

Exceptional sheaves on the Hirzebruch surface \mathbb{F}_2

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Abstract

We investigate exceptional sheaves on the Hirzebruch surface \mathbb{F}_2 , as the first attempt toward the classification of exceptional objects on weak del Pezzo surfaces.

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1 Introduction

Let X be a smooth projective variety over an algebraically closed field k . The bounded derived category of coherent sheaves on X , which we denote by $D(X) = D^b \text{coh } X$, admits a structure of k -linear triangulated category (see [Huy06]). An object \mathcal{E} of a k -linear triangulated category is called *exceptional* if it satisfies the condition

$$\text{Hom}(\mathcal{E}, \mathcal{E}[i]) = \begin{cases} 0 & i \neq 0 \\ k \cdot \text{id}_{\mathcal{E}} & i = 0 \end{cases}$$

(see [Huy06, Definition 1.57]). For example, when X is a Fano manifold in characteristic zero, the Kodaira vanishing theorem implies that line bundles on X are exceptional objects of $D(X)$. Classification of exceptional objects in a given triangulated category is basic but quite a nontrivial issue.

Exceptional objects of the derived category of del Pezzo surfaces (i.e., Fano manifolds of dimension 2) were thoroughly studied in the paper [KO94]. Among others, it was shown in [KO94, Propositions 2.9 and 2.10] that any exceptional object on such surfaces is isomorphic to a shift of an exceptional vector bundle or a line bundle on a (-1) -curve.

Since exceptional objects on del Pezzo surfaces are well understood, it is natural to work on *weak del Pezzo* surfaces; i.e., those non-singular surfaces with nef and big anti-canonical line bundles. The purpose of this paper is to study exceptional sheaves on the Hirzebruch surface of degree 2, i.e., $\mathbb{F}_2 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$, which is the easiest example of weak del Pezzo surfaces.

Unlike the case of del Pezzo surfaces the situation becomes much more involved. This is due to the existence of the non-standard autoequivalences of the derived category which are called *twist functors*.

Definition 1.1 ([ST01]). Let X be a smooth projective variety.

(1) We say that an object $\alpha \in D(X)$ is *spherical* if we have $\alpha \otimes \omega_X \cong \alpha$ and

$$\text{Hom}(\alpha, \alpha[i]) \cong \begin{cases} 0 & i \neq 0, \dim X \\ k \cdot \text{id}_{\alpha} & i = 0, \dim X. \end{cases}$$

(2) Let $\alpha \in D(X)$ be a spherical object. We consider the mapping cone

$$\mathcal{C} = \text{Cone}(\pi_1^* \alpha^\vee \otimes \pi_2^* \alpha \rightarrow \mathcal{O}_{\Delta})$$

of the natural evaluation $\pi_1^* \alpha^\vee \otimes \pi_2^* \alpha \rightarrow \mathcal{O}_\Delta$, where $\Delta \subset X \times X$ is the diagonal and π_i is the projection from $X \times X$ to the i th factor. Then the integral functor $T_\alpha := \Phi_{X \rightarrow X}^{\mathcal{C}}$ defines an autoequivalence of $D(X)$, called the *twist functor* along the spherical object α . By definition, for $\beta \in D(X)$, we have an exact triangle

$$\mathbb{R}\mathrm{Hom}(\alpha, \beta) \otimes \alpha \xrightarrow{\text{evaluation}} \beta \longrightarrow T_\alpha \beta. \quad (1.1)$$

We can also define the *inverse twist functor* T'_α so that it is a quasi-inverse of T_α and there exists an exact triangle

$$T'_\alpha \beta \longrightarrow \beta \xrightarrow{\text{coevaluation}} \mathbb{R}\mathrm{Hom}(\beta, \alpha)^\vee \otimes \alpha. \quad (1.2)$$

Remark 1.2. The triangle (1.1) yields the equality

$$[T_\alpha \beta] = [\beta] - \chi(\alpha, \beta)[\alpha]$$

in the K -group $K_0(X)$. Here $\chi(-, -)$ denotes the Euler pairing. From this and Definition 1.1 (1), we see that

$$\chi(\alpha, T_\alpha \beta) = (-1)^{\dim X + 1} \chi(\alpha, \beta).$$

As a consequence, if $\dim X$ is even, the (inverse) twist functor acts as a reflection on $K_0(X)$. In particular, if we do it twice, the action on $K_0(X)$ is trivial.

Note that any line bundle on the (-2) -curve on \mathbb{F}_2 provides an example of a spherical object in $D(\mathbb{F}_2)$. Twisting exceptional sheaves by those spherical objects, we can produce many more exceptional objects on \mathbb{F}_2 and they are not necessarily isomorphic to shifts of sheaves. Note that the group of autoequivalences of \mathbb{F}_2 is known by [BP14, Theorem 1] and satisfies

$$\mathrm{Auteq}(D(\mathbb{F}_2)) = (B(\mathbb{F}_2) \rtimes \mathbb{Z}\mathcal{O}(F)) \times \mathrm{Aut}(\mathbb{F}_2) \times \mathbb{Z}[1],$$

where F is a fiber of the \mathbb{P}^1 -bundle $\mathbb{F}_2 \rightarrow \mathbb{P}^1$ and $B(\mathbb{F}_2)$ is the subgroup generated by spherical twists. Here $[1]$ is the shift by 1 of complexes, and $\mathcal{O}(F)$ acts by tensoring with this line bundle. Since $\mathrm{Aut}(\mathbb{F}_2) = \mathrm{Aut}^0(\mathbb{F}_2)$, and line bundles and spherical objects are rigid, we see that elements of $\mathrm{Aut}(\mathbb{F}_2)$ commute with all other autoequivalences.

Despite this complication, we expect the following conjecture.

Conjecture 1.3. *For any exceptional object $\mathcal{E} \in D(\mathbb{F}_2)$, there exists an autoequivalence $\Phi \in \text{Auteq}(D(\mathbb{F}_2))$ such that $\Phi(\mathcal{E})$ is an exceptional vector bundle on \mathbb{F}_2 .*

As a special case, we prove the following theorem. We denote by $C \subset \mathbb{F}_2$ the unique (-2) -curve.

Theorem 1.4. *Let \mathcal{E} be an exceptional sheaf on \mathbb{F}_2 with a nontrivial torsion subsheaf. Consider the standard decomposition of the sheaf \mathcal{E}*

$$0 \rightarrow \mathcal{T}^- \rightarrow \mathcal{E} \rightarrow \mathcal{F}^- \rightarrow 0 \quad (1.3)$$

into the torsion part \mathcal{T}^- and the torsion-free part \mathcal{F}^- . Also let

$$0 \rightarrow \mathcal{F}^+ \rightarrow \mathcal{E} \rightarrow \mathcal{T}^+ \rightarrow 0 \quad (1.4)$$

be the standard decomposition of the sheaf \mathcal{E} into the restriction \mathcal{T}^+ of \mathcal{E} to C and the kernel \mathcal{F}^+ of the restriction map. Then the following hold.

(1) *There exists an integer a satisfying the following properties.*

- *The triangle (1.1) for $\alpha = \mathcal{O}_C(a)$ and $\beta = \mathcal{E}$ becomes a short exact sequence*

$$0 \rightarrow \text{Hom}(\mathcal{O}_C(a), \mathcal{E}) \otimes \mathcal{O}_C(a) \rightarrow \mathcal{E} \rightarrow T_{\mathcal{O}_C(a)}\mathcal{E} \rightarrow 0, \quad (1.5)$$

and is isomorphic to the sequence (1.3).

