ON TOPOLOGICAL UPPER-BOUNDS ON THE NUMBER OF SMALL CUSPIDAL EIGENVALUES

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Abstract. Let S be a noncompact, finite area hyperbolic surface of type (g, n). Let Δ_S denote the Laplace operator on S. As S varies over the *moduli space* $\mathcal{M}_{g,n}$ of finite area hyperbolic surfaces of type (g, n), we study, adapting methods of Lizhen Ji [Ji] and Scott Wolpert [Wo], the behavior of *small cuspidal eigenpairs* of Δ_S. In Theorem 2 we describe limiting behavior of these eigenpairs on surfaces $S_m \in \mathcal{M}_{g,n}$ when (S_m) converges to a point in $\overline{\mathcal{M}_{g,n}}$. Then we consider the i-th *cuspidal eigenvalue*, $\lambda_i^c(S)$, of $S \in \mathcal{M}_{g,n}$. Since *non-cuspidal* eigenfunctions (*residual eigenfunctions* or *generalized eigenfunctions*) may converge to cuspidal eigenfunctions, it is not known if $\lambda_i^c(S)$ is a continuous function. However, applying Theorem 2 we prove that, for all $k \geq 2g-2$, the sets

$$\mathcal{C}^{\frac{1}{4}}_{g,n}(k) = \{S \in \mathcal{M}_{g,n} : \lambda_k^c(S) > \frac{1}{4}\}$$

are open and contain a neighborhood of $\cup_{i=1}^{n} \mathcal{M}_{0,3} \cup \mathcal{M}_{g-1,2}$ in $\overline{\mathcal{M}_{g,n}}$. Moreover, using topological properties of nodal sets of *small eigenfunctions* from [O], we show that $\mathcal{C}^{\frac{1}{4}}_{g,n}(2g-1)$ contains a neighborhood of $\mathcal{M}_{0,n+1} \cup \mathcal{M}_{g,1}$ in $\overline{\mathcal{M}_{g,n}}$. These results provide evidence in support of a conjecture of Otal-Rosas [O-R].

1. Introduction

In this paper a *hyperbolic surface* is a two dimensional complete Riemannian manifold S with sectional curvature equal to -1. Such a surface is isomorphic to the quotient $\mathbb{H}/\Gamma$, of the Poincaré upper halfplane $\mathbb{H}$ by a *Fuchsian group* Γ, i.e. a discrete torsion-free subgroup of PSL(2,$\mathbb{R}$). The *Laplace operator* on $\mathbb{H}$ is the differential operator which associates to a C^2-function f the function

$$\Delta f(z) = y^2(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}).$$

Since the action of PSL(2, $\mathbb{R}$) on $\mathbb{H}$ leaves Δ invariant, Δ induces a differential operator on $S = \mathbb{H}/\Gamma$ which extends to a self-adjoint operator Δ_S densely defined on $L^2(S)$. It is a general fact that the Laplace operator is a non-positive operator whose spectrum is contained in the smallest interval $(-\infty, -\lambda_0(S)] \subset \mathbb{R}^- \cup \{0\}$ with $\lambda_0(S) \geq 0$.

Definition 1.1. Let $\lambda > 0$ be a real number and $f \in L^2(S)$ be a nonzero function on S. The pair (λ, f) is called an eigenpair of S if $\Delta_S f + \lambda f \equiv 0$ on S where λ and f are respectively called an *eigenvalue* and an *eigenfunction*

Date: June 5, 2014.
1991 *Mathematics Subject Classification.* ...
Key words and phrases. Laplace operator, eigenvalues, ...

(sometimes a λ-*eigenfunction*). When $0 < \lambda \leq 1/4$, we add the adjective *small* i.e. (λ, f), λ and f are respectively called a *small eigenpair*, a *small eigenvalue* and a *small eigenfunction.*

We begin with a noncompact, finite area hyperbolic surface S of type (g,n) i.e. $S \in \mathcal{M}_{g,n}$. The Laplace spectrum of such a surface is composed of two parts: *the discrete part* and *the continuous part* [I]. The continuous part covers the interval $[\frac{1}{4}, \infty)$ and is spanned by *Eisenstein series* with multiplicity n. Eisenstein series are not eigenfunctions although they satisfy

$$\Delta E(.,s) + s(1-s)E(.,s) = 0,$$

because they are not in L^2. For this reason, they are called *generalized eigenfunctions.* The discrete spectrum consists of eigenvalues. They are distinguished into two parts: *the residual spectrum* and *the cuspidal spectrum.* An eigenpair (λ, f) is called *residual* if f is a linear combination of residues of meromorphic continuations of Eisenstein series. Such λ and f are respectively called a *residual eigenvalue* and a *residual eigenfuction.* The residual spectrum is a finite set contained in $[0, \frac{1}{4})$. On the other hand, an eigenpair (λ, f) is called *cuspidal* if f tends to zero at each cusp. In this case λ and f are respectively called a *cuspidal eigenvalue* and a *cuspidal eigenfuction.* These eigenvalues with multiplicity are arranged by increasing order and we denote $\lambda_n^c(S)$ the n-th cuspidal eigenvalue of S. For an arbitrary Fuchsian group Γ, it is not known whether the cardinality of the set of cuspidal eigenvalues of $\mathbb{H}/\Gamma$ is infinite. However, a famous theorem of A. Selberg says that it is the case when Γ is arithmetic. Any cuspidal eigenpair (λ, f) with $\lambda \leq \frac{1}{4}$ is called a *small cuspidal eigenpair* and in that case, λ and f are respectively called a *small cuspidal eigenvalue* and a *small cuspidal eigenfunction.*

In [O-R], Jean-Pierre Otal and Eulalio Rosas proved that the total number of small eigenvalues of any hyperbolic surface of type (g,n) is at most $2g - 3 + n$. In the same paper they formulate the following:

Conjecture. *Let S be a noncompact, finite area hyperbolic surface of type (g,n). Then $\lambda_{2g-2}^c(S) > \frac{1}{4}$.*

This conjecture is motivated by the following two results

Proposition 1.2. ***(Huxley** [Hu]**, Otal** [O]**)** Let S be a finite area hyperbolic surface of genus 0 or 1. Then S does not carry any small cuspidal eigenpair.*

Proposition 1.3. ***(Otal** [O]**)** Let S be a finite area hyperbolic surface of type (g,n). Then the multiplicity of a small cuspidal eigenvalue of S is at most $2g-3$.*

The set $\mathcal{M}_{g,n}$ carries a topology for which two surfaces $\mathbb{H}/\Gamma$ and $\mathbb{H}/\Gamma'$ are close when the groups Γ and Γ' can be conjugated inside $\mathrm{PSL}(2,\mathbb{R})$ so that they have generators which are close. With this topology $\mathcal{M}_{g,n}$ is not compact. However it can be compactified by adjoining $\cup_i \mathcal{M}_{g_i,n_i}$'s for each $(g_1,n_1), ..., (g_k,n_k)$ with $2\sum_i^k (g_i - 2) + \sum_i^k n_i = 2g - 2 + n$. In this compactification a sequence $(S_m) \in \mathcal{M}_{g,n}$ converges to $S_\infty \in \overline{\mathcal{M}_{g,n}}$ if and only if for any given $\epsilon > 0$ the ϵ-thick part $(S_m^{[\epsilon,\infty)})$ converges to $S_\infty^{[\epsilon,\infty)}$ in the Gromov-Hausdorff topology. Recall that the ϵ-thick part of a surface S is the subset of those points of S where the *injectivity radius* is at least ϵ.

Recall also that the injectivity radius of a point $p \in S$ is the radius of the largest geodesic disc that can be embedded in S with center p.

For any $N \in \mathbb{N}$ and $t \in \mathbb{R}_{>0}$ we define the sets

$$\mathcal{C}^t_{g,n}(N) = \{S \in \mathcal{M}_{g,n} : \lambda^c_N(S) > t\}.$$

It is clear that $\mathcal{C}^{\frac{1}{4}}_{g,n}(k) \subset \mathcal{C}^{\frac{1}{4}}_{g,n}(k+1)$ for $k \geq 1$. With this notation the conjecture can be formulated by saying that

$$\mathcal{C}^{\frac{1}{4}}_{g,n}(2g-2) = \mathcal{M}_{g,n}.$$

In this paper, we study the sets $\mathcal{C}^{\frac{1}{4}}_{g,n}(k)$. The methods developed here are not sufficient to prove the conjecture but we show that the sets $\mathcal{C}^{\frac{1}{4}}_{g,n}(2g-2)$ and $\mathcal{C}^{\frac{1}{4}}_{g,n}(2g-1)$ ($\mathcal{C}^{\frac{1}{4}}_{g,n}(2g-2) \subseteq \mathcal{C}^{\frac{1}{4}}_{g,n}(2g-1)$) contains neighborhoods of certain strata in the compactification of $\mathcal{M}_{g,n}$.

Theorem 1.4. ***(i)*** *For any integer k, $\mathcal{C}^{\frac{1}{4}}_{g,n}(k)$ is an open subset of $\mathcal{M}_{g,n}$.*
(ii) *$\mathcal{C}^{\frac{1}{4}}_{g,n}(2g-2)$ contains a neighborhood of $\cup_{i=1}^n \mathcal{M}_{0,3} \cup \mathcal{M}_{g-1,2}$ in $\overline{\mathcal{M}_{g,n}}$.*
(iii) *$\mathcal{C}^{\frac{1}{4}}_{g,n}(2g-1)$ contains a neighborhood of $\mathcal{M}_{0,n+1} \cup \mathcal{M}_{g,1}$ in $\overline{\mathcal{M}_{g,n}}$.*

Observe that it is theoretically possible for a residual eigenfunction to converge to a cuspidal eigenfunction. Therefore indicating that λ^c_{2g-1} may not be continuous. Also, the result [P-S] suggest that λ^c_{2g-1} may not be continuous at those $S \in \mathcal{M}_{g,n}$ where it takes value strictly more than $\frac{1}{4}$. Therefore, the first assertion is not completely trivial.

The paper is organized as follows. In §1 we recall some preliminaries for convergence of hyperbolic surfaces in $\overline{\mathcal{M}_{g,n}}$. In §2 and §3 we study convergence properties of eigenpairs on converging hyperbolic surfaces. Similar study has already been carried out by Scott Wolpert [Wo], Lizhen Ji [Ji] and Christopher Judge [J]. We shall first make precise the notions of convergence in $\overline{\mathcal{M}_{g,n}}$ and the notion of convergence of a sequence of functions on a converging sequence of surfaces.

1.1. **Convergence of functions.** Let (S_m) be a sequence of surfaces in $\mathcal{M}_{g,n}$ converging to a surface S_∞ in the compactification $\overline{\mathcal{M}_{g,n}}$. Another way of understanding this convergence is as follows:

Let $S_m = \mathbb{H}/\Gamma_m$ and let $0 < c_0 < \epsilon_0$ (ϵ_0 is the Margulis constant; see thick/thin decomposition for details) be a fixed constant. Let $x_m \in S_m^{[c_0,\infty)}$. Up to a conjugation of Γ_m in $\mathrm{PSL}(2,\mathbb{R})$, one may assume that $i \in \mathbb{H}$ is mapped to x_m under the projection $\mathbb{H} \to \mathbb{H}/\Gamma_m$. Then up to extracting a subsequence we may suppose that Γ_m converges to some Funchsian group Γ_∞. We say that the pair $(\mathbb{H}/\Gamma_m, x_m)$ converges to $(\mathbb{H}/\Gamma_\infty, x_\infty)$ where x_∞ is the image of $i \in \mathbb{H}$ under the projection $\mathbb{H} \to \mathbb{H}/\Gamma_\infty$. Let S_∞ be the hyperbolic surface of finite area whose connected components are the $\mathbb{H}/\Gamma_\infty$'s for different choices of base point x_m in different connected components of $S_m^{[c_0,\infty)}$. The surface S_∞ does not depend, up to isometry, on the choice of the base point x_m in a fixed connected component of $S_m^{[c_0,\infty)}$ (i.e. if y_m be a point in the same connected component of $S_m^{[c_0,\infty)}$ as x_m then the corresponding limiting surfaces are isometric). One can check that $(S_m) \to S_\infty$ in $\overline{\mathcal{M}_{g,n}}$.

