathcal{Q}} = \mathrm{Sp}(2,\mathbf{C}) \times \mathrm{Sp}(2,\mathbf{C}) \cdot \mathfrak{b}_{\tilde{\mathcal{Q}}}$. In order to give an explicit description of the base points $\mathfrak{b}_{\mathcal{Q}}$, $\mathfrak{b}_{\tilde{\mathcal{Q}}}$ we need to introduce some notation.

Realize $\mathrm{Sp}(8,\mathbf{R})$ as the group of matrices which leave invariant the symplectic form determined by $J_{4,4} = \mathrm{antidiagonal}(-I_{4\times 4}, I_{4\times 4})$. Identify $\mathrm{Sp}(2,2)$ with the group of matrices in $\mathrm{Sp}(8,\mathbf{C})$ which leave invariant the Hermitian from of signature $(4,4)$ determined by $K_{2,2} = \mathrm{diag}(I_{2\times 2}, -I_{2\times 2}, I_{2\times 2}, -I_{2\times 2})$. Fix $\mathfrak{h} = \{\mathrm{diag}(t_1,t_2,t_3,t_4,-t_1,-t_2,-t_3,-t_4), t_i \in \mathbf{C}\} \subset \mathfrak{sp}(8,\mathbf{C})$ a Cartan subalgebra of $\mathfrak{sp}(8,\mathbf{C})$. Write, in the usual notation, $\Delta(\mathfrak{sp}(8,\mathbf{C}),\mathfrak{h}) = \{\pm(\epsilon_i \pm \epsilon_j 1 \leq i < j \leq 4\} \cup \{2\epsilon_i, 1 \leq i \leq 4\}$.

Description of the base point $\mathfrak{b}_{\mathcal{Q}}$.

Choose the set of simple roots $\sum^+ = \{\beta_1 = \epsilon_1 - \epsilon_2, \beta_2 = \epsilon_2 - \epsilon_3, \beta_3 = \epsilon_3 - \epsilon_4, \beta_4 = 2\epsilon_4\}$. Write $\mathfrak{b}$ for the Borel subalgebra of $\mathfrak{sp}(8,\mathbf{C})$ determined by $(\mathfrak{h}, \sum^+)$. Here $G_{\mathbf{R}} = \mathrm{Sp}(8,\mathbf{R})$ and the simple root $2\epsilon_4$ is a non-compact imaginary root. Let $c_{2\epsilon_4} : \mathfrak{sp}(8,\mathbf{C}) \to \mathfrak{sp}(8,\mathbf{C})$ be the Cayley transform through the root $2\epsilon_4$. Set $\mathfrak{h}_1 = c_{2\epsilon_4}(\mathfrak{h})$ and $c_{2\epsilon_4}(\sum^+) = \{\alpha_i : \alpha_i = c_{2\epsilon_4}(\beta_i)\ 1 \leq i \leq 4\}$. Write $W_1 = W(\mathfrak{sp}(8,\mathbf{C}), \mathfrak{h}_1)$ the corresponding Weyl group. Let $s_{\alpha_3} \in W_1$ denote the simple reflection through the root α_3. Then $\mathfrak{b}_{\mathcal{Q}} = s_{\alpha_3}(c_{2\epsilon_4} \cdot \mathfrak{b})$.

Description of the base point $\widetilde{\mathfrak{b}_{\mathcal{Q}}}$.

Now $G_{\mathbf{R}} = \mathrm{Sp}(2,2)$. Choose $\sum^+ = \{\beta_1 = \epsilon_1 - \epsilon_3, \beta_2 = \epsilon_3 - \epsilon_4, \beta_3 = \epsilon_4 - \epsilon_2, \beta_4 = 2\epsilon_2\}$ a set of simple roots for $\Delta(\mathfrak{sp}(8,\mathbf{C})\mathfrak{h})$. Observe that the roots β_1 and β_3 are imaginary non-compact. Let $c_{\beta_i} : \mathfrak{sp}(8,\mathbf{C}) \to \mathfrak{sp}(8,\mathbf{C})$, $i = 1,3$ be the Cayley transforms through the roots β_i with $i = 1,3$. Set $\mathfrak{h}' = c_{\beta_3} c_{\beta_1} \mathfrak{h}$ and $(c_{\beta_3} c_{\beta_1} \sum^+) = \{\alpha_i : \alpha_i = c_{\beta_3} c_{\beta_1} \beta_i\ 1 \leq i \leq 4\}$. Then $\widetilde{\mathfrak{b}_{\mathcal{Q}}} = s_{\alpha_1} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} c_{\beta_3} c_{\beta_1} \mathfrak{b}$.

5.3 Acknowlegments

I am indebted to Peter Trapa and Monty McGovern for many helpful suggestions and conversations during the course of this project. I also thank them for their continuous support. Their work has been an inspiration for this paper. I thank Roger Zierau for useful comments and conversations. I express my sincere gratitude to the Max-Planck Institute für Mathematik for the warm hospitality.

The author was partially supported by NSA grant AA-5-48903.

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