- *The triangle (1.2) for $\alpha = \mathcal{O}_C(a+1)$ and $\beta = \mathcal{E}$ becomes a short exact sequence*

$$0 \rightarrow T'_{\mathcal{O}_C(a+1)}\mathcal{E} \rightarrow \mathcal{E} \rightarrow \text{Hom}(\mathcal{E}, \mathcal{O}_C(a+1))^\vee \otimes \mathcal{O}_C(a+1) \rightarrow 0, \quad (1.6)$$

and is isomorphic to the sequence (1.4).

(2) *\mathcal{F}^+ and \mathcal{F}^- are exceptional vector bundles. Moreover they are related to each other by*

$$\mathcal{F}^- \cong \mathcal{F}^+ \otimes \mathcal{O}(C).$$

Remark 1.5. Conjecture 1.3 is not correct for Hirzebruch surfaces \mathbb{F}_n with $n \geq 3$. In fact, since those surfaces have no non-standard autoequivalences by [BP14, Theorem 1], shifts of sheaves are always sent to shifts of sheaves

under autoequivalences. On the other hand, we can construct exceptional objects which are genuine complexes as follows. Let F, C be the classes of a fiber and the negative curve, respectively. Then we easily see that

$$\mathcal{O}, \mathcal{L} := \mathcal{O}((n-1)F + C)$$

is an exceptional pair. The left mutation (see [BK89]) of the pair is included in the triangle

$$L_{\mathcal{O}}\mathcal{L} \rightarrow \mathbb{R}\mathrm{Hom}(\mathcal{O}, \mathcal{L}) \otimes \mathcal{O} \xrightarrow{\text{evaluation}} \mathcal{L},$$

and we can easily check that \mathcal{H}^0 and \mathcal{H}^1 of the complex $L_{\mathcal{O}}\mathcal{L}$ are nontrivial; to see this, note that \mathcal{L} is not globally generated since $\mathcal{L} \cdot \mathcal{O}(C) = -1 < 0$.

For a del Pezzo surface X , any exceptional object of $D(X)$ is, up to even shifts $\mathbb{Z}[2]$, uniquely determined by its class in $K_0(X)$ (see [Gor88, Corollary 2.5]). This is not the case for \mathbb{F}_2 (see Remark 4.7). Therefore we can also pose the following finer problem.

Problem 1.6. For each exceptional object $\tilde{\mathcal{E}} \in D(\mathbb{F}_2)$, classify all other exceptional objects \mathcal{E} such that $[\tilde{\mathcal{E}}] = [\mathcal{E}] \in K_0(\mathbb{F}_2)$.

As a partial answer to the problem, for any exceptional object $\tilde{\mathcal{E}}$ on \mathbb{F}_2 , we describe all the exceptional sheaves \mathcal{E} sharing the class with $\tilde{\mathcal{E}}$.

Theorem 1.7 (Corollary 3.7 + Proposition 4.1). *For any exceptional object $\tilde{\mathcal{E}} \in D(\mathbb{F}_2)$, there exists a unique exceptional vector bundle \mathcal{E} such that $[\tilde{\mathcal{E}}] = [\mathcal{E}]$ or $-\mathcal{E}] \in K_0(\mathbb{F}_2)$. Moreover the set of exceptional sheaves sharing the class with \mathcal{E} can be explicitly described as $\{\mathcal{E}_i \mid i \geq -1\}$ with $\mathcal{E}_{-1} = \mathcal{E}$ (see Section 3 for the definition of \mathcal{E}_i).*

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Notations and Conventions

We always work over an algebraically closed field k of an arbitrary characteristic.

To simplify the notations, we write ${}^i(\mathcal{E}, \mathcal{F})$ to indicate $\text{Ext}^i(\mathcal{E}, \mathcal{F}) = \text{Hom}(\mathcal{E}, \mathcal{F}[i])$. Even more, ${}^0(\mathcal{E}, \mathcal{F})$ is also denoted by $(\mathcal{E}, \mathcal{F})$. These symbols also indicate the dimensions of the respective vector spaces, depending on the context. The symbol $\chi(-, -)$ denotes the Euler pairing, which is defined by $\chi(\mathcal{E}, \mathcal{F}) = \sum_{i \in \mathbb{Z}} (-1)^i \dim {}^i(\mathcal{E}, \mathcal{F})$.

For each $n > 0$, we denote by $C \subset \mathbb{F}_n$ the unique negative curve with $C^2 = -n$. Exceptional objects on surfaces whose anti-canonical line bundle is big and has at most zero-dimensional base locus were systematically investigated in [Kul97]. Since the anti-canonical line bundle of \mathbb{F}_2 is globally generated, we freely quote the results of [Kul97] in this paper. In particular we use the notion of $(-K_{\mathbb{F}_2})$ -stability for torsion-free sheaves on this surface.

2 Proof of Theorem 1.4

We give a proof of Theorem 1.4 in this section. For simplicity, we put $\mathcal{F} = \mathcal{F}^-$ and $\mathcal{T} = \mathcal{T}^-$. Let us begin with some preparations.

Lemma 2.1. *For any sequence of the form (1.3) with \mathcal{E} exceptional, the dimensions of the Ext groups among $\mathcal{T}, \mathcal{E}, \mathcal{F}$ can be calculated as follows:*

$$\begin{array}{lll}
 {}^0(\mathcal{T}, \mathcal{T}) = t & {}^1(\mathcal{T}, \mathcal{T}) = 0 & {}^2(\mathcal{T}, \mathcal{T}) = t \\
 {}^0(\mathcal{T}, \mathcal{E}) = t & {}^1(\mathcal{T}, \mathcal{E}) = f - 1 & {}^2(\mathcal{T}, \mathcal{E}) = 0 \\
 {}^0(\mathcal{T}, \mathcal{F}) = 0 & {}^1(\mathcal{T}, \mathcal{F}) = f + t - 1 & {}^2(\mathcal{T}, \mathcal{F}) = 0 \\
 {}^0(\mathcal{E}, \mathcal{T}) = 0 & {}^1(\mathcal{E}, \mathcal{T}) = f - 1 & {}^2(\mathcal{E}, \mathcal{T}) = t \\
 {}^0(\mathcal{E}, \mathcal{E}) = 1 & {}^1(\mathcal{E}, \mathcal{E}) = 0 & {}^2(\mathcal{E}, \mathcal{E}) = 0 \\
 {}^0(\mathcal{E}, \mathcal{F}) = f & {}^1(\mathcal{E}, \mathcal{F}) = t & {}^2(\mathcal{E}, \mathcal{F}) = 0 \\
 {}^0(\mathcal{F}, \mathcal{T}) = 0 & {}^1(\mathcal{F}, \mathcal{T}) = f + t - 1 & {}^2(\mathcal{F}, \mathcal{T}) = 0 \\
 {}^0(\mathcal{F}, \mathcal{E}) = 0 & {}^1(\mathcal{F}, \mathcal{E}) = t - 1 & {}^2(\mathcal{F}, \mathcal{E}) = f - 1 \\
 {}^0(\mathcal{F}, \mathcal{F}) = f & {}^1(\mathcal{F}, \mathcal{F}) = 0 & {}^2(\mathcal{F}, \mathcal{F}) = f - 1
 \end{array}$$

Above we set $t = {}^0(\mathcal{T}, \mathcal{T})$ and $f = {}^0(\mathcal{F}, \mathcal{F})$.