Convergence of functions

Fix an $\epsilon > 0$ and choose a base point $x_m \in {S_m}^{[\epsilon,\infty)}$ for each m. Assume that the pair $(\mathbb{H}/\Gamma_m, x_m)$ converges to $(\mathbb{H}/\Gamma_\infty, x_\infty)$ where, for each $m \in \mathbb{N}\cup\{\infty\}$, the point $i \in \mathbb{H}$ maps to x_m under the projection $\mathbb{H} \to \mathbb{H}/\Gamma_m$.

For a C^∞ function f on S_m denote by $\widetilde{f}$ the lift of f under the projection $\mathbb{H} \to \mathbb{H}/\Gamma_m$. Let (f_m) be a sequence of functions in $C^\infty(S_m) \cap L^2(S_m)$. One says that (f_m) *converges to* a continuous function f_∞ if $\widetilde{f_m}$ converges, uniformly over compact subsets of $\mathbb{H}$, to $\widetilde{f_\infty}$ for each choice of base points $x_m \in {S_m}^{[\epsilon,\infty)}$ and for each $\epsilon < \epsilon_0$.

With the above understanding of convergence of functions we shall prove the following theorem which has close resemblance with [Ji, Theorem 1.2] and [Wo, Theorem 4.2]. However, our result does not follow from these. We would like to mention that a similar limiting theorem might not be true (see [Wo, p-71] if one considers $\lambda_m \geq \frac{1}{4}$ instead of $\lambda_m \leq \frac{1}{4}$ (see Theorem 2).

In the following, for a function $f \in L^2(S)$, we shall denote the L^2 norm of f by $\|f\|$. Also, for $f \in L^2(V)$ and $U \subset V$ we denote the L^2-norm of the restriction of f to U by $\|f\|_U$. A function $f \in L^2(V)$ will be called *normalized* if $\|f\| = 1$. An eigenpair (λ, ϕ) will be called normalized if ϕ is normalized.

Theorem 1.5. *Let $S_m \to S_\infty$ in $\overline{\mathcal{M}_{g,n}}$. Let (λ_m, ϕ_m) be a normalized small cuspidal eigenpair of S_m. Assume that λ_m converges to λ_∞. Then one of the following holds:*
(1) *There exist strictly positive constants ϵ, δ such that $\limsup \|\phi_m\|_{S_m^{[\epsilon,\infty)}} \geq \delta$. Then, up to extracting a subsequence, (ϕ_m) converges to a λ_∞-eigenfunction ϕ_∞ of S_∞.*
(2) *For each $\epsilon > 0$ the sequence $(\|\phi_m\|_{S_m^{[\epsilon,\infty)}}) \to 0$. Then $S_\infty \in \partial\mathcal{M}_{g,n}$ and $\lambda_\infty = \frac{1}{4}$. Moreover, there exist constants $K_m \to \infty$ such that, up to extracting a subsequence, $(K_m\phi_m)$ converges to a linear combination of Eisenstein series and (possibly) a cuspidal λ_∞-eigenfunction of S_∞.*

Remark 1.6. For $s = \frac{1}{2}$, by Eisenstein series we understand a linear combination of the following two:
(i) the classical (meromorphic continuation) Eisenstein series $E^i(., \frac{1}{2})$ corresponding to the cusps (i is the index for cusps) on the surface,
(ii) the derivatives $\frac{\partial}{\partial s}E^i(., s)|_{s=\frac{1}{2}}$ of $E^i(., s)$ at $s = \frac{1}{2}$.
The first Fourier coefficient of such functions in any cusp have the form $\alpha y^{\frac{1}{2}} + \beta y^{\frac{1}{2}} \log y$. Each moderate growth $\frac{1}{4}$-eigenfunction is a linear combination of Eisenstein series, in the above sense, and (possibly) a cuspidal eigenfunction.

Theorem 2 will be applied to prove all three statements of Theorem 1. The first one is a direct application; in §4 we prove:

Lemma 1 *For any $k \geq 1$, $\mathcal{C}_{g,n}^{\frac{1}{4}}(k)$ is an open subset of $\mathcal{M}_{g,n}$.*

The second statement of Theorem 1 is also an easy application of Theorem 2 and the Buser construction [Bu]: we explain it now since the proof is short. We argue by contradiction and assume that there is a sequence (S_m) in $\mathcal{M}_{g,n}$ such that S_m converges to $S_\infty \in \cup_{i=1}^n \mathcal{M}_{0,3} \cup \mathcal{M}_{g-1,2}$ and $\lambda_{2g-2}^c(S_m) \leq \frac{1}{4}$. Then S_∞ has exactly $n+1$ components of which exactly n are thrice punctured spheres. Observe that each component of S_∞ contains an *old cusp* i.e. cusps of S_∞ which are limits of cusps of S_m (see Proof of Theorem 2).

The construction used in the proof of [Bu, Theorem 8.1.3] implies that, for m large, S_m has at least n eigenvalues that converge to zero as m tends to infinity. Let us suppose by contradiction that one of the corresponding eigenfunctions ϕ_m is cuspidal. Then by Theorem 2, ϕ_m converges uniformly over compacta to a function ϕ and ϕ is an eigenfunction for the eigenvalue 0. So ϕ is constant in each component of S_∞. On those components of $S_\infty^{[\epsilon,\infty)}$ that contains an old cusp ϕ is necessarily zero because ϕ_m being cuspidal the average of ϕ_m over any horocycle is zero. On the the other component (the one that does not contain an old cusp) ϕ is zero because the mean of ϕ over S_∞ is equal to the mean of ϕ_m over S_m which is zero (follows from Theorem 3.36). Therefore, ϕ is the zero function which is a contradiction by Theorem 2. Hence, for large m each eigenfunction corresponding to any of the first n eigenvalues of S_m is necessarily residual. Now if $\lambda_{2g-2}^c(S_m) \leq \frac{1}{4}$ then each S_m has at least $2g-2+n$ small eigenvalues. This is a contradiction to [O-R, Theorem 2]. Therefore we have proved that $\mathcal{C}_{g,n}^{\frac{1}{4}}(2g-2)$ contains a neighborhood of $\cup_{i=1}^n \mathcal{M}_{0,3} \cup \mathcal{M}_{g-1,2}$ in $\overline{\mathcal{M}_{g,n}}$.

In the last section we prove the last statement of Theorem 1. We consider $\mathcal{M}_{g,1} \cup \mathcal{M}_{0,n+1}$ as a subset of $\partial\mathcal{M}_{g,n} = \overline{\mathcal{M}_{g,n}} \setminus \mathcal{M}_{g,n}$ and show the following

Proposition 1.7. *There exists a neighborhood $\mathcal{N}(\mathcal{M}_{g,1} \cup \mathcal{M}_{0,n+1})$ of $\mathcal{M}_{g,1} \cup \mathcal{M}_{0,n+1}$ in $\overline{\mathcal{M}_{g,n}}$ such that for each $S \in \mathcal{N}(\mathcal{M}_{g,1} \cup \mathcal{M}_{0,n+1})$: $\lambda_{2g-1}^c(S) > \frac{1}{4}$ i.e.*

$$\mathcal{N}(\mathcal{M}_{g,1} \cup \mathcal{M}_{0,n+1}) \subset \mathcal{C}_{g,n}^{\frac{1}{4}}(2g-1).$$

Now we briefly sketch a proof of this proposition. We argue by contradiction and consider a sequence (S_m) in $\mathcal{M}_{g,n}$ that converges to S_∞ in $(\mathcal{M}_{g,1} \cup \mathcal{M}_{0,n+1}) \subset \partial\mathcal{M}_{g,n}$ such that ${\lambda^c}_{2g-1}(S_m) \leq \frac{1}{4}$. Then, for $1 \leq i \leq 2g-1$ and for each m, we choose a small cuspidal eigenpair (λ_m^i, ϕ_m^i) of S_m such that

(i) $\{\phi_m^i\}_{i=1}^{2g-1}$ is an orthonormal family in $L^2(S_m)$,
(ii) λ_m^i is the i-th eigenvalue of S_m.

For $1 \leq i \leq 2g-1$ let (λ_m^i) converges to λ_∞^i as $m \to \infty$. By Theorem 2 there are two possible types of behavior that the sequence (ϕ_m^i) can exhibit. Either, for each $1 \leq i \leq 2g-1$ the sequence (ϕ_m^i) converges to a λ_∞^i-eigenfunction ϕ_∞^i on S_∞, or for some i the sequence (λ_m^i, ϕ_m^i) satisfies condition (2) in Theorem 2. However, in our case we have the following lemma:

Lemma 2 *For each i, $1 \leq i \leq 2g-1$, up to extracting a subsequence, the sequence (ϕ_m^i) converges to a λ_∞^i-eigenfunction ϕ_∞^i of S_∞. The limit functions ϕ_∞^i and ϕ_∞^j are orthogonal for $i \neq j$ i.e. S_∞ has at least $2g-1$ small eigenvalues. Moreover none of the ϕ_∞^i is residual.*

Then we count the number of small eigenvalues of S_∞ using [O-R] to conclude that at least one of ϕ^i_∞ is nonzero on the component of S_∞ of type $(0, n+1)$. This leads to a contradiction by Huxley [Hu] or [O, Proposition 2].

1.2. **Acknowledgement.** The author would like to express his sincere gratitude to his advisor Jean-Pierre Otal for his patience, encouragement and insight. The author was supported during this research by the Indo-French CEFIPRA-IFCPAR grant.

2. Preliminaries

In this section we shall recall some preliminary concepts that are important for our purpose. Metric convergence of a sequence $(S_m) \in \mathcal{M}_{g,n}$ to $S_\infty \in \overline{\mathcal{M}_{g,n}}$ is one of the prime aspects of our study. We start by explaining the thick/thin decomposition of a hyperbolic surface which is convenient to understand the metric convergence.

2.1. **The thick / thin decomposition of a hyperbolic surface.** Let $S \in \mathcal{M}_{g,n}$. Recall that for any $\epsilon > 0$, *the* ϵ*-thin part of* S, $S^{(0,\epsilon)}$, is the set of points of S with injectivity radius $< \epsilon$. The complement of $S^{(0,\epsilon)}$, *the* ϵ*-thick part of* S, denoted by $S^{[\epsilon,\infty)}$, is the set of points where the injectivity radius of S is $\geq \epsilon$.

2.1.1. *Cylinders.* Let γ be a simple closed geodesic on S. It can be viewed as the quotient of a geodesic in $\mathbb{H}$ by a hyperbolic isometry Υ fixing the geodesic. We may conjugate Υ such that the geodesic is the imaginary axis and the isometry is $\tau : z \to e^{2\pi l} z$, $2\pi l = l_\gamma$ being the length of the geodesic. We define the *hyperbolic cylinder* $\mathcal{C}$ *with core geodesic* γ as the quotient $\mathbb{H}/ < \tau >$. Recall that the *Fermi coordinates* on $\mathcal{C}$ assign to each point $p \in \mathcal{C}$ the pair $(r, \theta) \in \mathbb{R} \times \{\gamma\}$ where r is the signed distance of p from γ and θ is the projection of p on γ [Bu, p. 4]. These coordinates give a diffeomorphism of this hyperbolic cylinder to $\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$. In terms of these coordinates the hyperbolic metric is given by:

$$ds^2 = dr^2 + l^2 \cosh^2 r d\theta^2.$$

For $w \geq l$ we define the *collar* $\mathcal{C}^w$ around γ by

$$\mathcal{C}^w = \{(r, \theta) \in \mathcal{C} : l_\gamma \cosh r < w, 0 \leq \theta \leq 2\pi\}.$$

Then $\mathcal{C}^w$ is diffeomorphic to an annulus whose each boundary component has length w. The *Collar Theorem* of Linda Keen [Ke] says that $\mathcal{C}^1$ embeds in S (more precisely, $\mathcal{C}^{w(l_\gamma)}$ embeds in S where $w(l_\gamma) = l_\gamma \cosh(\sinh^{-1}(\frac{1}{\sinh \frac{l_\gamma}{2}})) > 1$ and $w(l_\gamma) \approx 2$).

