OPERADIC CATEGORIES AND DUOIDAL DELIGNE'S CONJECTURE

MICHAEL BATANIN AND MARTIN MARKL

Abstract. The purpose of this paper is two-fold. In **Part 1** we introduce a new theory of operadic categories and their operads. This theory is, in our opinion, of an independent value.

In **Part 2** we use this new theory together with our previous results to prove that multiplicative 1-operads in duoidal categories admit, under some mild conditions on the underlying monoidal category, natural actions of contractible 2-operads. The result of D. Tamarkin on the structure of dg-categories, as well as the classical Deligne conjecture for the Hochschild cohomology, is a particular case of this statement.

Contents

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Introduction

In [9] we proposed a notion of center and homotopy center of a monoid in a monoidal category enriched in a duoidal category. Examples include classical centers but also the 2-category of categories, the symmetric monoidal closed category **Gray** of 2-categories, 2-functors and pseudonatural transformations [22] and Tamarkin's homotopy 2-category of dg-categories, dg-functors and their coherent natural transformations [30].

It is well-known that the center of an associative algebra is a commutative algebra. A homotopical analogue of this statement is the famous Deligne conjecture which states that there is a natural action of an E_2-operad on the Hochschild complex of an associative algebra lifting the Gerstenhaber algebra structure from the cohomology to the chain level. We conjectured in [9] that our generalized (homotopical) center admits a closely related algebraic structure. Namely, there is a natural action of a contractible 2-operad in the sense of the first author [3] on the homotopical center of a monoid [9, Corollary 11.20]. We call this statement duoidal Deligne's conjecture. Tamarkin's main theorem from [30] is a particular case of this conjecture. Classical Deligne's conjecture also follows from the duoidal version [9, Corollary 11.22]. The **main goal** of this paper is to prove this conjecture under some mild homotopical conditions on the base symmetric monoidal category V.

Our **secondary goal** is to advertise a new theory which, as we believe, has an independent value. During our work on the proof of duoidal Deligne's conjecture we discovered that the existing language is not adequate for our purposes. Some operad-like structures that we wanted to use were not operads in any of the existing senses. To overcome these difficulties, we introduce a concept of an operadic category and of an operad corresponding to such a category.

Examples of operadic categories are abundant. They include categories like finite sets, finite ordinals, the categories of n-ordinals [4] and n-trees [3, 10], Barwick operator categories [2], ordered graphs and many other. Examples of the corresponding operads are (colored) classical symmetric and nonsymmetric operads, n-operads [3], hyperoperads of Getzler and Kapranov [21], charades of Kapranov [23], &c. As classical operads, our generalized operads have algebras, which now include other operad-like structures such as (wheeled) properads or PROPs, cyclic operads and (twisted) modular operads.

We believe that operadic categories admit a rich and interesting theory. They are closely connected to other existing and emerging approaches to generalized operad-like structures such as Feynman categories of Kaufmann [24], polynomial monads [6], moment categories of Berger [13] and operator categories of Barwick [2]. To keep the focus, we decided to choose a 'minimalistic' approach and to include as much or as little theory of operadic categories as necessary for the proof of Deligne's conjecture. A deeper theory, including 2-categorical aspects of operadic categories and relation to other notions with more applications, will be developed in subsequent papers.

Plan of the paper

According to our goals we decided to subdivide our paper into two parts. **Part 1** contains all necessary definitions and facts about operadic categories.

An operadic category is a category over the (skeletal) category of finite sets whose morphisms come with a finite set of objects of the same category, called fibers. The motivating example is the category of finite sets itself, where the set of fibers of a map $f : T \to S$ is the set of preimages $f^{-1}(i),\ i \in S$. The axioms of operadic category are designed to make the assignment of fibers to morphisms an abstract algebraic structure on a category. Operadic functors are functors which preserve fibers and some other elements of this structure.

With each operadic category one associates its category of operads with values in a symmetric monoidal category. In many respects the category of operads plays the rôle of the presheaf category over a small category, but other aspects of operadic categories make it closer to multicategories. We show that several standard categorical notions such as the left Kan extensions, discrete fibrations, the Grothendieck construction or Beck-Chevalley squares extend to operadic categories and operads. On the other hand, some important notions from the theory of classical operads, such as the Day-Street convolution, multitensors and the condensation can be easily carried over to the context of operadic categories.

Part 2 of our paper shows how all these notions work together in the proof of duoidal Deligne's conjecture. In Section 5 we describe an operadic category $\mathtt{LTr}$ and an $\mathtt{LTr}$-operad $\mathfrak{I}^{\mathtt{LTr}}$ that canonically acts on any multiplicative 1-operad in any duoidal category.

In the subsequent sections we demonstrate that this action induces an action of a colored 2-operad $\mathcal{T}m_2^{\mathbb{N}}$ on the same multiplicative 1-operad. This induced action is crucial and the most complicated part of our proof. The 2-operad $\mathcal{T}m_2^{\mathbb{N}}$ is constructed as a pullback of the Tamarkin-Tsygan colored symmetric operad $\mathcal{L}^{(2)}$ [31, 7] whose definition we recall in

Section 6. Algebras of $\mathcal{L}^{(2)}$ are multiplicative nonsymmetric operads. While this observation was enough to prove classical Deligne's conjecture [7], for the duoidal version we have to take into account that the two units of a duoidal category can be different, though connected by a noninvertible morphism. This asymmetry cannot be captured by the classical approach via symmetric operads. This was our reason for introducing operadic categories.

Once this induced action of $\mathcal{T}m_2^{\mathbb{N}}$ is constructed, we obtain the **proof** of duoidal Deligne's conjecture using the condensation of [7] generalized to the context of operadic categories. This is the subject of the last section of our paper.

Conventions

Throughout this article, V will be a complete, cocomplete closed symmetric monoidal category. A tree will always mean a rooted (i.e. directed) tree [27, II.1.5]. The arity of a vertex of a directed tree is the number of incoming edges of that vertex.

Categories will be denoted by typewriter letters $\mathtt{O}$, $\mathtt{P}$, $\mathtt{Tam}_2^{\mathbb{N}}$, $\Omega_2^{\mathbb{N}}$, &c. Exceptions are our basic monoidal category which we keep denoting V from historical reasons, and the basic duoidal category $\mathscr{D}$ used in Part 2. Operads in V will be denoted by the calligraphic letters $\mathcal{O}$, $\mathcal{P}$, &c., while a typical operad in $\mathscr{D}$ will be denoted by the script $\mathscr{A}$. A more specific notation used in Part 2 is summarized in Table 1. Finally, $\mathbb{N}$ denotes the set of natural numbers (inducing 0).

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Part 1. Theory of operadic categories

1. Operadic categories and their operads

Let $\mathtt{sFSet}$ be the skeletal category of finite sets. The objects of this category are linearly ordered sets $\bar{n} = \{1, \ldots, n\}, n \in \mathbb{N}$. Morphisms are arbitrary maps between these sets. We define the ith fiber $f^{-1}(i)$ of a morphism $f : T \to S$, $i \in S$, as the pullback of f along the map $\bar{1} \to S$ which picks up the element i, so this is an object $f^{-1}(i) = \bar{n}_i \in \mathtt{sFSet}$ which is isomorphic as a linearly ordered set to the preimage $\big\{j \in T \mid f(j) = i\big\}$. Any commutative diagram in $\mathtt{sFSet}$

$$\begin{array}{ccc} T & \xrightarrow{\;f\;} & S \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & R & \end{array}$$

then induces a map $f_i : h^{-1}(i) \to g^{-1}(i)$ for any $i \in R$. This assignment is a functor $Fib_i : \mathtt{sFSet}/R \to \mathtt{sFSet}$. Moreover, for any $j \in S$ we have the equality $f^{-1}(j) = f^{-1}_{g(j)}(j)$. The above structure on the category $\mathtt{sFSet}$ motivates the following abstract definition.

A *strict operadic category* is a category $\mathtt{O}$ equipped with a 'cardinality' functor $|\text{-}| : \mathtt{O} \to \mathtt{sFSet}$ having the following properties. We require that each connected component of $\mathtt{O}$ has a chosen terminal object U_c, $c \in \pi_0(\mathtt{O})$. We also assume that for every $f : T \to S$ in $\mathtt{O}$ and every element $i \in |S|$ there is given an object $f^{-1}(i) \in \mathtt{O}$, which we will call *the* i*th fiber* of f, such that $|f^{-1}(i)| = |f|^{-1}(i)$. We also require that

(i) For any $c \in \pi_0(\mathtt{O})$, $|U_c| = 1$.

A *trivial* morphism $f : T \to S$ in $\mathtt{O}$ is a morphism such that, for each $i \in |S|$, $f^{-1}(i) = U_{d_i}$ for some $d_i \in \pi_0(\mathtt{O})$.

The remaining axioms for a strict operadic category are:

(ii) The identity morphism $id : T \to T$ is trivial for any $T \in \mathtt{O}$;

(iii) For any commutative diagram in $\mathtt{O}$

(1)
$$\begin{array}{ccc} T & \xrightarrow{\ f\ } & S \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & R & \end{array}$$

and every $i \in |R|$ one is given a map

$$f_i : h^{-1}(i) \to g^{-1}(i)$$

such that $|f_i| : |h^{-1}(i)| \to |g^{-1}(i)|$ is the map $|h|^{-1}(i) \to |g|^{-1}(i)$ of sets induced by

$$\begin{array}{ccc} |T| & \xrightarrow{\ |f|\ } & |S| \\ & {\scriptstyle |h|}\searrow \quad \swarrow{\scriptstyle |g|} & \\ & |R| & \end{array} .$$

We moreover require that this assignment forms a functor $Fib_i : \mathtt{O}/R \to \mathtt{O}$. If $R = U_c$, the functor Fib_1 is required to be the domain functor $\mathtt{O}/R \to \mathtt{O}$.

(iv) In the situation of (iii), for any $j \in |S|$, one has the equality

(2)
$$f^{-1}(j) = f^{-1}_{|g|(j)}(j).$$

(v) Let

$$\begin{array}{ccccc} & & S & & \\ & {\scriptstyle f}\nearrow & \downarrow{\scriptstyle g} & \searrow{\scriptstyle a} & \\ T & & \xrightarrow{\qquad b \qquad} & & Q \\ & {\scriptstyle h}\searrow & \downarrow & \swarrow{\scriptstyle c} & \\ & & R & & \end{array}$$

be a commutative diagram in $\mathtt{O}$ and let $j \in |Q|, i = |c|(j)$. Then by axiom (iii) the diagram

$$\begin{array}{ccc} h^{-1}(i) & \xrightarrow{\ f_i\ } & g^{-1}(i) \\ & {\scriptstyle b_i}\searrow \quad \swarrow{\scriptstyle a_i} & \\ & c^{-1}(i) & \end{array}$$

commutes, so it induces a morphism $(f_i)_j : b_i^{-1}(j) \to a_i^{-1}(j)$. By axiom (iv) we have

$$a^{-1}(j) = a_i^{-1}(j) \text{ and } b^{-1}(j) = b_i^{-1}(j).$$

We then require the equality

$$f_i = (f_i)_j.$$

We will also assume that the set $\pi_0(\mathtt{O})$ of connected components is *small* with respect to a sufficiently big ambient universe.

Remark 1.1. It follows from axiom (iii) that the unique fiber of the canonical morphism $!_T : T \to U_c$ is T.

A *strict operadic functor* between two strict operadic categories is a functor $F : \mathtt{P} \to \mathtt{O}$ over $\mathtt{sFSet}$ which preserves fibers in the sense that $F\big(f^{-1}(i)\big) = F(f)^{-1}(i)$, for any $f : T \to S \in \mathtt{P}$ and $i \in |S| = |F(S)|$. We also require that F preserves the chosen terminal objects in each connected component, and equality (2). This gives the category $\mathtt{OpCat}$ of strict operadic categories and their strict operadic functors.

Remark 1.2. Our notion of operadic category is not invariant under categorical equivalences. It is, nevertheless, very convenient and sufficient for our applications. There is a more general non-strict version of the above definition which we are going to consider in a subsequent paper. We will also consider non-strict operadic functors and operadic natural transformations. We hope to prove a coherence theorem saying that every general operadic category is equivalent in an appropriate sense to a strict one. Since in this paper we only use strict operadic categories we will call them simply operadic categories for brevity.

Example 1.3. The terminal category $\mathtt{1}$ with the cardinality functor $|\text{-}| : \mathtt{1} \to \mathtt{sFSet}$ which sends the unique object of $\mathtt{1}$ to $\bar{1} \in \mathtt{sFSet}$ is operadic.

Example 1.4. The category Δ_{alg} of finite ordinals (including the empty one) has an obvious structure of an operadic category

Example 1.5. The categories $\mathtt{sFSet}$ and Δ_{alg} are examples of operator categories in the sense of Barwick [2]. It is easy to see that, in fact, any Barwick's operator category is equivalent to an operadic category in our sense. Recall that an *operator category* is an essentially small category Φ which satisfies the following conditions:

(i) the set of morphisms $\Phi(T, S)$ is finite for any pair of objects $T, S \in \Phi$,

(ii) the category Φ has a terminal object 1, and
(iii) there is a pullback $T \times_S i$ of i along f for any morphisms $f : T \to S$ and $i : 1 \to S$.

To find an equivalent operadic category we take a skeletal version of Φ, fix 1 as the chosen terminal object, take $\Phi(1, T)) = |T| \in \mathtt{sFSet}$ as the cardinality of T and choose pullbacks $T \times_S i, i \in |S|$, as the fiber functors. The rest of the structure is clear.

Example 1.6. Each category $\mathtt{C}$ determines the 'tautological' operadic category $\mathtt{C}+1$ which, as a category, is $\mathtt{C}$ with a formally added terminal object 1. The cardinality $|\text{-}| : \mathtt{C}+1 \to \mathtt{sFSet}$ is defined by

$$|T| := \begin{cases} \bar{0}, & \text{if } T \in \mathtt{C}, \text{ and} \\ \bar{1}, & \text{if } T = 1. \end{cases}$$

The axioms dictate that the only maps that have fibers are $!_T : T \to 1$ with the unique fiber T, and the identity $id : 1 \to 1$ whose fiber is 1. This construction constitutes a fully faithful embedding of the category of small categories to the category of operadic categories.

Example 1.7. Let $\mathfrak{C}$ be a set. A $\mathfrak{C}$-*bouquet* is a map $b : X+1 \to \mathfrak{C}$, where $X \in \mathtt{sFSet}$. In other words, a $\mathfrak{C}$-bouquet is an ordered $(k+1)$-tuple $(c_1, \ldots, c_k; c), X = \bar{k}$, of elements of $\mathfrak{C}$. It can also be thought of as a planar corolla whose all edges (including the root) are colored. The extra color $b(1) \in \mathfrak{C}$ is called the *root color*. The finite set X is the *underlying set* of the bouquet b.

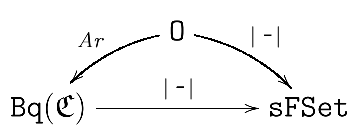
A map of $\mathfrak{C}$-bouquets $b \to c$ whose root colors coincide is an arbitrary map $f : X \to Y$ of their underlying sets. Otherwise there is no map between $\mathfrak{C}$-bouquets. We denote the resulting category of $\mathfrak{C}$-bouquets by $\mathtt{Bq}(\mathfrak{C})$.

The cardinality functor $|\text{-}| : \mathtt{Bq}(\mathfrak{C}) \to \mathtt{sFSet}$ assigns to a bouquet $b : X + 1 \to \mathfrak{C}$ its underlying set X. The fiber of a map $b \to c$ given by $f : X \to Y$ over an element $y \in Y$ is a $\mathfrak{C}$-bouquet whose underlying set is $f^{-1}(y)$, the root color coincides with the color of y and the colors of the elements are inherited from the colors of the elements of X.

It is easy to see that $\mathtt{Bq}(\mathfrak{C})$ is an operadic category with $\mathfrak{C}$ its set of connected components. It is an example of an operadic category whose fibers are not pullbacks. It has the following important property:

Proposition 1.8. *For each operadic category* $\mathtt{O}$ *with* $\pi_0(\mathtt{O}) = \mathfrak{C}$*, there is a canonical operadic 'arity' functor* $Ar : \mathtt{O} \to \mathtt{Bq}(\mathfrak{C})$ *giving rise to the factorization*

(3)



of the cardinality functor $|\text{-}| : \mathtt{O} \to \mathtt{sFSet}$.

Proof. Let the *source* $s(T)$ of $T \in \mathtt{O}$ be the set of fibers of the identity $id : T \to T$. We define $Ar(T) \in \mathtt{Bq}(\mathfrak{C})$ as the bouquet $b : s(T) + 1 \to \mathfrak{C}$, where b associates to each fiber $U_c \in s(T)$ the corresponding connected component $c \in \mathfrak{C}$, and $b(1) := \pi_0(T)$. We leave as an exercise to check that the assignment $T \mapsto Ar(T)$ extends into an operadic functor. $\square$

The assignment $\mathfrak{C} \mapsto \mathtt{Bq}(\mathfrak{C})$ extends to a functor $Bq : \mathtt{Set} \to \mathtt{OpCat}$, and the functor $Ar : \mathtt{O} \to \mathtt{Bq}(\mathfrak{C})$ is the initial object of the comma-category $\mathtt{O}/Bq$. This explains why the bouquets will play such a prominent rôle among operadic categories. Indeed, the arity functor will be used to define the endomorphism operads in Example 1.19.

Proposition 1.9. *Pullbacks in the category of categories create pullbacks in the category of operadic categories and operadic functors.*

Proof. Let us consider the ordinary pullback

(4)
$$\begin{array}{ccc} \mathtt{R} & \xrightarrow{\ r\ } & \mathtt{O} \\ {\scriptstyle \varpi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathtt{Q} & \xrightarrow{\ p\ } & \mathtt{P} \end{array}$$

of the diagram $\mathtt{Q} \xrightarrow{p} \mathtt{P} \xleftarrow{\pi} \mathtt{O}$ of operadic categories and their operadic functors. We may assume that objects of $\mathtt{R}$ are pairs (t,S), $t \in \mathtt{O}$, $S \in \mathtt{Q}$, such that $\pi(t) = p(S)$. Morphisms $(t,T) \to (s,S)$ in $\mathtt{R}$ are couples (σ, f), where $\sigma : t \to s$ is a morphism in $\mathtt{O}$ and $f : T \to S$ a morphism in $\mathtt{Q}$ such that $\pi(\sigma) = p(f)$. The functors $r : \mathtt{R} \to \mathtt{O}$ and $\varpi : \mathtt{R} \to \mathtt{Q}$ are the obvious projections to the first resp. the second factor.

We equip $\mathtt{R}$ with a structure of an operadic category as follows. We define the cardinality functor $|\text{-}| : \mathtt{R} \to \mathtt{sFSet}$ by $|(t,T)| := |t|$.[1] The chosen terminal objects are

$$U_{\pi_0(t,S)} := (U_{\pi_0(t)}, U_{\pi_0(S)}).$$

The fibers are defined componentwise, i.e. for a morphism $(\sigma, f) : (t,T) \to (s,S)$ we put

$$(\sigma, f)^{-1}(i) := \big(\sigma^{-1}(i), f^{-1}(i)\big),\ i \in |s| = |S|.$$

Notice that, since p and π are operadic functors,

$$\pi\big(\sigma^{-1}(i)\big) = (\pi\sigma)^{-1}(i) = (pf)^{-1}(i) = p\big(f^{-1}(i)\big),$$

so indeed $(\sigma, f)^{-1}(i) \in \mathtt{R}$.

We leave the verification that the diagram (4) is indeed a pullback in the category of operadic categories as an exercise. $\square$

[1]Since $|t| = |S|$ we could as well put $|(t,T)| := |T|$.

Pullbacks can be used to define colored versions of operadic categories. Given an operadic category $\mathtt{O}$ and a finite set $\mathfrak{C}$, we define the operadic category $\mathtt{O}^{\mathfrak{C}}$ of $\mathfrak{C}$-colored objects in $\mathtt{O}$ as the pullback

$$\begin{array}{ccc} \mathtt{O}^{\mathfrak{C}} & \longrightarrow & \mathtt{Bq}(\mathfrak{C}) \\ \downarrow & & \downarrow {\scriptstyle |\text{-}|} \\ \mathtt{O} & \xrightarrow{\ |\text{-}|\ } & \mathtt{sFSet} \end{array} \tag{5}$$

Notice that $\pi_0(\mathtt{O}^{\mathfrak{C}}) \cong \pi_0(\mathtt{O}) \times \mathfrak{C}$.

Remark 1.10. Since $\mathtt{sFSet}$ is the terminal object in the category of operadic categories, the pullback $\mathtt{O}^{\mathfrak{C}}$ is actually the product $\mathtt{O} \times \mathtt{Bq}(\mathfrak{C})$ in $\mathtt{OpCat}$.

A $\mathtt{O}$-*collection* in V is a collection $E = \{E(T)\}_{T \in \mathtt{O}}$ of objects of V indexed by the objects of the category $\mathtt{O}$. The category of $\mathtt{O}$-collections in V will be denoted $\mathtt{Coll}^{\mathtt{O}}(V)$. For a $\mathtt{O}$-collection E and a morphism $f : T \to S$ let

$$E(f) = \bigotimes_{i \in |S|} E(T_i)$$

In the following definition we tacitly use equalities (2).

Definition 1.11. An $\mathtt{O}$-operad is a collection $\mathcal{P} = \{\mathcal{P}(T)\}_{T \in \mathtt{O}}$ in V together with units

$$I \to \mathcal{P}(U_c),\ c \in \pi_0(\mathtt{O}),$$

and structure maps

$$\mu(f) : \mathcal{P}(f) \otimes \mathcal{P}(S) \to \mathcal{P}(T),\ f : T \to S,$$

satisfying the following axioms.

(i) Let $T \stackrel{f}{\to} S \stackrel{g}{\to} R$ be morphisms in $\mathtt{O}$ and $h := gf : T \to R$ as in (1). Then the following diagram of structure maps of $\mathcal{P}$ combined with the canonical isomorphisms of products in V commutes:

$$\begin{array}{ccc} \bigotimes_{i \in |R|} \mathcal{P}(f_i) \otimes \mathcal{P}(g) \otimes \mathcal{P}(R) & \xrightarrow{\ \bigotimes_i \mu(f_i) \otimes id\ } & \mathcal{P}(h) \otimes \mathcal{P}(R) \\ \downarrow {\scriptstyle id \otimes \mu(g)} & & \downarrow {\scriptstyle \mu(h)} \\ \bigotimes_{i \in |R|} \mathcal{P}(f_i) \otimes \mathcal{P}(S) \cong \mathcal{P}(f) \otimes \mathcal{P}(S) & \xrightarrow{\ \mu(f)\ } & \mathcal{P}(T) \end{array}.$$

(ii) The composition

$$\mathcal{P}(T) \longrightarrow \bigotimes_{i \in |T|} I \otimes \mathcal{P}(T) \longrightarrow \bigotimes_{i \in |T|} \mathcal{P}(U_{c_i}) \otimes \mathcal{P}(T) \xrightarrow{\ =\ } \mathcal{P}(id_T) \otimes \mathcal{P}(T) \xrightarrow{\ \mu(id)\ } \mathcal{P}(T)$$

is the identity for each $T \in \mathtt{O}$, as well as the identity is

(iii) the composition

$$\mathcal{P}(T)\otimes I \longrightarrow \mathcal{P}(T)\otimes\mathcal{P}(U_c) \xrightarrow{=} \mathcal{P}(!_T)\otimes\mathcal{P}(U_c) \xrightarrow{\mu(!_T)} \mathcal{P}(T),\ c := \pi_0(T).$$

Notice that for an arbitrary operad $\mathcal{P}$ and $c \in \pi_0(\mathtt{O})$, $\mathcal{P}(U_c)$ with the multiplication

$$\mu(id) : \mathcal{P}(U_c) \otimes \mathcal{P}(U_c) \to \mathcal{P}(U_c)$$

forms a unital monoid in V.

A *morphism* $\varsigma : \mathcal{P}' \to \mathcal{P}''$ of $\mathtt{O}$-operads in V is a collection $\{\varsigma_T\}_{T\in\mathtt{O}}$ of V-morphisms $\varsigma_T : \mathcal{P}'(T) \to \mathcal{P}''(T)$ commuting with the structure operations. We denote by $\mathtt{Op}^{\mathtt{O}}(V)$ the category of $\mathtt{O}$-operads in V. Each operadic functor $F : \mathtt{O} \to \mathtt{P}$ obviously induces the restriction $F^* : \mathtt{Op}^{P}(V) \to \mathtt{Op}^{\mathtt{O}}(V)$.