Proof. Consider the long exact sequences associated to the six functors

$$(\mathcal{E}, -), (-, \mathcal{E}), (\mathcal{F}, -), (-, \mathcal{F}), (\mathcal{T}, -), (-, \mathcal{T})$$

applied to the short exact sequence (1.3), and use the facts that

- \mathcal{E} is exceptional,
- $(\mathcal{T}, \mathcal{F}) = 0$,
- $\mathcal{T} \otimes \omega_{\mathbb{F}_2} \cong \mathcal{T}$ (= [Kul97, Lemma 2.2.5]), and
- \mathcal{F} and \mathcal{T} are rigid (this follows from Mukai's lemma [Kul97, Lemma 2.1.4. 2. (a)] and the rigidity of \mathcal{E}).

□

Lemma 2.2. *For any sequence of the form (1.3), the sheaf \mathcal{F} is an exceptional vector bundle and \mathcal{T} is a pure one-dimensional sheaf supported on C .*

Proof. First of all, [Kul97, Corollary 2.2.3] implies that the torsion-free part \mathcal{F} and the torsion part \mathcal{T} are both rigid, and that \mathcal{T} is a pure one-dimensional sheaf. Then [Kul97, Lemma 2.2.1] tells us that \mathcal{F} is a vector bundle.

Next, by [Kul97, Theorem 2.4.1. (1)] we see that \mathcal{F} is a direct sum

$$\mathcal{F} = \bigoplus_{i=1}^m \mathcal{F}_i$$

of $\mu_{(-K_{\mathbb{F}_2})}$ -semi-stable rigid bundles. If we assume $m > 1$, from [Kul97, Lemma 2.2.2] we obtain the inequality $(\mathcal{F}, \mathcal{F}) \geq {}^2(\mathcal{F}, \mathcal{F}) - 2$. Since Lemma 2.1 implies $(\mathcal{F}, \mathcal{F}) - {}^2(\mathcal{F}, \mathcal{F}) = 1$, this is a contradiction.

Hence we see that $m = 1$, so that \mathcal{F} itself is semi-stable. This implies ${}^2(\mathcal{F}, \mathcal{F}) = (\mathcal{F}, \mathcal{F} \otimes \omega_{\mathbb{F}_2}) = 0$, because of the inequality

$$\mu_{(-K_{\mathbb{F}_2})}(\mathcal{F}) > \mu_{(-K_{\mathbb{F}_2})}(\mathcal{F} \otimes \omega_{\mathbb{F}_2})$$

and [Kul97, Lemma 1.1.5]. Therefore we see ${}^2(\mathcal{F}, \mathcal{F}) = (\mathcal{F}, \mathcal{F}) - 1 = 0$, concluding that \mathcal{F} is an exceptional vector bundle. □

Remark 2.3. The above proof in particular tells us that the number $f = (\mathcal{F}, \mathcal{F})$ in Lemma 2.1 is one.

For a positive integer m , let $\iota: C \hookrightarrow mC$ be the natural closed immersion into the m -th thickening of C . We next consider the Harder-Narasimhan filtrations of pure sheaves on mC .

Lemma 2.4. *Let \mathcal{T} be a pure one-dimensional sheaf on the scheme mC . Then the subquotients of the Harder-Narasimhan filtration*

$$0 = \mathcal{T}^0 \subset \mathcal{T}^1 \subset \dots \subset \mathcal{T}^n = \mathcal{T}$$

of \mathcal{T} are of the form

$$\{\mathcal{T}^1/\mathcal{T}^0, \mathcal{T}^2/\mathcal{T}^1, \dots, \mathcal{T}^n/\mathcal{T}^{n-1}\} = \{\mathcal{O}_C(a_1)^{\oplus r_1}, \mathcal{O}_C(a_2)^{\oplus r_2}, \dots, \mathcal{O}_C(a_n)^{\oplus r_n}\}$$

with $a_1 > a_2 > \dots > a_n$ and $r_i > 0$.

Proof. A stable sheaf S on mC is simple [HL10, Corollary 1.2.8], and thus is isomorphic to a coherent \mathcal{O}_C -module by [IU05, Lemma 4.8]. Since C is isomorphic to \mathbb{P}^1 and S is stable, S has to be a line bundle on C .

Recall that any semi-stable sheaf has a Jordan-Hölder filtration [HL10, Section 1.5]. By definition, its subquotients are stable with the same slope. On the other hand, for any a and $m > 0$ we have

$$\begin{aligned} & \text{Ext}_{mC}^1(\iota_*\mathcal{O}_C(a), \iota_*\mathcal{O}_C(a)) \\ & \cong \text{Hom}_{mC}(\iota_*\mathcal{O}_C(a), \iota_*\mathcal{O}_C(a) \otimes \omega_{mC})^\vee \\ & \cong \text{Hom}_C(\mathcal{O}_C(a), \mathcal{O}_C(a - 2m))^\vee = 0. \end{aligned}$$

Therefore any semi-stable sheaf on mC turns out to be polystable; i.e., isomorphic to the direct sum of its Jordan-Hölder factors. Thus we conclude the proof. \square

Now let \mathcal{T} be the torsion sheaf in (1.3). Since it is pure by Lemma 2.2, we can apply Lemma 2.4. Then we can prove the following

Claim 2.5. *The number n of the Harder-Narasimhan factors of \mathcal{T} is 1, so that $\mathcal{T} \cong \mathcal{O}_C(a)^{\oplus r}$ holds for some integers a and $r > 0$.*

Proof. Suppose for a contradiction that $n > 1$. Consider the short exact sequence

$$0 \rightarrow \mathcal{T}^n/\mathcal{T}^{n-1} \rightarrow \mathcal{E}/\mathcal{T}^{n-1} \rightarrow \mathcal{E}/\mathcal{T}^n = \mathcal{F} \rightarrow 0$$

and apply the functor $(\mathcal{O}_C(a_{n-1}), -)$ to it. As part of the associated long exact sequence we obtain

$${}^1(\mathcal{O}_C(a_{n-1}), \mathcal{F}) \xrightarrow{\delta} {}^2(\mathcal{O}_C(a_{n-1}), \mathcal{T}^n/\mathcal{T}^{n-1}) \rightarrow {}^2(\mathcal{O}_C(a_{n-1}), \mathcal{E}/\mathcal{T}^{n-1}).$$

The third term is trivial. In fact we have the series of inequalities

$$\begin{aligned}
& {}^2(\mathcal{O}_C(a_{n-1}), \mathcal{E}/\mathcal{T}^{n-1})^\vee \\
&= (\mathcal{E}/\mathcal{T}^{n-1}, \mathcal{O}_C(a_{n-1})) \quad (\text{Serre duality}) \\
&\leq (\mathcal{E}/\mathcal{T}^{n-1}, \mathcal{O}_C(a_1)) \quad (a_{n-1} < a_1) \\
&\leq (\mathcal{E}/\mathcal{T}^{n-1}, \mathcal{T}) \quad (\mathcal{O}_C(a_1) \hookrightarrow \mathcal{T}) \\
&\leq (\mathcal{E}, \mathcal{T}) \quad (\mathcal{E} \twoheadrightarrow \mathcal{E}/\mathcal{T}^{n-1}) \\
&= 0 \quad (\text{Lemma 2.1}).
\end{aligned}$$

Therefore δ is surjective and we obtain the inequalities

$${}^1(\mathcal{O}_C(a_{n-1}), \mathcal{F}) \geq {}^2(\mathcal{O}_C(a_{n-1}), \mathcal{T}^n/\mathcal{T}^{n-1}) = r_n(a_{n-1} - a_n + 1) \geq 2r_n. \quad (2.1)$$

On the other hand, we can prove the two inequalities

$${}^1(\mathcal{O}_C(a_{n-1}), \mathcal{F}) < {}^1(\mathcal{O}_C(a_n), \mathcal{F}) \quad (2.2)$$

and

$${}^1(\mathcal{O}_C(a_n), \mathcal{F}) \leq r_n, \quad (2.3)$$

to obtain a contradiction.