2.1.2. *Cusps.* S has n ends called *punctures*. *Cusps* are particular neighborhood of the punctures. Denote by ι the parabolic isometry $\iota : z \to z + 2\pi$. For a choice of $t > 0$, a cusp $\mathcal{P}^t$ is the half-infinite cylinder $\{z = x + iy : y > \frac{2\pi}{t}\}/ < \iota >$. The boundary curve $\{y = \frac{2\pi}{t}\}$ is a *horocycle* of length t that we identify with $\mathbb{R}/t\mathbb{Z}$. One can parametrize $\mathcal{P}^t$ using the *horocycle coordinates* [Bu, p. 4] with respect to its boundary horocycle $\{y = \frac{2\pi}{t}\}$. The horocycle coordinates assigns to a point $p \in \mathcal{P}^t$ the pair $(r, \theta) \in \mathbb{R}_{\geq 0} \times \{\mathbb{R}/t\mathbb{Z}\}$ where

r is the distance from p to the horocycle and θ the projection of p on the horocycle. In terms of these coordinates the hyperbolic metric takes the form:

$$ds^2 = dr^2 + (\frac{t}{2\pi})^2 e^{-2r} d\theta^2.$$

Recall that the cusp $\mathcal{P}^1$ (in fact $\mathcal{P}^2$) around each puncture embeds in S and that those cusps corresponding to distinct punctures have disjoint interiors (ref. [Bu, Chapter 4]). We call them *standard cusps*. Observe that the area and boundary length of a standard cusp is equal to 1. For $t \leq 1$ denote the disjoint union $\bigcup_{c \in S} \mathcal{P}^t$ by $S_c^{(0,t)}$ where c ranges over distinct cusps in S.

2.1.3. *The decomposition.* By Margulis lemma there exists a constant $\epsilon_0 > 0$, the Margulis constant, such that for all $\epsilon \leq \epsilon_0$, $S^{(0,\epsilon)}$ is a disjoint union of embedded collars, one for each geodesic of length less than 2ϵ, and of embedded cusps, one for each puncture. The collar around a geodesic of length $\leq \epsilon$ is called a *Margulis tube.*

2.2. **Metric degeneration of a collar to a pair of cusps.** We describe how a collar around a geodesic of length $l_\gamma = 2\pi l$ converges as l tends to zero to a pair of cusps. First shift the origin of the Fermi coordinates of $\mathcal{C}^{w(l_\gamma)}$ to the right boundary of $\mathcal{C}^{w(l_\gamma)}$ by making the change of variable $t = r - \sinh^{-1}(\frac{1}{\sinh \frac{l_\gamma}{2}})$. In the shifted Fermi coordinates the metric on $\mathcal{C}^{w(l_\gamma)}$ is equal to

$$ds^2 = dr^2 + l^2 \cosh^2(r - \sinh^{-1}(\frac{1}{\sinh \frac{l_\gamma}{2}}))d\theta^2.$$

For r in a compact region we have the limiting

$$\lim_{l \to 0} l \cosh(r - \sinh^{-1}(\frac{1}{\sinh \frac{l_\gamma}{2}})) = \frac{e^{-r}}{\pi}.$$

Now the hyperbolic metric on $\mathcal{P}^2$ is equal to

$$ds^2 = dr^2 + \frac{e^{-2r}}{\pi^2} d\theta^2$$

with respect to the boundary horocycle $\{y = \pi\}$ of $\mathcal{P}^2$.

Choose a base point p_l on the right half of $\mathcal{C}^{w(l_\gamma)}{}^{[\epsilon,\infty)}$. Then by above, as $l \to 0$, the pair $(\mathcal{C}^{w(l_\gamma)}, p_l)$ converges, up to extracting a subsequence, to $(\mathcal{P}^2, p)$ where $p \in \mathcal{P}^{2[\epsilon,\infty)}$. Since one can choose the base point on the left half of $\mathcal{C}^{w(l_\gamma)}$ also, the metric limit of $\mathcal{C}^{w(l_\gamma)}$ is a pair of $\mathcal{P}^2$.

3. Mass distribution of small cuspidal functions over thin parts

Our goal is to study the behavior of sequences of small cuspidal eigenpairs (λ_n, f_n) of $S_n \in \mathcal{M}_{g,n}$ when (S_n) converges to $S_\infty \in \overline{\mathcal{M}_{g,n}}$ and finally to prove Theorem 1. For this we need to understand how the *mass* (L^2 norm) of a small eigenfunction is distributed over the surface, and in particular how it is distributed with respect to the *thin/thick* decomposition. Let $S \in \mathcal{M}_{g,n}$. Recall that for any $\epsilon \leq \epsilon_0$ the ϵ-thin part, $S^{(0,\epsilon)}$, of S consists of cusps and Margulis tubes. We separately study the mass distribution of a small cuspidal eigenfunction over these two different types of domains.

3.1. **Mass distribution over cusps.** For $2\pi \leq a < b$ consider the annulus $\mathcal{P}(a,b) = \{(x,y) \in \mathcal{P}^1 : a \leq y < b\}$ contained in a cusp $\mathcal{P}^1$ and bounded by two horocycles of length $\frac{2\pi}{a}$ and $\frac{2\pi}{b}$. We begin our study with the following lemma.

Lemma 3.1. *For any $b > 2\pi$ there exists $K(b) < \infty$ such that for any small cuspidal eigenpair (λ, f) of $\mathcal{P}^1$ one has*

$$\|f\|_{\mathcal{P}(b,\infty)} < K(b)\|f\|_{\mathcal{P}(2\pi,b)}. \tag{3.2}$$

If $\lambda < \frac{1}{4} - \eta$ for some $\eta > 0$ then there exists a constant $T(b,\eta) < \infty$ depending on b and η such that for any small eigenpair (λ, f) one has

$$\|f\|_{\mathcal{P}(b,\infty)} < T(b,\eta)\|f\|_{\mathcal{P}(2\pi,b)}. \tag{3.3}$$

Furthermore, $K(b), T(b,\eta) \to 0$ as $b \to \infty$.

Proof. We begin with the first part. Since f is cuspidal inside $\mathcal{P}^1$ it can be expressed as

$$f(z) = \sum_{n \in \mathbb{Z}^*} f_n W_s(nz) \tag{3.4}$$

where $s(1-s) = \lambda$ and W_s is the *Whittaker function* (see [I, Proposition 1.5]). The meaning of (3.4) is that the right hand series converges to f in $L^2(\mathcal{P}^1)$ and that the convergence is uniform over compact subsets. Recall also that for $n \in \mathbb{Z}^*$ the Whittaker functions is defined by

$$W_s(nz) = 2(|n|y)^{\frac{1}{2}} K_{s-\frac{1}{2}}(|n|y) e^{inx}$$

where K_ϵ is the *McDonald's function* and that for any ϵ (see [Le, p. 119])

$$K_\epsilon(y) = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-y\cosh u - \epsilon u} du \tag{3.5}$$

whenever the integral makes sense. From the expression it is clear that the functions $(W_s(n.))$ form an orthogonal family over $\mathcal{P}(a,b)$ (independent of the choices of a and b). Hence (1) will follow from the following claim.

Claim 3.6. *Let $s \in [\frac{1}{2}, 1]$. Then for any $b > 2\pi$ there exists $K(b) < \infty$ such that for all $n \in \mathbb{Z}^*$*

$$\|W_s(nz)\|_{\mathcal{P}(b,\infty)} \leq K(b)\|W_s(nz)\|_{\mathcal{P}(2\pi,b)}.$$

Furthermore, $K(b) \to 0$ as $b \to \infty$.

Proof. From the expression of W_s we have

$$\|W_s(nz)\|_{\mathcal{P}(a,b)} = 2\pi\left(\int_a^b 4|n|y K_{s-\frac{1}{2}}(|n|y)^2 \frac{dy}{y^2}\right).$$

To prove the claim we may suppose that $n \geq 1$. Our next objective is to obtain bounds for the functions $K_{s-\frac{1}{2}}(y)$ for $s \in [\frac{1}{2}, 1]$. We start from the above integral representation of $K_\epsilon(y)$. We write $K_\epsilon(y) = \frac{1}{2}\{c(\epsilon,y)+d(\epsilon,y)\}$ where

$$c(\epsilon,y) = \int_{-1}^{1} e^{-y\cosh u - \epsilon u}du \tag{3.7}$$

and

$$d(\epsilon,y) = \int_{-\infty}^{-1} e^{-y\cosh u-\epsilon u}du + \int_{1}^{\infty} e^{-y\cosh u-\epsilon u}du. \tag{3.8}$$

Now we treat $c(\epsilon,y)$ and $d(\epsilon,y)$ separately.

Bounding $c(\epsilon,y)$:

We have

$$c(\epsilon,y) = \int_{-1}^{1} e^{-y\cosh u}.e^{-\epsilon u}du \le e^{\epsilon}.\int_{-1}^{1} e^{-y\cosh u}du = e^{\epsilon}\int_{-1}^{1} e^{-y(1+\frac{u^2}{2!}+\frac{u^4}{4!}+...)}du$$

$$= e^{\epsilon}.e^{-y}\int_{-1}^{1} e^{-y(\frac{u^2}{2!}+\frac{u^4}{4!}+...)}du \le 2e^{\epsilon}.e^{-y}\int_{0}^{1} e^{-y\frac{u^2}{2!}}du.$$

Since $e^{\frac{yu^2}{2}} > 1+\frac{yu^2}{2}$ for $u>0$, we have:

$$\int_0^1 e^{-y\frac{u^2}{2!}}du < \int_0^1 \frac{du}{1+\frac{yu^2}{2}} = \frac{2}{y}\tan^{-1}(\frac{y}{2}) \le \frac{2}{y}.\frac{\pi}{2}.$$

Therefore

$$c(\epsilon,y) \le 2\pi e^{\epsilon}\frac{e^{-y}}{y}.$$

To obtain a lower bound, we write

$$\int_{-1}^{1} e^{-y\cosh u}.e^{-\epsilon u}du \ge e^{-\epsilon}.\int_{-1}^{1} e^{-y\cosh u}du = e^{-\epsilon}\int_{-1}^{1} e^{-y(1+\frac{u^2}{2!}+\frac{u^4}{4!}+...)}du$$

$$= 2e^{-\epsilon}.e^{-y}\int_0^1 e^{-y(\frac{u^2}{2!}+\frac{u^4}{4!}+...)}du.$$

Since for all $u\in(0,1]$ one has

$$\frac{u^2}{2!}+\frac{u^4}{4!}+... < u(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...) = u.$$

Hence

$$c(\epsilon,y) \ge 2e^{-\epsilon}.e^{-y}\int_0^1 e^{-uy}du = 2e^{-\epsilon}\frac{e^{-y}}{y}(1-e^{-y}).$$

Combining the above two inequalities

$$2e^{-\epsilon}\frac{e^{-y}}{y}(1-e^{-y}) \le c(\epsilon,y) \le 2\pi e^{\epsilon}\frac{e^{-y}}{y}.$$

Bounding $d(\epsilon,y)$:

$$d(\epsilon,y) = \int_{-\infty}^{-1} e^{-y\cosh u-\epsilon u}du + \int_1^{\infty} e^{-y\cosh u-\epsilon u}du$$

$$= \int_1^{\infty} e^{-y\cosh u-\epsilon u}du + \int_1^{\infty} e^{-y\cosh u+\epsilon u}du.$$

Now for any $u>1$,

$$\frac{u^2}{2!}+\frac{u^4}{4!}+... > \gamma_0 u^2 > \gamma_0 u$$

where $\gamma_0 = \sum_{n=1}^{\infty} \frac{1}{(2n)!}$.