We can put the definition of $\mathtt{O}$-operad in a 2-categorical context as follows[2]. Let ΣV denote the symmetric monoidal bicategory with one object $\star$ and V as its category of morphisms $\star \to \star$. Recall that a part of a lax-functor structure on $\mathcal{P}$ from a category $\mathtt{O}$ to the bicategory ΣV are morphisms

$$\mathcal{P}(f) \otimes \mathcal{P}(g) \to \mathcal{P}(h)$$

for each commutative diagram like (1), as well as morphisms $I \to \mathcal{P}(id)$. For such a lax-functor and an object $T \in \mathtt{O}$ we denote $\mathcal{P}(T) := \mathcal{P}(T \stackrel{!_T}{\to} U_c)$.

Definition 1.12. An *operad-like* functor from $\mathtt{O}$ to V is a lax-functor $\mathcal{P} : \mathtt{O} \to \Sigma V$ equipped, for each $f : T \to S$ with fibers $T_i := f^{-1}(i)$, $i \in |S|$, with an isomorphism

$$\mathcal{P}(f) \cong \bigotimes_{i\in|S|} \mathcal{P}(T_i) \tag{6}$$

which satisfies the following axioms:

(i) For any commutative diagram (1) the following diagram commutes

$$\begin{array}{ccc}
\mathcal{P}(f)\otimes\mathcal{P}(g) & \longrightarrow & \mathcal{P}(h) \\
\downarrow\cong & & \downarrow\cong \\
\bigotimes_{j\in|S|}\mathcal{P}\big(f^{-1}(j)\big)\otimes\bigotimes_{i\in|R|}\mathcal{P}\big(g^{-1}(i)\big) & & \bigotimes_{i\in|R|}\mathcal{P}\big(h^{-1}(i)\big) \\
\downarrow id & & \uparrow \\
\bigotimes_{j\in|S|}\mathcal{P}\big(f^{-1}_{|g|(j)}(j)\big)\otimes\bigotimes_{i\in|R|}\mathcal{P}\big(g^{-1}(i)\big) & \xrightarrow{\cong} & \bigotimes_{j\in|R|}\Big(\bigotimes_{|g|(j)=i}\mathcal{P}\big(f_i^{-1}(j)\big)\otimes\mathcal{P}\big(g^{-1}(i)\big)\Big)
\end{array}$$

where the bottom vertical arrow on the left side is an identity due to equality

$$\mathcal{P}\big(f^{-1}_{|g|(j)}(j)\big) \xrightarrow{=} \mathcal{P}\big(f^{-1}(j)\big) \tag{7}$$

[2] We were inspired by the definition of a non-symmetric operad as a strict monoidal lax-functor $\Delta_{alg} \to \Sigma V$ given by Day and Street in [19].

and the up-going right vertical arrow is given by the lax-constraints induced by commutative diagrams:

$$\begin{array}{ccc} h^{-1}(i) & \xrightarrow{f_i} & g^{-1}(i) \\ & {\scriptstyle !}\searrow \quad \swarrow{\scriptstyle !} & \\ & U_c & \end{array} .$$

(ii) For each object $T \in \mathtt{O}$, the following diagram commutes

$$\begin{array}{ccc} I & \longrightarrow & \mathcal{P}(id_T) \\ \downarrow & & \downarrow \cong \\ \bigotimes_{i\in|T|} \mathcal{P}(U_{c_i}) & \xrightarrow{=} & \bigotimes_{i\in|T|} \mathcal{P}(id_T^{-1}(i)) \end{array} .$$

The proof of the following lemma is straightforward:

Lemma 1.13. *The category $\mathtt{Op}^{\mathtt{O}}(V)$ is equivalent to the category of operad-like functors from $\mathtt{O}$ to V and their lax-natural transformations which commute with the structure isomorphisms (6).*

Example 1.14. Operads over the terminal category $\mathtt{1}$ of Example 1.3 are monoids in V.

Example 1.15. The category of operads over the category $\mathtt{sFSet}$ is isomorphic to the category of classical symmetric operads. Operads over Δ_{alg} are ordinary non-Σ operads [5, Sec. 3, Prop. 3.1]. Applying the construction of diagram (5) to the operadic category Δ_{alg} we obtain the category $\Delta^{\mathfrak{C}}_{alg}$ describing $\mathfrak{C}$-colored non-Σ-operads. More examples of this construction will be given in Section 5.

Example 1.16. An operad over the category $\mathtt{C}+1$ from Example 1.6 is the same as a monoid $M = \mathcal{P}(1)$ in V, together with the 'actions'

$$\mu(!_T) : M \otimes \mathcal{P}(T) \to \mathcal{P}(T),\ T \in \mathtt{C}, \tag{8}$$

and a contravariant functor $\Phi : \mathtt{C} \to V$ such that the maps

$$\Phi(f) := \mathcal{P}(f) : \mathcal{P}(T) \to \mathcal{P}(S),\ f : T \to S \in \mathtt{C},$$

commute with the actions (8). In particular, $\mathtt{C}+1$-operads with $\mathcal{P}(1) = I$ are precisely presheaves $\mathtt{C}^{\mathtt{op}} \to V$.

Example 1.17. Operads over the category $\mathtt{Bq}(\mathfrak{C})$ of $\mathfrak{C}$-bouquets introduced in Example 1.7 are ordinary $\mathfrak{C}$-colored operads. Therefore, for each $\mathfrak{C}$-colored collection $E = \{E_c\}_{c\in\mathfrak{C}}$ of objects of V one has the *endomorphism* $\mathtt{Bq}(\mathfrak{C})$*-operad* $\mathcal{E}nd_E^{\mathtt{Bq}(\mathfrak{C})}$, namely the ordinary colored endomorphism operad [15, §1.2].

Example 1.18. The category $\mathtt{Op}^{\mathtt{O}}(V)$ of $\mathtt{O}$-operads in $V = (V, \otimes, I)$ has a monoidal structure given by the 'componentwise' multiplication in V. The unit for this structure is the operad $\mathfrak{I}^{\mathtt{O}}$ with $\mathfrak{I}^{\mathtt{O}}(T) := I$ for each $T \in \mathtt{O}$. Clearly $F^*(\mathfrak{I}^P) = \mathfrak{I}^{\mathtt{O}}$ for any operadic functor $F : \mathtt{O} \to P$.

Example 1.19. For a $\mathfrak{C}$-colored collection $E = \{E_c\}_{c\in\mathfrak{C}}$ in V and an operadic category $\mathtt{O}$ with $\pi_0(\mathtt{O}) = \mathfrak{C}$, one defines the *endomorphism* $\mathtt{O}$*-operad* $\mathcal{E}nd^{\mathtt{O}}_E$ as the restriction

$$\mathcal{E}nd^{\mathtt{O}}_E := Ar^*\big(\mathcal{E}nd^{\mathtt{Bq}(\mathfrak{C})}_E\big)$$

of the $\mathtt{Bq}(\mathfrak{C})$-endomorphism operad of Example 1.17 along the arity functor Ar of Proposition 1.8.

Definition 1.20. An *algebra* over an $\mathtt{O}$-operad $\mathcal{P}$ in V is a collection $A = \{A_c\}_{c\in\pi_0(\mathtt{O})}$, $A_c \in V$, equipped with an $\mathtt{O}$-operad map $\alpha : \mathcal{P} \to \mathcal{E}nd^{\mathtt{O}}_A$.

An algebra is thus given by suitable structure maps

$$\alpha_T : \bigotimes_{c\in\pi_0(s(T))} A_c \otimes \mathcal{P}(T) \to A_{\pi_0(T)},\ T \in \mathtt{O},$$

where $s(T)$ denotes, as before, the set of fibers of the identity $id : T \to T$. This notion of $\mathcal{P}$-algebras will further be generalized in Section 3.

Example 1.21. The category Γ of stable labelled graphs [26, Section 7] is an operadic category. Morphisms are given by contractions of subgraphs. The cardinality functor associates to a graph its set of vertices. Fibers of a morphism are the subgraphs contracted to a vertex.

If V is the category of differential graded vector spaces, then Γ-operads are precisely *hyperoperads* in the sense of [21]. Algebras over these operads are (twisted) modular operads, see [21] or [27, Def. II.5.5].

Example 1.22. This is our only example of a large operadic category. Let $\mathtt{A}$ be an abelian category and let $\mathtt{Epi}(\mathtt{A})$ be its subcategory of epimorphisms. The cardinality functor on $\mathtt{Epi}(\mathtt{A})$ maps all objects to the one point set $\bar{1}$. The (unique) fiber of any morphism is its (chosen) kernel. It is easy to check that this defines an operadic category structure on $\mathtt{Epi}(\mathtt{A})$. An $\mathtt{Epi}(\mathtt{A})$-operad in the category of vector spaces is the same as a charade over $\mathtt{A}$ in Kapranov's sense [23, Definition 3.2].[3]

[3]Generally speaking, $\mathtt{Epi}(\mathtt{A})$ is not a strict operadic category but we can easily "strictify" it if we use a skeletal version.

1.1. **The category of k-trees.** We are going to recall briefly the category Ω_k of k-trees, for $k \geq 0$; the details can be found in [3, Sec. 3, Example 8] or [10]. The category of 0-*trees* Ω_0 is the terminal category 1. Its unique object is denoted by $\mathcal{U}_0$.

The category of 1-*trees* Ω_1 is the category of finite ordinals $(n) := \{1, \ldots, n\}$, $n \geq 0$, and their order-preserving maps. As usual, we interpret $\{1, \ldots, n\}$ for $n = 0$ as the empty set. The terminal object of Ω_1 is $\mathcal{U}_1 := (1)$. When the meaning is clear from the context, we will simplify the notation and denote the object $(n) \in \Omega_1$ simply by n. The category Ω_1 is isomorphic to the operadic category Δ_{alg} recalled in Example 1.4.

Let $k \geq 2$. A k-*tree* is a chain

$$\mathcal{T} = (\ n_k \xrightarrow{t_{k-1}} n_{k-1} \xrightarrow{t_{k-2}} \ \cdots \ \xrightarrow{t_1} n_1\ ) \tag{9}$$

of morphisms in Ω_1. A morphism

$$\sigma : (\ n_k \xrightarrow{t_{k-1}} n_{k-1} \xrightarrow{t_{k-2}} \ \cdots \ \xrightarrow{t_1} n_1\ ) \longrightarrow (\ m_k \xrightarrow{s_{k-1}} m_{k-1} \xrightarrow{s_{k-2}} \ \cdots \ \xrightarrow{s_1} m_1\ ) \tag{10}$$

of k-trees is a diagram in $\mathtt{Set}$

$$\begin{array}{ccccccc} n_k & \xrightarrow{t_{k-1}} & n_{k-1} & \xrightarrow{t_{k-2}} & \cdots & \xrightarrow{t_1} & n_1 \\ \downarrow{\scriptstyle \sigma_k} & & \downarrow{\scriptstyle \sigma_{k-1}} & & & & \downarrow{\scriptstyle \sigma_1} \\ m_k & \xrightarrow{s_{k-1}} & m_{k-1} & \xrightarrow{s_{k-2}} & \cdots & \xrightarrow{s_1} & m_1 \end{array}$$

such that

(i) σ_1 is order preserving and
(ii) for any p, $k > p \geq 1$, and $i \in n_p$, the restriction of σ_{p+1} to $t_p^{-1}(i)$ is order-preserving.

We denote by Ω_k the category of k-trees and their morphisms as defined above. Its terminal object is the k-tree $\mathcal{U}_k := (1 \to 1 \to \cdots \to 1)$, its initial object is $z^k\mathcal{U}_0 := (0 \to 0 \to \cdots \to 0)$. Notice that we have the obvious *truncation* functors

$$\Omega_k \xrightarrow{\mathrm{tr}_{k-1}} \Omega_{k-1} \xrightarrow{\mathrm{tr}_{k-2}} \cdots \xrightarrow{\mathrm{tr}_2} \Omega_2 \xrightarrow{\mathrm{tr}_1} \Omega_1 = \Delta_{alg}\ .$$

One also has the *suspension* functor $z : \Omega_k \to \Omega_{k+1}$, $k \geq 0$, that to an k-tree as in (9) associates the $(k+1)$-tree

$$z\mathcal{T} := (\ n_k \xrightarrow{t_{k-1}} n_{k-1} \xrightarrow{t_{k-2}} \ \cdots \ \xrightarrow{t_1} n_1 \longrightarrow 1\ ).$$

An s-*leaf* (or a *leaf of height* s) of a k-tree $\mathcal{T}$ as in (9) is, for $s = k$, by definition an element of n_k. For $1 \leq s < k$ an s-leaf is an element $i \in n_s$ such that $t_s^{-1}(i) = \emptyset$. We denote by $Lf_s(\mathcal{T})$ the set of all s-leaves of $\mathcal{T}$.

Let $\sigma : \mathcal{T} \to \mathcal{S}$ be a map of k-trees as in (10) and $i \in m_k = Lf_k(\mathcal{S})$ a k-leaf of $\mathcal{S}$. Let us define the fiber $\sigma^{-1}(i)$ over i as the chain

$$\sigma^{-1}(i) := \big(\ \sigma_k^{-1}(i) \longrightarrow \sigma_{k-1}^{-1} s_{k-1}(i)) \longrightarrow \ \cdots \ \longrightarrow \sigma_1^{-1}(s_1 \cdots s_{k-1}(i))\ \big) \tag{11}$$

of the restrictions of the maps in (9). Analogously one may define also fibers over the s-leaves for $s < k$, but we will not use them in this article.

The category Ω_k with the cardinality functor $|\mathcal{T}| := Lf_k(\mathcal{T})$, with the fibers defined as above and the chosen terminal object the k-tree $\mathcal{U}_k$, is an operadic category. Operads in $\mathtt{Op}^{\Omega_k}(V)$ are precisely k-operads in the monoidal globular k-category $\Sigma^k V$, see [9, §11.3].

1.2. **The category of k-ordinals.** Let, as in the Section 1.1 , $k \geq 0$. Recall [4, Sec. II] that a *k-ordinal* is a finite set $\mathcal{O}$ equipped with k binary relations $<_0, \ldots, <_{k-1}$ such that

(i) $<_p$ is nonreflexive,
(ii) for every pair a, b of distinct elements of $\mathcal{O}$ there exists exactly one p such that
$$a <_p b \ \text{ or } \ b <_p a,$$
(iii) if $a <_p b$ and $b <_q c$ then $a <_{\min(p,q)} c$.

A morphism of k-ordinals $\sigma : \mathcal{O} \to \mathcal{N}$ is a map of the underlying sets such that $i <_p j$ in $\mathcal{O}$ implies that

(i) $\sigma(i) <_r \sigma(j)$ for some $r \geq p$, or
(ii) $\sigma(i) = \sigma(j)$, or
(iii) $\sigma(j) <_r \sigma(i)$ for $r > p$.

Let $\mathtt{Ord}_k$ be the skeletal category of k-ordinals and their morphisms. The category $\mathtt{Ord}_k$ is operadic. The cardinality $|\text{-}| : \mathtt{Ord}_k \to \mathtt{sFSet}$ associates to a k-ordinal $\mathcal{O}$ its underlying set (denoted $\mathcal{O}$ again). The fiber of a map $\sigma : \mathcal{O} \to \mathcal{N}$ over $i \in \mathcal{N}$ is the preimage $\sigma^{-1}(i)$ with the induced structure of a k-ordinal. The category $\mathtt{Ord}_k$ is connected, the unique terminal object being the one-point k-ordinal 1_k. Operads in $\mathtt{Op}^{\mathtt{Ord}_k}(V)$ are pruned k-operads in the monoidal globular n-category $\Sigma^k V$ [4].

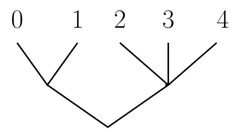
There is a natural k-ordinal structure on the set of k-leaves of each k-tree in Ω_k. Let $\mathcal{T}$ be as in (9) and $a, b \in n_k = Lf_k(\mathcal{T})$ its distinct k-leaves. We say that $a <_p b$ if a precedes b in n_k and p is such that
$$t_p t_{p-1} \cdots t_{k-1}(a) = t_p t_{p-1} \cdots t_{k-1}(b)$$
but
$$t_{p-1} \cdots t_{k-1}(a) \neq t_{p-1} \cdots t_{k-1}(b).$$
If such a p does not exist, we put $a <_0 b$. It is easy to show that this construction extends to an operadic functor

$$p : \Omega_k \to \mathtt{Ord}_k. \tag{12}$$

On the other hand, $\mathtt{Ord}_k$ can be identified with the full subcategory of Ω_k consisting of pruned trees. Recall that a k-tree $\mathcal{T} \in \Omega_k$ as in (9) is *pruned* if all $t_{k-1}, \ldots, t_1$ are epimorphisms. Equivalently, all leaves of $\mathcal{T}$ are its k-leaves, so $\mathcal{T}$ is "fully grown." For example, the 2-ordinal

$$0 <_0 2,\ 0 <_0 3,\ 0 <_0 4,\ 1 <_0 2,\ 1 <_0 3,\ 1 <_0 4,\ 0 <_1 1,\ 2 <_1 3,\ 2 <_1 4,\ 3 <_1 4,$$

is represented by the following pruned tree



See [4, Theorem 2.1] for a more detailed discussion. We thus have an inclusion of categories $l : \mathtt{Ord}_k \hookrightarrow \Omega_k$ which is left adjoint to the pruning functor (12).

It will sometimes be useful to identify a k-ordinal $\mathcal{O} \in \mathtt{Ord}_k$ with the corresponding pruned tree $l(\mathcal{O}) \in \Omega_k$. The functor $p : \Omega_k \to \mathtt{Ord}_k$ then appears as the *pruning* associating to each $\mathcal{T} \in \Omega_k$ its maximal pruned subtree. We must emphasize that $l : \mathtt{Ord}_k \hookrightarrow \Omega_k$ is not an operadic functor, since it does not preserve fibers in general. It is only lax operadic in an appropriate sense.

2. Discrete fibrations of operadic categories

Definition 2.1. An operadic functor $F : \mathtt{O} \to \mathtt{P}$ is called a *discrete operadic fibration* if

(i) F induces an epimorphism $\pi_0(\mathtt{O}) \twoheadrightarrow \pi_0(\mathtt{P})$ and
(ii) for any morphism $f : T \to S$ in $\mathtt{P}$ and any $t_i, s \in \mathtt{O}$, where $i \in |S|$, such that

$$F(s) = S \text{ and } F(t_i) = f^{-1}(i) \text{ for } i \in |S|,$$

there exists a unique $\sigma : t \to s$ in $\mathtt{O}$ such that

$$F(\sigma) = f \text{ and } t_i = \sigma^{-1}(i) \quad \text{for } i \in |S|.$$ [4]

We have the following simple

Lemma 2.2. *A discrete operadic fibration $F : \mathtt{O} \to \mathtt{P}$ induces an isomorphism $\pi_0(\mathtt{O}) \xrightarrow{\cong} \pi_0(\mathtt{P})$.*

Proof. Assume that $V_{a'}$, $V_{a''}$, $a', a'' \in \pi_0(\mathtt{O})$, are chosen terminal objects in components of the category $\mathtt{O}$ such that $F(V_{a'}) = F(V_{a''}) = U_c$ for $c \in \pi_0(\mathtt{P})$. Then (ii) of Definition 2.1 taken with $T = S = U_c$, $f : U_c \to U_c$ the identity map, $t_1 = V_{a'}$ and $s = V_{a''}$ produces a map $\sigma : t \to V_{a''}$ whose unique fiber $\sigma^{-1}(1)$ is $V_{a'}$. Since $Fib_1 : \mathtt{O}/V_{a''} \to \mathtt{O}$ is the domain functor by (iii) of the definition of the operadic category, $t = V_{a'}$, so σ is in fact a map $V_{a'} \to V_{a''}$, therefore $V_{a'}$ and $V_{a''}$ belong to the same component of $\mathtt{O}$, i.e. $a' = a''$. □

[4] Notice that $F(s) = S$ implies $|s| = |S|$ and $F(\sigma) = f$ implies $F(t) = T$.

For an $\mathtt{O}$-operad $\mathcal{O} \in \mathtt{Op}^{\mathtt{O}}(V)$ and $T \in \mathtt{P}$ put

$$F_!\mathcal{O}(T) := \coprod_{F(t)=T} \mathcal{O}(t). \tag{13}$$

Proposition 2.3. *Assume that F is a discrete operadic fibration. Then (13) is the underlying collection of a naturally defined* $\mathtt{P}$*-operad.*

Proof. For each $f : T \to S \in \mathtt{P}$ and $T_i := f^{-1}(i)$, $i \in |S|$, we need the structure map

$$\mu(f) : \bigotimes_{i\in|S|} F_!\mathcal{O}(T_i) \otimes F_!\mathcal{O}(S) \to F_!\mathcal{O}(T).$$

Expanding the definition of $F_!\mathcal{O}$ and invoking the distributivity of the monoidal product over coproducts we see that it is the same as to give a map

$$\mu(f) : \coprod_{\substack{F(t_j)=T_j,\ j\in|S| \\ F(s)=S}} \bigotimes_{i\in|S|} \mathcal{O}(t_i) \otimes \mathcal{O}(s) \longrightarrow \coprod_{F(w)=T} \mathcal{O}(w).$$

It clearly amounts to specifying, for each t_i's and s as above, a map

$$\bigotimes_{i\in|S|} \mathcal{O}(t_i) \otimes \mathcal{O}(s) \longrightarrow \coprod_{F(w)=T} \mathcal{O}(w). \tag{14}$$

The defining property of discrete operadic fibrations provides a unique $t \in \mathtt{O}$ with $F(t) = T$, and a morphism $\sigma : t \to s$ such that $\sigma^{-1}(i) = t_i$ for $i \in |S|$. We then choose (14) to be the composition

$$\bigotimes_{i\in|S|} \mathcal{O}(t_i) \otimes \mathcal{O}(s) \xrightarrow{\mu(\sigma)} \mathcal{O}(t) \xrightarrow{\iota_t} \coprod_{F(w)=T} \mathcal{O}(w),$$

where $\mu(\sigma)$ is the structure map of the $\mathtt{O}$-operad $\mathcal{O}$ and ι_t the canonical map to the coproduct.

Let U_c, $c \in \pi_0(\mathtt{P})$, be a chosen terminal object of a component of $\mathtt{P}$. By Lemma 2.2, there is a unique chosen terminal object V_a, $a \in \pi_0(\mathtt{O})$, such that $F(V_a) = U_c$. We define the unit $I \to F_!\mathcal{O}(U_c)$ as the composition

$$I \longrightarrow \mathcal{O}(V_a) \longrightarrow \coprod_{F(t)=U_c} \mathcal{O}(t) = F_!\mathcal{O}(U_c).$$

of the unit map for $\mathcal{O}$ with the coprojection. Since all constructions were functorial lifts of the operad structure of $\mathcal{O}$, the resulting structure is an operad again. □

Theorem 2.4. *Let $F : \mathtt{O} \to \mathtt{P}$ be a discrete operadic fibration of operadic categories. Then the assignment $\mathcal{O} \mapsto F_!\mathcal{O}$ described above defines a left adjoint $F_! : \mathtt{Op}^{\mathtt{O}}(V) \to \mathtt{Op}^{\mathtt{P}}(V)$ to the restriction functor $F^* : \mathtt{Op}^{\mathtt{P}}(V) \to \mathtt{Op}^{\mathtt{O}}(V)$.*

Proof. We need to establish, for $\mathcal{O} \in \mathtt{Op}^{\mathtt{O}}(V)$ and $\mathcal{P} \in \mathtt{Op}^{\mathtt{P}}(V)$, a natural isomorphism

$$\mathtt{Op}^{\mathtt{P}}(V)\big(F_!\mathcal{O}, \mathcal{P}\big) \cong \mathtt{Op}^{\mathtt{O}}(V)\big(\mathcal{O}, F^*\mathcal{P}\big).$$

There is an isomorphism of the sets of morphisms of collections

$$\mathtt{Coll}^{\mathtt{P}}(V)\big(F_!\mathcal{O}, \mathcal{P}\big) \cong \mathtt{Coll}^{\mathtt{O}}(V)\big(\mathcal{O}, F^*\mathcal{P}\big). \tag{15}$$

This follows immediately since (13) is the formula for the left Kan extension along the induced functor $Ob(F) : Ob(\mathtt{O}) \to Ob(\mathtt{P})$ between the discrete categories of objects.

The proof is finished by showing that a morphism of collections in the left hand side of (15) is an operad morphism (i.e. it commutes with the operad structure maps) if and only if the corresponding morphism in the right hand side of (15) does. We leave this as an exercise. □

Let now V be the monoidal category $\mathtt{Set}$ of sets, $F : \mathtt{O} \to \mathtt{P}$ a discrete operadic fibration and $\mathfrak{I}^{\mathtt{O}} \in \mathtt{Op}^{\mathtt{O}}(\mathtt{Set})$ the terminal $\mathtt{O}$-operad with $\mathfrak{I}^{\mathtt{O}}(t) = \{t\}$. Theorem 2.4 gives the operad

$$\mathcal{O} := F_!(\mathfrak{I}^{\mathtt{O}}) \in \mathtt{Op}^{\mathtt{P}}(\mathtt{Set}) \tag{16}$$

with $\mathcal{O}(T) = \big\{t \in \mathtt{O};\ F(t) = T\big\}$.