Let us show first the inequality (2.2). Due to the Serre dualities on \mathbb{F}_2 and C , (2.2) can be rewritten as

$$h^0(C, \mathcal{O}_C(-2 - a_{n-1}) \otimes \mathcal{F}|_C) < h^0(C, \mathcal{O}_C(-2 - a_n) \otimes \mathcal{F}|_C). \quad (2.4)$$

Since $r_n > 0$ by the assumption, combining with (2.1), we see that the LHS of (2.4) = the LHS of (2.2) is positive. In particular, $\mathcal{O}_C(-2 - a_{n-1}) \otimes \mathcal{F}|_C$ contains at least one line bundle with non-negative degree as a direct summand. Thus the strict inequality $a_{n-1} > a_n$ implies the strict inequality (2.4).

In the remainder we prove (2.3). Note first that (2.3) is equivalent to

$${}^1(\mathcal{T}/\mathcal{T}^{n-1}, \mathcal{F}) \leq {}^2(\mathcal{T}/\mathcal{T}^{n-1}, \mathcal{T}/\mathcal{T}^{n-1}). \quad (2.5)$$

In fact, note that the LHS of (2.5) equals $r_n \cdot {}^1(\mathcal{O}_C(a_n), \mathcal{F})$ and the RHS can be calculated as

$$\begin{aligned}
& {}^2(\mathcal{T}/\mathcal{T}^{n-1}, \mathcal{T}/\mathcal{T}^{n-1}) = (\mathcal{T}/\mathcal{T}^{n-1}, \mathcal{T}/\mathcal{T}^{n-1}) \quad (\text{Serre duality}) \\
&= (\mathcal{O}_C(a_n)^{\oplus r_n}, \mathcal{O}_C(a_n)^{\oplus r_n}) = r_n^2.
\end{aligned}$$

In order to show (2.5), consider the following short exact sequence

$$0 \rightarrow \mathcal{T}/\mathcal{T}^{n-1} \rightarrow \mathcal{E}/\mathcal{T}^{n-1} \rightarrow \mathcal{E}/\mathcal{T} = \mathcal{F} \rightarrow 0. \quad (2.6)$$

By applying the functor $(\mathcal{T}/\mathcal{T}^{n-1}, -)$ we obtain the exact sequence

$${}^1(\mathcal{T}/\mathcal{T}^{n-1}, \mathcal{E}/\mathcal{T}^{n-1}) \rightarrow {}^1(\mathcal{T}/\mathcal{T}^{n-1}, \mathcal{F}) \rightarrow {}^2(\mathcal{T}/\mathcal{T}^{n-1}, \mathcal{T}/\mathcal{T}^{n-1}).$$

Hence it is enough to show that ${}^1(\mathcal{T}/\mathcal{T}^{n-1}, \mathcal{E}/\mathcal{T}^{n-1}) = 0$, which is equivalent to

$${}^1(\mathcal{E}/\mathcal{T}^{n-1}, \mathcal{T}/\mathcal{T}^{n-1}) = 0 \quad (2.7)$$

by the Serre duality. To show the latter, we apply the functor $(\mathcal{E}/\mathcal{T}^{n-1}, -)$ to the sequence (2.6) to obtain the exact sequence

$$\begin{aligned} & (\mathcal{E}/\mathcal{T}^{n-1}, \mathcal{E}/\mathcal{T}^{n-1}) \xrightarrow{\epsilon} (\mathcal{E}/\mathcal{T}^{n-1}, \mathcal{F}) \rightarrow {}^1(\mathcal{E}/\mathcal{T}^{n-1}, \mathcal{T}/\mathcal{T}^{n-1}) \\ \rightarrow & {}^1(\mathcal{E}/\mathcal{T}^{n-1}, \mathcal{E}/\mathcal{T}^{n-1}) = 0. \end{aligned}$$

The vanishing of the last entry, namely the rigidity of $\mathcal{E}/\mathcal{T}^{n-1}$, can be checked by applying Mukai's lemma [Kul97, Lemma 2.1.4. 2.(a)] to the short exact sequence

$$0 \rightarrow \mathcal{T}^{n-1} \rightarrow \mathcal{E} \rightarrow \mathcal{E}/\mathcal{T}^{n-1} \rightarrow 0$$

and using the rigidity of \mathcal{E} . For this we have to check the vanishing of $(\mathcal{T}^{n-1}, \mathcal{E}/\mathcal{T}^{n-1})$, and this follows from the fact that \mathcal{T}^{n-1} is an extension of the line bundles $\mathcal{O}_C(d)$ with $d > a_n$ and that $\mathcal{E}/\mathcal{T}^{n-1}$ is an extension of line bundles $\mathcal{O}_C(a_n)$ and the torsion free sheaf \mathcal{F} .

Now since the map ϵ above is nontrivial, it is enough to show $(\mathcal{E}/\mathcal{T}^{n-1}, \mathcal{F}) = 1$ for the vanishing (2.7). For this, we apply the functor $(-, \mathcal{F})$ to (2.6) to obtain the exact sequence

$$0 \rightarrow (\mathcal{F}, \mathcal{F}) \rightarrow (\mathcal{E}/\mathcal{T}^{n-1}, \mathcal{F}) \rightarrow (\mathcal{T}/\mathcal{T}^{n-1}, \mathcal{F}).$$

Since $\mathcal{T}/\mathcal{T}^{n-1}$ is torsion and \mathcal{F} is torsion free, we see $(\mathcal{T}/\mathcal{T}^{n-1}, \mathcal{F}) = 0$. Combining this with the fact \mathcal{F} is exceptional (see Lemma 2.2), we obtain $(\mathcal{E}/\mathcal{T}^{n-1}, \mathcal{F}) = 1$ and thus conclude the proof. \square

Let us now conclude the proof of Theorem 1.4. So far we have seen that the sequence (1.3) is always of the form

$$0 \rightarrow \mathcal{O}_C(a)^{\oplus r} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0 \quad (2.8)$$

for some $a \in \mathbb{Z}$ and $r \in \mathbb{Z}_{>0}$, with \mathcal{F} an exceptional vector bundle. Moreover by ${}^1(\mathcal{O}_C(a), \mathcal{E}) = {}^2(\mathcal{O}_C(a), \mathcal{E}) = 0$ in Lemma 2.1, we see that the exact triangle (1.1) for $\alpha = \mathcal{O}_C(a)$ and $\beta = \mathcal{E}$ gives rise to a short exact sequence (1.5).

Let us check that the sequence (2.8) is isomorphic to (1.5). For this note first that

$$(\mathcal{O}_C(a), \mathcal{E}) = \frac{t}{r} = r$$

follows from Lemma 2.1 and $f = 1$. On the other hand since $(\mathcal{E}, \mathcal{F}) \cong (\mathcal{F}, \mathcal{F}) = k$ by Lemma 2.1, the two morphisms $\mathcal{E} \rightarrow \mathcal{F}$ in (2.8) and (1.5) should be isomorphic. Thus we obtain an isomorphism between these two short exact sequences. In particular we obtain $\mathcal{E} \cong T'_{\mathcal{O}_C(a)}\mathcal{F}$. Thus we obtain the first half of Theorem 1.4 (1).