Thus

$$\begin{aligned} d(\epsilon, y) &= e^{-y} \int_1^{\infty} \{e^{-y(\frac{u^2}{2!}+\frac{u^4}{4!}+\ldots)-\epsilon u} + e^{-y(\frac{u^2}{2!}+\frac{u^4}{4!}+\ldots)+\epsilon u}\} du \\ &\leq e^{-y} \int_1^{\infty} \{e^{-y\gamma_0 u-\epsilon u} + e^{-y\gamma_0 u+\epsilon u}\} du \\ &= \frac{e^{-y}}{y} \bigg(\frac{e^{-(y\gamma_0+\epsilon)}}{\gamma_0 + \frac{\epsilon}{y}} + \frac{e^{-(y\gamma_0-\epsilon)}}{\gamma_0 - \frac{\epsilon}{y}} \bigg). \end{aligned}$$

Thus combining the estimates for $c(\epsilon, y)$ and $d(\epsilon, y)$ we obtain

$$2e^{-\epsilon} \frac{e^{-y}}{y}(1 - e^{-y}) < K_\epsilon(y) < 2\pi e^{\epsilon} \frac{e^{-y}}{y} + \frac{e^{-y}}{y} \bigg(\frac{e^{-(y\gamma_0+\epsilon)}}{\gamma_0 + \frac{\epsilon}{y}} + \frac{e^{-(y\gamma_0-\epsilon)}}{\gamma_0 - \frac{\epsilon}{y}} \bigg).$$

Let

$$\delta(\epsilon, y) = \frac{e^{-(y\gamma_0+\epsilon)}}{\gamma_0 + \frac{\epsilon}{y}} + \frac{e^{-(y\gamma_0-\epsilon)}}{\gamma_0 - \frac{\epsilon}{y}}.$$

Observe that for $\epsilon < 1$ and $y \geq \frac{2}{\gamma_0}$

$$\delta(\epsilon, y) < \frac{4\cosh 1}{\gamma_0} e^{-\gamma_0 y} = \delta_0(y).$$

So, for $y \geq \frac{2}{\gamma_0}$ large enough

$$2e^{-\epsilon} \frac{e^{-y}}{y} < K_\epsilon(y) < \frac{e^{-y}}{y} \bigg(2\pi e^{\epsilon} + \delta_0(y) \bigg). \tag{3.9}$$

Going back to the expression of W_s, for $s \in [\frac{1}{2}, 1]$, we find:

$$\begin{aligned} \frac{1}{2\pi} \|W_s(nz)\|^2_{\mathcal{P}(2\pi,b)} &= \int_{2\pi}^{b} 4nyK_{s-\frac{1}{2}}(ny)^2 \frac{dy}{y^2} = \int_{2\pi}^{b} 4nK_{s-\frac{1}{2}}(ny)^2 \frac{dy}{y} \\ \geq \int_{2\pi}^{b} \frac{4n}{b} K_{s-\frac{1}{2}}(ny)^2 dy &> \frac{16ne^{1-2s}}{b} \int_{2\pi}^{b} \frac{e^{-2ny}}{(ny)^2} dy = \frac{16ne^{1-2s}}{n^2 b} \int_{2\pi}^{b} \frac{e^{-2ny}}{y^2} dy \\ = \frac{16ne^{1-2s}}{n^2 b} \bigg(\int_{2\pi}^{\frac{b}{2}} \frac{e^{-2ny}}{y^2} dy + \int_{\frac{b}{2}}^{b} \frac{e^{-2ny}}{y^2} dy \bigg) &> \frac{16ne^{1-2s}}{n^2 b} \bigg(\int_{\frac{b}{2}}^{b} \frac{e^{-2ny}}{y^2} dy \bigg) \\ &= \frac{16e^{1-2s}}{nb} \frac{e^{-nb}}{n\frac{b^2}{4}} \{1 + O(e^{-nb} + \frac{2}{b})\} \end{aligned}$$

i.e.

$$\|W_s(nz)\|^2_{\mathcal{P}(2\pi,b)} > 2\pi \frac{16e^{1-2s}}{nb} \frac{e^{-nb}}{n\frac{b^2}{4}} \{1 + O(e^{-nb} + \frac{1}{b})\} \tag{3.10}$$

Also,

$$\begin{aligned} \frac{1}{2\pi} \|W_s(nz)\|^2_{\mathcal{P}(b,\infty)} &= \int_b^{\infty} 4nyK_{s-\frac{1}{2}}(ny)^2 \frac{dy}{y^2} = \int_b^{\infty} 4nK_{s-\frac{1}{2}}(ny)^2 \frac{dy}{y} \\ &\leq \int_b^{\infty} \frac{4n}{b} K_{s-\frac{1}{2}}(ny)^2 dy \leq \frac{4n(2\pi e^{(s-\frac{1}{2})} + \delta_0(b))^2}{b} \int_b^{\infty} \frac{e^{-2ny}}{(ny)^2} dy \\ &= \frac{4\big(2\pi e^{(s-\frac{1}{2})} + \delta_0(b)\big)^2}{nb} \frac{e^{-2nb}}{2nb^2} \{1 + O(\frac{1}{b})\} \end{aligned}$$

i.e.

$$\|W_s(nz)\|^2_{\mathcal{P}(b,\infty)} \leq 2\pi \frac{2(2\pi e^{(s-\frac{1}{2})} + \delta_0(b))^2}{nb} \frac{e^{-2nb}}{nb^2}\{1 + O(\frac{1}{b})\} \tag{3.11}$$

In the last inequality, we used the following estimate from [Le, Section 3.2]:

$$\int_{t_1}^{t_2} \frac{e^{-2\alpha y}}{y^2} dy = \frac{e^{-2\alpha t_1}}{2\alpha {t_1}^2}\{1 + O(e^{2(t_1-t_2)} + {t_1}^{-1})\}$$

with an absolute constant for the O-term for $\alpha > 1$.

Comparing (3.10) and (3.11) we get, for any $n \in \mathbb{Z}^*$

$$\|W_s(nz)\|_{\mathcal{P}(b,\infty)} \leq K(b)\|W_s(nz)\|_{\mathcal{P}(2\pi,b)} \tag{3.12}$$

where

$$K^2(b) = \frac{e^{2s-1}}{8}(2\pi e^{(s-\frac{1}{2})} + \delta_0(b))^2 e^{-|n|b} \frac{(1 + O(\frac{1}{b}))}{1 + O(e^{-|n|b} + \frac{2}{b})}.$$

From the expression it is clear that K is bounded independent of n, b (once b is large enough) and $s \in [\frac{1}{2}, 1]$. So we obtain the claim by choosing some $b > \frac{2}{\gamma_0}$ sufficiently large (once and for all) such that the O-terms in the expression of T are small enough. It is also clear from the expression that when $b \to \infty$, $K(b) \to 0$. This proves the Claim 3.6 and hence the first part of Lemma 3.1.

Now we prove the second part. Let $\lambda < \frac{1}{4} - \eta$ for some $\eta > 0$ and let (λ, f) be a residual eigenpair. The Fourier expansion of f inside $\mathcal{P}^1$ has the form

$$f(z) = f_0 y^s + \sum_{n\in\mathbb{Z}^*} f_n W_s(nz) = f_0 y^s + g(z) \tag{3.13}$$

where $s(1-s) = \lambda$, $s \in (0, \frac{1}{2})$ (see [I]) and $g(z) = \sum_{n\in\mathbb{Z}^*} f_n W_s(nz)$. Since $f_0 y^s$ and g are orthogonal and since the first part can be applied to g, one needs only to prove the lemma for the term $f_0 y^s$. So we calculate:

$$\int_a^c y^{2s} \frac{dy}{y^2} = \frac{1}{1-2s}\left(\frac{1}{a^{1-2s}} - \frac{1}{c^{1-2s}}\right).$$

Therefore, for $b > 2\pi$,

$$\|f_0 y^s\|^2_{\mathcal{P}(b,\infty)} = \frac{1}{(\frac{b}{2\pi})^{1-2s} - 1}\|f_0 y^s\|^2_{\mathcal{P}(2\pi,b)}. \tag{3.14}$$

The lemma is satisfied by $T_2(b, \eta)$ such that

$$T_2^2(b, \eta) = \max\left(K^2(b), \frac{1}{(\frac{b}{2\pi})^{1-2s} - 1}\right).$$

From the expression it is clear that $T_2(b, \eta)$ depends only on two quantities: b and $\frac{1}{2} - s$. Since $\frac{1}{2} - s > \sqrt{\eta} > 0$, $\frac{1}{(\frac{b}{2\pi})^{1-2s}-1} \to 0$ when $b \to \infty$. This proves the second part.

3.2. **Mass distribution over Margulis tubes.** Now we study the distribution of the mass of a small eigenfunction over Margulis tubes. Let γ be a simple closed geodesic of length $l_\gamma = 2\pi l$. Recall that $\mathcal{C}^a$ denotes the collar around γ bounded by two equidistant curves of length a. Any $f \in L^2(\mathcal{C}^1)$ can be written as a Fourier series in the θ-coordinate:

$$f(r,\theta) = a_0(r) + \sum_{j=1}^{\infty}\Big(a_j(r)\cos j\theta + b_j(r)\sin j\theta\Big). \tag{3.15}$$

The functions $a_j = a_j(r)$ and $b_j = b_j(r)$ are defined on $[-\cosh^{-1}(\frac{1}{l_\gamma}), \cosh^{-1}(\frac{1}{l_\gamma})]$ and are called the *j-th Fourier coefficients of f* (in $\mathcal{C}^1$). When f is a λ-eigenfunction, a_j and b_j are solutions of the differential equation

$$\frac{d^2\phi}{dr^2} + \tanh r\frac{d\phi}{dr} + (\lambda - \frac{j^2}{l^2\cosh^2 r})\phi = 0. \tag{3.16}$$

We set $[f]_0 = a_0(r)$ and $[f]_1 = f - [f]_0$. The following lemma concerns the distribution of masses of $[f]_0$ and $[f]_1$ inside $\mathcal{C}^1$.

Lemma 3.17. *For any $l_\gamma < \epsilon \leq \epsilon_0$ there exist constants $T_1(\epsilon), T_2(\epsilon) < \infty$, depending only on ϵ, such that for any small eigenpair (λ, f) of $\mathcal{C}^1$ the following inequalities hold:*

$$\|[f]_1\|_{\mathcal{C}^\epsilon} < T_1(\epsilon)\|[f]_1\|_{\mathcal{C}^1\setminus\mathcal{C}^\epsilon} \tag{3.18}$$

and

$$\|[f]_0\|_{\mathcal{C}^{\epsilon_0}\setminus\mathcal{C}^\epsilon} < T_2(\epsilon)\|[f]_0\|_{\mathcal{C}^1\setminus\mathcal{C}^{\epsilon_0}}. \tag{3.19}$$

Therefore, for any $l_\gamma < \epsilon \leq \epsilon_0$ and any small eigenpair (λ, f) of $\mathcal{C}^1$ one has

$$\|f\|_{\mathcal{C}^{\epsilon_0}\setminus\mathcal{C}^\epsilon} < max\ \{T_1(\epsilon_0), T_2(\epsilon)\}\|f\|_{\mathcal{C}^1\setminus\mathcal{C}^{\epsilon_0}}. \tag{3.20}$$

If $\lambda < \frac{1}{4} - \eta$ for some $\eta > 0$ then there exists a constant $T_0(\epsilon, \eta) < \infty$, depending only on η and ϵ, such that

$$\|[f]_0\|_{\mathcal{C}^\epsilon} < T_0(\epsilon,\eta)\|[f]_0\|_{\mathcal{C}^1\setminus\mathcal{C}^\epsilon}. \tag{3.21}$$

Furthermore, $T_1(\epsilon), T_0(\epsilon,\eta) \to 0$ as $\epsilon \to 0$.