Vice versa, assume that one is given an operad $\mathcal{O} \in \mathtt{Op}^{\mathtt{P}}(\mathtt{Set})$. One then has the category $\mathtt{O}$ whose objects are $t \in \mathcal{O}(T)$ for some $T \in \mathtt{P}$. A morphism $\sigma : t \to s$ from $t \in \mathcal{O}(T)$ to $s \in \mathcal{O}(S)$ is a couple (ε, f) consisting of a morphism $f : T \to S$ in $\mathtt{P}$ and of some $\varepsilon \in \bigtimes_{i \in |S|} \mathcal{O}\big(t_i\big)$, $t_i := f^{-1}(i)$, such that

$$\mu(f)(\varepsilon, s) = t,$$

where μ is the structure map of the operad $\mathcal{O}$. Compositions of morphisms are defined in the obvious manner. The category $\mathtt{O}$ thus constructed is clearly an operadic category such that the functor $F : \mathtt{O} \to \mathtt{P}$ given by

$$F(t) := T \text{ for } t \in \mathcal{O}(T) \text{ and } F(\varepsilon, f) := f$$

is a discrete operadic fibration. We call this construction the *operadic Grothendieck construction.* It is a direct generalization of the classical Grothendieck construction for presheaves [29, p. 44] as the following proposition shows.

Proposition 2.5. *The above construction establishes an equivalence between the category* $\mathtt{Op}^{\mathtt{P}}(\mathtt{Set})$ *of* $\mathtt{P}$*-operads in* $\mathtt{Set}$ *and the category of discrete operadic fibrations over* $\mathtt{P}$*.*

Proof. A direct verification. □

Example 2.6. As we saw in Example 1.16, $\mathtt{Set}$-operads $\mathcal{P}$ in the operadic category $\mathtt{C}+1$ of Example 1.6 such that $\mathcal{P}(1) = 1$ are presheaves on $\mathtt{O}$. The restriction of the correspondence of Proposition 2.5 to operads $\mathcal{P}$ with this property is an equivalence between the category of presheaves on $\mathtt{O}$ and the category of discrete fibrations $\mathtt{P} \to \mathtt{C}$ of categories. Proposition 2.5 therefore indeed generalizes the discrete version of the Grothendieck construction.

Proposition 2.7. *Assume that in the pullback (4), the functor* $\pi : \mathtt{O} \to \mathtt{P}$ *is a discrete operadic fibration. Then* $\varpi : \mathtt{R} \to \mathtt{Q}$ *is a discrete operadic fibration, too.*

Proof. We rely on the notation in the proof of Proposition 1.9. Suppose we are given a morphism $f : T \to S$ in $\mathtt{Q}$ with fibers $T_i := f^{-1}(i)$, $i \in |S|$. Suppose we are also given objects $(t_i, \widetilde{T}_i)$ and $(s, \widetilde{S})$ of the category $\mathtt{R}$ such that $S = \varpi(s, \widetilde{S})$ and $T_i = \varpi(t_i, \widetilde{T}_i)$ for each $i \in |S|$. We must find a unique $(\sigma, \widetilde{f}) : (t, \widetilde{T}) \to (s, \widetilde{S})$ in $\mathtt{R}$ such that $\varpi(\sigma, \widetilde{f}) = f$ and $(\sigma, \widetilde{f})^{-1}(i) = (t_i, \widetilde{T}_i)$ for each $i \in |S|$.

It follows from definitions that $S = \varpi(s, \widetilde{S})$ implies $\widetilde{S} = S$, $T_i = \varpi(t_i, \widetilde{T}_i)$ implies $\widetilde{T}_i = T_i$ and $\varpi(\sigma, \widetilde{f}) = f$ implies $\widetilde{f} = f$. Since $(s, \widetilde{S})$ and $(t_i, \widetilde{T}_i)$ are objects of $\mathtt{R}$ we see that $\pi(s) = p(S)$, and $\pi(t_i) = p(T_i)$ for all $i \in |S|$. Similarly, we conclude that $\pi(\sigma) = p(f)$.

We therefore need to prove the following statement. Given $f : T \to S$ in $\mathtt{Q}$ with fibers $T_i := f^{-1}(i)$ and objects t_i, s of $\mathtt{O}$, $i \in |S|$, such that $\pi(s) = p(S)$, and $\pi(t_i) = p(T_i)$, there exists a unique $\sigma : t \to s$ in $\mathtt{O}$ such that $\sigma^{-1}(i) = t_i$ and $\pi(\sigma) = p(f)$. The above statement however follows from the lifting property in the discrete operadic fibration $\pi : \mathtt{O} \to \mathtt{P}$ applied to the data $p(f) : p(T) \to p(S) \in \mathtt{P}$ and $s, t_i \in \mathtt{O}$, $i \in |S|$. This finishes the proof. □

We close this section by proving that squares of adjoint functors between the associated categories of operads induced by pullbacks of discrete operadic fibrations satisfy the Beck-Chevalley property [29, p. 205]. Therefore operads over operadic categories behave similarly as presheaves over small categories.

Proposition 2.8. *If in the pullback of operadic categories (4) the functor* $\pi : \mathtt{O} \to \mathtt{P}$ *is a discrete operadic fibration, then the induced functors between the associated categories of operads satisfy the Beck-Chevalley condition, meaning that there is a natural isomorphism*

$$\varpi_!(r^*(\mathcal{P})) \cong p^*(\pi_!(\mathcal{P})) \tag{17}$$

for any operad $\mathcal{P} \in \mathtt{Op}^{\mathtt{O}}(V)$.

Proof. Notice that the functor $\varpi : \mathtt{R} \to \mathtt{Q}$ is a discrete operadic fibration by Proposition 2.7. We will use the explicit description of the left adjoint along discrete fibrations provided by Theorem 2.4. For $T \in \mathtt{P}$ it gives

$$\pi_!(\mathcal{P})(T) = \coprod_{\pi(t)=T} \mathcal{P}(t),$$

therefore, for $S \in \mathtt{Q}$,

$$p_*\big(\pi_!(\mathcal{P})\big)(S) = \pi_!(\mathcal{P})(p(S)) \cong \coprod_{\pi(t)=p(S)} \mathcal{P}(t).$$

On the other hand,

$$\varpi_!\big(r^*(\mathcal{P})\big)(S) = \coprod_{\varpi(t,\widetilde{S})=S} r^*(\mathcal{P})(t, \widetilde{S}) \cong \coprod_{\varpi(t,\widetilde{S})=S} \mathcal{P}\big(r(t, \widetilde{S})\big) = \coprod_{\pi(t)=p(S)} \mathcal{P}(t),$$

therefore indeed $p_*\big(\pi_!(\mathcal{P})\big)(S) \cong \varpi_!\big(r^*(\mathcal{P})\big)(S)$ for each $S \in \mathtt{Q}$. In the last display we used the fact that $\varpi(t, \widetilde{S}) = S$ implies $\widetilde{S} = S$ so, since $(t, \widetilde{S}) \in \mathtt{R}$, we have the equality $\pi(t) = p(S)$. $\square$

3. $\mathtt{O}$-multi(co)tensors and generalized algebras of $\mathtt{O}$-operads

We start by showing how the standard notion of a multitensor on a V-category $\mathtt{C}$, see [11, Def. 2.1] or [5, 18], generalizes to the realm of operadic categories. Let $\mathcal{E}nd_{\mathtt{C}}$ be the endomorphism $\mathtt{sFSet}$-operad of $\mathtt{C}$, so $\mathcal{E}nd_{\mathtt{C}}(n)$ is, for $n \geq 0$, the category of V-functors $\mathtt{C}^{\otimes n} \to \mathtt{C}$. The restriction along the cardinality functor $|\text{-}| : \mathtt{O} \to \mathtt{sFSet}$ gives a categorical $\mathtt{O}$-operad $|\mathcal{E}nd_{\mathtt{C}}|^*$. Let $\mathit{1}_{\mathtt{O}}$ be the terminal categorical $\mathtt{O}$-operad. An $\mathtt{O}$-multitensor on $\mathtt{C}$ is then a lax-morphism of categorical operads $\mathit{1}_{\mathtt{O}} \to |\mathcal{E}nd_{\mathtt{C}}|^*$. Unpacking this definition we obtain:

Definition 3.1. An $\mathtt{O}$-*multitensor* on a V-category $\mathtt{C}$ is an $\mathtt{O}$-collection $E = \{E_T\}_{T\in\mathtt{O}}$ of V-functors

$$E_T : \underbrace{\mathtt{C} \otimes \cdots \otimes \mathtt{C}}_{|T|-times} \to \mathtt{C},\ \mathtt{T} \in \mathtt{O},$$

equipped with

(i) V-natural transformations

$$\mu_f : E_S(E_{T_1}, \ldots, E_{T_k}) \to E_T$$

defined for any $f : T \to S$ in $\mathtt{O}$ with fibers $T_1, \ldots, T_k$, and

(ii) V-natural transformations (the units)

$$\eta_c : id \to E_{U_c},\ c \in \pi_0(\mathtt{O}),$$

satisfying the obvious associativity and unitality conditions.

Multitensors create operads, as shown in the following lemma whose simple proof we leave to the reader. For an object $T \in \mathtt{O}$, let $\pi_0 s(T)$ denote the set of connected components of the fibers of the identity $id : T \to T$, and $\pi_0(T)$ the connected component of T, cf. the proof of Proposition 1.8.

Lemma 3.2. *Let E be an $\mathtt{O}$-multitensor on a V-category $\mathtt{C}$ and $\delta = \{\delta(i)\}_{i\in\pi_0(\mathtt{O})}$ an arbitrary collection in $\mathtt{C}$. Then the collection*

$$co\mathcal{E}nd^E_\delta = \big\{co\mathcal{E}nd^E_\delta(T)\big\}_{T\in\mathtt{O}}$$

with $co\mathcal{E}nd^E_\delta(T)$ the enriched hom

$$\mathtt{C}\big(\delta(i), E_T(\delta(i_1), \ldots, \delta(i_k))\big),\ i = \pi_0(T),\ \{i_1, \ldots, i_k\} = \pi_0 s(T),$$

is a natural $\mathtt{O}$-operad in V.

Dually, we introduce $\mathtt{O}$-*multicotensors* on $\mathtt{C}$ as colax-morphisms of the categorical $\mathtt{O}$-operads $1_{\mathtt{O}} \to |\mathcal{E}nd_{\mathtt{C}}|^*$. An explicit definition can be obtained by inverting arrows in Definition 3.1. Multicotensors create operads in a similar way as multitensors do:

Lemma 3.3. *Let D be an $\mathtt{O}$-multicotensor on a V-category $\mathtt{C}$ and $X = \{X(i)\}_{i\in\pi_0(\mathtt{O})}$ an arbitrary collection of objects of $\mathtt{C}$. Then the collection*

$$\mathcal{E}nd_X^D = \big\{\mathcal{E}nd_X^D(T)\big\}_{T\in\mathtt{O}}$$

with $\mathcal{E}nd_X^E(T) := \mathtt{C}\big(D_T(X(i_1),\ldots,X(i_k)),X(i)\big)$ and $i_1,\ldots,i_k,i \in \pi_0(\mathtt{O})$ having the meaning as in Lemma 3.2, is a natural $\mathtt{O}$-operad in V.

Example 3.4. Let $\mathtt{1}$ be the terminal category of Example 1.3. A $\mathtt{1}$-(co)multitensor on $\mathtt{C}$ is the same as a V-(co)monad on $\mathtt{C}$.

For an arbitrary operadic category $\mathtt{O}$ and $c \in \pi_0(\mathtt{O})$, there is an operadic functor $\mathtt{1} \to \mathtt{O}$ which sends the unique object of $\mathtt{1}$ to U_c. Restricting along this functor we verify that E_{U_c} (resp. D_{U_c}) is a V-monad (resp. V-comonad) on $\mathtt{C}$ for an arbitrary $\mathtt{O}$-multitensor E (resp. multicotensor D).

Example 3.5. The tensor product $\odot$ of a symmetric monoidal V-category $\mathtt{C}$ gives rise to a $\mathtt{sFSet}$-multitensor $\bigodot$ on $\mathtt{C}$, which is simultaneously a $\mathtt{sFSet}$-multicotensor on $\mathtt{C}$. Namely, for a finite set S of cardinality n and $X_1,\ldots,X_n \in \mathtt{C}$ we put

$$\textstyle\bigodot_T(X_1,\ldots,X_n) := \bigodot_{i\in S} X_i.$$

For any operadic category $\mathtt{O}$ we then have an $\mathtt{O}$-multitensor $\bigodot^{\mathtt{O}}$ on $\mathtt{C}$ (which is also a $\mathtt{O}$-multicotensor) given by restricting $\bigodot$ along the cardinality functor:

$$\textstyle\bigodot^{\mathtt{O}}_T(X_1,\ldots,X_n) := \bigodot_{|T|}(X_1,\ldots,X_k),\ T\in\mathtt{O}.$$

The case $\mathtt{C} = V$ of the above construction along with Lemmas 3.2 and 3.3 explains the standard fact that an object of a symmetric monoidal category V has both the (classical) endomorphism and coendomorphism operads, see [27], Definitions II.1.7 and II.1.9.

Definition 3.6. Let D be a fixed $\mathtt{O}$-multicotensor and $\mathcal{P}$ an $\mathtt{O}$-operad. An *algebra of $\mathcal{P}$ in $\mathtt{C}$* is a $\pi_0(\mathtt{O})$-collection A in $\mathtt{C}$ with a morphism of $\mathtt{O}$-operads

$$\mathcal{P} \to \mathcal{E}nd_A^D.$$

Example 3.7. If $\mathtt{C} = V$ and $D = \bigodot^{\mathtt{O}}$, then $\mathcal{P}$-algebras in the sense of the above definition are the same as $\mathcal{P}$-algebras of Definition 1.20.

Assume that D is a $\mathtt{O}$-multicotensor and suppose that $\mathtt{C}$ is cocomplete as a V-category. This means, in particular, that there is a left action functor $V \otimes \mathtt{C} \to \mathtt{C}$ which we denote by $\otimes$, believing that it would not lead to confusion. We have an adjunction

$$\mathtt{C}(B \otimes X, Y) \cong V(B, \mathtt{C}(X, Y))$$

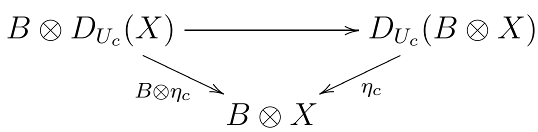
for any $B \in V$ and $X, Y \in \mathtt{C}$.

Since D is a V-functor in each variable, there is for each $T \in \mathtt{O}$, $|T| = n$, $X_1, \ldots, X_n \in \mathtt{C}$, and $B \in V$, a V-natural transformation

$$B \otimes D_T(X_1, \ldots, X_i, \ldots, X_n) \longrightarrow D_T(X_1, \ldots, B \otimes X_i, \ldots, X_n),\ 1 \leq i \leq n,$$

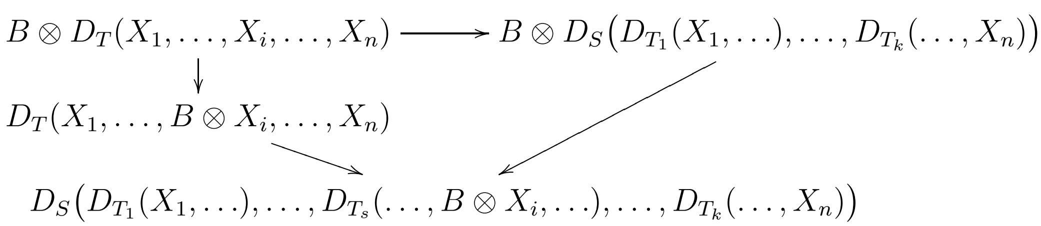
called the *strength* of D. We will require that D interacts with the left V-action in such a way that the following conditions are satisfied:

(i) The diagram



commutes for any $c \in \pi_0(\mathtt{O})$, $X \in \mathtt{C}$ and $B \in V$.

(ii) Let $f : T \to S$ be a morphism in $\mathtt{O}$ with fibers $T_1, \ldots, T_k$. Let $n = |T|$, and $1 \leq i \leq n$, $1 \leq s \leq k$ be such that $i \in |T|$ belongs to $|T_s|$. Then the diagram



commutes for any $B \in V$ and $X_1, \ldots, X_n \in \mathtt{C}$.

Definition 3.8. An $\mathtt{O}$-multicotensor D on a cocomplete V-category $\mathtt{C}$ satisfying properties (i) and (ii) above will be called *strong*. We will call a V-category $\mathtt{C}$ equipped with a strong multitensor D for which the comonad D_{U_c} of Example 3.4 is the identity comonad for each $c \in \pi_0(\mathtt{O})$ a *colax* $\mathtt{O}$*-monoidal* V*-category*.

Remark 3.9. The terminology above has been adapted from the classical definition of a strong monad on a closed monoidal category [25]. Indeed, when $\mathtt{O}$ is the terminal category $\mathtt{1}$, the notion of a strong $\mathtt{O}$-multicotensor coincides with the notion of a strong comonad, cf. Example 3.4.

For a $\pi_0(\mathtt{O})$-collection X in $\mathtt{C}$ and $T \in \mathtt{O}$ denote

$$X^T := D_T\big(X(i_1), \ldots, X(i_k)\big),\ \{i_1, \ldots, i_k\} = \pi_0 s(T).$$

The structure of a $\mathcal{P}$-algebra on a $\pi_0(\mathtt{O})$-collection A in a colax monoidal V-category $\mathtt{C}$ can be expressed in terms of an action defined as a collection of morphisms

$$\mathcal{P}(T)\otimes A^T \to A(i),\ T\in \mathtt{O},\ i=\pi_0(T),$$

satisfying the following conditions:

(i) The square

$$\begin{array}{ccc} I\otimes A^{U_c} & \xrightarrow{\ \cong\ } & A^{U_c} \\ \big\downarrow & & \big\downarrow{\scriptstyle \eta_c} \\ \mathcal{P}(U_c)\otimes A^{U_c} & \longrightarrow & A \end{array}$$

in which η_c is the counit of the comonad $D_{U_c}(-) = (-)^{U_c}$, commutes for any $c\in\pi_0(\mathtt{O})$.

(ii) For any morphism $f : T \to S$ in $\mathtt{O}$ with fibers $T_1,\ldots,T_k$, the following diagram in which $i = \pi_0(T)$ commutes

$$\begin{array}{ccc} \mathcal{P}(S)\otimes\mathcal{P}(T_1)\otimes\cdots\otimes\mathcal{P}(T_k)\otimes A^T & \longrightarrow & \mathcal{P}(S)\otimes\mathcal{P}(T_1)\otimes\cdots\otimes\mathcal{P}(T_k)\otimes D_S(A^{T_1},\ldots,A^{T_k}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{P}(T)\otimes A^T & & \mathcal{P}(S)\otimes D_S(\mathcal{P}(T_1)\otimes A^{T_1},\ldots,\mathcal{P}(T_k)\otimes A^{T_k}) \\ \big\downarrow & & \big\downarrow \\ A(i) & \longleftarrow & \mathcal{P}(S)\otimes A^S \end{array}$$

4. Convolution and condensation

The condensation described in this section creates, in a controlled manner, out of colored operads and their algebras, non-colored ones. The main statement of this section is Proposition 4.6.

Fix a finite set $\mathfrak{C}$ and consider the pullback $\mathtt{O}^{\mathfrak{C}}$ in (5). Objects of $\mathtt{O}^{\mathfrak{C}}$ can be interpreted as objects of $\mathtt{O}$ colored by elements of $\mathfrak{C}$. A typical object of $\mathtt{O}^{\mathfrak{C}}$ will therefore be denoted by $T(i_1,\ldots,i_k;i)$, where $k := |T|$ and $i_1,\ldots,i_k,i\in\mathfrak{C}$.

The operadic category $\mathtt{O}^{\mathfrak{C}}$ contains a full operadic subcategory $\mathtt{LO}^{\mathfrak{C}}$[5] whose objects are objects of $\mathtt{O}^{\mathfrak{C}}$ of the form $U_c(i;j)$, $i,j\in\mathfrak{C}$, $c\in\pi_0(\mathtt{O})$. It is easy to see that $\mathtt{LO}^{\mathfrak{C}}$-operads are precisely $\pi_0(\mathtt{O})$-families of V-enriched categories with the set of objects $\mathfrak{C}$. In other words, $\mathtt{Op}^{\mathtt{LO}^{\mathfrak{C}}}(V)\cong \mathtt{Cat}_{\mathfrak{C}}(V)^{\pi_0(\mathtt{O})}$, where $\mathtt{Cat}_{\mathfrak{C}}(V)$ denotes the category of V-categories with the set of objects $\mathfrak{C}$, and the set $\pi_0(\mathtt{O})$ is considered as a discrete category.

The inclusion $\iota : \mathtt{LO}^{\mathfrak{C}} \hookrightarrow \mathtt{O}^{\mathfrak{C}}$ induces the restriction functor $\iota^* : \mathtt{Op}^{\mathtt{O}^{\mathfrak{C}}}(V) \to \mathtt{Op}^{\mathtt{LO}^{\mathfrak{C}}}(V)$ between the categories of operads. We define the functor $\mathtt{U} : \mathtt{Op}^{\mathtt{O}^{\mathfrak{C}}}(V)\to\mathtt{Cat}(V)$ as the

[5] "L" abbreviating "linear."

composition

$$\mathtt{U}:\ \mathtt{Op}^{\mathtt{O}^{\mathfrak{C}}}(V) \xrightarrow{\iota^*} \mathtt{Op}^{\mathtt{LO}^{\mathfrak{C}}}(V) \cong \mathtt{Cat}_{\mathfrak{C}}(V)^{\pi_0(\mathtt{O})} \xrightarrow{\coprod_{c\in\pi_0(\mathtt{O})}} \mathtt{Cat}(V)\ ,$$

where the coproduct exists because V is cocomplete.

Definition 4.1. We will call the V-enriched category $\mathtt{U}(\mathcal{P})$ the *underlying category* of an $\mathtt{O}^{\mathfrak{C}}$-operad $\mathcal{P}$ in V.

Explicitly, the set of objects of $\mathtt{U}(\mathcal{P})$ is $\pi_0(\mathtt{O})\times\mathfrak{C}$, while the enriched hom-set between $(c,i)\in\pi_0(\mathtt{O})\times\mathfrak{C}$ and $(d,j)\in\pi_0(\mathtt{O})\times\mathfrak{C}$ is

$$\mathtt{U}(\mathcal{P})\big((c,i),(d,j)\big) = \begin{cases} \text{the initial object } 0 \text{ of } V, & \text{if } c\neq d, \text{ and} \\ \mathcal{P}\big(U_c(i;j)\big), & \text{otherwise.} \end{cases}$$

Since $U_c(i;i)$ are the chosen terminal objects of $\mathtt{O}^{\mathfrak{C}}$, we have the identity morphisms

$$I \to \mathtt{U}(\mathcal{P})\big((c,i),(c,i)\big),\ (c,i)\in\pi_0(\mathtt{O})\times\mathfrak{C},$$

given by the operad units of $\mathcal{P}$. The composition in $\mathtt{U}(\mathcal{P})$ is induced by the (unique) maps between objects of the form $U_c(i;j)$, $i,j\in\mathfrak{C}$, $c\in\pi_0(\mathtt{O})$.