Consider the sequence (2.8) and its restriction to C . Then we obtain the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_C(a)^{\oplus r} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_C(a)^{\oplus r} & \longrightarrow & \mathcal{E}|_C & \longrightarrow & \mathcal{F}|_C \longrightarrow 0. \end{array}$$

We see from the snake lemma that the sequence (1.4) is of the form

$$0 \rightarrow \mathcal{F}(-C) \rightarrow \mathcal{E} \rightarrow \mathcal{E}|_C \rightarrow 0. \quad (2.9)$$

This means that $\mathcal{F}(-C) \cong \mathcal{F}^+$, which is Theorem 1.4. (2). On the other hand, by [IU05, Lemma 4.15] we have

$$T'_{\mathcal{O}_C(a+1)}\mathcal{E} \cong T'_{\mathcal{O}_C(a+1)} \circ T'_{\mathcal{O}_C(a)}\mathcal{F} \cong \mathcal{F}(-C).$$

Therefore the exact triangle (1.2) for $\alpha = \mathcal{O}_C(a+1)$ and $\beta = \mathcal{E}$ gives rise to a short exact sequence

$$0 \rightarrow \mathcal{F}(-C) \rightarrow \mathcal{E} \rightarrow (\mathcal{E}, \mathcal{O}_C(a+1))^\vee \otimes \mathcal{O}_C(a+1) \rightarrow 0. \quad (2.10)$$

Here note that since the sheaf ${}^1(\mathcal{E}, \mathcal{O}_C(a+1))^\vee \otimes \mathcal{O}_C(a+1)$ appears as a subsheaf of the torsion-free sheaf $\mathcal{F}(-C)$, it has to vanish.

Since there is no morphism from $\mathcal{F}(-C)$ to $\mathcal{O}_C(a+1)$ by the equality $(\mathcal{F}, \mathcal{T}) = 0$ in Lemma 2.1, we see $(\mathcal{F}(-C), \mathcal{E}) \cong (\mathcal{F}(-C), \mathcal{F}(-C)) = k$. Hence the dual of the arguments before tells us that the two morphisms $\mathcal{F}(-C) \hookrightarrow \mathcal{E}$ in (2.9) and (2.10) are isomorphic to each other. This implies that the sequence (1.4) is isomorphic to the sequence (1.6) as desired. \square

3 Exceptional sheaves sharing classes in K_0

Recall that every exceptional object on a del Pezzo surface X is isomorphic to a shift of either an exceptional vector bundle or a line bundle on a (-1) -curve. If \mathcal{E} and \mathcal{E}' are exceptional vector bundles on X satisfying the equality

$$\frac{c_1(\mathcal{E})}{\text{rank } \mathcal{E}} = \frac{c_1(\mathcal{E}')}{\text{rank } \mathcal{E}'},$$

then by [Gor88, Corollary 2.5] they are isomorphic. Therefore exceptional objects on X are uniquely determined by their classes in $K_0(X)$, up to even shifts $\mathbb{Z}[2]$.

Such uniqueness is no longer true for exceptional objects on \mathbb{F}_2 . In Subsections 3.1 and 3.3, starting with any exceptional sheaf \mathcal{E} on \mathbb{F}_2 , we construct exceptional objects \mathcal{E}_i for $i \in \mathbb{Z}$ sharing the classes in $K_0(\mathbb{F}_2)$ with \mathcal{E} . Some of them are sheaves, and in Subsection 3.4 we prove that thus constructed sheaves exhaust the set of exceptional sheaves sharing the class with \mathcal{E} .

3.1 Construction of \mathcal{E}_i for exceptional sheaves with nontrivial torsion part

Suppose that \mathcal{E} is an exceptional sheaf whose torsion part is nontrivial. In this case, by Theorem 1.4, we have a short exact sequence

$$0 \rightarrow \mathcal{O}_C(a_{\mathcal{E}})^{\oplus r_{\mathcal{E}}} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \cong T_{\mathcal{O}_C(a_{\mathcal{E}})}\mathcal{E} \rightarrow 0$$

for some integers $a_{\mathcal{E}}$ and $r_{\mathcal{E}} > 0$. Let us define integers $b_{\mathcal{E}}$ and s ($0 < s \leq \text{rank } \mathcal{E}$) by the isomorphism

$$\mathcal{F}|_C \cong \mathcal{O}_C(b_{\mathcal{E}})^{\oplus s} \oplus \mathcal{O}_C(b_{\mathcal{E}} + 1)^{\oplus \text{rank } \mathcal{E} - s}$$

(again use [Kul97, Remark 2.3.4]).

Since $r_{\mathcal{E}} = {}^1(\mathcal{F}, \mathcal{O}_C(a_{\mathcal{E}}))$, we get the equality

$$r_{\mathcal{E}} = (b_{\mathcal{E}} - a_{\mathcal{E}}) \cdot \text{rank } \mathcal{E} - s.$$

For each integer i , set

$$\begin{aligned} \mathcal{F}_i &:= \mathcal{F} \otimes \mathcal{O}((b_{\mathcal{E}} - a_{\mathcal{E}} - 1 - i)C) \text{ and} \\ b_0 &:= 2a_{\mathcal{E}} - b_{\mathcal{E}} + 2. \end{aligned}$$

Then we see $\mathcal{F} = \mathcal{F}_{b_{\mathcal{E}} - a_{\mathcal{E}} - 1}$ and obtain the isomorphism

$$\mathcal{F}_i|_C \cong \mathcal{O}_C(b_0 + 2i)^{\oplus s} \oplus \mathcal{O}_C(b_0 + 2i + 1)^{\oplus \text{rank } \mathcal{E} - s}.$$

Finally we set

$$\mathcal{E}_i := T_{\mathcal{O}_C(b_0+i)} \mathcal{F}_{i+1}.$$

Then notice that we have an isomorphism

$$\mathcal{E}_i \cong T'_{\mathcal{O}_C(b_0+i-1)} \mathcal{F}_i$$

by the isomorphisms of the functors

$$T_{\mathcal{O}_C(a-1)} \circ T_{\mathcal{O}_C(a)} \cong \otimes \mathcal{O}(C)$$

for any $a \in \mathbb{Z}$ ([IU05, Lemma 4.15]). Since $a_{\mathcal{E}} = b_0 + (b_{\mathcal{E}} - a_{\mathcal{E}} - 1) - 1$ and $\mathcal{F} = \mathcal{F}_{b_{\mathcal{E}} - a_{\mathcal{E}} - 1}$, we obtain the isomorphisms

$$\mathcal{E} \cong T'_{\mathcal{O}_C(a_{\mathcal{E}})} \mathcal{F} \cong \mathcal{E}_{b_{\mathcal{E}} - a_{\mathcal{E}} - 1}.$$

3.2 Relations among the objects \mathcal{E}_i

Let \mathcal{E} be an exceptional sheaf with nontrivial torsion part on \mathbb{F}_2 as in Subsection 3.1. We study properties of \mathcal{E}_i constructed in Subsection 3.1.