Before starting the proof of the above lemma we make a few observations about the solutions of (3.16). The change of variable $u(r) = \cosh^{\frac{1}{2}}(r)\phi(r)$ transforms (3.16) into

$$\frac{d^2u}{dr^2} = \Big((\frac{1}{4} - \lambda) + \frac{1}{4\cosh^2 r} + \frac{j^2}{l^2\cosh^2 r}\Big)u. \tag{3.22}$$

Let s_j (resp. c_j) be the solution of (3.22) satisfying the conditions: $s_j(0) = 0$ and $s_j^{'}(0) = 1$ (resp. $c_j(0) = 1$ and $c_j^{'}(0) = 0$). Since (3.22) is invariant under $r \to -r$ one has: $s_j(-r) = -s_j(r)$ and $c_j(-r) = c_j(r)$ for all $j \geq 0$. Therefore there exists $t > 0$ such that $s_j > 0$ and $c_j^{'} > 0$ on $(0, t]$. Now we prove the following claim.

Claim 3.23. *Let $L > 0$. Let $g : [0, L] \to \mathbb{R}$ be a C^2-function which satisfies the inequality:*

$$\frac{d^2g}{dr^2} > \delta^2 g$$

for some $\delta > 0$*. If* $g^{'}(0) \geq 0$ *then* $\dfrac{g(r)}{\cosh \delta r}$ *is a monotone increasing function of* r *in* $(0, L]$*.*

Proof. Observe that

$$\left(\frac{g(r)}{\cosh \delta r}\right)^{'} = \frac{g^{'}(r)\cosh \delta r - \delta g(r) \sinh \delta r}{\cosh^2(\delta r)}.$$

Consider the function H defined on $[0, L]$ by

$$H(r) = g^{'}(r)\cosh \delta r - \delta g(r) \sinh \delta r.$$

Since g is a C^2 function H is continuous on $[0, L]$. Observe that the claim follows if $H(r) > 0$ in $(0, L]$. Now for any $r \in (0, L]$

$$H^{'}(r) = g^{''}(r)\cosh \delta r - \delta^2 g(r) \cosh \delta r = (g^{''}(r) - \delta^2 g(r)) \cosh \delta r > 0.$$

Therefore for $r > 0$, $H(r) > H(0) = g^{'}(0) \geq 0$. Hence the claim.□

Proof of Lemma 3.17. We need to estimate, for $l_\gamma \leq t < w \leq 1$, the quantities:

$$\|[f]_1\|^2_{\mathcal{C}^w\setminus\mathcal{C}^t} = l_\gamma \int_{-L_w}^{-L_t} \left(\sum_{j=1}^{\infty} \alpha_j^2 + \beta_j^2\right) dr + l_\gamma \int_{L_t}^{L_w} \left(\sum_{j=1}^{\infty} \alpha_j^2 + \beta_j^2\right) dr$$

and

$$\|[f]_0\|^2_{\mathcal{C}^w\setminus\mathcal{C}^t} = l_\gamma \int_{-L_w}^{-L_t} \alpha_0^2 dr + l_\gamma \int_{L_t}^{L_w} \alpha_0^2 dr$$

where $\alpha_0(r) = \cosh^{\frac{1}{2}}(r) a_0(r)$, $\alpha_j(r) = a_j(r)\cosh^{\frac{1}{2}}(r)$, $\beta_j(r) = b_j(r)\cosh^{\frac{1}{2}}(r)$ and $L_u = \cosh^{-1}(\frac{u}{l_\gamma})$. Since s_j is odd and c_j is even, for any symmetric subset $U \subset [-L_1, L_1]$, s_j and c_j are orthogonal in $L^2(U)$. Now α_j and β_j are linear combinations of s_j and c_j for $j \geq 1$ and α_0 is a linear combination of s_0 and c_0. Therefore, since s_j and c_j are orthogonal, it is enough to prove the lemma with s_j and c_j instead of $[f]_1$ and with s_0 and c_0 instead of $[f]_0$. We detail the computations for s_j. The computations for c_j are similar. Let us choose ϵ such that $l_\gamma < \epsilon < \epsilon_0$. The lemma reduces to find $K_1(\epsilon), K_2(\epsilon) < \infty$, depending on ϵ, and $K_0(\epsilon, \eta) < \infty$, depending on ϵ, η (> 0), such that

$$\|s_j\|_{\mathcal{C}^\epsilon} < K_1(\epsilon)\|s_j\|_{\mathcal{C}^1\setminus\mathcal{C}^\epsilon},\ \|s_0\|_{\mathcal{C}^{\epsilon_0}\setminus\mathcal{C}^\epsilon} < K_2(\epsilon)\|s_0\|_{\mathcal{C}^1\setminus\mathcal{C}^{\epsilon_0}}$$

and

$$\|s_0\|_{\mathcal{C}^\epsilon} < K_0(\epsilon, \eta)\|s_0\|_{\mathcal{C}^1\setminus\mathcal{C}^\epsilon}.$$

Let $\eta < \frac{1}{4} - \lambda$ and set $\delta_0 = \sqrt{\eta}$ and set for $j \geq 1$, $\delta_j = 1$. Notice that $l \cosh r < 1$ on $[0, L_1)$. Hence by (3.22) $s_j : [0, L_1) \to \mathbb{R}$ satisfies the inequality:

$$\frac{d^2 s_j}{dr^2} > \delta_j^2 s_j.$$

Hence by Claim 3.23 $h_j(r) = \frac{s_j(r)}{\cosh r}$, for $j \geq 1$, is strictly increasing on $(0, L_1)$. The same is true for $h_0 = \frac{s_0(r)}{\cosh \delta_0 r}$ (even when $\delta_0 = 0$).

We begin with the proof of the second part of the Lemma. So we assume $\eta > 0$. For $0 \leq a < b$ consider the integral:

$$\int_a^b s_0^2(r)dr = \int_a^b h_0^2(r)\cosh^2(\delta_0 r)dr.$$

Since h_0 is strictly increasing we have

$$h_0^2(a)\int_a^b \cosh^2(\delta_0 r)dr < \int_a^b s_0^2(r)dr < h_0^2(b)\int_a^b \cosh^2(\delta_0 r)dr. \tag{3.24}$$

Now choosing $a = 0$ and $b = L_\epsilon$ the last inequality in (3.24) gives

$$\|s_0\|^2_{\mathcal{C}^\epsilon} < 2l_\gamma h_0^2(L_\epsilon)\int_0^{L_\epsilon} \cosh^2(\delta_0 r)dr. \tag{3.25}$$

Next choosing $a = L_\epsilon$ and $b = L_1$ the first inequality in (3.24) gives

$$\|s_0\|^2_{\mathcal{C}^1\setminus\mathcal{C}^\epsilon} > 2l_\gamma h_0^2(L_\epsilon)\int_{L_\epsilon}^{L_1} \cosh^2(\delta_0 r)dr. \tag{3.26}$$

Therefore

$$\|s_0\|_{\mathcal{C}^\epsilon} < T_0\|s_0\|_{\mathcal{C}^1\setminus\mathcal{C}^\epsilon} \tag{3.27}$$

where

$$T_0^2 = \frac{\sinh 2\delta_0 L_\epsilon + 2\delta_0 L_\epsilon}{\sinh 2\delta_0 L_1 - \sinh 2\delta_0 L_\epsilon + 2\delta_0(L_1 - L_\epsilon)}. \tag{3.28}$$

We see that T_0 depends only on ϵ, δ_0 and l_γ. Now $L_\epsilon = \cosh^{-1}(\frac{\epsilon}{l_\gamma}) = \log(\frac{\epsilon}{l_\gamma} + \sqrt{(\frac{\epsilon}{l_\gamma})^2 - 1})$. Therefore, for ϵ and $\delta_0^2 = \eta > 0$ fixed, and l_γ small

$$T_0^2 < K_0\frac{1}{\epsilon^{-2\delta_0} - 1},$$

and the constant K_0 is independent of l_γ as soon as l_γ is small compared to ϵ. Thus we can choose $T_0(\epsilon, \eta)$ independent of l_γ satisfying (3.27). This proves (3.21)

For s_j, $j \geq 1$, exactly the same computations for s_0 work with δ_0 replaced by $\delta_j = 1$. Hence in this case our constant,

$$T_1^2(\epsilon) < K_1\frac{1}{\epsilon^{-2} - 1},$$

depends only on ϵ. This proves (3.18).

Now we prove (3.20). Since $s_0 : [0, L_1] \to \mathbb{R}^+$ is strictly increasing we have:

$$\int_{L_\epsilon}^{L_{\epsilon_0}} s_0^2(r)dr < s_0^2(L_{\epsilon_0})(L_{\epsilon_0} - L_\epsilon) \text{ and } \int_{L_{\epsilon_0}}^{L_1} s_0^2(r)dr > s_0^2(L_{\epsilon_0})(L_1 - L_{\epsilon_0}).$$

Combining the two inequalities we obtain

$$\|s_0\|_{\mathcal{C}^{\epsilon_0}\setminus\mathcal{C}^\epsilon} < T_2(\epsilon)\|s_0\|_{\mathcal{C}^1\setminus\mathcal{C}^{\epsilon_0}} \tag{3.29}$$

where

$$T_2^2(\epsilon) = \frac{L_{\epsilon_0} - L_\epsilon}{L_1 - L_{\epsilon_0}} < K_2\left(\frac{\log\frac{1}{\epsilon}}{\log\frac{1}{\epsilon_0}} - 1\right). \tag{3.30}$$

The constant K_2 is independent of l_γ as soon as l_γ is small compared to ϵ. Thus we can choose $T_2(\epsilon)$ independent of l_γ satisfying (3.29). This proves (3.20).

3.3. **Applications.** Let S be a finite area hyperbolic surface with n punctures. Denote by $\mathcal{P}_i$ the standard cusp around the i-th puncture. Recall that $\mathcal{P}_i$'s have disjoint interiors and that each of them is isometric to the half-infinite annulus $\mathcal{P}^1$ (see 2.1.2). Applying Lemma 3.1 in each $\mathcal{P}_i$ separately we obtain the following corollary which will be useful in our analysis.

Corollary 3.31. *For any $0 < \epsilon < \epsilon_0$ there exists $T(\epsilon) < \infty$, depending only on ϵ, such that for any small cuspidal eigenpair (λ, f) of S one has*

$$\|f\|_{S_c^{(0,\epsilon)}} < T(\epsilon)\|f\|_{S_c^{(0,1]}\setminus S_c^{(0,\epsilon]}}. \tag{3.32}$$

If $\lambda < \frac{1}{4} - \eta$ for some $\eta > 0$ then for any $0 < \epsilon < \epsilon_0$ there exists $T_1(\epsilon, \eta) < \infty$, depending only on ϵ and η, such that for any λ-eigenfunction f of S one has

$$\|f\|_{S_c^{(0,\epsilon)}} < T_1(\epsilon, \eta)\|f\|_{S_c^{(0,1]}\setminus S_c^{(0,\epsilon]}}. \tag{3.33}$$

Furthermore, $T(\epsilon)$ and $T_1(\epsilon, \eta)$ tends to zero as $\epsilon \to 0$.