Let $\mathtt{C}$ be a colax $\mathtt{O}$-monoidal V-category with a multicotensor D as in Definition 3.8. We are going to define the convolution product on the category $\mathtt{C}^{\mathtt{U}(\mathcal{P})}$. Assume first for simplicity that $\mathtt{O}$ is connected so that the objects of $\mathtt{U}(\mathcal{P})$ are elements of $\mathfrak{C}$. For any $T\in\mathtt{O}$, $k=|T|$, the operad $\mathcal{P}$ generates a V-functor

$$\mathcal{P}(T): \underbrace{\mathtt{U}(\mathcal{P})^{op}\otimes\cdots\otimes\mathtt{U}(\mathcal{P})^{op}}_{k-times}\otimes\mathtt{U}(\mathcal{P})\to V$$

defined on objects $i_1,\ldots,i_k,i\in(\mathtt{U}(\mathcal{P})^{op})^{\otimes k}\otimes\mathtt{U}(\mathcal{P})$ by

$$\mathcal{P}(T)(i_1,\ldots,i_k;i) := \mathcal{P}\big(T(i_1,\ldots,i_k;i)\big). \tag{18}$$

To see how $\mathcal{P}$ acts on morphisms, we observe that in $\mathtt{O}^{\mathfrak{C}}$ we have morphisms of the form

$$T(i_1,\ldots,i_k;i)\longrightarrow T(j_1,\ldots,j_k;i),\ i_1,\ldots,i_k,j_1,\ldots,j_k,i\in\mathfrak{C},$$

whose underlying morphism in $\mathtt{O}$ is the identity of T. The multiplication in the $\mathtt{O}^{\mathfrak{C}}$-operad $\mathcal{P}$ with respect to these morphisms induces the contravariant part of the functor $\mathcal{P}(T)$. The covariant part is induced by the multiplication in $\mathcal{P}$ corresponding to the morphisms in $\mathtt{O}^{\mathfrak{C}}$ of the form

$$T(i_1,\ldots,i_k;i)\to U_c(n;i),\ i_1,\ldots,i_k,i,n\in\mathfrak{C}.$$

For T as above and $X_1,\ldots,X_k\in\mathtt{C}^{\mathtt{U}(\mathcal{P})}$, the *convolution product* ${}^{D}E^{\mathcal{P}}_T(X_1,\ldots,X_k)\in\mathtt{C}^{\mathtt{U}(\mathcal{P})}$ is given by the coend in $\mathtt{C}$:

$${}^{D}E^{\mathcal{P}}_T(X_1,\ldots,X_k)(i) = \int^{\mathtt{U}(\mathcal{P})^{\otimes n}} \mathcal{P}(T)(i_1,\ldots,i_k;i)\otimes D_T\big(X_1(i_1),\ldots,X_k(i_k)\big),\ i\in\mathtt{U}(\mathcal{P}). \tag{19}$$

The above constructions easily extend to the case of an arbitrary $\pi_0(\mathtt{O})$, we just take the products of the corresponding constructions for the individual connected components.

The convolution (19) generalizes the Day-Street convolution product [18] of a substitude related to an ordinary (that is $\mathtt{sFSet}$-) operad. If $\mathtt{C}$ is a symmetric monoidal V-category and $D = \bigodot^{\mathtt{O}}$ is the $\mathtt{O}$-multicotensor from Example 3.5, many standard facts about the convolution remain valid in this generalized situation. For example we have

Proposition 4.2. *The convolution product determines an* $\mathtt{O}$*-multitensor* ${}^{\odot}E^{\mathcal{P}} = \{{}^{\odot}E^{\mathcal{P}}_T\}_{T\in\mathtt{O}}$ *on* $\mathtt{C}^{\mathtt{U}(\mathcal{P})}$ *whose unital part* $\prod_{c\in\pi_0(\mathtt{O})} {}^{\odot}E^{\mathcal{P}}_{U_c}$ *is the identity monad.*

Remark 4.3. The $\mathtt{O}$-multitensor structure of ${}^{\odot}E^{\mathcal{P}}$ is induced by the $\mathtt{O}$-multitensor structure of the $\mathtt{O}$-multicotensor $\bigodot^{\mathtt{O}}$. The fact that $\bigodot^{\mathtt{O}}$ is both a multicotensor *and* a multitensor makes this situation very special – the convolution product (19) need not be a $\mathtt{O}$-multitensor for general D. To simplify the notation, we will sometimes write $E^{\mathcal{P}}$ instead of ${}^{\odot}E^{\mathcal{P}}$.

An $\mathtt{O}$-multicotensor D on $\mathtt{C}$ induces an $\mathtt{O}^{\mathfrak{C}}$-multicotensor $D^{\mathfrak{C}}$ on $\mathtt{C}$ by 'forgetting the colors'

$$D^{\mathfrak{C}}_{T(i_1,\ldots,i_k;i)} := D_T,\ T(i_1,\ldots,i_k;i) \in \mathtt{O}^{\mathfrak{C}} \tag{20}$$

which in turn restricts to an $\mathtt{LO}^{\mathfrak{C}}$-multicotensor $LD^{\mathfrak{C}}$. Since, by assumption, D_{U_c} is the identity comonad for each $c \in \pi_0(\mathtt{O})$, the colax unitality of multicotensors gives natural morphisms

$$LD^{\mathfrak{C}}_{U_c(i;j)}(X) = D_{U_c}(X) \stackrel{\cong}{\longrightarrow} X,\ U_c(i;j) \in \mathtt{LO}^{\mathfrak{C}}. \tag{21}$$

Let now A be an algebra of an $\mathtt{O}^{\mathfrak{C}}$-operad $\mathcal{P}$ in $\mathtt{C}$ as in Definition 3.6. The action $\alpha : \mathcal{P} \to \mathcal{E}nd^{D^{\mathfrak{C}}}_A$ induces, via the restriction along the inclusion $\iota : \mathtt{LO}^{\mathfrak{C}} \hookrightarrow \mathtt{O}^{\mathfrak{C}}$ combined with the canonical transformations (21), an action

$$\iota^*(\alpha) : \iota^*(\mathcal{P}) \longrightarrow \iota^*(\mathcal{E}nd^{D^{\mathfrak{C}}}_A) = \mathcal{E}nd^{LD^{\mathfrak{C}}}_A \longrightarrow \mathcal{E}nd^{id}_A,$$

where id is the obvious identity $\mathtt{LO}^{\mathfrak{C}}$-multitensor on $\mathtt{C}$. This gives rise to a functor

$$\iota^* : \mathtt{Alg}_{\mathcal{P}}(\mathtt{C}) \to \mathtt{C}^{\mathtt{U}(\mathcal{P})} \tag{22}$$

from the category of $\mathcal{P}$-algebras in $\mathtt{C}$ to the functor category $\mathtt{C}^{\mathtt{U}(\mathcal{P})}$. If $\mathtt{O}$ is connected, the algebra A is a collection $\{A(i)\}_{i\in\mathfrak{C}}$ in V, the functor $\iota^*(\alpha)$ takes $i \in \mathfrak{C}$ to $A(i)$, and the map

$$\iota^*(\alpha) : \mathcal{P}\big(U(i;j)\big) \to \mathtt{C}\big(A(i), A(j)\big)$$

of the enriched hom-sets is given by the $\mathcal{P}$-algebra structure of A. The description of $\iota^*(\alpha)$ for a general $\pi_0(\mathtt{O})$ is similar.

In the particular case when $\mathtt{C}$ is a symmetric monoidal V-category, $E^{\mathcal{P}} = {}^{\odot}E^{\mathcal{P}}$ is a multitensor on $\mathtt{C}^{\mathtt{U}(\mathcal{P})}$ by Proposition 4.2. It turns out that $\iota^*(A)$ is an algebra over $E^{\mathcal{P}}$ in the sense of the following definition.

Definition 4.4. An *algebra over an* $\mathtt{O}$*-multitensor* E on a V-category $\mathtt{C}$ is an object $Z \in \mathtt{C}$ equipped with a family of morphisms

$$\alpha_T : E_T(Z, \dots, Z) \to Z,\ T \in \mathtt{O},$$

such that

(i) the composition $Z \xrightarrow{\eta_c} E_{U_c}(Z) \xrightarrow{\alpha_{U_c}} Z$ is the identity for each $c \in \pi_0(\mathtt{O})$, and
(ii) the diagram

$$\begin{array}{ccc} E_S\big(E_{T_1}(Z,\dots,Z),\dots,E_{T_k}(Z,\dots,Z)\big) & \xrightarrow{\mu_\sigma} & E_T(Z,\dots,Z) \\ {\scriptstyle E_S(\alpha_{T_1},\dots,\alpha_{T_k})}\downarrow & & \downarrow{\scriptstyle \alpha_T} \\ E_S(Z,\dots,Z) & \xrightarrow{\alpha_S} & Z \end{array}$$

commutes for any morphism $\sigma : T \to S$ in $\mathtt{O}$ with fibers $T_1, \dots, T_k$.

As in the classical case [7, Proposition 1.8] we obtain

Proposition 4.5. *The functor* $\iota^* : \mathtt{Alg}_{\mathcal{P}}(\mathtt{C}) \to \mathtt{C}^{\mathtt{U}(\mathcal{P})}$ *induces an isomorphism between the category of* $\mathcal{P}$*-algebras in* $\mathtt{C}$ *and the category of* $E^{\mathcal{P}}$*-algebras in* $\mathtt{C}^{\mathtt{U}(\mathcal{P})}$.

Let us return to a general colax $\mathtt{O}$-monoidal V-category $\mathtt{C}$. Assume that, in addition, $\mathtt{C}$ is complete as a V-category, so we have cotensors $Y^\alpha \in \mathtt{C}$ such that

$$\mathtt{C}(X, Y^\alpha) \cong V\big(\alpha, \mathtt{C}(X,Y)\big) \ \text{ for } \ X, Y \in \mathtt{C},\ \alpha \in V.$$

For an $\mathcal{P}$-algebra A in $\mathtt{C}$ and an object δ of the functor category $V^{\mathtt{U}(\mathcal{P})}$ we define the δ-*totalization* of A as the end

$$Tot_\delta(A) := \int_{i \in \mathtt{U}(\mathcal{P})} \big(\iota^*(A)(i)\big)^{\delta(i)}. \tag{23}$$

Since $E^{\mathcal{P}} = {}^{\odot}E^{\mathcal{P}}$ is a multitensor, the $\mathtt{O}$-collection $co\mathcal{E}nd_\delta^{\mathcal{P}}$ with

$$co\mathcal{E}nd_\delta^{\mathcal{P}}(T) := \int_{i \in \mathtt{U}(\mathcal{P})} V\big(\delta(i), E_T^{\mathcal{P}}(\delta, \dots, \delta)(i)\big)$$

is an $\mathtt{O}$-operad in V by Lemma 3.2. The main statement of this section reads:

Proposition 4.6. *Let* $\mathcal{P}$ *be an* $\mathtt{O}^{\mathfrak{C}}$*-operad in* V *and* A *a* $\mathcal{P}$*-algebra in* $\mathtt{C}$*. Then the* $\pi_0(\mathtt{O})$*-collection* $Tot_\delta(A)$ *is a natural* $co\mathcal{E}nd_\delta^{\mathcal{P}}$*-algebra.*

Definition 4.7. We will call the $\mathtt{O}$-operad $co\mathcal{E}nd_\delta^{\mathcal{P}}$ the δ-*condensation* of $\mathcal{P}$ and its algebra $Tot_\delta(A)$ the δ-*totalization* of A.

Remark 4.8. If $\mathtt{C} = V$, the δ-totalization (23) can be given by a simplified formula

$$Tot_\delta(A) := \int_{i \in \mathtt{U}(\mathcal{P})} V\big(\delta(i), \iota^*(A)(i)\big).$$

Proposition 4.6 in this case can be proved using Proposition 4.5 by a straightforward generalization of [7, Proposition 1.5], see also the appendix to [8].

Proof of Proposition 4.6. We will assume for simplicity that $\mathtt{O}$ is connected, the general case can be handled similarly. The action

$$coEnd^{\mathcal{P}}_{\delta}(T) \otimes Tot_{\delta}(A)^T \longrightarrow Tot_{\delta}(A),\ T \in \mathtt{O}, \tag{24}$$

where $Tot_{\delta}(A)^T = D_T\big(Tot_{\delta}(A), \ldots, Tot_{\delta}(A)\big)$, will be constructed using a natural morphism

$$\mho_T : Tot_{\delta}(A)^T \longrightarrow \int_{i \in \mathtt{U}(\mathcal{P})} {}^{D}E^{\mathcal{P}}_T(\iota^* A, \ldots, \iota^* A)(i)^{E^{\mathcal{P}}_T(\delta,\ldots,\delta)(i)}. \tag{25}$$

To simplify the notation, we will implicitly assume that the symbols $i_1, \ldots, i_n,\ j_1, \ldots, j_n$ and i denote objects of the underlying category $\mathtt{U}(\mathcal{P})$. We will also drop ι^* from the notation, writing A instead of $\iota^* A$. Notice that the target of (25) is equal to

$$\int_i \int_{i_1,\ldots,i_n} {}^{D}E^{\mathcal{P}}_T(A, \ldots, A)(i)^{\mathcal{P}(T)(i_1,\ldots,i_n;i) \otimes \delta(i_1) \otimes \ldots \otimes \delta(i_n)},\ n = |T|,$$

so the morphism (25) is the same as a family of morphisms

$$Tot_{\delta}(A)^T \to {}^{D}E^{\mathcal{P}}_T(A, \ldots, A)(i)^{\mathcal{P}(T)(i_1,\ldots,i_n;i) \otimes \delta(i_1) \otimes \ldots \otimes \delta(i_n)},$$

which satisfy some obvious naturality conditions. By adjunction, this amounts to a family of morphisms

$$\begin{aligned} &\mathcal{P}(T)(i_1, \ldots, i_n; i) \otimes \delta(i_1) \otimes \cdots \otimes \delta(i_n) \otimes Tot_{\delta}(A)^T \\ &\qquad \longrightarrow \mathcal{P}(T)(i_1, \ldots, i_n; i) \otimes D_T\Big(\delta(i_1) \otimes \int_{j_1} A(j_1)^{\delta(j_1)}, \ldots, \delta(i_n) \otimes \int_{j_n} A(j_n)^{\delta(j_n)}\Big) \\ &\qquad \longrightarrow \int^{i_1,\ldots i_n} \mathcal{P}(T)(i_1, \ldots, i_n; i) \otimes D_T\big(A(i_1), \ldots, A(i_n)\big). \end{aligned}$$

We also have, for each $1 \leq k \leq n$, the evaluation morphisms

$$\delta(i_k) \otimes \int_{j_k} A(j_k)^{\delta(j_k)} \longrightarrow A(i_k)$$

which induce a map

$$\begin{aligned} \mathcal{P}(T)(i_1, \ldots, i_n; i) \otimes D_T\Big(\delta(i_1) \otimes \int_{j_1} A(j_1)^{\delta(j_1)}, \ldots, \delta(i_n) \otimes \int_{j_n} A(j_n)^{\delta(j_n)}\Big)& \\ \longrightarrow \mathcal{P}(T)(i_1, \ldots, i_n; i) \otimes D_T\big(A(i_1), \ldots, A(i_n)\big)&. \end{aligned}$$

Composing this map with the canonical coprojection to the coend

$$\begin{aligned} &\mathcal{P}(T)(i_1, \ldots, i_n; i) \otimes D_T\big(A(i_1), \ldots, A(i_n)\big) \\ &\qquad \longrightarrow \int^{i_1,\ldots,i_n} \mathcal{P}(T)(i_1, \ldots, i_n; i) \otimes D_T\big(A(i_1), \ldots, A(i_n)\big) \end{aligned}$$

we obtain the required morphism (25). The necessary naturality conditions for the above construction is obvious.

Let us finally explain how the map $\mho_T$ in (25) leads to the action (24). To this end, observe that a $\mathcal{P}$-algebra structure on A generates a morphism

$$
{}^{D}E_T^{\mathcal{P}}(A,\ldots,A)(i) = \int^{i_1,\ldots,i_n} \mathcal{P}(T)(i_1,\ldots,i_n;i) \otimes D_T\big(A(i_1),\ldots,A(i_n)\big) \to A(i). \tag{26}
$$

The action (24 is then the composition

$$
\begin{aligned}
co\mathcal{E}nd_\delta^{\mathcal{P}}(T) \otimes Tot_\delta(A)^T = \Big(\int_i V\big(\delta(i), E_T^{\mathcal{P}}(\delta,\ldots,\delta)(i)\big)\Big) \otimes Tot_\delta(A)^T \xrightarrow{id\otimes\mho_T} \\
\int_i V\big(\delta(i), E_T^{\mathcal{P}}(\delta,\ldots,\delta)(i)\big) \otimes \int_i {}^{D}E_T^{\mathcal{P}}(A,\ldots,A)(i)^{E_T^{\mathcal{P}}(\delta,\ldots,\delta)(i)} \longrightarrow \\
\int_i {}^{D}E_T^{\mathcal{P}}(A,\ldots,A)(i)^{\delta(i)} \longrightarrow \int_i A(i)^{\delta(i)} = Tot_\delta(A)
\end{aligned}
$$

whose last arrow is generated by (26). We leave a straightforward but tedious verification that this composition is, indeed, a $\mathcal{P}$-algebra structure to the reader. □

Part 2. Duoidal Deligne's conjecture

Reminders

In this part we work with many particular examples of operadic categories and their operads. We included Table 1 to simplify the reader's navigation through them. We also briefly recall, following [9] closely, duoidal categories and the necessary related notions.

A *duoidal V-category* is a pseudomonoid in the 2-category of monoidal V-categories, lax-monoidal V-functors and their monoidal V-transformations. Explicitly, a duoidal V-category is a quintuple $\mathscr{D} = (\mathscr{D}, \square_0, \square_1, e, v)$ such that

(i) $(\mathscr{D}, \square_0, e)$ and $(\mathscr{D}, \square_1, v)$ are monoidal V-categories, equipped with
(ii) a V-natural interchange transformation

$$
(X \square_1 Y) \square_0 (Z \square_1 W) \to (X \square_0 Z) \square_1 (Y \square_0 W), \tag{27}
$$

(iii) a map $e \to e \square_1 e$,
(iv) a map $v \square_0 v \to v$, and
(v) a map $e \to v$.

The above data should enjoy the coherence properties listed e.g. in [9, p. 1816]. Moreover, we require that v is a monoid in $(\mathscr{D}, \square_0, e)$ and e a comonoid in $(\mathscr{D}, \square_1, v)$.

A duoidal category $\mathscr{D}$ is called *strict* if both monoidal categories $(\mathscr{D}, \square_0, e)$ and $(\mathscr{D}, \square_1, v)$ are strict monoidal categories. Since every duoidal category is equivalent to a strict one by [9, Theorem 2.16], we assume that all duoidal categories in this article are strict.

A 1-*operad* in a duoidal category $\mathscr{D} = (\mathscr{D}, \square_0, \square_1, e, v)$ is a collection $\mathscr{A} = \{\mathscr{A}(n)\}_{n\geq 0}$ of objects of $\mathscr{D}$ such that

Categories

Name:	type of objects:	introduced in:	typical object:
Ω_k	Batanin's k-trees	Section 1.1	$\mathcal{T}, \mathcal{S}, \ldots$
$\Omega_k^{\mathbb{N}}$	$\mathbb{N}$-colored k-trees	Section 5.1	$\boldsymbol{\mathcal{T}} = (\mathcal{T}, c),\ \boldsymbol{\mathcal{S}} = (\mathcal{S}, c)$
$\mathtt{Ord}_2$	2-ordinals	Section 1.2	$\mathcal{O}, \mathcal{N}$
$\mathtt{Ord}_2^{\mathbb{N}}$	$\mathbb{N}$-colored 2-ordinals	Section 5.1	$\boldsymbol{\mathcal{O}} = (\mathcal{O}, c),\ \boldsymbol{\mathcal{N}} = (\mathcal{N}, c)$
$\mathtt{LTr}$	trees with levels	Section 5.2	$\beta, \alpha, \ldots$
$\mathtt{Tam}_2^{\mathbb{N}}$	$\mathtt{Ord}_2^{\mathbb{N}}$-labelled trees	Section 6	$(\boldsymbol{\mathcal{O}}, \delta),\ (\boldsymbol{\mathcal{N}}, \gamma)$
$\mathtt{Tm}_2^{\mathbb{N}}$	$\Omega_2^{\mathbb{N}}$-labelled trees	Section 6	$(\boldsymbol{\mathcal{T}}, \delta),\ (\boldsymbol{\mathcal{S}}, \gamma)$
$\widetilde{\mathtt{Tam}}_2^{\mathbb{N}}$	trees with labeled $\circ$-vertices	Proof of Lemma 7.5	ζ
$\widetilde{\mathtt{Tm}}_2^{\mathbb{N}}$	trees with labeled $\circ$-vertices	Proof of Lemma 7.5	ξ

Operads:

Name:	type of operad:	introduced in:	typical element:
$\mathcal{L}^{(2)}$	$\mathbb{N}$-colored Σ-operad	Section 6	tree $\delta, \gamma, \ldots$
$\mathcal{T}m_2^{\mathbb{N}}$	$\mathbb{N}$-colored 2-operad	Definition 6.1	$\Omega_2^{\mathbb{N}}$-labelled tree $(\boldsymbol{\mathcal{T}}, \delta)$
$\mathcal{T}am_2^{\mathbb{N}}$	pruned $\mathbb{N}$-colored 2-operad	pullback (54)	$\mathtt{Ord}_2^{\mathbb{N}}$-labelled tree $(\boldsymbol{\mathcal{O}}, \delta)$

TABLE 1. Notation.

(i) for each integers $n \geq 1$, $k_1, \ldots, k_n \geq 0$, one is given a structure morphism

$$\gamma : \big(\mathscr{A}(k_1) \,\square_1 \cdots \square_1 \mathscr{A}(k_n)\big) \,\square_0 \mathscr{A}(n) \to \mathscr{A}(k_1 + \cdots + k_n), \tag{28}$$

(ii) one is given a map $j : e \to \mathscr{A}(1)$ (the unit) and

(iii) a left v-module structure $v \,\square_0 \mathscr{A}(0) \to \mathscr{A}(0)$

such that appropriate axioms are satisfied, see [9, p. 1825] for details. An example is the operad $\underline{\mathscr{A}ss}$ with $\underline{\mathscr{A}ss}(n) = v$ for each $n \in \mathbb{N}$ [9, Example 4.4]. Recall finally that a *multiplicative* 1-*operad* is a 1-operad $\mathscr{A}$ equipped with an operad morphism

$$\alpha : \underline{\mathscr{A}ss} \to \mathscr{A}. \tag{29}$$

For a duoidal category $\mathscr{D}$ we denote by $\mathtt{Cat}(\mathscr{D})$ the 2-category of $(\mathscr{D}, \square_0, e)$-enriched categories. As observed by Forcey [20], $\mathtt{Cat}(\mathscr{D})$ has a monoidal structure. The tensor product $\times_1$ of two $\mathscr{D}$-categories $\mathscr{K}$ and $\mathscr{L}$ is given by the cartesian product on the objects level while

$$(\mathscr{K} \times_1 \mathscr{L})\big((X,Y),(Z,W)\big) := \mathscr{K}(X,Z) \,\square_1 \mathscr{L}(Y,W),\ \text{for } X, Z \in \mathscr{K},\ Y, W \in \mathscr{L}.$$

The unit for this tensor product is the category $\mathbf{1}_v$ which has one object $*$ and $\mathbf{1}_v(*,*) = v$.

A *monoidal $\mathscr{D}$-category* $\mathscr{K} = (\mathscr{K}, \odot, \eta)$ is defined as a pseudomonoid in the monoidal 2-category $(\mathtt{Cat}(\mathscr{D}), \times_1, \mathbf{1}_v)$, see [9, pp. 1820–21] for a detailed description of this structure. Each object $X \in \mathscr{K}$ has its *endomorphism* 1-*operad* $\mathcal{E}nd_X$ in $\mathscr{D}$ with components

$$\mathcal{E}nd_X(n) := \mathscr{K}(\textstyle\bigodot^n X, X),\ n \geq 1,$$

see [9, Def. 4.7].

A *monoid in* $\mathscr{K}$ is a lax monoidal functor $\mathbf{1}_v \to \mathscr{K}$. More explicitly, a monoid in $\mathscr{K}$ is an object $\mathsf{M} \in \mathscr{K}$ together with:

(i) a morphism (neutral element) $i : \eta \to \mathsf{M}$,
(ii) a morphism (multiplication) $m : \mathsf{M} \odot \mathsf{M} \to \mathsf{M}$ and
(iii) a morphism (the *unit*) $u : v \to \mathscr{K}(\mathsf{M}, \mathsf{M})$ in $\mathscr{D}$.

These data should satisfy axioms listed in [9, p. 1823]. The endomorphism 1-operad $\mathcal{E}nd_{\mathsf{M}}$ of a monoid $\mathsf{M} \in \mathscr{K}$ is multiplicative by [9, Prop. 4.9].

The *center* of a monoid M is the following equalizer in $\mathscr{D}$:

$$Z(\mathsf{M}) \to \mathscr{K}(\eta, \mathsf{M}) \rightrightarrows \mathscr{K}(\mathsf{M}, \mathsf{M}). \tag{30}$$

The center has a natural structure of a *duoid* (*double monoid* in the terminology of [1]) in $\mathscr{D}$. This is, by definition, an object $\mathsf{D} \in \mathscr{D}$ together with

(i) a structure of a monoid $\mathsf{D} \,\Box_0 \mathsf{D} \to \mathsf{D}$, $e \to \mathsf{D}$ with respect to the first monoidal structure of $\mathscr{D}$, and
(ii) a structure of a monoid $\mathsf{D} \,\Box_1 \mathsf{D} \to \mathsf{D}$, $v \to \mathsf{D}$ with respect to the second monoidal structure of $\mathscr{D}$

such that suitable axioms listed in [9, p. 1818] are satisfied.

5. Operadic categories of trees and ordinals.

In this section we introduce several operadic categories of colored trees and ordinals, and study various functors between them. We also define endomorphism operads of collections in duoidal categories and prove, in Theorem 5.4, the existence of canonical actions on multiplicative 1-operads.