The relationship among the objects \mathcal{F}_i and \mathcal{E}_i is summarized in the following diagram.

$$\begin{array}{ccccccc}
 & & \xrightarrow{\otimes \mathcal{O}(C)} & & \xrightarrow{\otimes \mathcal{O}(C)} & & \\
 & \mathcal{F}_{i+1} & & \mathcal{F}_i & & \mathcal{F}_{i-1} & \\
 \nearrow T_{\mathcal{O}_C(b_0+i)} & & \searrow T_{\mathcal{O}_C(b_0+i)} & \nearrow T_{\mathcal{O}_C(b_0+i-1)} & \searrow T_{\mathcal{O}_C(b_0+i-1)} & \nearrow T_{\mathcal{O}_C(b_0+i-2)} & \searrow T_{\mathcal{O}_C(b_0+i-2)} \\
 \mathcal{E}_{i+1} & & \mathcal{E}_i & & \mathcal{E}_{i-1} & & \mathcal{E}_{i-2}
 \end{array}$$

Claim 3.1. *We have $[\mathcal{E}_i] = [\mathcal{E}]$ in $K_0(\mathbb{F}_2)$ for any $i \in \mathbb{Z}$.*

Proof. As we have seen above, there exists an isomorphism $\mathcal{E} \cong \mathcal{E}_i$ for some $i \in \mathbb{Z}$. Since $T_{\mathcal{O}_C(a)}^2$ acts on $K_0(\mathbb{F}_2)$ trivially (see Remark 1.2), the result follows from the isomorphisms $T_{\mathcal{O}_C(b_0+i)}^2 \mathcal{E}_{i+1} \cong \mathcal{E}_i$ for all $i \in \mathbb{Z}$. \square

Claim 3.2. (1) *For any integer $i \geq -1$, \mathcal{E}_i is an exceptional sheaf.*

- (2) For any integer $i < -1$, \mathcal{E}_i is an exceptional object of length 1.
- (3) \mathcal{E}_i is a vector bundle if and only if either $i = -1$, or $i = 0$ and $s = \text{rank } \mathcal{E}$.
- (4) There exists an isomorphism

$$\mathcal{E}_{-1}|_C \cong \mathcal{O}_C(b_0 - 1)^{\oplus \text{rank } \mathcal{E} - s} \oplus \mathcal{O}_C(b_0)^{\oplus s}.$$

Proof. We start with some preparatory computations for the twist functors.

Put $r_i = (i + 1) \text{rank } \mathcal{E} - s \in \mathbb{Z}$. For each $i \geq 0$ we have $r_i \geq 0$, and $r_i = 0$ occurs precisely when $i = 0$ and $s = \text{rank } \mathcal{E}$. The following calculation

$$\begin{aligned} & \mathbb{R}\text{Hom}_{D(\mathbb{F}_2)}(\mathcal{F}_i, \mathcal{O}_C(b_0 + i - 1)) \\ & \cong \text{Ext}_C^1(\mathcal{O}_C(b_0 + 2i)^{\oplus s} \oplus \mathcal{O}_C(b_0 + 2i + 1)^{\oplus \text{rank } \mathcal{E} - s}, \mathcal{O}_C(b_0 + i - 1))[-1] \\ & \cong k^{\oplus r_i}[-1] \end{aligned} \quad (3.1)$$

tells us that the defining exact triangles of the twist functor $T_{\mathcal{O}_C(b_0+i-1)}$ and its quasi-inverse $T'_{\mathcal{O}_C(b_0+i-1)}$, respectively, are equivalent to the following short exact sequences.

$$0 \rightarrow \mathcal{F}_i \rightarrow \mathcal{E}_{i-1} = T_{\mathcal{O}_C(b_0+i-1)}\mathcal{F}_i \rightarrow \mathcal{O}_C(b_0 + i - 1)^{\oplus r_i} \rightarrow 0 \quad (3.2)$$

$$0 \rightarrow \mathcal{O}_C(b_0 + i - 1)^{\oplus r_i} \rightarrow \mathcal{E}_i \cong T'_{\mathcal{O}_C(b_0+i-1)}\mathcal{F}_i \rightarrow \mathcal{F}_i \rightarrow 0 \quad (3.3)$$

For each $i < 0$, we have $r_i < 0$. In this case the calculation

$$\begin{aligned} & \mathbb{R}\text{Hom}_{D(\mathbb{F}_2)}(\mathcal{F}_i, \mathcal{O}_C(b_0 + i - 1)) \\ & \cong \text{Hom}_C(\mathcal{O}_C(b_0 + 2i)^{\oplus s} \oplus \mathcal{O}_C(b_0 + 2i + 1)^{\oplus \text{rank } \mathcal{E} - s}, \mathcal{O}_C(b_0 + i - 1)) \\ & \cong k^{\oplus -r_i} \end{aligned}$$

tells us that the defining triangles of the functors $T_{\mathcal{O}_C(b_0+i-1)}$ and $T'_{\mathcal{O}_C(b_0+i-1)}$ are as follows.

$$\mathcal{O}_C(b_0 + i - 1)^{\oplus -r_i}[-2] \rightarrow \mathcal{F}_i \rightarrow \mathcal{E}_{i-1} = T_{\mathcal{O}_C(b_0+i-1)}\mathcal{F}_i \quad (3.4)$$

$$\mathcal{E}_i \cong T'_{\mathcal{O}_C(b_0+i-1)}\mathcal{F}_i \rightarrow \mathcal{F}_i \rightarrow \mathcal{O}_C(b_0 + i - 1)^{\oplus -r_i} \quad (3.5)$$

Now we use all these results to obtain the conclusions. The statement (1) follows from the exact sequence (3.2).

To see (2), note that (3.4) implies

$$\mathcal{H}^p(\mathcal{E}_{i-1}) = \begin{cases} \mathcal{F}_i & (p = 0) \\ \mathcal{O}_C(b_0 + i - 1)^{\oplus -r_i} & (p = 1) \\ 0 & (\text{otherwise}). \end{cases}$$

Next we check the assertion (3). If $i \geq 1$, (3.3) implies that \mathcal{E}_i has torsion. If $i = -1$, from (3.5) we see that \mathcal{E}_{-1} has no torsion and hence is a vector bundle. When $i = 0$, if we further assume $r_0 = 0 \iff s = \text{rank } \mathcal{E}$, from (3.3) and (3.2) we obtain $\mathcal{E}_0 \cong \mathcal{F}_0 \cong \mathcal{E}_{-1}$. Hence \mathcal{E}_0 is a vector bundle in this case. Finally if $r_0 > 0$, then (3.3) forces \mathcal{E}_0 to have torsion.

To show the last assertion (4), use the isomorphism $\mathcal{T}or_1^{\mathbb{F}_2}(\mathcal{O}_C, \mathcal{O}_C(b_0 - 1)) \cong \mathcal{O}_C(b_0 + 1)$ and the restriction of (3.2) for $i = 0$ to C . \square

Remark 3.3. By the above proof, we know that $\mathcal{E}_i \cong \mathcal{E}_j$ if and only if either $i = j$ or $\{i, j\} = \{-1, 0\}$ and $s = \text{rank } \mathcal{E}$.

3.3 Construction of \mathcal{E}_i for exceptional vector bundles

In Subsection 3.1, we constructed an exceptional vector bundle \mathcal{E}_{-1} which shares the class in $K_0(\mathbb{F}_2)$ with a given exceptional sheaf \mathcal{E} with nontrivial torsion part. In this section, let us follow the procedure in an opposite direction; starting with an exceptional vector bundle \mathcal{E}_{-1} on \mathbb{F}_2 , let us recover exceptional objects \mathcal{E}_i for each i . See the precise statement in Remark 3.4.