Using this corollary and (3.20) we deduce the following

Corollary 3.34. *For any $0 < \epsilon < \epsilon_0$ there exists a constant $L(\epsilon) < \infty$, depending only on ϵ, such that for any small cuspidal eigenfuction f of S one has*

$$\|f\|_{S^{[\epsilon,\infty)}} < L(\epsilon)\|f\|_{S^{[\epsilon_0,\infty)}}. \tag{3.35}$$

Now we give a new proof of the following theorem of D. Hejhal [H].

Theorem 3.36. *Consider a sequence $(S_m) \in \mathcal{M}_{g,n}$ converging to $S_\infty \in \overline{\mathcal{M}_{g,n}}$. Let (λ_m, ϕ_m) be a normalized small eigenpair of S_m such that $\lambda_m \to \lambda_\infty$. If $\lambda_\infty < \frac{1}{4}$ then, up to extracting a subsequence, ϕ_m converges to a normalized λ_∞-eigenfunction ϕ_∞ of S_∞.*

D. Hejhal's proof uses *convergence of Green's functions* of S_m to that of S_∞. Our approach is more elementary and uses the above estimates on the mass distribution of eigenfunctions over thin part of surfaces.

Proof of Theorem 3.36. First we prove that, up to extracting a subsequence, ϕ_m converges to a λ_∞-eigenfunction ϕ_∞ of S_∞. By Theorem 2 (which will be proven in §3) it is enough to prove that there exist $\epsilon, \delta > 0$ such that $\|\phi_m\|_{S_m^{[\epsilon,\infty)}} \geq \delta$ up to extracting a subsequence. We argue by contradiction. Suppose that for any $\epsilon > 0$ the sequence $\|\phi_m\|_{S_m^{[\epsilon,\infty)}} \to 0$ as $m \to \infty$. Let $\eta > 0$, such that $\lambda_m < \frac{1}{4} - \eta$ for all $m \geq 1$. By Lemma 3.17 we have

$$\|\phi_m\|_{\mathcal{C}^\epsilon} < \max\{T_0(\epsilon, \eta), T_1(\epsilon)\}\|\phi_m\|_{\mathcal{C}^1\setminus\mathcal{C}^\epsilon}. \tag{3.37}$$

Therefore from (3.33) and (3.37) we have

$$\|\phi_m\|_{S_m^{(0,\epsilon)}} < \max\{T_0(\epsilon, \eta), T_1(\epsilon), T_1(\epsilon, \eta)\}\|\phi_m\|_{S_m^{[\epsilon,\infty)}}. \tag{3.38}$$

Hence if $\|\phi_m\|_{S_m^{[\epsilon,\infty)}} \to 0$ as $m \to \infty$ then $\|\phi_m\| \to 0$ as $m \to \infty$. This is a contradiction to the fact that each ϕ_m is normalized i.e. $\|\phi_m\| = 1$.

Next we prove that $\|\phi_\infty\| = 1$. By uniform convergence over compacta, in each cusp and in each pinching collar, the Fourier coefficients of ϕ_m will converge to the corresponding Fourier coefficients of ϕ_∞. Therefore, by

(3.18), (3.21) and (3.33), ϕ_m's are uniformly integrable: for any $\delta > 0$ there exist $\epsilon > 0$ such that for all large values of m

$$\|\phi_m\|_{S_m^{[\epsilon,\infty)}} > 1 - \delta. \tag{3.39}$$

Hence $\|\phi_\infty\| = 1$. This finishes the proof. □

4. Proof of Theorem 2

Let (S_m) be a sequence in $\mathcal{M}_{g,n}$ which converges in $\overline{\mathcal{M}_{g,n}}$ to S_∞. Let Γ_m, Γ_∞ be such that $S_m = \mathbb{H}/\Gamma_m$ and $S_\infty = \mathbb{H}/\Gamma_\infty$. Recall that the convergence $S_m \to S_\infty$ means that for any fixed positive constant $\epsilon_1 \leq \epsilon_0$ (ϵ_0 is the Margulis constant) and a choice of base point $p_m \in S_m^{[\epsilon_1,\infty)}$, after conjugating Γ_m so that the projection $\mathbb{H} \to \mathbb{H}/\Gamma_m$ maps i to p_m, $(\mathbb{H}/\Gamma_m, p_m)$ converges to a component $(\mathbb{H}/\Gamma_\infty, p_\infty)$ of S_∞. We begin by fixing some $\epsilon < \epsilon_0$ and $p_m \in S_m^{[\epsilon,\infty)}$. In the following we assume that ϵ_1, p_m, Γ_m, p_∞ and Γ_∞ satisfy the previous statement.

To simplify notations we shall assume that only one closed geodesic γ_m gets pinched as $S_m \to S_\infty \in \partial\mathcal{M}_{g,n}$. In particular the limit surface S_∞ (which may be disconnected) has two new cusps. Denote the standard cusps of S_m by $\mathcal{P}_1(m)$, $\mathcal{P}_2(m), ..., \mathcal{P}_n(m)$ and the limits of these in $S_\infty \in \partial\mathcal{M}_{g,n}$ by $\mathcal{P}_1(\infty)$,...,$\mathcal{P}_n(\infty)$ and denote by $\mathcal{P}_{n+1}(\infty), \mathcal{P}_{n+2}(\infty)$ the *new cusps* which arise due to the pinching of γ. The cusps $\mathcal{P}_i(\infty)$ for $1 \leq i \leq n$ will be called *old cusps.*

Recall that we have a sequence of small cuspidal eigenpairs (λ_m, ϕ_m) of $S_m = \mathbb{H}/\Gamma_m$ such that the L^2-norm of ϕ_m is 1 and $\lambda_m \to \lambda_\infty \leq \frac{1}{4}$.

Notation 4.1. *In what follows $d\mu_m$ will denote the area measure on S_m for $m \in \mathbb{N} \cup \{\infty\}$ and $d\mu_{\mathbb{H}}$ will denote the area measure on $\mathbb{H}$. The lift of $f \in L^2(S_m)$ to $\mathbb{H}$ under the projection $\mathbb{H} \to \mathbb{H}/\Gamma_m$, defined as above, will be denoted by $\widetilde{f}$.*

By Green's formula one has:

$$\int_{S_m} |\nabla\phi_m|^2 d\mu_m = \lambda_m \int_{S_m} |\phi_m|^2 d\mu_m = \lambda_m.$$

Let $K \subset \mathbb{H}$ be compact. One can cover K by finitely many geodesic balls of radius ρ. If ρ is sufficiently small then each of these balls maps injectively to S_m since $\Gamma_m \to \Gamma_\infty$. Therefore, since $\|\phi_m\| = 1$ $\|\widetilde{\phi_m}|_K\|$ is bounded depending only on K. From the mean value formula [F, Corollary 1.3] there exists a constant $\Lambda(\lambda_\infty, \rho)$ such that for λ_m close to λ_∞,

$$|\widetilde{\phi_m}(q)| \leq \Lambda(\lambda_\infty, \rho) \int_{N(K,\frac{\rho}{2})} |\widetilde{\phi_m}| d\mu_{\mathbb{H}}$$

for each $q \in K$ where $N(K, r)$ denotes the closed neighborhood of radius r of K in $\mathbb{H}$. Next we use the L^p-Schauder estimates [B-J-S, Theorem 4, Sect. II.5.5] to obtain a uniform bound for $\nabla\widetilde{\phi_m}$ on $N(K, \frac{\rho}{2})$. This makes $(\widetilde{\phi_m}|_K)$ an equicontinuous family. So, by Arzela-Ascoli theorem, up to extracting a subsequence, $(\widetilde{\phi_m})$ converges to a continuous function $\widetilde{\phi_\infty}$ on K. By a diagonalization argument one may suppose that the sequence works for all compact subsets of $\mathbb{H}$. Therefore, up to extracting a subsequence, $\widetilde{\phi_m} \to \widetilde{\phi_\infty}$

uniformly over compacta. By this uniform convergence it is clear that $\widetilde{\phi_\infty}$ is a *weak solution* of the Laplace equation: $\Delta u + \lambda_\infty u = 0$. Therefore, by elliptic regularity, $\widetilde{\phi_\infty}$ indeed a smooth and satisfies

$$\Delta\widetilde{\phi_\infty} + \lambda_\infty\widetilde{\phi_\infty} = 0.$$

Also by the convergence $\widetilde{\phi_\infty}$ induces a function ϕ_∞ on S_∞ that satisfies

$$\Delta\phi_\infty + \lambda_\infty\phi_\infty = 0.$$

However, ϕ_∞ may not be an eigenfunction since it could be the zero function. In order to discuss this point, we shall consider two cases according to whether the L^2-norm $\|\phi_m\|_{S_m^{[\epsilon,\infty)}}$ of the restriction of ϕ_m to $S_m^{[\epsilon,\infty)}$ is bounded below by a positive constant or not.

Case 1: $\exists\ \epsilon, \delta > 0$ *such that* $\limsup \|\phi_m\|_{S_m^{[\epsilon,\infty)}} \geq \delta$. We may assume that $\lim \|\phi_m\|_{S_m^{[\epsilon,\infty)}} \geq \delta$. Then by the uniform convergence of $\widetilde{\phi_m} \to \widetilde{\phi_\infty}$ over compacta,

$$\int_{S_\infty^{[\epsilon,\infty)}} \phi_\infty^2 d\mu_\infty = \lim_{m_j\to\infty}\int_{S_{m_j}^{[\epsilon,\infty)}} \phi_{m_j}^2 d\mu_{m_j} \geq \delta > 0.$$

Therefore ϕ_∞ is not the zero function and its L^2 norm is less than 1. Therefore it is a λ_∞-eigenfunction.

Case 2: For any $\epsilon > 0$ *the sequence* $\|\phi_m\|_{S_m^{[\epsilon,\infty)}} \to 0$. Then we will prove the following statements:
(i) $S_\infty \in \partial\mathcal{M}_{g,n}$,
(ii) $\lambda_\infty = \frac{1}{4}$ and
(iii) $\exists$ constants K_m such that, up to extracting a subsequence, $(K_m\widetilde{\phi_m})$ converges uniformly to a function which is a linear combination of Eisenstein series and (possibly) a $\frac{1}{4}$-cuspidal eigenfunction.

(i) Suppose by contradiction that $S_\infty \in \mathcal{M}_{g,n}$. Then all the cusps of S_∞ are old cusps. Let $s(S_\infty)$ denote the *systole* of S_∞. Then, for $0 < \epsilon < \frac{s(S_\infty)}{2}$ and for m large enough, we have $S_m^{(0,\epsilon)} \subset \cup_{i=1}^n \mathcal{P}_i(m)$. Therefore, applying Corollary 3.31, the assumption $\|\phi_m\|_{S_m^{[\epsilon,\infty)}} \to 0$ implies that $\|\phi_m\| \to 0$. This is a contradiction since each ϕ_m is normalized. Thus $S_\infty \in \partial\mathcal{M}_{g,n}$.

(ii) follows from Theorem 3.36.