5.1. **The category of coloured 2-trees.** We will need the $\mathbb{N}$-colored version $\Omega_k^{\mathbb{N}}$ of the category Ω of k-trees recalled in Section 1.1. It is constructed by taking $\mathtt{O} = \Omega_k$ and $\mathfrak{C} = \mathbb{N}$ in the pullback (5). Explicitly, an $\mathbb{N}$-*colored k-tree* is a couple $\boldsymbol{\mathcal{T}} = (\mathcal{T}, c)$ consisting of a k-tree $\mathcal{T} \in \Omega_k$ and of a coloring $c : Lf_k(\mathcal{T})+1 \to \mathbb{N}$ of its set of k-leaves plus one more color $c(1)$ interpreted as the 'output' color of $\mathcal{T}$. Morphisms $(\mathcal{T}, c') \to (\mathcal{S}, c'')$ are morphisms $\mathcal{T} \to \mathcal{S}$ of the underlying k-trees if $c'(1) = c''(1)$, while there are no morphisms if $c'(1) \neq c''(1)$. This explicit description follows the ideas of Tamarkin's [30].

The category $\Omega_k^{\mathbb{N}}$ with $|\boldsymbol{\mathcal{T}}| := Lf_k(\mathcal{T})$ is operadic. The fiber $\boldsymbol{\mathcal{T}}_i$ of a map $f : \boldsymbol{\mathcal{T}} = (\mathcal{T}, c') \to \boldsymbol{\mathcal{S}} = (\mathcal{S}, c'')$ over a k-leaf $i \in |\mathcal{S}|$ is the fiber as in (11), with the coloring of its k-leaves induced by c' and the output color $c''(i)$. One has $\pi_0(\Omega_k^{\mathbb{N}}) \cong \mathbb{N}$, the chosen terminal object for $n \in \mathbb{N}$ being the terminal k-tree $\boldsymbol{\mathcal{U}}_k^n$ with its unique k-leaf colored by n and the output color n.

One analogously defines an $\mathbb{N}$-colored version $\mathtt{Ord}_k^{\mathbb{N}}$ of the operadic category $\mathtt{Ord}_k$ of k-ordinals of Example 1.2. We leave its detailed description to the reader. One has $\pi_0(\mathtt{Ord}_k^{\mathbb{N}}) \cong \mathbb{N}$, the chosen terminal object for $n \in \mathbb{N}$ being the terminal k-ordinal $\mathbf{1}_k^n$ colored by n, with the output color n.

In the rest of this article we will however need the categories Ω_k and $\mathtt{Ord}_k$, resp. their colored versions $\Omega_k^{\mathbb{N}}$ and $\mathtt{Ord}_k^{\mathbb{N}}$, only for $k \leq 2$.

Definition 5.1. We will call Ω_2-operads, resp. $\Omega_2^{\mathbb{N}}$-operads, *2-operads*, resp. $\mathbb{N}$-*colored 2-operads*. Similarly, $\mathtt{Ord}_2$-operads, resp. $\mathtt{Ord}_2^{\mathbb{N}}$-operads, will be called *pruned 2-operads*, resp. *pruned* $\mathbb{N}$-*colored 2-operads*.

Proposition 5.2. *There is a natural Ω_2-multicotensor $\square$ on any duoidal V-category $\mathscr{D} = (\mathscr{D}, \square_0, \square_1, e, v)$. If $\mathscr{D}$ is a cocomplete V-category then the multicotensor $\square$ is strong.*

In particular, each $\mathbb{N}$-colored collection $\mathscr{E} = \{\mathscr{E}(n)\}_{n \geq 0}$ of objects in $\mathscr{D}$ admits its $\mathbb{N}$-colored endomorphism 2-operad $\mathcal{E}nd_{\mathscr{E}}^{\Omega_2^{\mathbb{N}}} := \mathcal{E}nd_{\mathscr{E}}^{\square^{\mathbb{N}}} \in \mathtt{Op}^{\Omega_2^{\mathbb{N}}}(V)$.

Proof. The proof is based on a simple modification of a construction given in [9, p. 1853]. Let $\mathcal{T} \in \Omega_2$ be a 2-tree. Let the 1-truncation of $\mathcal{T}$ be $\{1, \ldots, t\}$, with its set of 2-leaves over the 1-vertex $d \in \{1, \ldots, t\}$ being $\{v_1^d, \ldots, v_{q_d}^d\}$, and let $\mathcal{X} = \{X_{c,i}\}_{1 \leq d \leq t,\ 1 \leq i \leq q_d}$ be a family of objects in $\mathscr{D}$.

We then define, for $1 \leq d \leq t$,

$$\mathcal{X}_{\mathcal{T}}^{\square_1^{q_d}} := \begin{cases} X_{d,1} \square_1 \cdots \square_1 X_{d,q_d}, & \text{if } q_d > 1 \\ v, & \text{if } q_d = 0. \end{cases} \tag{31}$$

With this notation,

$$\square_{\mathcal{T}}(X_{1,1}, \ldots, X_{t,q_t}) := \mathcal{X}_{\mathcal{T}}^{\square_1^{q_t}} \square_0 \mathcal{X}_{\mathcal{T}}^{\square_1^{q_{t-1}}} \square_0 \cdots \square_0 \mathcal{X}_{\mathcal{T}}^{\square_1^{q_2}} \square_0 \mathcal{X}_{\mathcal{T}}^{\square_1^{q_1}}.$$ [6]

[6] Observe the reversed order of the factors.

The counit of the multicotensor $\square$ is the identity. The comultiplication

$$\mu_\sigma : \ \square_{\mathcal{T}} \longrightarrow \square_{\mathcal{S}}(\square_{\mathcal{T}_1}, \dots, \square_{\mathcal{T}_k}) \tag{32}$$

corresponding to a morphism $\sigma : \mathcal{T} \to \mathcal{S}$ in Ω_2 with fibers $\mathcal{T}_1, \dots, \mathcal{T}_k$ can be described by induction. We do not provide the details here but refer to the construction of the morphism X^σ in the proof of [9, Lemma 11.13] which is exactly the comultiplication μ_σ in the case when all members of the family $\mathcal{X}$ are equal to the same object X. The argument however does not depend on this difference. We will repeat the same kind of construction in the description of the endomorphism 2-operad $\mathcal{E}nd_{\mathscr{E}}^{\Omega_2^{\mathbb{N}}}$ in the second part of this proof.

The proof that $\square$ is a strong multitensor also goes by an induction following the construction of the comultiplication, using the fact that both $\square_0$, $\square_1$ are V-functors and the interchange morphism (27) is a V-natural transformation. We leave the details to the reader.

As shown in (20), the Ω_2-multicotensor $\square$ induces an $\Omega_2^{\mathbb{N}}$-multicotensor $\square^{\mathbb{N}}$. The $\mathbb{N}$-colored 2-operad $\mathcal{E}nd_{\mathscr{E}}^{\Omega_2^{\mathbb{N}}} \in \mathtt{Op}^{\Omega_2^{\mathbb{N}}}(V)$ of an $\mathbb{N}$-collection $\mathscr{E} = \{\mathscr{E}(n)\}_{n\geq 0}$ in $\mathscr{D}$ is the endomorphism operad related to this multicotensor as in Lemma 3.3. We describe it in detail because we will need this description later.

We start by associating, to each $\mathbb{N}$-colored 2-tree $\boldsymbol{\mathcal{T}} = (\mathcal{T}, c) \in \Omega_2^{\mathbb{N}}$, the $\square$-*power* $\mathscr{E}^{\boldsymbol{\mathcal{T}}}$ of $\mathscr{E}$ as follows. Let the 1-truncation of the underlying 2-tree $\mathcal{T}$ be $\{1, \dots, t\}$, with its set of 2-leaves over the 1-vertex $d \in \{1, \dots, t\}$ being $\{v_1^d, \dots, v_{q_d}^d\}$.

We then define, for $1 \leq d \leq t$,

$$\mathscr{E}_{\boldsymbol{\mathcal{T}}}^{\square_1^{q_d}} := \begin{cases} \mathscr{E}\big(c(v_1^d)\big) \,\square_1 \cdots \square_1 \mathscr{E}\big(c(v_{q_d}^d)\big), & \text{if } q_d > 1 \\ v, & \text{if } q_d = 0. \end{cases} \tag{33}$$

With this notation,

$$\mathscr{E}^{\boldsymbol{\mathcal{T}}} := \mathscr{E}_{\boldsymbol{\mathcal{T}}}^{\square_1^{q_t}} \square_0 \mathscr{E}_{\boldsymbol{\mathcal{T}}}^{\square_1^{q_{t-1}}} \square_0 \cdots \square_0 \mathscr{E}_{\boldsymbol{\mathcal{T}}}^{\square_1^{q_2}} \square_0 \mathscr{E}_{\boldsymbol{\mathcal{T}}}^{\square_1^{q_1}}.$$

We believe that the portrait of $\mathscr{E}^{\boldsymbol{\mathcal{T}}}$ in Figure 1 borrowed from [9] clarifies our definition. Define finally

$$\mathcal{E}nd_{\mathscr{E}}^{\Omega_2^{\mathbb{N}}}(\boldsymbol{\mathcal{T}}) := \mathscr{D}\big(\mathscr{E}^{\boldsymbol{\mathcal{T}}}, \mathscr{E}(n)\big),$$

where n is the output color of $\boldsymbol{\mathcal{T}}$. Let us describe the operad multiplication

$$\mu(f) : \ \mathcal{E}nd_{\mathscr{E}}^{\Omega_2^{\mathbb{N}}}(\boldsymbol{\mathcal{T}}_1) \otimes \cdots \otimes \mathcal{E}nd_{\mathscr{E}}^{\Omega_2^{\mathbb{N}}}(\boldsymbol{\mathcal{T}}_k) \otimes \mathcal{E}nd_{\mathscr{E}}^{\Omega_2^{\mathbb{N}}}(\boldsymbol{\mathcal{S}}) \longrightarrow \mathcal{E}nd_{\mathscr{E}}^{\Omega_2^{\mathbb{N}}}(\boldsymbol{\mathcal{T}}) \tag{34}$$

corresponding to a morphism $f : \boldsymbol{\mathcal{T}} \to \boldsymbol{\mathcal{S}}$ in $\Omega_2^{\mathbb{N}}$ with fibers $\boldsymbol{\mathcal{T}}_1, \dots, \boldsymbol{\mathcal{T}}_k$. For

$$\phi : \mathscr{E}^{\boldsymbol{\mathcal{S}}} \to \mathscr{E}(n) \in \mathcal{E}nd_{\mathscr{E}}^{\Omega_2^{\mathbb{N}}}(\boldsymbol{\mathcal{S}}) \ \text{ and } \ \phi_i : \mathscr{E}^{\boldsymbol{\mathcal{T}}_i} \to \mathscr{E}(n_i) \in \mathcal{E}nd_{\mathscr{E}}^{\Omega_2^{\mathbb{N}}}(\boldsymbol{\mathcal{T}}_i), \ i \in |\boldsymbol{\mathcal{S}}|,$$

we use an auxiliary natural morphism

$$\Phi^f(\phi_1, \dots, \phi_k) : \mathscr{E}^{\boldsymbol{\mathcal{T}}} \longrightarrow \mathscr{E}^{\boldsymbol{\mathcal{S}}} \tag{35}$$

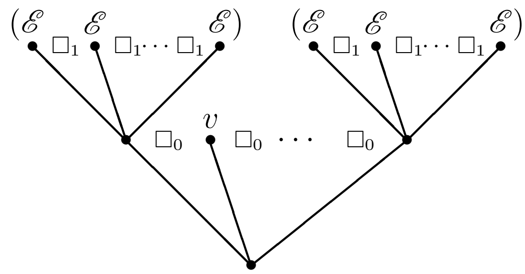


FIGURE 1. An 'ideological' picture of $\mathscr{E}^{\mathfrak{T}}$. Leaves of height 2 (resp. 1) are decorated by $\mathscr{E} = \{\mathscr{E}(n)\}_{n \geq 0}$ (resp. v). The decorations of vertices of height 2 (resp. 1) are then multiplied by $\square_1$ (resp. $\square_0$), with the $\square_1$-multiplication performed first. For brevity we did not show the colors.

described below, and define $\mu(f)$ by the formula

$$\mu(f)(\phi_1 \otimes \cdots \phi_k \otimes \phi) := \phi \circ \Phi^f(\phi_1, \ldots, \phi_k).$$

Let us define (35). Suppose first that $\mathrm{tr}_1(\mathfrak{S}) = (1)$, so $\mathfrak{S}$ is the suspension of the 1-tree $(1, \ldots, k)$. In this case the exchange rule (27) in $\mathscr{D}$ induces a natural map

$$\mathscr{E}^f : \mathscr{E}^{\mathfrak{T}} \longrightarrow \mathscr{E}^{\mathfrak{T}_1} \square_1 \cdots \square_1 \mathscr{E}^{\mathfrak{T}_k}$$

and $\Phi^f(\phi_1, \ldots, \phi_k)$ is the composite

$$\mathscr{E}^{\mathfrak{T}} \xrightarrow{\mathscr{E}^f} \mathscr{E}^{\mathfrak{T}_1} \square_1 \cdots \square_1 \mathscr{E}^{\mathfrak{T}_k} \xrightarrow{\phi_1 \square_1 \cdots \square_1 \phi_k} \mathscr{E}(n_1) \square_1 \cdots \square_1 \mathscr{E}(n_k) = \mathscr{E}^{\mathfrak{S}}.$$

To address the general case denote, for $\mathfrak{T}_1, \mathfrak{T}_2 \in \Omega_2^{\mathbb{N}}$, by $\mathfrak{T}_1 \vee \mathfrak{T}_2$ the colored 2-tree obtained by identifying the root of $\mathfrak{T}_1$ with the root of $\mathfrak{T}_2$.[7] Observe that

$$\mathscr{E}^{\mathfrak{T}_1 \vee \mathfrak{T}_2} = \mathscr{E}^{\mathfrak{T}_1} \square_0 \mathscr{E}^{\mathfrak{T}_2}. \tag{36}$$

A general $\mathfrak{S}$ uniquely decomposes into the product $\mathfrak{S}_1 \vee \cdots \vee \mathfrak{S}_p$ of the suspensions of 1-trees; $f : \mathfrak{T} \to \mathfrak{S}$ is then of the form

$$f = f_1 \vee \cdots \vee f_p : \mathfrak{T} = \mathfrak{T}_1 \vee \cdots \vee \mathfrak{T}_p \longrightarrow \mathfrak{S}_1 \vee \cdots \vee \mathfrak{S}_p = \mathfrak{S}. \tag{37}$$

Suppose the fibers of $f_a : \mathfrak{T}_a \to \mathfrak{S}_a$ are $\mathfrak{T}^a_1, \ldots, \mathfrak{T}^a_{p_a}$, $1 \leq a \leq s$, then the fibers of f are

$$\mathfrak{T}_1, \ldots, \mathfrak{T}_k = \mathfrak{T}^1_1, \ldots, \mathfrak{T}^1_{p_1}, \ldots, \mathfrak{T}^s_1, \ldots, \mathfrak{T}^s_{p_s}.$$

For $\phi^a_b : \mathscr{E}^{\mathfrak{T}^a_b} \to \mathscr{E}(n^a_b)$ we define $\Phi^f(\phi^1_1, \ldots, \phi^1_{p_1}, \ldots, \phi^s_1, \ldots, \phi^s_{p_s}) : \mathscr{E}^{\mathfrak{T}} \to \mathscr{E}^{\mathfrak{S}}$ by

$$\Phi^f(\phi^1_1, \ldots, \phi^1_{p_1}, \ldots, \phi^s_1, \ldots, \phi^s_{p_s}) := \Phi^{f_1}(\phi^1_1, \ldots, \phi^1_{p_1}) \square_0 \cdots \square_0 \Phi^{f_s}(\phi^s_1, \ldots, \phi^s_{p_s}), \tag{38}$$

where we tacitly used (36). This defines (35) for arbitrary trees. $\square$

[7] In [9] we denoted this operation by $\mathfrak{T}_1 + \mathfrak{T}_2$.

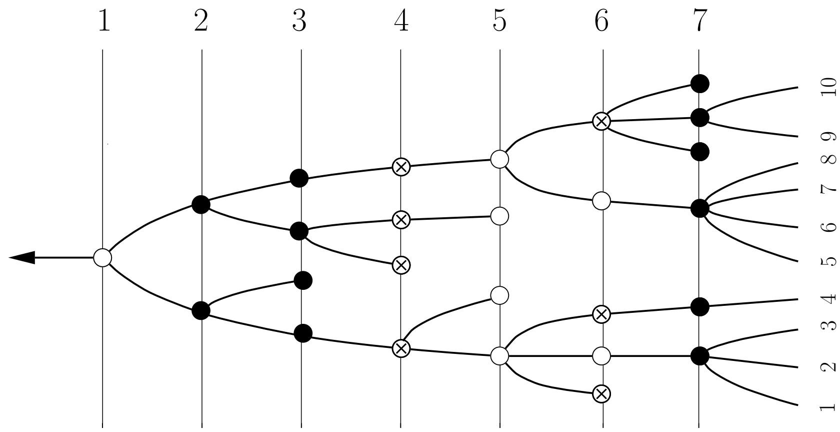


FIGURE 2. A tree in LTr. It has 7 levels numbered from the left to the right, 4 of type (i), 3 of type (ii). The vertices on the same level and the input leaves are ordered from the bottom up.

5.2. **The category of levelled trees.** The category LTr has objects planar rooted trees with three types of vertices: 'white' vertices $\circ$, 'vertical' vertices $\otimes$ and 'horizontal' vertices $\bullet$.[8] These vertices may have arbitrary arities ≥ 0 and are lined up into levels of two types:

(i) levels consisting of white vertices $\circ$ and/or vertical vertices $\otimes$, and
(ii) levels consisting only of horizontal vertices $\bullet$.

An example of a tree in LTr is given in Figure 2 which uses the convention that levelled trees are drawn horizontally, with the root on the left. Morphisms in LTr are generated by three types of 'elementary' morphisms:

Type 1. Maps of trees $f : \beta \to \alpha$, where α is obtained from β by choosing two adjacent levels and contracting all edges connecting vertices in these two chosen levels. This contraction results in a single level with vertices determined by the following rules:

(i) contracting an edge adjacent to a $\circ$-vertex produces a $\circ$-vertex,
(ii) contracting an edge connecting two $\otimes$-vertices or a $\bullet$-vertex with a $\otimes$-vertex produces a $\otimes$-vertex and
(iii) contracting an edge connecting two $\bullet$-vertices produces a $\bullet$-vertex.

If we 'order' the types of vertices by

$$\bullet \prec \otimes \prec \circ \tag{39}$$

then the above rules say that the 'higher takes everything.'

Type 2. Maps of trees $f : \beta \to \alpha$, where α is obtained from β by replacing a $\otimes$-vertex by a $\circ$-vertex of the same arity, or by replacing all $\bullet$-vertices in the same level by $\otimes$-vertices. Therefore only replacements that increase order (39) are allowed.

[8]The terminology will be explained in Section 5.3.

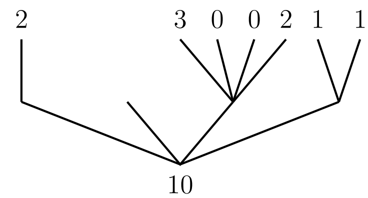


FIGURE 3. An $\mathbb{N}$-colored 2-tree corresponding to the leveled tree of Figure 2.

Type 3. Maps of trees $f : \beta \to \alpha$, where α is obtained from β by introducing a new level consisting only of $\circ$-vertices of arity 1.

We require the following relations between these elementary morphisms: a contraction of two levels does not depend on the order, contractions commute with replacement of vertices, and introducing a level and then contracting it is an identity morphism.

The category $\mathtt{LTr}$ is operadic. The cardinality functor $\mathtt{LTr} \to \mathtt{FSet}$ assigns to $\beta \in \mathtt{LTr}$ the set $\circ(\beta)$ of its white vertices. The fibers of a map $\sigma : \beta \to \alpha$ are preimages of white vertices of α. Trees in $\mathtt{LTr}$ belong to the same connected component if they have the same arity (= number of input edges), one therefore clearly has $\pi_0(\mathtt{LTr}) \cong \mathbb{N}$. The (in this case unique) terminal objects are white corollas $\circ^n$ of arity $n \in \mathbb{N}$.

There is an operadic functor

$$\Omega : \mathtt{LTr} \to \Omega_2^{\mathbb{N}} \tag{40}$$

assigning to $\beta \in \mathtt{LTr}$ an $\mathbb{N}$-colored 2-tree $\boldsymbol{\mathcal{T}} = (\mathcal{T}, c)$ defined as follows. Assume that the white and vertical vertices of β are lined up, from the root down, as in the table

$$\begin{matrix} v_1^1 & v_2^1 & \cdots & v_{q_1}^1 \\ v_1^2 & v_2^2 & \cdots & v_{q_2}^2 \\ \vdots & \vdots & & \vdots \\ v_1^t & v_2^t & \cdots & v_{q_t}^t \end{matrix} \tag{41}$$

(the table therefore does not show type (ii) levels). The 1-truncation of $\mathcal{T}$ is the 1-ordinal $\{1, \ldots, t\}$ represented by the corolla with t leaves numbered from the left to the right. The leaves of $\mathcal{T}$ are elements of the set

$$\big\{ v_d^c,\ 1 \leq c \leq t,\ 1 \leq d \leq q_c,\ v_d^c \text{ is white} \big\}.^9 \tag{42}$$

The leaf v_d^c is connected to the cth 1-vertex c. Finally, the color $c(v_d^c)$ of the leaf v_d^c is the arity of the vertex v_d^c, while the output color $c(1)$ of $\mathcal{T}$ is the arity of β. We put $\Omega(\beta) := (\mathcal{T}, c)$. An example of this construction is in Figure 3. Observe that the functor Ω in (40) does not see horizontal vertices.

[9] So we use the same symbols for both the white vertices of β and for the leaves of $\mathcal{T}$.

5.3. **Endomorphism object for levelled trees.** By Proposition 5.2, each collection $\mathscr{E} = \{\mathscr{E}(n)\}_{n\geq 0}$ of objects of a duoidal category $\mathscr{D} = (\mathscr{D}, \square_0, \square_1, e, v)$ has its $\mathbb{N}$-colored endomorphism 2-operad $\mathcal{E}nd_{\mathscr{E}}^{\Omega_2^{\mathbb{N}}}$. Likewise, one can define a $\mathtt{LTr}$-collection $\mathcal{E}nd_{\mathscr{E}}^{\mathtt{LTr}} = \big\{\mathcal{E}nd_{\mathscr{E}}^{\mathtt{LTr}}(\beta)\big\}_{\beta\in\mathtt{LTr}}$ which becomes, under the mild assumption of the commutativity of diagram (49) below, an operad in $\mathtt{Op}^{\mathtt{LTr}}(V)$.

Its construction is similar to the one in the proof of Proposition 5.2. Assume that all (not only $\circ$- and $\otimes$-) vertices of a leveled tree $\beta \in \mathtt{LTr}$ are organized as in the table

$$\begin{matrix} u_1^1 & u_2^1 & \cdots & u_{p_1}^1 \\ u_1^2 & u_2^2 & \cdots & u_{p_2}^2 \\ \vdots & \vdots & & \vdots \\ u_1^\ell & u_2^\ell & \cdots & u_{p_\ell}^\ell \end{matrix} \tag{43}$$

Notice that necessarily $p_1 = 1$. Then we define, for $1 \leq b \leq \ell$,

$$\mathscr{E}_\beta^{\square_1^{p_b}} := \mathscr{E}(u_1^b)\,\square_1 \cdots \square_1 \mathscr{E}(u_{p_b}^b),$$

where, for $1 \leq c \leq p_b$,

$$\mathscr{E}(u_c^b) := \begin{cases} \mathscr{E}(n_c^b), & \text{if } u_c^b \text{ is a white vertex of arity } n_c^b, \\ v, & \text{if } u_c^b \text{ is a vertical vertex, and} \\ e, & \text{if } u_c^b \text{ is a horizontal vertex.} \end{cases} \tag{44}$$

Vertical vertices $\otimes$ are therefore represented by the vertical unit v and the horizontal vertices $\bullet$ by the horizontal unit e, which explains the terminology. Finally we put

$$\mathscr{E}^\beta := \mathscr{E}_\beta^{\square_1^{p_\ell}}\,\square_0 \cdots \square_0 \mathscr{E}_\beta^{\square_1^{p_1}}$$

or, in the expanded form,

$$\mathscr{E}^\beta = \big(\mathscr{E}(u_1^\ell)\,\square_1 \cdots \square_1 \mathscr{E}(u_{p_\ell}^\ell)\big)\,\square_0 \cdots \square_0 \big(\mathscr{E}(u_1^1)\,\square_1 \cdots \square_1 \mathscr{E}(u_{p_1}^1)\big), \tag{45}$$

and define $\mathcal{E}nd_{\mathscr{E}}^{\mathtt{LTr}}(\beta) := \mathscr{D}\big(\mathscr{E}^\beta, \mathscr{E}(n)\big)$ with n the arity of β.