Suppose that an exceptional vector bundle \mathcal{E}' is given. Set

$$\mathcal{E}'_{-1} := \mathcal{E}'.$$

Since $\mathcal{E}'_{-1}|_C$ is rigid by [Kul97, Remark 2.3.4], there exist integers b'_0 and s' ($0 < s' \leq \text{rank } \mathcal{E}'$) such that

$$\mathcal{E}'_{-1}|_C \cong \mathcal{O}_C(b'_0 - 1)^{\oplus \text{rank } \mathcal{E}' - s'} \oplus \mathcal{O}_C(b'_0)^{\oplus s'}.$$

Next let us define

$$\mathcal{F}'_0 := T'_{\mathcal{O}_C(b'_0 - 1)} \mathcal{E}'_{-1}.$$

Then we see

$$\begin{aligned} & \mathbb{R}\text{Hom}_{D(\mathbb{F}_2)}(\mathcal{E}'_{-1}, \mathcal{O}_C(b'_0 - 1)) \\ & \cong \left(\mathcal{O}_C(b'_0 - 1)^{\oplus \text{rank } \mathcal{E}' - s'} \oplus \mathcal{O}_C(b'_0)^{\oplus s'}, \mathcal{O}_C(b'_0 - 1) \right) \\ & \cong_k^{\oplus \text{rank } \mathcal{E}' - s'}, \end{aligned}$$

so as to obtain the following short exact sequence

$$0 \rightarrow \mathcal{F}'_0 = T'_{\mathcal{O}_C(b'_0 - 1)} \mathcal{E}'_{-1} \rightarrow \mathcal{E}'_{-1} \rightarrow \mathcal{O}_C(b'_0 - 1)^{\oplus \text{rank } \mathcal{E}' - s'} \rightarrow 0. \quad (3.6)$$

This implies that \mathcal{F}'_0 is also an exceptional vector bundle. Successively we define

$$\mathcal{F}'_i := \mathcal{F}'_0(-iC), \quad \mathcal{E}'_i := T_{\mathcal{O}_C(b'_0+i)}\mathcal{F}'_{i+1}$$

for any $i \in \mathbb{Z}$. Using the isomorphism $\mathcal{T}or_1^{\mathbb{F}_2}(\mathcal{O}_C, \mathcal{O}_C(b_0 - 1)) \cong \mathcal{O}_C(b_0 + 1)$ and restricting (3.6) to C , we obtain

$$\mathcal{F}'_0|_C \cong \mathcal{O}_C(b'_0)^{\oplus s'} \oplus \mathcal{O}_C(b'_0 + 1)^{\oplus \text{rank } \mathcal{E}' - s'},$$

and

$$\mathcal{F}'_i|_C \cong \mathcal{O}_C(b'_0 + 2i)^{\oplus s'} \oplus \mathcal{O}_C(b'_0 + 2i + 1)^{\oplus \text{rank } \mathcal{E}' - s'}.$$

Then a direct computation as in (3.1) yields that \mathcal{E}'_i is a sheaf if and only if $i \geq -1$.

Remark 3.4. We can see, from the definitions, that the constructions of \mathcal{E}_i and \mathcal{E}'_i given in Subsections 3.1 and 3.3, respectively, are inverses of each other in the following sense.

Given an exceptional sheaf \mathcal{E} with nontrivial torsion part, construct the exceptional objects \mathcal{E}_i as in Subsection 3.1. From the exceptional vector bundle $\mathcal{E}' := \mathcal{E}_{-1}$, we can also construct the exceptional objects \mathcal{E}'_i as in Subsection 3.3. Then we see $\mathcal{E}_i \cong \mathcal{E}'_i$ and $\mathcal{F}_i \cong \mathcal{F}'_i$ for each $i \in \mathbb{Z}$. In fact, $\mathcal{E}_{-1} \cong \mathcal{E}'_{-1}$ holds by definition. Since the objects $\mathcal{E}_i, \mathcal{F}_i$ for $i \in \mathbb{Z}$ and $\mathcal{E}'_i, \mathcal{F}'_i$ for $i \in \mathbb{Z}$ satisfy the same recursive relations, we obtain the conclusion; note that the numbers b_0 and b'_0 in Subsections 3.1 and 3.3, respectively, depend only on the restrictions to C of \mathcal{E}_{-1} and \mathcal{E}'_{-1} . Since these two sheaves are known to be isomorphic, we obtain $b_0 = b'_0$.

Similarly, given an exceptional vector bundle \mathcal{E}' , construct the exceptional objects \mathcal{E}'_i as in Subsection 3.3. Choose an integer $i \geq 0$ such that $\mathcal{E} := \mathcal{E}'_i$ is an exceptional sheaf with nontrivial torsion part, and construct the exceptional objects \mathcal{E}_i as in Subsection 3.1. Then we see $\mathcal{E}_i \cong \mathcal{E}'_i$ and $\mathcal{F}_i \cong \mathcal{F}'_i$ for each $i \in \mathbb{Z}$. To see this, it is again enough to check $\mathcal{E}_{-1} \cong \mathcal{E}'_{-1}$. Note, by their constructions, that they are exceptional vector bundles whose classes in $K_0(\mathbb{F}_2)$ are the same. Then we can use Lemma 3.5 to see that these two vector bundles should be isomorphic.

3.4 Exceptional sheaves sharing the same class

We start with the following lemma.

Lemma 3.5. *Let \mathcal{E} and \mathcal{E}' be exceptional vector bundles on \mathbb{F}_2 . Suppose that the equality $\frac{c_1(\mathcal{E})}{\text{rank } \mathcal{E}} = \frac{c_1(\mathcal{E}')}{\text{rank } \mathcal{E}'}$ holds in $\text{Pic}(\mathbb{F}_2)_{\mathbb{Q}}$. Then $\mathcal{E} \cong \mathcal{E}'$.*

Proof. The proof is analogous to that of [Gor88, Corollary 2.5] in the case of del Pezzo surfaces. The $(-K)$ -stability of exceptional vector bundle on \mathbb{F}_2 follows from [Kul97, Corollary 2.2.9]. \square

Now we give the main result of this section.

Theorem 3.6. *Suppose that \mathcal{E} and \mathcal{E}' are exceptional sheaves on \mathbb{F}_2 . Then the following conditions are equivalent.*

- (1) $\frac{c_1(\mathcal{E})}{\text{rank } \mathcal{E}} = \frac{c_1(\mathcal{E}')}{\text{rank } \mathcal{E}'} \in \text{Pic}(\mathbb{F}_2)_{\mathbb{Q}}$.
- (2) $\mathcal{E}_i \cong \mathcal{E}'_i$ for any $i \in \mathbb{Z}$.
- (3) $[\mathcal{E}] = [\mathcal{E}'] \in K_0(\mathbb{F}_2)$.
- (4) $\mathcal{E}' \cong \mathcal{E}_i$ for some $i \geq -1$.

Proof. (4) \Rightarrow (3) easily follows from Claim 3.1. (3) \Rightarrow (1) is also obvious, since the Chern character map factors through the K -group.

As we have seen in Sections 3.1 and 3.3, $\mathcal{E}' \cong \mathcal{E}'_i$ always holds for some i . This implies (2) \Rightarrow (4).

Finally let us assume (1). By Claim 3.1 we obtain the equalities $[\mathcal{E}] = [\mathcal{E}_{-1}]$ and $[\mathcal{E}'] = [\mathcal{E}'_{-1}]$. Hence we see $\frac{c_1(\mathcal{E}_{-1})}{\text{rank } \mathcal{E}_{-1}} = \frac{c_1(\mathcal{E}'_{-1})}{\text{rank } \mathcal{E}'_{-1}}$. Since \mathcal{E}_{-1} and \mathcal{E}'_{-1} are vector bundles, we obtain an isomorphism $\mathcal{E}_{-1} \cong \mathcal{E}'_{-1}$ by Lemma 3.5. Thus we obtain the condition (2). \square

We summarize the results of this section.

Corollary 3.7. *Let \mathcal{E} be an exceptional sheaf on \mathbb{F}_2 . Then the set of isomorphism classes of exceptional sheaves \mathcal{E}' with $[\mathcal{E}] = [\mathcal{E}'] \in K_0(\mathbb{F}_2)$ is just $\{\mathcal{E}_i \mid i \geq -1\}$. In this set, \mathcal{E}_{-1} is the unique vector bundle up to isomorphism. Furthermore, if $i \neq j$, $\mathcal{E}_i \cong \mathcal{E}_j$ occurs if and only if $\{i, j\} = \{-1, 0\}$ and $\mathcal{E}_{-1}|_C \cong \mathcal{O}_C(b)^{\oplus \text{rank } \mathcal{E}_{-1}}$ for some $b \in \mathbb{Z}$.*

4 Some results obtained via deformation to del Pezzo surfaces

We explain a couple of results about exceptional objects on \mathbb{F}_2 , which are obtained by using its deformation to $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$. Since \mathbb{F}_0 is a del Pezzo surface, exceptional objects on it are well known (see [KO94] and [GK04]). The idea is to use that knowledge to understand the exceptional objects on \mathbb{F}_2 .