(iii) Fix some ϵ, $0 < \epsilon < \epsilon_0$. Choose constants $K_m \geq 1$ such that

$$\int_{S_m^{[\epsilon,\infty)}} |K_m\phi_m|^2 d\mu_m = 1.$$

Therefore the sequence (K_m) must diverge to ∞. Using mean value formula [F, Corollary 1.3], L^p-Schauder estimates [B-J-S] and elliptic regularity, as earlier, and Corollary 3.34 we obtain that, up to extracting a subsequence, $(\widetilde{K_m\phi_m})$ converges, uniformly over compacta, to a C^∞ function $\widetilde{\phi_\infty}$ that satisfies

$$\Delta\widetilde{\phi_\infty} + \frac{1}{4}\widetilde{\phi_\infty} = 0.$$

Moreover, $\widetilde{\phi_\infty}$ induces a function ϕ_∞ on S_∞ that satisfies

$$\Delta\phi_\infty + \frac{1}{4}\phi_\infty = 0. \tag{4.2}$$

Using the uniform convergence over compacta we have

$$\int_{S_\infty^{[\epsilon,\infty)}} \phi_\infty^2 d\mu_\infty = \lim_{m\to\infty}\int_{S_m^{[\epsilon,\infty)}} K_m\phi_m^2 d\mu_m = 1.$$

Therefore ϕ_∞ is not the zero function. From Lemma 3.1 and Lemma 3.17 (3.18) we deduce that ϕ_∞ satisfies *moderate growth* condition [Wo, p. 80] in each cusp. It is known that for any $\lambda \geq \frac{1}{4}$ the space of *moderate growth* λ*-eigenfunctions* of S_∞ is spanned by Eisenstein series and (possibly) λ-cuspidal eigenfunctions (see §3 in [Wo]). In particular, ϕ_∞ is a linear combination of Eisenstein series and (possibly) a cuspidal eigenfunction. This finishes the proof of *(iii)*.□

5. Proof of Theorem 1

We begin by proving Lemma 1 which says that $\mathcal{C}_{g,n}^{\frac{1}{4}}(k)$ is open in $\mathcal{M}_{g,n}$.

5.1. **Proof of Lemma 1.** Empty set is open by convention. Therefore, we argue by contradiction and assume that there exists a $S \in \mathcal{C}_{g,n}^{\frac{1}{4}}(k)$ such that every neighborhood of S contains points from $\mathcal{M}_{g,n} \setminus \mathcal{C}_{g,n}^{\frac{1}{4}}(k)$. In other words, there exists a sequence $(S_m) \subseteq \mathcal{M}_{g,n}$ that converges to S and, for all m, $\lambda_k^c(S_m) \leq \frac{1}{4}$. For $1 \leq i \leq k$, let us denote by ϕ_m^i a normalized $\lambda_i^c(S_m)$-cuspidal eigenfunction such that $\{\phi_m^i\}_{i=1}^k$ is an orthonormal family in $L^2(S_m)$. Since we are considering small eigenvalues, up to extracting a subsequence, the sequence $(\lambda_i^c(S_m))$ converges. For simplicity we assume that, for $1 \leq i \leq k$, the sequence $(\lambda_i^c(S_m))$ converges and denote by λ_∞^i its limit. Observe that, for $1 \leq i \leq k$, $\lambda_\infty^i \leq \frac{1}{4}$. Now, since $S \in \mathcal{M}_{g,n}$ by Theorem 2, up to extracting a subsequence, (ϕ_m^i) converge to λ_∞^i-eigenfunction ϕ_∞^i of S. Moreover, by the result about uniform integrability inside cusps in Corollary 3.31: $\|\phi_\infty^i\| = 1$. Hence $\{\phi_\infty^i\}_{i=1}^k$ is an orthonormal family in $L^2(S)$ so that the k-th cuspidal eigenvalue $\lambda_k^c(S)$ of S is below $\frac{1}{4}$. This is a contradiction because by our assumption $\lambda_k^c(S) > \frac{1}{4}$ as $S \in \mathcal{C}_{g,n}^{\frac{1}{4}}(k)$.□

Now we give a proof of Proposition 1.7 which says

Proposition 1.7 *There exists a neighborhood $\mathcal{N}(\mathcal{M}_{g,1}\cup\mathcal{M}_{0,n+1})$ of $\mathcal{M}_{g,1}\cup \mathcal{M}_{0,n+1}$ in $\overline{\mathcal{M}_{g,n}}$ such that for each $S \in \mathcal{N}(\mathcal{M}_{g,1}\cup\mathcal{M}_{0,n+1})$: $\lambda_{2g-1}^c(S) > \frac{1}{4}$ i.e.*

$$\mathcal{N}(\mathcal{M}_{g,1}\cup\mathcal{M}_{0,n+1}) \subset \mathcal{C}_{g,n}^{\frac{1}{4}}(2g-1).$$

5.2. **Proof of Proposition.** We argue by contradiction and assume that there is a sequence $S_m \in \mathcal{M}_{g,n}$ converging to $S_\infty \in \mathcal{M}_{g,1}\cup\mathcal{M}_{0,n+1} \subset \partial\mathcal{M}_{g,n}$ such that $\lambda_{2g-1}^c(S_m) \leq \frac{1}{4}$. For $1 \leq i \leq 2g-1$ and for each m we choose small cuspidal eigenpairs (λ_m^i, ϕ_m^i) of S_m such that

(i) $\{\phi_m^i\}_{i=1}^{2g-1}$ is an orthonormal family in $L^2(S_m)$,

(ii) λ_m^i is the i-th eigenvalue of S_m.

Theorem 2 provides two possible behaviors of the sequence (ϕ_m^i). However in our case we have Lemma 2:

Lemma 2 *For each i, $1 \leq i \leq 2g-1$, up to extracting a subsequence, the sequence (ϕ_m^i) converges to a λ_∞^i-eigenfunction ϕ_∞^i of S_∞. The limit functions ϕ_∞^i and ϕ_∞^j are orthogonal for $i \neq j$ i.e. S_∞ has at least $2g-1$ small eigenvalues. Moreover none of the ϕ_∞^i is residual.*

5.2.1. *Proof of Lemma 2.* By uniform convergence of ϕ_m^i to ϕ_∞^i, we have $\|\phi_\infty^i\| \leq 1$. To prove the first two statements of the lemma it is enough to prove that, for $1 \leq i \leq 2g-1$, $\|\phi_\infty^i\| = 1$ because this will imply that ϕ_∞^i is not the zero function and that (ϕ_m^i) is uniformly integrable over the thick parts: for any $t > 0$ there exists ϵ such that for all m one has,

$$\|\phi_m\|_{S_m^{[\epsilon,\infty)}} > 1 - t.$$

To prove that, for each $1 \leq i \leq 2g-1$, $\|\phi_\infty^i\| = 1$ we argue by contradiction and assume that for some $1 \leq i \leq 2g-1$, $\|\phi_\infty^i\| = 1-\delta$. To simplify the notation, denote the sequence (λ_m^i, ϕ_m^i) by (λ_m, ϕ_m) and the limit $(\lambda_\infty^i, \phi_\infty^i)$ by $(\lambda_\infty, \phi_\infty)$. By Corollary 3.31 the functions ϕ_m are uniformly integrable over the union of cusps of S_m: for any $t > 0$ there exists $\epsilon > 0$ such that for all m one has:

$$\|\phi_m\|_{S_{m,c}^{(0,\epsilon)}} < t. \tag{5.1}$$

Since $S_\infty \in \mathcal{M}_{g,1} \cup \mathcal{M}_{0,n+1}$ there is only one closed geodesic, $\gamma_m \subset S_m$, whose length l_{γ_m} tends to zero. For any $l \leq 1$ and for m large enough such that $l_{\gamma_m} < l$ denote by $\mathcal{C}_m^l \subset S_m$ the collar around γ_m bounded by two equidistant curves of length l. In view of the uniform integrability inside cusps (5.1), there exists $\epsilon_0 > 0$ such that for any $\epsilon \leq \epsilon_0$ there exists $m(\epsilon)$ such that for $m \geq m(\epsilon)$ we have:

$$\|\phi_m\|_{\mathcal{C}_m^\epsilon} > \frac{\delta}{2}. \tag{5.2}$$

Now we distinguish again two cases depending on whether $\lambda_\infty < \frac{1}{4}$ or $\lambda_\infty = \frac{1}{4}$. If $\lambda_\infty < \frac{1}{4}$ then we have a contradiction since $\|\phi_\infty\| = 1$ by Theorem 3.36. Hence we may suppose that $\lambda_\infty = \frac{1}{4}$. So, by Theorem 2 either ϕ_∞ is the zero function or, for instance by [I, Theorem 3.2], ϕ_∞ is cuspidal. Now recall that by lemma 3.17 we have uniform integrability of $[\phi_m]_1$: for any t there exists ϵ such that for all m:

$$\|[\phi_m]_1\|_{\mathcal{C}_m^\epsilon} < t.$$

Hence by (5.2), there exists ϵ_1 such that for any $\epsilon \leq \epsilon_1$ the exists $m_1(\epsilon)$ such that for $m \geq m_1(\epsilon)$ one has:

$$\|[\phi_m]_0\|_{\mathcal{C}_m^\epsilon} > \frac{\delta}{4} \tag{5.3}$$

In particular, if $c(\epsilon, m) = \sup_{z \in \mathcal{C}_m^\epsilon} |[\phi_m]_0|$ then, since area of $\mathcal{C}_m^\epsilon$ is less than 1, we have for any $\epsilon \leq \epsilon_1$ and $m \geq m_1(\epsilon)$:

$$c(\epsilon, m) > \frac{\delta}{4}. \tag{5.4}$$

Now we prove that $[\phi_m]_1$ is uniformly small inside $\mathcal{C}_m^\epsilon$. More precisely,

Lemma 5.5. *Let ϵ be such that $0 < \epsilon < 1$. There exists a constant $K < \infty$, independent of ϵ, and $m_2(\epsilon) \in \mathbb{N}$ such that for $m \geq m_2(\epsilon)$ and $z \in \mathcal{C}_m^{\epsilon}$:*

$$|[\phi_m]_1|(z) < K\frac{\epsilon^{\frac{1}{2}}}{1-\epsilon}.$$

Proof. Consider the expansion of ϕ_m inside $\mathcal{C}_m^1$ with respect to the Fermi coordinates (see 2.1.1):

$$\phi_m(r,\theta) = a_0^m(r) + \sum_{j=1}^{\infty}\left(a_j^m(r)\cos j\theta + b_j^m(r)\sin j\theta\right). \tag{5.6}$$

Here, for each $j \geq 0$, (a_j^m, b_j^m) are the j-th Fourier coefficients of ϕ_m inside $\mathcal{C}_m^1$ and are defined for all $|r| \leq L_{1,m}$. Recall that, for any $\epsilon \in [l_{\gamma_m}, 1]$ we denote by $L_{\epsilon,m}$ the number $\cosh^{-1}(\frac{\epsilon}{l_{\gamma_m}})$. Recall also that since ϕ_m is a λ_m-eigenfunction, a_j^m and b_j^m satisfy (3.16) with $2\pi l = l_{\gamma_m}$ and $\lambda = \lambda_m$. Therefore, for $j \geq 1$, one can express:

$$(1)\ a_j^m(r) = a_{m,j}s_{m,j}(r) + b_{m,j}c_{m,j}(r)$$
$$(2)\ b_j^m(r) = a_{m,j}{}'s_{m,j}(r) + b_{m,j}{}'c_{m,j}(r) \tag{5.7}$$

where $s_{m,j}(r)$ and $c_{m,j}(r)$ are the two linearly independent solutions of (3.16) with $l = l(\gamma_m)$ and $\lambda = \lambda_m$.