One has a natural morphism of $\mathtt{LTr}$-collections

$$\Lambda : \mathcal{E}nd_{\mathscr{E}}^{\mathtt{LTr}} \to \Omega^* \mathcal{E}nd_{\mathscr{E}}^{\Omega_2^{\mathbb{N}}} \tag{46}$$

whose components

$$\Lambda_\beta : \mathcal{E}nd_{\mathscr{E}}^{\mathtt{LTr}}(\beta) \to \mathcal{E}nd_{\mathscr{E}}^{\Omega_2^{\mathbb{N}}}(\mathbf{\mathfrak{T}}),\ \beta \in \mathtt{LTr},\ \mathbf{\mathfrak{T}} := \Omega(\beta),$$

are induced by a natural map

$$\theta_\beta : \mathscr{E}^{\mathbf{\mathfrak{T}}} \to \mathscr{E}^\beta \tag{47}$$

defined as follows. Let us introduce first an auxiliary reduced $\square$-power

$$\overline{\mathscr{E}}^\beta := \overline{\mathscr{E}}_\beta^{\square_1^{p_\ell}}\,\square_0 \mathscr{E}_\beta^{\square_1^{p_{\ell-1}}}\,\square_0 \cdots \square_0 \overline{\mathscr{E}}_\beta^{\square_1^{p_2}}\,\square_0 \overline{\mathscr{E}}_\beta^{\square_1^{p_1}}$$

whose factors are, for $1 \leq b \leq \ell$, given as

$$\overline{\mathscr{E}}^{\,\square_1^{p_b}} := \begin{cases} \mathscr{E}^{\,\square_1^{p_b}}, & \text{if the } b\text{th level of } \beta \text{ is of type (i), and} \\ e, & \text{if the } b\text{th level of } \beta \text{ is of type (ii).} \end{cases}$$

Clearly, $\overline{\mathscr{E}}^{\beta}$ is obtained from $\mathscr{E}^{\beta}$ by replacing multiple $\square_1$-powers of e by a single instance of e. The comonoid structure

$$e \to e \,\square_1 e \tag{48}$$

therefore gives rise to a canonical map $\varpi : \overline{\mathscr{E}}^{\beta} \to \mathscr{E}^{\beta}$. More precisely,

$$\varpi = \varpi^{\ell} \,\square_0 \cdots \square_0 \varpi^1 : \overline{\mathscr{E}}^{\beta} = \overline{\mathscr{E}}_{\beta}^{\,\square_1^{p_\ell}} \,\square_0 \cdots \square_0 \overline{\mathscr{E}}_{\beta}^{\,\square_1^{p_1}} \longrightarrow \mathscr{E}^{\beta} = \mathscr{E}_{\beta}^{\,\square_1^{p_\ell}} \,\square_0 \cdots \square_0 \mathscr{E}_{\beta}^{\,\square_1^{p_1}},$$

where $\varpi^b : \overline{\mathscr{E}}_{\mathfrak{T}}^{\,\square_1^{p_b}} \to \mathscr{E}_{\beta}^{\,\square_1^{p_b}}$, $1 \leq b \leq \ell$, are the following canonical morphisms.

If the bth level of β is of type (i), then ϖ^b is the identity $\overline{\mathscr{E}}_{\beta}^{\,\square_1^{p_b}} = \mathscr{E}_{\beta}^{\,\square_1^{p_b}}$. When the bth level is of type (ii), then by definition $\overline{\mathscr{E}}_{\beta}^{\,\square_1^{p_b}} = e$ while $\mathscr{E}_{\beta}^{\,\square_1^{p_b}} = e \,\square_1 \cdots \square_1 e$ (p_b-times). In this case we take as ϖ_b the map given by iterating the comonoid structure of e.

Comparing the definitions of $\overline{\mathscr{E}}^{\beta}$ and $\mathscr{E}^{\mathfrak{T}}$ we see that $\overline{\mathscr{E}}^{\beta}$ differs from $\mathscr{E}^{\mathfrak{T}}$ by (possibly multiple) $\square_0$-products with e and/or $\square_1$-products with v. Recalling that e (resp. v) is the unit for the horizontal (resp. vertical) multiplication in $\mathscr{D}$, we conclude that $\overline{\mathscr{E}}^{\beta}$ and $\mathscr{E}^{\mathfrak{T}}$ are canonically isomorphic. The map Λ_β in (47) is the composition $\mathscr{E}^{\mathfrak{T}} \cong \overline{\mathscr{E}}^{\beta} \xrightarrow{\varpi} \mathscr{E}^{\beta}$.

Remark 5.3. One can check that if the diagram

$$\begin{array}{ccc} e \,\square_1 e & \longrightarrow & e \,\square_1 v \\ \downarrow & & \downarrow \\ v \,\square_1 e & \longrightarrow & e \end{array} \tag{49}$$

induced by the canonical map $e \to v$ and by the unit constraint for v commutes, the construction of Section 5.3 gives rise to an `LTr`-multicotensor that induces a natural `LTr`-operad structure on $\mathcal{E}nd_{\mathscr{E}}^{\mathtt{LTr}}$.

This in particular happens when the comultiplication (48) is an isomorphism, in which case ϖ is an isomorphism, too, and Λ is an isomorphism of `LTr`-operads. An important instance of such a situation is when $\mathscr{D}$ is the duoidal category $\mathbb{S}p_2(\mathtt{C}, V)$ of span V-objects over a small category `C` [9, Def. 6.3].

Theorem 5.4. *For each multiplicative* 1*-operad* $\mathscr{A}$ *in a duoidal category* $\mathscr{D}$ *there exists a natural map of* `LTr`*-collections*

$$\Psi : \mathfrak{I}^{\mathtt{LTr}} \to \mathcal{E}nd_{\mathscr{A}}^{\mathtt{LTr}} \tag{50}$$

such that the composite

$$\Xi : \mathfrak{I}^{\mathtt{LTr}} \xrightarrow{\Psi} \mathcal{E}nd_{\mathscr{A}}^{\mathtt{LTr}} \xrightarrow{\Lambda} \Omega^* \mathcal{E}nd_{\mathscr{A}}^{\Omega_2^{\mathbb{N}}} \tag{51}$$

is a morphism of LTr*-operads. If the diagram (49) commutes, all maps in (51) are operad morphisms.*

Proof. The map (29) gives rise to a map $\alpha_v : v \to \mathscr{A}(n)$ and, when precomposed with the canonical map $e \to v$, to a map $\alpha_e : e \to \mathscr{A}(n)$ for each $n \geq 0$. By the definition of $\mathfrak{J}^{\mathtt{LTr}}$, morphism (50) is determined by specifying a map $\Psi_\beta : \mathscr{A}^\beta \to \mathscr{A}(n)$ for each $\beta \in \mathtt{LTr}$, where n is the arity of β. Assume again that the vertices of β are as in table (43) and denote, only for purposes of this proof,

$$\mathscr{A}(\beta) := \big(\mathscr{A}(n_1^\ell) \,\square_1 \cdots \square_1 \mathscr{A}(n_{p_\ell}^\ell)\big) \,\square_0 \cdots \square_0 \big(\mathscr{A}(n_1^1) \,\square_1 \cdots \square_1 \mathscr{A}(n_{p_1}^1)\big) \tag{52}$$

where n_c^b is the arity of the vertex v_c^b, $1 \leq b \leq \ell$, $1 \leq c \leq p_b$. It is clear that the structure morphisms (28) of the 1-operad $\mathscr{A}$ give rise to a map $\gamma_\beta : \mathscr{A}(\beta) \to \mathscr{A}(n)$.

For $\mathscr{A}(u_c^b)$ as in (44) define $\omega_c^b : \mathscr{A}(u_c^b) \to \mathscr{A}(n_c^b)$ by

$$\omega_c^b := \begin{cases} \text{the identity } id : \mathscr{A}(n_c^b) \to \mathscr{A}(n_c^b), & \text{if } u_c^b \text{ is white} \\ \text{the map } \alpha_v : v \to \mathscr{A}(n_c^b), & \text{if } u_c^b \text{ is vertical, and} \\ \text{the map } \alpha_e : e \to \mathscr{A}(n_c^b), & \text{if } u_c^b \text{ is horizontal.} \end{cases}$$

Comparing (52) and (45), we see that the above maps assemble to a morphism

$$\omega_\beta : \mathscr{A}^\beta \to \mathscr{A}(\beta) \tag{53}$$

We finally define $\Psi_\beta : \mathscr{A}^\beta \to \mathscr{A}(n)$ as the composite

$$\Psi_\beta : \mathscr{A}^\beta \xrightarrow{\omega_\beta} \mathscr{A}(\beta) \xrightarrow{\gamma_\beta} \mathscr{A}(n).$$

The category LTr was designed to model the 'pasting schemes' for multiplicative 1-operads and all related constructions were 'tautological.' This makes the desired properties of the above objects obvious. □

6. The Tamarkin operad

We are going to give an alternative definition of Tamarkin's operad **seq** acting on dg-categories and study various related categories and operads. The second part of this section is devoted to the construction of the functor (56) needed in Section 7. We consider as in the introduction to [7] the pruned $\mathbb{N}$-colored 2-operad $\mathcal{T}am_2^{\mathbb{N}}$ given by the pullback

$$\begin{array}{ccc} \mathcal{T}am_2^{\mathbb{N}} & \longrightarrow & \mathcal{K}^{(2)}/a_2 \\ \big\downarrow & & \big\downarrow \\ Des_2(\mathcal{L}^{(2)}) & \xrightarrow{Des_2(c)} & Des_2(\mathcal{K}^{(2)}) \end{array} \tag{54}$$

where $\mathcal{L}^{(2)}$ is the second filtration of the lattice-path operad, $\mathcal{K}^{(2)}$ is the second filtration of the complete graph operad of Berger [12], $c : \mathcal{L}^{(2)} \to \mathcal{K}^{(2)}$ is the complexity index functor and a_2 is the canonical internal 2-operad in $\mathcal{K}^{(2)}$ consisting of 2-ordinals. The prominent rôle of

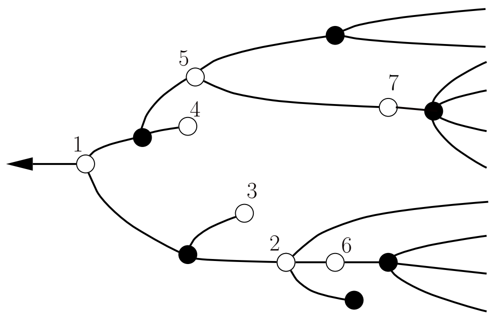


FIGURE 4. A tree $\delta \in \mathcal{L}^{(2)}(2,3,0,0,2,1,1;10)$.

$\mathcal{T}am_2^{\mathbb{N}}$ is given by the fact that the operad **seq** of [30, §5.6.1] acting on dg-categories equals its restriction along the pruning functor (12). To keep the notation compatible with the rest of the paper, we denote the Tamarkin operad by $\mathcal{T}m_2^{\mathbb{N}}$ and take the above observations as its definition.

Definition 6.1. The *Tamarkin operad* $\mathcal{T}m_2^{\mathbb{N}}$ is the restriction $p^*\mathcal{T}am_2^{\mathbb{N}}$ of the pullback (54) along the pruning functor $p : \Omega_2^{\mathbb{N}} \to \mathtt{Ord}_2^{\mathbb{N}}$.

An alternative description of $\mathcal{T}m_2^{\mathbb{N}}$ is given in Remark 7.2. Without going into details of all objects above, we give below an explicit description of $\mathcal{T}am_2^{\mathbb{N}}$ and the related objects.

It was shown in [8, Prop. 4.10] or [7, Prop. 2.14] that $\mathcal{L}^{(2)}$ is isomorphic to the Tamarkin-Tsygan operad [31]. Our exposition will use this description of $\mathcal{L}^{(2)}$. The operad $\mathcal{L}^{(2)}$ is an ordinary symmetric $\mathbb{N}$-colored operad in the monoidal category `Set` of sets. Its component $\mathcal{L}^{(2)}(n_1,\ldots,n_k;m)$ is the set of planar rooted trees δ with m input leaves and two types of vertices:

(i) white vertices $\circ$ labelled $1,\ldots,k$ such that the ith vertex has arity n_i, and
(ii) black vertices $\bullet$ of arbitrary arities different from 1.

We moreover require that δ has no internal edge connecting two black vertices. An example is given in Figure 4.

The symmetric groups permute the labels of white vertices. The $\circ_i$-operation

$$\mathcal{L}^{(2)}(n_1,\ldots,n_k;m)\times\mathcal{L}^{(2)}(m_1,\ldots,m_l;n_i) \to \mathcal{L}^{(2)}(n_1,\ldots,n_{i-1},m_1,\ldots,m_l,n_{i+1},\ldots,n_k;m)$$

inserts the tree $\gamma \in \mathcal{L}^{(2)}(m_1,\ldots,m_l;n_i)$ to the ith vertex of $\delta \in \mathcal{L}^{(2)}(n_1,\ldots,n_k;m)$ and contracts the edges connecting two black vertices if necessary. The units are represented by planar corollas whose unique vertex is white.

There is a canonical *complementary order* [4, Def. 2.2] on the set $\circ(\delta)$ of white vertices of a tree $\delta \in \mathcal{L}^{(2)}$ given as follows. For $i,j \in \circ(\delta)$,

(i) $i \lhd_0 j$ if and only if there is a directed path in δ from i to j, and

(ii) $i \lhd_1 j$ if and only if the edge path connecting i with the root lies on the left from the path connecting j to the root, where 'left' refers to the planar structure of δ.

The following definition will be useful in the sequel; recall that $\mathbb{N}$-colored 2-ordinals were reviewed in Section 5.1.

Definition 6.2. An $\mathtt{Ord}_2^{\mathbb{N}}$-*labelled tree* is a couple $(\mathbf{O}, \delta)$, where $\delta \in \mathcal{L}^{(2)}$ and $\mathbf{O} \in \mathtt{Ord}_2^{\mathbb{N}}$ is an $\mathbb{N}$-colored 2-ordinal whose underlying set is $\circ(\delta)$ and the coloring given by the arity of the corresponding vertex of δ.

Let $\lhd$ be some complementary order on the underlying set of a 2-ordinal $\mathcal{O}$. We say that $\mathcal{O}$ *dominates* $\lhd$ if, for all $i, j \in \mathcal{O}$,

$$i \not\lhd_0 j \text{ if } i <_0 j, \text{ and } i \lhd_1 j \text{ if } i <_1 j. \tag{55}$$

The pullback (54) represents the pruned 2-operad $\mathcal{T}am_2^{\mathbb{N}}$ as a suboperad of the desymmetrisation of the operad $\mathcal{L}^{(2)}$ whose arity- $\mathbf{O}$ operations are $\mathtt{Ord}_2^{\mathbb{N}}$-labelled trees $(\mathbf{O}, \delta)$ such that $\mathbf{O}$ dominates the canonical complementary order on the set $\circ(\delta)$ of white vertices of δ.

Let us define the operadic category $\mathtt{Tam}_2^{\mathbb{N}}$ as the Grothendieck construction of Proposition 2.5 applied to the pruned 2-operad $\mathcal{T}am_2^{\mathbb{N}}$. By definition, the objects of $\mathtt{Tam}_2^{\mathbb{N}}$ are pairs $(\mathbf{O}, \delta) \in \mathcal{T}am_2^{\mathbb{N}}(\mathbf{O})$ of $\mathtt{Ord}_2^{\mathbb{N}}$-labeled trees such that $\mathbf{O}$ dominates the canonical complementary order on $\circ(\delta)$. The rest of this section will be devoted to the definition of a functor

$$u : \mathtt{LTr} \to \mathtt{Tam}_2^{\mathbb{N}}, \tag{56}$$

where $\mathtt{LTr}$ is the category of leveled trees introduced in Section 5.2. This functor will play a key rôle in Section 7.

We start by noticing that the set $\circ(\beta)$ of $\circ$-vertices of $\beta \in \mathtt{LTr}$ has a natural lexicographic order defined by saying that $u < v$ if the level of u is closer to the root than the level of v; if u lies on the same level as v then $u < v$ if and only if u is on the left from v in the sense of the planar structure of β. Given a leveled tree $\beta \in \mathtt{LTr}$, we consider an $\mathtt{Ord}_2^{\mathbb{N}}$-labelled tree $u(\beta) = (\mathbf{O}, \bar{\beta})$ with $\mathbf{O} := p\big(\Omega(\beta)\big)$, where $\Omega : \mathtt{LTr} \to \Omega_2^{\mathbb{N}}$ is the functor in (40), $p : \Omega_2^{\mathbb{N}} \to \mathtt{Ord}_2^{\mathbb{N}}$ the pruning functor, and $\bar{\beta}$ produced from β in four steps:

(i) labelling the white vertices of β by $1, \ldots, k$ using the above lexicographical order,
(ii) forgetting the level structure of β,
(iii) converting all $\otimes$-vertices of β to $\bullet$-vertices and
(iv) contracting edges connecting two $\bullet$-vertices and erasing all $\bullet$-vertices of arity one.

As an exercise, we recommend to check that applying the above steps to the tree β in Figure 2 produces the tree in Figure 4. To show that indeed $u(\beta) = (\mathbf{O}, \bar{\beta}) \in \mathtt{Tam}_2^{\mathbb{N}}$ means to verify that the 2-ordinal $\mathbf{O}$ dominates the complementary order generated by $\bar{\beta}$.

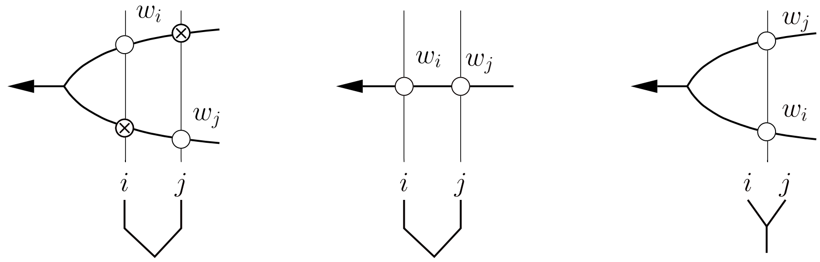


FIGURE 5. The cases $i <_0 j$ (two left pictures) and $i <_1 j$ (rightmost picture).

Let $i <_0 j$ in $\mathbf{O}$ and let w_i, w_j be the corresponding white vertices from $\bar{\beta}$. The relation $i <_0 j$ means that, in β, the vertex w_j lies on the level closer to the root than the level of w_i. There are two possibilities: either w_i and w_j are connected by a directed path and then $w_i \lhd_0 w_j$ in $\circ(\bar{\beta})$, or there is no such a directed path, in which case either $w_i \lhd_1 w_j$ or $w_j \lhd_1 w_1$ in $\bar{\beta}$. The domination condition (55) clearly holds for the pair i, j in this case.

Assume that $i <_1 j$ in $\mathbf{O}$. This means that w_i precedes w_j in the lexicographical order and also that there is no directed path connecting w_i and w_j in β. Hence $w_i \lhd_1 w_j$ in $\circ(\bar{\beta})$ which finishes the verification that $\mathbf{O}$ dominates $\bar{\beta}$. The idea is indicated in Figure 5.

A morphism $\phi : (\mathbf{O}, \delta) \to (\mathbf{N}, \gamma)$ in $\mathtt{Tam}_2^{\mathbb{N}}$ is, by definition of the Grothendieck construction, an $|\mathbf{N}|+1$-tuple

$$\big(\sigma, (\mathbf{O}_1, \delta_1), \ldots, (\mathbf{O}_k, \delta_k)\big) \tag{57}$$

where $\sigma : \mathbf{O} \to \mathbf{N}$ is a morphism of 2-ordinals, $\mathbf{O}_i = \sigma^{-1}(i)$ and $(\mathbf{O}_i, \delta_i) \in \mathcal{T}am_2^{\mathbb{N}}(\mathbf{O}_i)$, $i \in |\mathbf{N}|$, are such that

$$m\big((\mathbf{O}_1, \delta_1), \ldots, (\mathbf{O}_k, \delta_k); (\mathbf{N}, \gamma)\big) = (\mathbf{O}, \delta), \tag{58}$$

where m is the multiplication in the pruned 2-operad $\mathcal{T}am_2^{\mathbb{N}}$.

For any morphism $f : \beta \to \alpha$ in $\mathtt{LTr}$, the functor $\Omega : \mathtt{LTr} \to \Omega_2^{\mathbb{N}}$ induces a map of 2-trees $\Omega(f) : \Omega(\beta) \to \Omega(\alpha)$ so, denoting $\mathbf{O} := p\big(\Omega(\beta)\big)$ and $\mathbf{N} := p\big(\Omega(\alpha)\big)$, we have the induced map

$$\sigma : \mathbf{O} \longrightarrow \mathbf{N},\ \sigma := p(\Omega(f)) \tag{59}$$

of $\mathbb{N}$-colored 2-ordinals.

To define the functor $u : \mathtt{LTr} \to \mathtt{Tam}_2^{\mathbb{N}}$ on morphisms, it suffices to specify it on elementary morphisms listed in Section 5.2. Let $f : \beta \to \alpha$ be an elementary morphism of Type 1 collapsing two consecutive levels. Each $\circ$-vertex w of α has the fiber $f^{-1}(w) \in \mathtt{LTr}$. Such a w also corresponds to a unique element $i = i(w)$ of the 2-ordinal presented $\mathbf{N} = p(\Omega(\alpha))$. So, we associate to w the element $(\mathbf{O}_i, \delta_i) := \big(\sigma^{-1}(i), \overline{f^{-1}(w)}\big) \in \mathtt{Tam}_2^{\mathbb{N}}\big(\sigma^{-1}(i)\big)$. It is easy to check that then (58) with $(\mathbf{O}_i, \delta_i)$ as above, $(\mathbf{O}, \delta) := u(\beta)$ and $(\mathbf{N}, \gamma) := u(\alpha)$ is satisfied, so we constructed a morphism $u(f) : u(\beta) \to u(\alpha)$ in $\mathtt{Tam}_2^{\mathbb{N}}$.

If $f : \beta \to \alpha$ is an elementary morphism of Type 2 that replaces a level of $\bullet$-vertices by a level of $\otimes$-vertices, then clearly $u(\beta) = u(\alpha)$ and we define $u(f)$ to be the identity morphism.

Assume that $f : \beta \to \alpha$ replaces a $\otimes$-vertex of β by a $\bigcirc$-vertex w and denote by $e = e(w)$ the corresponding 'exceptional' element of the 2-ordinal $\mathbf{N}$. To describe the fibers of $f : \beta \to \alpha$ and of the induced map (59) denote, for $c \in \mathbb{N}$, by $\mathbf{1}^c \in \mathtt{Ord}_2^{\mathbb{N}}$ the terminal 2-ordinal whose input and output colors equal c, $\mathbf{0}_2^c \in \mathtt{Ord}_2^{\mathbb{N}}$ the initial 2-ordinal whose output color is c, and by $\bigcirc^c$ (resp. $\otimes^c$) the $\bigcirc$- (resp. $\otimes$-) corolla of arity c. With this notation,

$$f^{-1}(u) = \begin{cases} \bigcirc^{c_v}, & \text{if } v \neq w, \text{ and} \\ \otimes^{c_v}, & \text{if } v = w, \end{cases}$$

where c_v is the arity of a $\bigcirc$-vertex v of α. The induced map (59) is an inclusion and

$$\sigma^{-1}(u) = \begin{cases} \mathbf{1}_2^{c_i}, & \text{if } i \neq e, \text{ and} \\ \mathbf{0}_2^{c_i}, & \text{if } i = e, \end{cases}$$

where $c_i \in \mathbb{N}$ denotes the color of $i \in \mathcal{S}$. We define $u(f) : u(\beta) \to u(\alpha)$ by taking in (57)

$$(\mathbf{O}_i, \delta_i) := \begin{cases} (\mathbf{1}_2^{c_i}, \bigcirc^{c_i}), & \text{if } i \neq e, \text{ and} \\ (\mathbf{0}_2^{c_i}, \otimes^{c_i}), & \text{if } i = e. \end{cases}$$

The discussion of Type 3 elementary morphisms $f : \beta \to \alpha$ is analogous. The exceptional fibers of f are now the exceptional trees $|$ with no vertices, while the exceptional fibers of the induced map $\sigma : \mathbf{O} \to \mathbf{N}$ are the initial of 2-ordinals $\mathbf{1}_2$ with the output color 1 This completes the definition of the functor u.

7. Action of the Tamarkin operad

The aim of this section is to prove the following statement in which $\mathcal{T}m_2^{\mathbb{N}}$ is the Tamarkin operad recalled in Definition 6.1.

Theorem 7.1. *For any multiplicative* 1*-operad $\mathscr{A}$ in a duoidal category one has a natural action*

(60) $$\mathcal{T}m_2^{\mathbb{N}} \to \mathcal{E}nd_{\mathscr{A}}^{\Omega_2^{\mathbb{N}}}.$$

Proposition 2.5 associates to the operad $\mathcal{T}am_2^{\mathbb{N}}$ the operadic category $\mathtt{Tam}_2^{\mathbb{N}}$, together with a discrete operadic fibration

$$s : \mathtt{Tam}_2^{\mathbb{N}} \to \mathtt{Ord}_2^{\mathbb{N}}.$$

By (16), $\mathcal{T}am_2^{\mathbb{N}} = s_!(\mathfrak{I}^{\mathtt{Tam}_2^{\mathbb{N}}})$, therefore $\mathcal{T}m_2^{\mathbb{N}} = p^*(\mathcal{T}am_2^{\mathbb{N}}) = p^*\big(s_!(\mathfrak{I}^{\mathtt{Tam}_2^{\mathbb{N}}})\big)$, so the map in (60) is the same as a morphism

(61) $$p^*\big(s_!(\mathfrak{I}^{\mathtt{Tam}_2^{\mathbb{N}}})\big) \to \mathcal{E}nd_{\mathscr{A}}^{\Omega_2^{\mathbb{N}}}.$$

On the other hand, define the operadic category $\mathtt{Tm}_2^{\mathbb{N}}$ using Proposition 1.9 as the pullback of operadic categories:

(62)
$$\begin{array}{ccc} \mathtt{Tm}_2^{\mathbb{N}} & \xrightarrow{\ r\ } & \mathtt{Tam}_2^{\mathbb{N}} \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ \Omega_2^{\mathbb{N}} & \xrightarrow{\ p\ } & \mathtt{Ord}_2^{\mathbb{N}}. \end{array}$$

The induced functors of the associated categories of operads enjoy the Beck-Chevalley property by Proposition 2.8, so

(63)
$$p^*\big(s_!(\mathfrak{I}^{\mathtt{Tam}_2^{\mathbb{N}}})\big) \cong t_!\big(r^*(\mathfrak{I}^{\mathtt{Tam}_2^{\mathbb{N}}})\big).$$

The map in (61) is thus the same as a morphism

$$t_!\big(r^*(\mathfrak{I}^{\mathtt{Tam}_2^{\mathbb{N}}})\big) \to \mathcal{E}nd_{\mathscr{A}}^{\Omega_2^{\mathbb{N}}}$$

which is, by adjunction, the same as an operad morphism

$$r^*(\mathfrak{I}^{\mathtt{Tam}_2^{\mathbb{N}}}) \to t^*(\mathcal{E}nd_{\mathscr{A}}^{\Omega_2^{\mathbb{N}}}).$$

Since clearly $r^*(\mathfrak{I}^{\mathtt{Tam}_2^{\mathbb{N}}}) = \mathfrak{I}^{\mathtt{Tm}_2^{\mathbb{N}}}$, Theorem 7.1 will be proved if we construct a natural morphism.

(64)
$$\mathfrak{I}^{\mathtt{Tm}_2^{\mathbb{N}}} \to t^*(\mathcal{E}nd_{\mathscr{A}}^{\Omega_2^{\mathbb{N}}}).$$

Remark 7.2. Notice that $r^*(\mathfrak{I}^{\mathtt{Tam}_2^{\mathbb{N}}}) = \mathfrak{I}^{\mathtt{Tm}_2^{\mathbb{N}}}$ together with (63) implies that $\mathcal{T}m_2^{\mathbb{N}} = t_!(\mathfrak{I}^{\mathtt{Tm}_2^{\mathbb{N}}})$, so $\mathcal{T}m_2^{\mathbb{N}}$ is the result of the application of the functor inverse to the Grothendieck construction to the discrete fibration $\mathtt{Tm}_2^{\mathbb{N}} \to \Omega_2^{\mathbb{N}}$.

It is obvious that the diagram

$$\begin{array}{ccc} \mathtt{LTr} & \xrightarrow{\ u\ } & \mathtt{Tam}_2^{\mathbb{N}} \\ {\scriptstyle \Omega}\downarrow & & \downarrow{\scriptstyle s} \\ \Omega_2^{\mathbb{N}} & \xrightarrow{\ p\ } & \mathtt{Ord}_2^{\mathbb{N}} \end{array}$$

in which $\Omega : \mathtt{LTr} \to \Omega_2^{\mathbb{N}}$ is the functor (40) and $u : \mathtt{LTr} \to \mathtt{Tam}_2^{\mathbb{N}}$ the functor (56) commutes, so we have the induced operadic functor $w : \mathtt{LTr} \to \mathtt{Tm}_2^{\mathbb{N}}$ to the pullback of (62) as in the commutative diagram

(65)
$$\begin{array}{ccccc} \mathtt{LTr} & & \xrightarrow{\qquad u \qquad} & & \\ & \searrow^{w} & & & \\ & & \mathtt{Tm}_2^{\mathbb{N}} & \xrightarrow{\ r\ } & \mathtt{Tam}_2^{\mathbb{N}} \\ & {\scriptstyle \Omega} & {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ & & \Omega_2^{\mathbb{N}} & \xrightarrow{\ p\ } & \mathtt{Ord}_2^{\mathbb{N}}. \end{array}$$

Recall that we already constructed a canonical morphism (51) of $\mathtt{LTr}$-operads

$$\Xi : \mathfrak{I}^{\mathtt{LTr}} \to \Omega^*(\mathcal{E}nd_{\mathscr{A}}^{\Omega_2^{\mathbb{N}}}).$$

By (65), $tw = \Omega$, so $\Omega^*(\mathcal{E}nd_{\mathscr{A}}^{\Omega_2^{\mathbb{N}}}) = w^*(t^*(\mathcal{E}nd_{\mathscr{A}}^{\Omega_2^{\mathbb{N}}}))$. Noticing the isomorphism $\mathfrak{I}^{\mathtt{LTr}} = w^*(\mathfrak{I}^{\mathtt{Tm}_2^{\mathbb{N}}})$, we see that we therefore also have a natural morphism of $\mathtt{LTr}$-operads

$$\Upsilon : w^*(\mathfrak{I}^{\mathtt{Tm}_2^{\mathbb{N}}}) \longrightarrow w^*\big(t^*(\mathcal{E}nd_{\mathscr{A}}^{\Omega_2^{\mathbb{N}}})\big). \tag{66}$$

The requisite map in (64) will be constructed by 'inverting w^*' in (66), using:

Lemma 7.3. *Let $F : \mathtt{O} \to \mathtt{P}$ be an operadic functor and $\mathcal{P}, \mathcal{O} \in \mathtt{Op}^{\mathtt{P}}(V)$. Assume that F is surjective on objects. Suppose we are also given a morphism $\varsigma : F^*(\mathcal{P}) \to F^*(\mathcal{O})$ of $\mathtt{O}$-operads such that, for arbitrary $Q', Q'' \in \mathtt{O}$ such that $F(Q') = F(Q'')$,*

$$\varsigma_{Q'} : F^*\mathcal{P}(Q') \to F^*\mathcal{O}(Q') \ \ \text{equals} \ \ \varsigma_{Q''} : F^*\mathcal{P}(Q'') \to F^*\mathcal{O}(Q''). \tag{67}$$

Then there exist a unique morphism $\rho : \mathcal{P} \to \mathcal{O}$ of $\mathtt{P}$-operads satisfying $\varsigma = F^*(\rho)$.

Proof Lemma 7.3. For $T \in \mathtt{P}$ choose $Q \in \mathtt{O}$ such that $T = F(Q)$. Since $F^*\mathcal{P}(Q) = \mathcal{P}(T)$ and $F^*\mathcal{O}(Q) = \mathcal{O}(T)$, we may define $\rho_T : \mathcal{P}(T) \to \mathcal{O}(T)$ by $\rho_T := \varsigma_Q$. We leave as an exercise to verify that the collection $\rho = \{\rho_T\}$ is a well-defined morphism of operads. □

We wish to apply the lemma to the situation when $\mathtt{O} = \mathtt{LTr}$, $\mathtt{P} = \mathtt{Tm}_2^{\mathbb{N}}$, F is the functor w in (65), $\mathcal{P} = \mathfrak{I}^{\mathtt{Tm}_2^{\mathbb{N}}}$, $\mathcal{O} = t^*(\mathcal{E}nd_{\mathscr{A}}^{\Omega_2^{\mathbb{N}}})$ and ς the morphism ε of (66). Since w is clearly surjective on objects, we only need to verify (67). In this particular case it means that, given $\beta', \beta'' \in \mathtt{LTr}$ such that $w(\beta') = w(\beta'')$, $\Upsilon_{\beta'} = \Upsilon_{\beta''}$. Recalling again that $\mathfrak{I}^{\mathtt{LTr}} = w^*(\mathfrak{I}^{\mathtt{Tm}_2^{\mathbb{N}}})$ and $tw = \Omega$, we easily see that it is enough to prove:

Lemma 7.4. *Let $\Xi = \{\Xi_\beta\} : \mathfrak{I}^{\mathtt{LTr}} \to \Omega^*(\mathcal{E}nd_{\mathscr{A}}^{\Omega_2^{\mathbb{N}}})$ be the composite (51). If $\beta', \beta'' \in \mathtt{LTr}$ are such that $w(\beta') = w(\beta'')$, then $\Psi_{\beta'} = \Psi_{\beta''}$.*

As the first step in proving Lemma 7.4 we characterize, in Lemma 7.6 below, pairs $\beta', \beta'' \in \mathtt{LTr}$ having the same w-image in $\mathtt{Tm}_2^{\mathbb{N}}$. For this, the following alternative description of objects of the categories $\mathtt{Tam}_2^{\mathbb{N}}$ and $\mathtt{Tm}_2^{\mathbb{N}}$ will be useful.

Lemma 7.5. *Objects of $\mathtt{Tam}_2^{\mathbb{N}}$ can be described as the isomorphism classes of planar rooted trees ζ with white vertices $\circ$, vertical vertices $\otimes$ and horizontal vertices $\bullet$. While $\circ$-vertices have arbitrary arities ≥ 0, $\bullet$-vertices have either arity ≥ 2 or 0, and all $\otimes$-vertices are of arity 1.*

We moreover require that ζ has no internal edge connecting two $\bullet$-vertices and no internal edge starting from a $\bullet$-vertex and ending in a $\otimes$-vertex, i.e. the following edges

$$\bullet\!\!\longleftarrow\!\!\bullet \quad \textit{or} \quad \otimes\!\!\longleftarrow\!\!\bullet \tag{68}$$

are not allowed. Finally, $\circ$- and $\otimes$-vertices are lined up in levels such that each level contains at least one $\circ$-vertex.

Objects of $\mathtt{Tm}_2^{\mathbb{N}}$ can similarly be identified with the isomorphisms classes of trees as above, but this time allowing also levels consisting solely of $\otimes$-vertices.

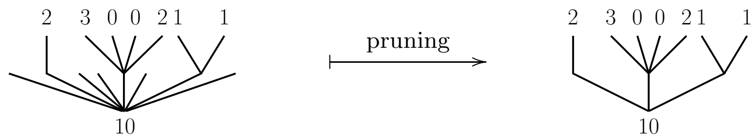


FIGURE 6. A 2-tree $\mathfrak{T} \in \Omega_2^{\mathbb{N}}$ and its pruning $\mathfrak{O} = p(\mathfrak{T}) \in \mathtt{Ord}_2^{\mathbb{N}}$.

Proof. Let us denote provisionally by $\widetilde{\mathtt{Tam}}_2^{\mathbb{N}}$ the set of isomorphism classes of trees which, according to the lemma, should describe objects of $\mathtt{Tam}_2^{\mathbb{N}}$, and let $\widetilde{\mathtt{Tm}}_2^{\mathbb{N}}$ be similarly related to $\mathtt{Tm}_2^{\mathbb{N}}$. We are going to construct two couples of mutually inverse maps,

$$\widetilde{\mathtt{Tam}}_2^{\mathbb{N}} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \mathtt{Tam}_2^{\mathbb{N}} \qquad \text{and} \quad \widetilde{\mathtt{Tm}}_2^{\mathbb{N}} \underset{\varsigma}{\overset{\varrho}{\rightleftarrows}} \mathtt{Tm}_2^{\mathbb{N}}\ .$$

While the definitions of $\phi : \widetilde{\mathtt{Tam}}_2^{\mathbb{N}} \to \mathtt{Tam}_2^{\mathbb{N}}$ and $\varrho : \widetilde{\mathtt{Tm}}_2^{\mathbb{N}} \to \mathtt{Tm}_2^{\mathbb{N}}$ are very simple, our constructions of their inverses will involve intuitive geometric arguments. A formal combinatorial construction should use a straightforward but lengthy induction on the number of vertices of the trees involved. We leave it to the interested reader.

Before we begin, notice that each planar rooted tree ω with levels of $\circ$- and $\otimes$-vertices determines an $\mathbb{N}$-colored 2-tree $\Omega(\omega) \in \Omega_2^{\mathbb{N}}$ by same procedure as described for the leveled trees of $\mathtt{LTr}$ in Section 5.2. That is, we organize $\circ$- and $\otimes$-vertices of ω to table (41); the 1-truncation of $\Omega(\omega)$ will then be the set $\{1, \ldots, t\}$ and its 2-leaves the same as in (42). The $\mathbb{N}$-coloring of the 2-leaves of $\Omega(\omega)$ is given, as always in this article, by the arities of the corresponding $\circ$-vertices. It is clear that $\Omega(\omega)$ is pruned if and only if ω does not have levels consisting solely of $\otimes$-vertices.

Let us describe $\phi : \widetilde{\mathtt{Tam}}_2^{\mathbb{N}} \to \mathtt{Tam}_2^{\mathbb{N}}$. As in Section 6, the level structure of $\zeta \in \widetilde{\mathtt{Tam}}_2^{\mathbb{N}}$ induces the lexicographic order on the set $\circ(\zeta)$ of its white vertices. We label the $\circ$-vertices of ζ by $\{1, \ldots, k\}$ accordingly, remove the levels and denote the resulting tree by $\overline{\zeta}$.

Since, by assumption, all levels of ζ contain at least one white vertex, the 2-tree $\Omega(\zeta)$ is pruned and can therefore be interpreted as a 2-ordinal. Then, by construction, $\phi(\zeta) := \big(\Omega(\zeta), \bar{\zeta}\big)$ is an $\mathtt{Ord}_2^{\mathbb{N}}$-labelled tree. It is moreover clear that the 2-ordinal $\Omega(\zeta)$ dominates the canonical complementary order on $\circ(\overline{\zeta})$, so in fact $\phi(\zeta) \in \mathtt{Tam}_2^{\mathbb{N}}$.

On the other hand, take $(\mathfrak{O}, \delta) \in \mathtt{Tam}_2^{\mathbb{N}}$ and define $\zeta = \psi(\mathfrak{O}, \delta) \in \widetilde{\mathtt{Tam}}_2^{\mathbb{N}}$ as follows. First, organize the $\circ$-vertices of δ to levels such that $\Omega(\delta) = \mathfrak{O}$. The domination condition for $(\mathfrak{O}, \delta)$ guarantees that it is possible. Then move the $\bullet$-vertices of δ so close to the root that none of the edges $\otimes\!\longleftarrow\!\bullet$ intersect a level line and that none of the $\bullet$-vertices lies on a level line. Finally, introduce unary $\otimes$-vertices at the intersection points of the level lines with the edges of δ. Then $\phi : \widetilde{\mathtt{Tam}}_2^{\mathbb{N}} \to \mathtt{Tam}_2^{\mathbb{N}}$ and $\psi : \mathtt{Tam}_2^{\mathbb{N}} \to \widetilde{\mathtt{Tam}}_2^{\mathbb{N}}$ are obviously mutual inverses.

For instance, if $\mathfrak{O}$ is the $\mathbb{N}$-colored 2-ordinal represented by the pruned $\mathbb{N}$-colored 2-tree in the right side of Figure 6 and δ the tree in Figure 4, then $(\mathfrak{O}, \delta) \in \mathtt{Tam}_2^{\mathbb{N}}$, and $\psi(\mathfrak{O}, \delta) \in \widetilde{\mathtt{Tam}}_2^{\mathbb{N}}$

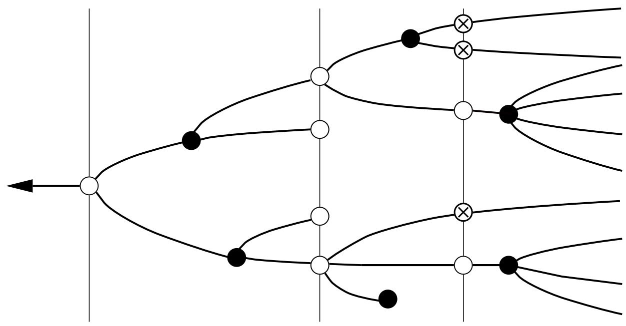


FIGURE 7. The tree $\zeta = \psi(\boldsymbol{\mathcal{O}}, \delta) \in \widetilde{\mathtt{Tam}}_2^{\mathbb{N}}$.

is the tree in Figure 7.

Let us describe $\varrho : \widetilde{\mathtt{Tm}}_2^{\mathbb{N}} \to \mathtt{Tm}_2^{\mathbb{N}}$. Notice that elements of $\mathtt{Tm}_2^{\mathbb{N}}$ are, by the definition via the pullback in (62), couples $(\boldsymbol{\mathfrak{T}}, \delta)$ such that $\boldsymbol{\mathfrak{T}} \in \Omega_2^{\mathbb{N}}$ and $\big(p(\boldsymbol{\mathfrak{T}}), \delta)\big) \in \mathtt{Tam}_2^{\mathbb{N}}$, where $p(\boldsymbol{\mathfrak{T}}) \in \mathtt{Ord}_2^{\mathbb{N}}$ is the pruning of the $\mathbb{N}$-colored 2-tree $\boldsymbol{\mathfrak{T}}$. We can call couples $(\boldsymbol{\mathfrak{T}}, \delta) \in \mathtt{Tm}_2^{\mathbb{N}}$ $\Omega_2^{\mathbb{N}}$*-labelled trees*.

For $\xi \in \widetilde{\mathtt{Tm}}_2^{\mathbb{N}}$ denote by $p(\xi)$ be the tree obtained from ξ by removing levels all consisting only of $\otimes$-vertices. Clearly $p(\xi) \in \widetilde{\mathtt{Tam}}_2^{\mathbb{N}}$, so it makes sense to put $\overline{\xi} := \phi\big(p(\xi)\big)$. It is then clear that the rule $\xi \mapsto \big(\Omega(\xi), \phi(p(\xi))\big)$ defines a map $\varrho : \widetilde{\mathtt{Tm}}_2^{\mathbb{N}} \to \mathtt{Tm}_2^{\mathbb{N}}$.

Let us construct its inverse $\varsigma : \mathtt{Tm}_2^{\mathbb{N}} \to \widetilde{\mathtt{Tm}}_2^{\mathbb{N}}$. Suppose that $(\boldsymbol{\mathfrak{T}}, \delta) \in \mathtt{Tm}_2^{\mathbb{N}}$. As we already observed, $(p(\boldsymbol{\mathfrak{T}}), \delta) \in \mathtt{Tam}_2^{\mathbb{N}}$, so we may use the previous construction and consider, as an intermediate step, the tree $\zeta := \psi\big(p(\boldsymbol{\mathfrak{T}}), \delta\big) \in \widetilde{\mathtt{Tam}}_2^{\mathbb{N}}$. The tree $\xi = \varsigma(\boldsymbol{\mathfrak{T}}, \delta)$ will be constructed by adding additional levels of $\otimes$-vertices of arity 1 to ζ as follows.

Let $\mathrm{tr}_1(\boldsymbol{\mathfrak{T}}) = \{1, \ldots, u\}$ and $\mathrm{tr}_1\big(p(\boldsymbol{\mathfrak{T}})\big) = \{1, \ldots, t\}$. If $t = u$, there is nothing to do as $\boldsymbol{\mathfrak{T}}$ is pruned; in this case we take $\xi := \zeta$. Assume therefore that $t < u$.

Since $\mathrm{tr}_1\big(p(\boldsymbol{\mathfrak{T}})\big)$ is a subset of $\mathrm{tr}_1(\boldsymbol{\mathfrak{T}})$, we have an inclusion $\iota : \{1, \ldots, t\} \hookrightarrow \{1, \ldots, u\}$. The complement $\{1, \ldots, u\} \setminus \mathrm{Im}(\iota)$ is the disjoint union $S_1 \cup \cdots \cup S_k$ of non-empty intervals. For instance, for $\boldsymbol{\mathfrak{T}}$ as Figure 6, $\mathrm{tr}_1(\boldsymbol{\mathfrak{T}}) = \{1, \ldots, 8\}$, $\mathrm{tr}_1\big(p(\boldsymbol{\mathfrak{T}})\big) = \{1, 2, 3\}$, $\mathrm{Im}(\iota) = \{2, 5, 7\}$, so

$$\{1, \ldots, 8\} \setminus \mathrm{Im}(\iota) = (1) \cup (3, 4) \cup (6) \cup (8).$$

For i, $1 \leq i \leq k$, such that $t \notin S_i$, let $r_i := \iota^{-1}(\max(S_i) + 1)$. In the example above, $r_1 = 1$, $r_2 = 2$ and $r_3 = 3$. For each such an i we add to ζ $\mathrm{card}(S_i)$ new levels consisting of $\otimes$-vertices of arity 1 placed above the r_ith level of ζ so close to it that all vertices of ζ above this level are also above these newly introduced levels. If $t \in S_i$[10] we introduce $\mathrm{card}(S_i)$ new levels of $\otimes$-vertices of arity 1 intersecting the input leaves of ζ.

We denote the resulting tree by ξ and define $\varsigma(\boldsymbol{\mathfrak{T}}, \delta) := \xi$. We believe that Figure 8 makes the construction of β out of ζ obvious. It is also clear that the maps $\varrho : \widetilde{\mathtt{Tm}}_2^{\mathbb{N}} \to \mathtt{Tm}_2^{\mathbb{N}}$ and $\varsigma : \mathtt{Tm}_2^{\mathbb{N}} \to \widetilde{\mathtt{Tm}}_2^{\mathbb{N}}$ are inverse to each other, showing that $\widetilde{\mathtt{Tm}}_2^{\mathbb{N}} \cong \mathtt{Tm}_2^{\mathbb{N}}$. This finishes the proof. □

[10] This may obviously happen only when $i = k$.

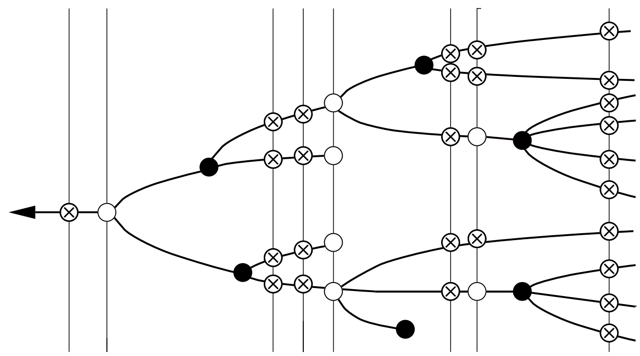
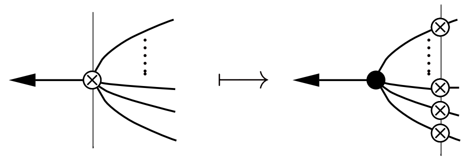
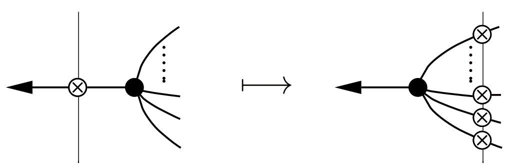


FIGURE 8. The tree $\xi = \varsigma(\mathfrak{T}, \delta) \in \widetilde{\mathtt{Tm}}_2^{\mathbb{N}}$.

In the description of Lemma 7.5, the w-image of a leveled tree $\beta \in \mathtt{LTr}$ in $\mathtt{Tm}_2^{\mathbb{N}}$ can be obtained in four steps. First we forget all type-(ii) levels of β, so the $\bullet$-vertices are allowed to move freely. In the second step we split all $\otimes$-vertices of arity $n > 1$ into one $\bullet$-vertex of arity 1 followed by n $\otimes$-vertices of arities 1, graphically:



The third step removes all internal edges starting at a $\bullet$-vertex and ending at a $\otimes$-vertex by allowing $\bullet$-vertices to penetrate through $\otimes$-vertices as in:



In the final step we contract all edges connecting two horizontal vertices and remove horizontal vertices of arity 1. The result is the image $w(\beta)$.

We leave as an exercise to show that if we apply the above steps to the tree β in Figure 2, we get the tree $w(\beta)$ in Figure 9. The following lemma describes all β's in $\mathtt{LTr}$ with the same w-image.