Recall that $s_{m,j}(r)\cosh^{\frac{1}{2}}(r)$ and $c_{m,j}(r)\cosh^{\frac{1}{2}}(r)$ satisfy:

$$\frac{d^2u}{dr^2} = \left(\frac{1}{4\cosh^2 r} + \frac{j^2}{l^2\cosh^2 r}\right)u.$$

Since, for $r \leq L_{\epsilon,m}$, $l^2\cosh^2 r \leq 1$ by Claim 3.23, for each $j \geq 1$, there exists strictly increasing functions $h_{m,j} : [0, L_{1,m}] \to \mathbb{R}_{>0}$ and $k_{m,j} : [0, L_{1,m}] \to \mathbb{R}_{>0}$ such that

$$(i)\ s_{m,j}(r)\sqrt{\cosh(r)} = h_{m,j}(r)\cosh jr$$
$$(ii)\ c_{m,j}(r)\sqrt{\cosh(r)} = k_{m,j}(r)\cosh jr. \tag{5.8}$$

We denote by $\mathcal{P}_{n+1}(\infty)$ and $\mathcal{P}_{n+2}(\infty)$ the two new cusps of S_∞ that appear as the limit of $\mathcal{C}_m^1$ as $m \to \infty$. Now, let us assume:

$$\sup_{z\in\partial\mathcal{P}_{n+1}(\infty)\cup\partial\mathcal{P}_{n+2}(\infty)}|\phi_\infty|(z) < \frac{t}{4}.$$

Then, by the uniform convergence of ϕ_m to ϕ_∞ over compacta, we have a $N \in \mathbb{N}$ such that for $m \geq N$ and $z \in \partial\mathcal{C}_m^1$:

$$|\phi_m|(z) < \frac{t}{4}.$$

By (5.6) for any $j \geq 1$:

$$|a_j^m|(\pm L_{1,m}) = \frac{1}{\pi}\left|\int_0^{2\pi}\phi_m(\pm L_{1,m},\theta)\cos j\theta d\theta\right| \leq \frac{t}{2}. \tag{5.9}$$

Similar calculations for b_j^m provide: $|b_j^m|(\pm L_{1,m}) \leq \frac{t}{2}$. Recall that $s_{m,j}$ is odd and $c_{m,j}$ is even. So by (5.2.1) and (5.2.1):

$$(i)\ a_j^m(L_{1,m}) + a_j^m(-L_{1,m}) = 2b_{m,j}k_j(L_{1,m})\frac{\cosh jL_{1,m}}{\sqrt{\cosh L_{1,m}}}$$

$$(ii)\ a_j^m(L_{1,m}) - a_j^m(-L_{1,m}) = 2a_{m,j}h_j(L_{1,m})\frac{\cosh jL_{1,m}}{\sqrt{\cosh L_{1,m}}}. \tag{5.10}$$

Therefore, by (5.9) and (5.2.1):

$$\begin{aligned}(i)\ |b_{m,j}|k_j(L_{1,m})\frac{\cosh jL_{1,m}}{\sqrt{\cosh L_{1,m}}} &< \frac{t}{2}\\ (ii)\ |a_{m,j}|h_j(L_{1,m})\frac{\cosh jL_{1,m}}{\sqrt{\cosh L_{1,m}}} &< \frac{t}{2}.\end{aligned} \tag{5.11}$$

Therefore, for any $r \leq L_{1,m}$:

$$|a_j^m|(r) = |a_{m,j}s_{m,j}(r) + b_{m,j}c_{m,j}(r)| < |a_{m,j}|s_{m,j}(r) + |b_{m,j}|c_{m,j}(r).$$

The last term of the inequality is

$$|a_{m,j}|h_{m,j}(r)\frac{\cosh jr}{\sqrt{\cosh r}} + |b_{m,j}|k_{m,j}(r)\frac{\cosh jr}{\sqrt{\cosh r}} < t\frac{\cosh jr}{\sqrt{\cosh r}}\frac{\sqrt{\cosh L_{1,m}}}{\cosh jL_{1,m}}$$

since $h_{m,j}$ and $k_{m,j}$ are strictly increasing functions (by (5.2.1)). Similarly,

$$|b_j^m|(r) < t\frac{\cosh jr}{\sqrt{\cosh r}}\frac{\sqrt{\cosh L_{1,m}}}{\cosh jL_{1,m}}.$$

Hence

$$|[\phi_m]_1|(r,\theta) < 2t\sum_{j=1}^{\infty}\frac{\cosh jr}{\sqrt{\cosh r}}\frac{\sqrt{\cosh L_{1,m}}}{\cosh jL_{1,m}}. \tag{5.12}$$

Since, for $j \geq 1$, the function $\frac{\cosh jr}{\sqrt{\cosh r}}$ is strictly increasing, for any $r \leq L_{\epsilon,m}$:

$$\sum_{j=1}^{\infty}\frac{\cosh jr}{\sqrt{\cosh r}}\frac{\sqrt{\cosh L_{1,m}}}{\cosh jL_{1,m}} < \sum_{j=1}^{\infty}\frac{\cosh jL_{\epsilon,m}}{\sqrt{\cosh L_{\epsilon,m}}}\frac{\sqrt{\cosh L_{1,m}}}{\cosh jL_{1,m}} \tag{5.13}$$

Now fix an ϵ such that $0 < \epsilon < 1$. Observe that $L_{\epsilon,m} = \log(\frac{\epsilon}{l_{\gamma_m}} + \sqrt{(\frac{\epsilon}{l_{\gamma_m}})^2 - 1})$. So, for m large such that l_{γ_m} is small compared to ϵ:

$$\sum_{j=1}^{\infty}\frac{\cosh jL_{\epsilon,m}}{\sqrt{\cosh L_{\epsilon,m}}}\frac{\sqrt{\cosh L_{1,m}}}{\cosh jL_{1,m}} < K'\sum_{j=1}^{\infty}\epsilon^j\epsilon^{-\frac{1}{2}} = K'\frac{\epsilon^{\frac{1}{2}}}{1-\epsilon} \tag{5.14}$$

where the constant K' can be chosen independently of ϵ as soon as m is larger than some number $m_2(\epsilon) \in \mathbb{N}$. Therefore, by (5.12) and (5.14), for $m \geq m_2(\epsilon)$ and $(r,\theta) \in \mathcal{C}_m^{\epsilon}$

$$|[\phi_m]_1|(r,\theta) < 2tK'\frac{\epsilon^{\frac{1}{2}}}{1-\epsilon}. \tag{5.15}$$

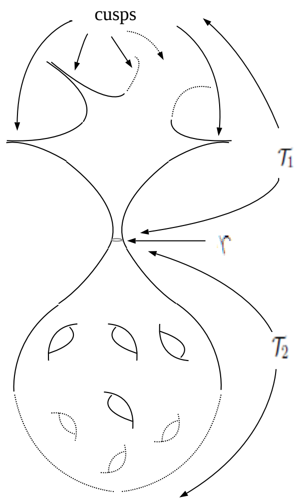
This proves the lemma.□

Now fix $\epsilon < \epsilon_1$ (see (5.3)) such that $K\frac{\epsilon^{\frac{1}{2}}}{1-\epsilon} < \frac{\delta}{4}$ and choose $m \geq \max\{m_1(\epsilon), m_2(\epsilon)\}$. Then by Lemma 5.5 and (5.4): for each $z \in \mathcal{C}_m^{\epsilon}$

$$c(\epsilon, m) > |[\phi_m]_1|(z). \tag{5.16}$$

So the parallel curve α_m with distance r_0 ($\leq L_{\epsilon,m}$) from γ_m such that $c = |[\phi_m]_0|(r_0)$ has the property that ϕ_m has constant sign on it. In other words, the nodal set $\mathcal{Z}(\phi_m)$ does not intersect this curve. This is a contradiction to the next lemma.

Lemma 5.17. *Let S be a noncompact, finite area hyperbolic surface of type (g, n). Let γ be a simple closed geodesic that separates S into two connected components $\mathcal{T}_1$ and $\mathcal{T}_2$ such that $\mathcal{T}_1$ is topologically a sphere with $n+1$ punctures and $\mathcal{T}_2$ is topologically a genus g surface with one puncture. Let f be a small cuspidal eigenfunction of S. Then the zero set $\mathcal{Z}(f)$ of f intersects every curve homotopic to γ.*



Proof. Recall that $\mathcal{Z}(f)$ is a locally finite graph [Ch]. Let us assume that $\mathcal{Z}(f)$ does not intersect some curve τ homotopic to γ. We have $S \setminus \tau = \mathcal{T}_1 \cup \mathcal{T}_2$ and all the punctures of S are contained in $\mathcal{T}_1$. Consider the components of $\mathcal{T}_1 \setminus \mathcal{Z}(f)$. Recall that since f is cuspidal $\mathcal{Z}(f)$ contains all the punctures of S and therefore these components give rise to a cell decomposition of a once punctured sphere. The Euler characteristic of the component $\mathcal{F}$ containing τ as a puncture is either negative or zero (since γ and each component of $\mathcal{Z}(f)$ are essential; see [O]). Each component of $\mathcal{T}_1 \setminus \mathcal{Z}(f)$ other than $\mathcal{F}$ (at least one such exists since g changes sign in $\mathcal{T}_1$) is a *nodal domain* of f and hence has negative Euler characteristic [O]. Also $\mathcal{Z}(f)$ being a graph has non-positive Euler characteristic. Let C^+ (resp. C^-) be the union of the nodal domains contained in $\mathcal{T}_1$ which are different from $\mathcal{F}$ and where f is positive (resp. negative). Denote by $\chi(X)$ the Euler characteristic of the topological space X. Since the Euler characteristic of a once punctured sphere is 1, by the Euler-Poincaré formula one has:

$$1 = \chi(\mathcal{F}) + \chi(C^+) + \chi(C^-) + \chi(\mathcal{Z}(f)).$$

This is a contradiction because the right hand side of the equality is strictly negative.□

Now we prove that ϕ_∞ is not a residual eigenfunction. It is clear from the uniform convergence that ϕ_∞ is cuspidal at the old cusps. If ϕ_∞ is a residual eigenfunction then the only possibility is that ϕ_∞ is not cupsidal at one of the two new cusps. Let us assume that ϕ_∞ is residual in $\mathcal{P}_{n+1}$. Then, for sufficiently large t, ϕ_∞ has constant sign in $\mathcal{P}^t_{n+1}$. Therefore, by the uniform convergence $\phi_m|_{S_m^{[\epsilon,\infty)}} \to \phi_\infty|_{S_\infty^{[\epsilon,\infty)}}$ it follows that, for all m large, ϕ_m has constant sign on a component of $\partial\mathcal{C}_m^{\frac{1}{t}}$. Since this component is homotopic to γ_m this leads to a contradiction to Lemma 5.17 as well. This finishes the proof of Lemma 2.□

5.2.2. *Continuation of Proof of Proposition.* Let us denote the two components of S_∞ by $\mathcal{N}_1$ and $\mathcal{N}_2$ such that $\mathcal{N}_1 \in \mathcal{M}_{g,1}$ and $\mathcal{N}_2 \in \mathcal{M}_{0,n+1}$. Lemma 2 says that S_∞ must have at least $2g-1$ many small cuspidal eigenvalues. By [O-R, Théoréme 0.2] the number of non-zero small eigenvalues of $\mathcal{N}_1$ is at most $2g-2$. In particular, the number of small cuspidal eigenvalues of $\mathcal{N}_1$ is at most $2g-2$. Thus for some i, $1 \leq i \leq 2g-1$, ϕ_∞^i is not the zero function when restricted to $\mathcal{N}_2$ i.e. ϕ_∞^i is a cuspidal eigenfunction of $\mathcal{N}_2$. This is a contradiction because $\mathcal{N}_2$ does not have any small cuspidal eigenfunction by [H] or [O]. □

Remark 5.18. The arguments in the proof of Proposition are applicable to more general settings. In particular, let (S_m) be a sequence in $\mathcal{M}_{g,n}$ that converges to $S_\infty \in \partial\mathcal{M}_{g,n}$. Let (λ_m, ϕ_m) be a normalized small eigenpair of S_m. Let $\lambda_m \to \lambda_\infty$ as m tends to infinity. The arguments show the following: If $\liminf_{m\to\infty} \|\phi_m\| < 1$ then there exists a curve α_m, homotopic to a geodesic of length tending to zero, on which, up to extracting a subsequence, ϕ_m has constant sign.

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