Lemma 7.6. *Let $\beta', \beta'' \in \mathtt{LTr}$. Then $w(\beta') = w(\beta'')$ if and only if β'' is obtained from β' by a finite sequence of the following elementary moves and their inverses:*

(i) *introducing a new level of horizontal vertices of arity one,*
(ii) *choosing two adjacent levels of horizontal vertices and contracting all edges connecting vertices in these two chosen levels, creating one level of horizontal vertices,*
(iii) *replacing an arity-1 vertical vertex followed by an arity-n horizontal vertex with an arity-n vertical vertex followed by an arity 1 horizontal vertex:*

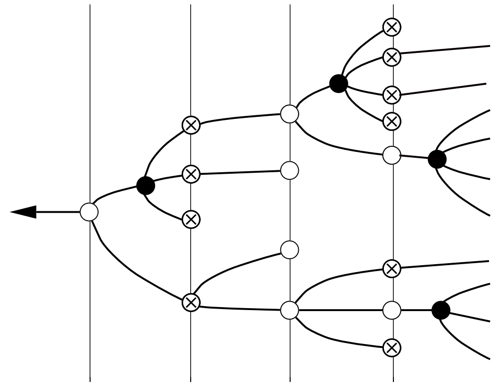
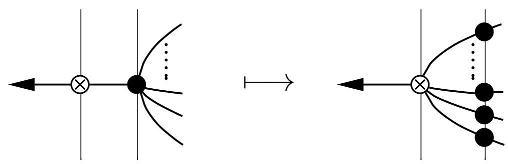
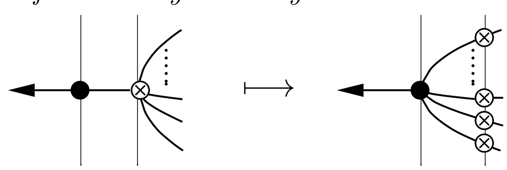


FIGURE 9. The w-image of the tree $\beta \in \mathtt{LTr}$ from Figure 2.



(iv) *replacing an arity-*1 *horizontal vertex followed by an arity-n vertical vertex with an arity-n horizontal vertex followed by an arity* 1 *vertical vertex:*



Notice that moves (iii) and (iv) are 'local' in that they do not change the level structure of β and that one can be obtained from the other by interchanging the rôles of $\otimes$ and $\bullet$.

Proof of Lemma 7.6. It is immediate to see that none of the moves changes the w-images. Therefore, if β' and β'' differ by a sequence of the moves and their inverses, $w(\beta') = w(\beta'')$.

To prove the opposite implication, let us say that a leveled tree $\beta \in \mathtt{LTr}$ is in the *canonical form*, if β has no levels consisting only of $\bullet$-vertices of arity 1, and no internal edges as in (68). It is obvious that, if $\beta', \beta'' \in \mathtt{LTr}$ are in the canonical form, then $w(\beta') = w(\beta'')$ if and only if $\beta' = \beta''$ in $\mathtt{LTr}$. The proof is finished by observing that each $\beta \in \mathtt{LTr}$ can be brought to the canonical form by a finite sequence of moves (i)–(iv) and their inverses. $\square$

Lemma 7.7. *Let $\mathscr{A}$ be a multiplicative* 1*-operad $\mathscr{A}$ in a duoidal category $\mathscr{D}$. If two leveled trees $\beta', \beta'' \in \mathtt{LTr}$ differ by a finite sequence of elementary moves listed in Lemma 7.6, then the structure morphisms*

$$\Xi_{\beta'}(I) : \mathscr{A}^{\mathfrak{T}} \to \mathscr{A}(n) \quad \text{and} \quad \Xi_{\beta''}(I) : \mathscr{A}^{\mathfrak{T}} \to \mathscr{A}(n),$$

where $\mathfrak{T} := \Omega(\beta') = \Omega(\beta'')$ and $\Xi = \{\Xi_\beta\}_{\beta \in \mathtt{LTr}}$ is the morphism (51), are equal.

Proof. Expanding the definitions we see that we must establish that the diagram

(69)
$$\begin{array}{ccccccc} & & \mathscr{A}^{\beta'} & \xrightarrow{\ \omega_{\beta'}\ } & \mathscr{A}(\beta') & & \\ & \overset{\theta_{\beta''}}{\nearrow} & & & & \overset{\gamma_{\beta'}}{\searrow} & \\ \mathscr{A}^{\mathfrak{T}} & & & & & & \mathscr{A}(n) \\ & \underset{\theta_{\beta'}}{\searrow} & & & & \underset{\gamma_{\beta''}}{\nearrow} & \\ & & \mathscr{A}^{\beta''} & \xrightarrow{\ \omega_{\beta''}\ } & \mathscr{A}(\beta'') & & \end{array}$$

in which $\theta_{\beta'}, \theta_{\beta''}$ are the maps (47), $\omega_{\beta'}, \omega_{\beta''}$ the maps (53) and $\gamma_{\beta'}, \gamma_{\beta''}$ the operad compositions, commutes for each move of Lemma 7.6.

Move (i). Assume that β'' is obtained from β' by adding a level of horizontal vertices of arity 1. It follows from the defining formula (45) that there are some $\mathscr{A}_l^{\beta'}, \mathscr{A}_r^{\beta'} \in \mathscr{D}$ such that

$$\mathscr{A}^{\beta'} = \mathscr{A}_l^{\beta'} \Box_0 \mathscr{A}_r^{\beta'} \ \text{ while } \ \mathscr{A}^{\beta''} = \mathscr{A}_l^{\beta'} \Box_0 (e \Box_1 \cdots \Box_1 e) \Box_0 \mathscr{A}_r^{\beta'}.$$

Likewise, it follows from (52) that there are some $\mathscr{A}_l(\beta'), \mathscr{A}_r(\beta') \in \mathscr{D}$ such that

$$\mathscr{A}(\beta') = \mathscr{A}_l(\beta') \Box_0 \mathscr{A}_r(\beta') \ \text{ while } \mathscr{A}(\beta'') = \mathscr{A}_l(\beta') \Box_0 \big(\mathscr{A}(1) \Box_1 \cdots \Box_1 \mathscr{A}(1)\big) \Box_0 \mathscr{A}_r(\beta').$$

The unitality axiom [9, Def. 4.1] for 1-operads then implies the commutativity of the diagram

$$\begin{array}{ccccc} \mathscr{A}_l^{\beta'} \Box_0 \mathscr{A}_r^{\beta'} & \xrightarrow{\ \omega\ } & \mathscr{A}_l(\beta') \Box_0 \mathscr{A}_r(\beta') & & \\ \downarrow \cong & & & \overset{\gamma}{\searrow} & \\ \mathscr{A}_l^{\beta'} \Box_0 e \Box_0 \mathscr{A}_r^{\beta'} & & & & \mathscr{A}(n) \\ \downarrow & & & \underset{\gamma}{\nearrow} & \\ \mathscr{A}_l^{\beta'} \Box_0 (e \Box_1 \cdots \Box_1 e) \Box_0 \mathscr{A}_r^{\beta'} & \xrightarrow{\ \omega\ } & \mathscr{A}_l(\beta') \Box_0 \big(\mathscr{A}(1) \Box_1 \cdots \Box_1 \mathscr{A}(1)\big) \Box_0 \mathscr{A}_r(\beta') & & \end{array}$$

which, along with the obvious commutativity of

$$\begin{array}{ccc} & & \mathscr{A}_l^{\beta'} \Box_0 \mathscr{A}_r^{\beta'} \\ & \overset{\theta_{\beta'}}{\nearrow} & \downarrow \cong \\ \mathscr{A}^{\mathfrak{T}} & & \mathscr{A}_l^{\beta'} \Box_0 e \Box_0 \mathscr{A}_r^{\beta'} \\ & \underset{\theta_{\beta''}}{\searrow} & \downarrow \\ & & \mathscr{A}_l^{\beta'} \Box_0 (e \Box_1 \cdots \Box_1 e) \Box_0 \mathscr{A}_r^{\beta'} \end{array}$$

implies the commutativity of (69).

Move (ii). By analyzing the definitions of the objects involved in (69), we easily see that its commutativity would follow from the commutativity of the diagram

(70)
$$\begin{array}{ccc} (e \Box_1 \cdots \Box_1 e) \Box_0 e & \longrightarrow & (v \Box_1 \cdots \Box_1 v) \Box_0 v \\ \downarrow & & \downarrow \cong \\ e & \longrightarrow & v \end{array}$$

whose left vertical arrow is constructed in [9, Example 4.5], and the remaining arrows are induced by the monoid structure of v and by the canonical map $e \to v$.

Move (iii). The commutativity of (69) would in this case follow from the commutativity of

$$
\begin{array}{ccc}
(e \,\square_1 \cdots \square_1 e) \,\square_0 v & \longrightarrow & (v \,\square_1 \cdots \square_1 v) \,\square_0 v \\
\uparrow & & \downarrow{\scriptstyle \cong} \\
e \,\square_0 v & \longrightarrow & v \,\square_0 v
\end{array}
\tag{71}
$$

whose maps are induced by the comonoid structure of e, monoid structure of v and the canonical map $e \to v$.

Move (iv). The commutativity of (69) would follow from the commutativity of the diagram

$$
\begin{array}{ccc}
(v \,\square_1 \cdots \square_1 v) \,\square_0 e & \longrightarrow & (v \,\square_1 \cdots \square_1 v) \,\square_0 v \\
{\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\
v \,\square_0 e & \longrightarrow & v \,\square_0 v
\end{array}
\tag{72}
$$

whose arrows are given by the module structure of v and the canonical map $e \to v$. The commutativity of diagrams (70)–(72) above is however an easy consequence of general properties of duoidal categories. □

8. Duoidal Deligne's conjecture

Let $\mathscr{D}$ be a complete V-category and $\delta : \Delta \to V$ a cosimplicial object in V. Recall [9, §5.2] that the *δ-totalization* of a cosimplicial object $\phi : \Delta \to \mathscr{D}$ is the V-enriched end

$$Tot_\delta(\phi) := \int_{n \in \Delta} \phi(n)^{\delta(n)} \in \mathscr{D}.$$

By Proposition 5.2 of [9], any multiplicative 1-operad $\mathscr{A}$ in $\mathscr{D}$ bears a canonical structure of a cosimplicial object $\mathring{A} = \{\mathscr{A}(n), n \geq 0\}$ in $\mathscr{D}$. In Definition 5.3 of [9] we introduced the *Hochschild δ-object* of a $\mathscr{A}$ as the δ-totalization

$$CH_\delta(\mathscr{A}) := Tot_\delta(\mathring{A}).$$

We claim that the canonical cosimplicial structure on $\mathscr{A}$ is induced by the action (60) of the 2-operad $\mathcal{T}m_2^{\mathbb{N}}$. By this we mean that the underlying category $\mathtt{U}(\mathcal{T}m_2^{\mathbb{N}})$ is the simplicial category Δ and that $\mathring{A} = \iota^*(\mathscr{A})$, where ι^* is the functor (22) with $\mathcal{P} = \mathcal{T}m_2^{\mathbb{N}}$ and $\mathtt{C} = \mathscr{D}$.

It follows from definition that the objects of the underlying category $\mathtt{U}(\mathcal{T}m_2^{\mathbb{N}})$ are natural numbers. Its hom-sets are

$$\mathtt{U}(\mathcal{T}m_2^{\mathbb{N}})(i,n) = \mathtt{Tm}_2^{\mathbb{N}}\big(\mathcal{U}_2(i,n)\big),\ \ i,n \in \mathbb{N},$$

where $\mathcal{U}_2(i,n)$ is the terminal 2-tree $\mathcal{U}_2$ with its unique 2-leaf colored by i and the root by n. Morphisms in $\mathtt{U}(\mathcal{T}m_2^{\mathbb{N}})(i,j)$ are thus represented by trees as in Lemma 7.5 with one $\bigcirc$-vertex of arity i, no $\otimes$-vertices, and n leaves. It is obvious from this description that

$\mathcal{T}m_2^{\mathbb{N}}$ and $\mathcal{L}^{(2)}$ have isomorphic underlying categories, while $\mathsf{U}(\mathcal{L}^{(2)}) \cong \Delta$ by [7, Lemma 2.5]. The identification $\mathring{A} = \iota^*(\mathscr{A})$ is now a simple exercise. We conclude that

$$CH_\delta(\mathscr{A}) \cong Tot_\delta(\mathscr{A}),$$

where $Tot_\delta(\mathscr{A})$ is given by (23).

As we already recalled from [9, Prop. 4.9], the endomorphism 1-operad $\mathcal{E}nd_{\mathsf{M}}$ of a monoid M in a $\mathscr{D}$-monoidal category is multiplicative. The Hochschild δ-object of $\mathcal{E}nd_{\mathsf{M}}$ was called the *δ-center* of M and denoted

$$CH_\delta(\mathsf{M},\mathsf{M}) := CH_\delta(\mathcal{E}nd_{\mathsf{M}}).$$

Since V is a cocomplete symmetric monoidal category it has Set-tensors. For a set S the tensor $S \otimes X$ is equal to the coproduct $\coprod_S X$ of S-copies of X. For any operadic category $\mathtt{O}$ and any $\mathtt{O}$-operad $\mathcal{P}$ in Set we can construct then an enrichment of $\mathcal{P}$ in V which on an object $T \in \mathtt{O}$ takes value $\mathcal{P}(T) \otimes I$. By abusing notations we will denote such an enriched $\mathtt{O}$-operad by the same letter $\mathcal{P}$. In particular we will consider the operad $\mathcal{T}m_2^{\mathbb{N}}$ as a colored 2-operad in V for any cocomplete V.

Proposition 4.6 immediately gives:

Theorem 8.1. *Let δ be a cosimplicial object in V. Then there is a canonical action of the 2-operad $co\mathcal{E}nd_\delta^{\mathcal{T}m_2^{\mathbb{N}}}$ on the Hochschild object $CH_\delta(\mathscr{A})$ of a multiplicative 1-operad $\mathscr{A}$. In particular, there is a canonical action of $co\mathcal{E}nd_\delta^{\mathcal{T}m_2^{\mathbb{N}}}$ on the δ-center $CH_\delta(\mathsf{M},\mathsf{M})$ of any monoid M in any $\mathscr{D}$-monoidal category*

When V is the category of chain complexes, $co\mathcal{E}nd_\delta^{\mathcal{T}m_2^{\mathbb{N}}}$ is the chain 2-operad $\emptyset$ considered by Tamarkin in [30, §5.2]. Let I be the constant cosimplicial object whose all terms equal the unit object $I \in V$.

Proposition 8.2. *The 2-operad $co\mathcal{E}nd_I^{\mathcal{T}m_2^{\mathbb{N}}}$ is isomorphic to the canonical 2-operad $\mathfrak{I}^{\Omega_2}$.*

Proof. It is sufficient to observe that the value of the multitensor $E_{\mathfrak{T}}^{\mathcal{T}m_2^{\mathbb{N}}}(I,\ldots,I)$ is the constant cosimplicial object I for any $\mathfrak{T} \in \Omega_2$. Indeed, if it is so, then clearly

$$co\mathcal{E}nd_I^{\mathcal{T}m_2^{\mathbb{N}}} = Nat(I,I) = V(I,I) \cong I$$

where $Nat(I,I)$ means the space of natural transformations (i.e. cosimplicial maps) between the constant cosimplicial objects I.

It is clear that for each $n \geq 0$ the coend $E_{\mathfrak{T}}^{\mathcal{T}m_2^{\mathbb{N}}}(I,\ldots,I)(n)$ in (19) equals the colimit of the k-simplicial object $\mathcal{T}m_2^{\mathbb{N}}(\mathfrak{T})(\bullet,\ldots,\bullet;n) \otimes I$, $k := |\mathfrak{T}|$, so it is enough to check that the colimit of the k-simplicial set $\mathcal{T}m_2^{\mathbb{N}}(\mathfrak{T})(\bullet,\ldots,\bullet;n)$ is a one point set. This boils down to verification that the equivalence relation generated by the simplicial operators on $\mathcal{T}m_2^{\mathbb{N}}(\mathfrak{T})(0,\ldots,0;n)$ has only one equivalence class.

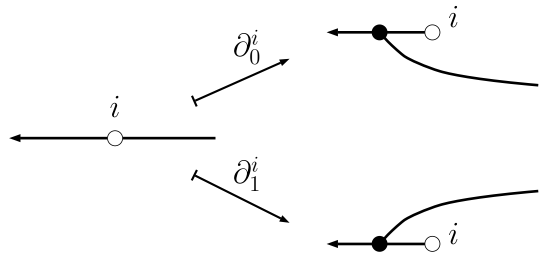


FIGURE 10. The local surgery defining the operators $\partial_0^i, \partial_1^i$ in (73).

Notice that, by the definition (18),

$$\mathcal{T}m_2^{\mathbb{N}}(\mathfrak{T})(i_1, \ldots, i_k; n) = \mathcal{T}m_2^{\mathbb{N}}(\mathbf{\mathfrak{T}}_n^{i_1,\ldots,i_k}),$$

where $\mathbf{\mathfrak{T}}_n^{i_1,\ldots,i_k} \in \Omega_2^{\mathbb{N}}$ is the 2-tree $\mathfrak{T} \in \Omega_2$ with its 2-leaves colored by $i_1, \ldots, i_k \in \mathbb{N}$ and the root by n. The elements of $\mathcal{T}m_2^{\mathbb{N}}(\mathfrak{T})(0, \ldots, 0; n)$ are thus represented by $\mathfrak{T}$-labelled leveled trees whose $\circ$-vertices have no incoming edges. In this description, the two simplicial operators

$$\partial_0^i, \partial_1^i : \mathcal{T}m_2^{\mathbb{N}}(\mathbf{\mathfrak{T}}_n^{0,\ldots,1,\ldots,0}) \to \mathcal{T}m_2^{\mathbb{N}}(\mathbf{\mathfrak{T}}_n^{0,\ldots,0}),\ 1 \leq i \leq k, \tag{73}$$

in i-th direction are local operations acting on a tree $\xi \in \mathcal{T}m_2^{\mathbb{N}}(\mathbf{\mathfrak{T}}_n^{0,\ldots,0})$ by the following surgery: 'cut off' the unique incoming edge of the ith $\circ$-vertex of ξ and then 'glue' it to the outcoming edge of this vertex in two possible ways introducing a new $\bullet$-vertex, as indicated in Figure 10 – compare with the differential in the brace operad [8, Example 5.8]. It is simple to prove by induction that any two trees from $\mathcal{T}m_2^{\mathbb{N}}(\mathfrak{T})(0, \ldots, 0; n)$ can be connected by a zig-zag of such elementary surgery operations, so the colimit of $\mathcal{T}m_2^{\mathbb{N}}(\mathfrak{T})(\bullet, \ldots, \bullet; n)$ is a one-point set as claimed. □

The center of a monoid M in a $\mathscr{D}$-monoidal category $\mathscr{K}$, defined as $Z(M) := CH_I(\mathsf{M}, \mathsf{M})$ or, more explicitly, as the equalized (30), has a canonical structure of a duoid in $\mathscr{D}$ by [9, Theorem 5.6]. On the other hand, according to [9, Example 11.15], duoids in $\mathscr{D}$ are the same as $\mathfrak{I}^{\Omega_2}$-algebras in $\mathscr{D}$.[11] The following proposition is easy to prove.

Proposition 8.3. *The action of the operad* $co\mathcal{E}nd_I^{\mathcal{T}m_2^{\mathbb{N}}} \cong \mathfrak{I}^{\Omega_2}$ *equips* $Z(\mathsf{M})$ *with its canonical duoid structure.*

Let now V be a monoidal model category and δ be a *standard system of simplices* for V in the sense of [16, Definition A.6]. Recall that this means that

(i) δ is cofibrant for the Reedy model structure on V^{Δ},
(ii) δ^0 is the unit object I of V and the simplicial operators $[m] \to [n]$ act via weak equivalences $\delta^m \to \delta^n$ in V, and

[11] The operad $\mathfrak{I}^{\Omega_2}$ was denoted $\underline{\mathcal{A}ss}_2$ in [9].

(iii) the simplicial realization functor $|\text{-}|_\delta = (\text{-}) \otimes_{\Delta} \delta : V^{\Delta^{op}} \to V$ is a symmetric monoidal functor whose structural maps

$$|X|_\delta \otimes_V |Y|_\delta \to |X \otimes_V Y|_\delta$$

are weak equivalences for Reedy-cofibrant objects $X, Y \in V^{\Delta^{op}}$.

Since $E_{\mathfrak{T}}^{\mathcal{T}m_2^{\mathbb{N}}}(I, \ldots, I) = I$, the canonical map of cosimplicial objects $\delta \to I$ induces a map of 2-operads

$$co\mathcal{E}nd_\delta^{\mathcal{T}m_2^{\mathbb{N}}} \longrightarrow co\mathcal{E}nd_I^{\mathcal{T}m_2^{\mathbb{N}}}.$$

Theorem 8.4. *Let δ be a standard system of simplices for a monoidal model category V such that the lattice path operad is strongly δ-reductive in the sense of* [7, Definition 3.7]. *Then the canonical morphism of 2-operads*

$$co\mathcal{E}nd_\delta^{\mathcal{T}m_2^{\mathbb{N}}} \longrightarrow co\mathcal{E}nd_I^{\mathcal{T}m_2^{\mathbb{N}}} \cong \mathfrak{I}^{\Omega_2} \tag{74}$$

is a weak equivalence. In other words, the 2-operad $co\mathcal{E}nd_\delta^{\mathcal{T}m_2^{\mathbb{N}}}$ is contractible.

Proof. The proof follows closely the proof of Theorem 3.8 from [7], with the simplification that we do not need to take a colimit over the complete graph operads. The only fact we should know is that the map of 0-objects

$$f^0 : (E_{\mathfrak{T}}^{Tam_2^{\mathbb{N}}}(\delta, \ldots, \delta))^0 \to I^0 = I$$

is a weak equivalence for every $\mathfrak{T}$. In the notation used in the proof of [7, Theorem 3.8], the object on the left side is the same as $\xi_{(\mu,\sigma)}(\delta)^0$ for (μ, σ) equal to $a_{\mathfrak{T}} \in \mathcal{K}^{(2)}$. It is shown at the end of the proof of [7, Theorem 3.8] that the map f^0 is a weak equivalence. □

The map $\delta \to I$ induces a canonical map

$$Z(M) = CH_I(\mathsf{M}, \mathsf{M}) \longrightarrow CH_\delta(\mathsf{M}, \mathsf{M}), \tag{75}$$

The map (74) equips the duoid $Z(\mathsf{M})$ with a structure of $co\mathcal{E}nd_\delta^{\mathcal{T}m_2^{\mathbb{N}}}$-algebra such that (75) becomes a map of $co\mathcal{E}nd_\delta^{\mathcal{T}m_2^{\mathbb{N}}}$-algebras.

Let $F(\mathsf{M})$ be a fibrant replacement of M in the category of monoids with the projective model structure, and δ a standard system of simplices for V. The δ-center $CH_\delta\big(F(\mathsf{M}), F(\mathsf{M})\big)$ of $F(\mathsf{M})$ was called in [9] the *homotopy center* of M. The above considerations imply the central result of our paper:

Corollary 8.5 (Duoidal Deligne's conjecture)**.** *Under the assumptions of Theorem 8.4, the Hochschild δ-object of a multiplicative 1-operad in a duoidal category $\mathscr{D}$ admits an action of a contractible 2-operad.*

The homotopy center of a monoid M in a multiplicative $\mathscr{D}$-category admits an action of a contractible 2-operad that lifts the duoid structure on the center $Z(\mathsf{M})$.

The assumptions of Theorem 8.4 are satisfied for instance when V is the category of compactly generated topological spaces or chain complexes over a commutative ring, and δ the cosimplicial space of topological simplices or normalized cellular chains on topological simplices, respectively, see [7, Examples 3.10(a),(c)]. It is also not difficult to show that these assumptions are satisfied for $V = \mathtt{Cat}$ with the Joyal-Tirney model structure and δ the cosimplicial chaotic groupoid on finite sets, cf. [9, Example 5.10].

On the other hand, it was shown in [7, Example 3.10(b)] that for the category of simplicial sets and $\delta = \delta_{Yon}$ the cosimplicial simplicial set of representables, the assumption of strongly δ-reductivity of the lattice operad fails. We however believe that the second part of Corollary 8.5 remains true even without this assumption, because taking fibrant replacement of M should counterweight the poor homotopical property of δ. We leave this refined version of Deligne's conjecture as a subject for a future work.

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Macquarie University, NSW 2109, Australia

E-mail address: `michaelbatanin@mq.edu.au`

Mathematical Institute of the Academy, Žitná 25, 115 67 Prague 1, The Czech Republic

Faculty of Mathematics and Physics, Charles University, 186 75 Sokolovská 83, Prague 8, The Czech Republic

E-mail address: `markl@math.cas.cz`