4.1 Any exceptional object has, up to sign, the same class as an exceptional bundle

Proposition 4.1. *Let n be an even non-negative integer. Then for any exceptional object \mathcal{E} on the Hirzebruch surface \mathbb{F}_n , there exists an exceptional vector bundle \mathcal{F} on \mathbb{F}_n such that*

$$[\mathcal{E}] = [\mathcal{F}] \text{ or } -[\mathcal{F}] \in K_0(\mathbb{F}_n).$$

We need some preparations for the proof. The assumption that n is even is used only in the proof of Corollary 4.4.

Lemma 4.2. *Let n be a non-negative integer and $m = 1$ or 0 , with the same parity as n . Then there exists a smooth projective morphism*

$$f: \mathcal{X} \rightarrow \mathbb{A}^1 \tag{4.1}$$

such that

- $\mathcal{X}_0 \cong \mathbb{F}_n$,
- $f^{-1}(\mathbb{A}^1 \setminus \{0\}) \cong \mathbb{F}_m \times (\mathbb{A}^1 \setminus \{0\})$ over $\mathbb{A}^1 \setminus \{0\}$.

Proof. See [Kod86, Example 2.16]. we can see that the construction works in any characteristics. □

Lemma 4.3. *Let*

$$\mathcal{X}_R \rightarrow \text{Spec } R$$

be a smooth projective (for simplicity) morphism over the spectrum of a complete discrete valuation ring R . Let \mathcal{X}_0 be the central fiber and take an object $\mathcal{E}_0 \in D(\mathcal{X}_0)$.

- When $\mathrm{Ext}_{\mathcal{X}_0}^2(\mathcal{E}_0, \mathcal{E}_0) = 0$, \mathcal{E}_0 extends to a bounded complex of coherent sheaves on \mathcal{X}_R .
- When $\mathrm{Ext}_{\mathcal{X}_0}^1(\mathcal{E}_0, \mathcal{E}_0) = 0$, the extension of \mathcal{E}_0 to \mathcal{X} , if exists, is unique.

Proof. By [HT10, Corollary 3.4], we see that the vanishing of $\mathrm{Ext}_{\mathcal{X}_0}^2(\mathcal{E}_0, \mathcal{E}_0)$ implies that \mathcal{E}_0 can be extended to a perfect complex on the formal neighborhood of \mathcal{X}_0 . By the algebraization [FGI⁺05, Theorem 8.4.2], we can extend the perfect complex to \mathcal{X} . For the uniqueness, see again [HT10, Corollary 3.4]. \square

Let R be the completion of the local ring $\mathcal{O}_{0, \mathbb{A}^1}$ by the maximal ideal, and K the field of fractions of R . Let

$$\mathcal{X}_R \rightarrow \mathrm{Spec} R \tag{4.2}$$

be the base change of the morphism (4.1) by the natural morphism $\mathrm{Spec} R \rightarrow \mathbb{A}^1$.

Corollary 4.4. *Assume that n is an even non-negative integer. Then any exceptional object \mathcal{E}_0 on $\mathcal{X}_0 \cong \mathbb{F}_n$ uniquely deforms to a family of exceptional objects \mathcal{E} on \mathcal{X}_R over R . Moreover, the restriction $\mathcal{E}_K := \mathcal{E}|_{\mathcal{X}_K}$ to the generic fiber is isomorphic to a shift of an exceptional vector bundle.*

Proof. Although K is not algebraically closed, by passing to the algebraic closure, we can check that any exceptional object on \mathcal{X}_K is isomorphic to a shift of a vector bundle by applying [KO94, Propositions 2.9 and 2.10]. Note that this is not the case for the Hirzebruch surface \mathbb{F}_1 , since line bundles on the (-1) -curve are also exceptional; this is why we assumed that n is even.

The rest is a direct consequence of Lemma 4.3. \square

Next we recall a fact on the relative moduli space of semi-stable sheaves.

Lemma 4.5 (= [Lan04, Theorem 4.1]). *Let $\mathcal{Y} \rightarrow B$ be a projective morphism of schemes of finite type over a universally Japanese ring, with a relatively ample line bundle L on \mathcal{Y} . Fix an integer-valued polynomial $P(t) \in \mathbb{Q}[t]$. Then there exists a projective scheme $\overline{\mathcal{M}} \rightarrow B$ such that for any $b \in B$, the fiber $\overline{\mathcal{M}}_b$ is the moduli scheme of L_b -semi-stable sheaves on \mathcal{Y}_b with the Hilbert polynomial $P(t)$.*

Lemma 4.6. *Fix a non-negative integer n . Let*

$$\mathcal{X}_R \rightarrow \text{Spec } R$$

be the morphism (4.2) and \mathcal{E}_K an exceptional vector bundle on the generic fiber \mathcal{X}_K . Then there exists an exceptional vector bundle \mathcal{F}_0 on the central fiber \mathcal{X}_0 such that

$$[\mathcal{E}_K] = [\mathcal{F}_0] \tag{4.3}$$

under the canonical isomorphism $K_0((\mathbb{F}_n)_K) \cong K_0(\mathbb{F}_m)$.

Proof. Note first that there exists a canonical isomorphism

$$\text{Pic}((\mathbb{F}_m)_K) \cong \text{Pic}(\mathbb{F}_n),$$

so that we can identify the \mathbb{R} -divisors on $(\mathbb{F}_n)_K$ with those on \mathbb{F}_m .

Note second that \mathcal{E}_K is $(-K_{(\mathbb{F}_m)_K})$ -stable (see [KO94, Theorem 5.2]). Take an ample divisor $H \in \text{Pic}(\mathbb{F}_n) \otimes \mathbb{R} \cong \text{Pic}((\mathbb{F}_m)_K) \otimes \mathbb{R}$ which is sufficiently close to $-K_{(\mathbb{F}_m)_K}$ so that \mathcal{E}_K is H -stable. Moreover by the local finiteness of the walls (see [HL10, Section 4C]), we can choose H generically so that there exists no H -strictly semi-stable sheaf $\mathcal{G} \in \text{Coh}(\mathbb{F}_n)$ with $[\mathcal{G}] = [\mathcal{E}_K]$; here the equality should be considered under the canonical isomorphism $K_0(\mathbb{F}_n) \cong K_0((\mathbb{F}_m)_K)$.

Consider the relative moduli space of H -semi-stable sheaves

$$\overline{M} \rightarrow \text{Spec } R$$

obtained by applying Lemma 4.5 to (4.2) for $\mathcal{Y} = \mathcal{X}_R$ and $B = \text{Spec } R$. Since R is a DVR, we can find a section $s: \text{Spec } R \rightarrow \overline{M}$ such that $s(\text{Spec } K)$ represents the sheaf \mathcal{E}_K . By the genericity of the polarization H , it holds that the section s factors through the stable locus $M \subset \overline{M}$. Hence, pulling back by $s \times \text{id}_{\mathcal{X}_R}$ the quasi-universal family on M , we obtain a flat family of coherent sheaves \mathcal{F}' on \mathcal{X}_R such that $\mathcal{F}'_K \cong \mathcal{E}_K^{\oplus r}$ for some $r > 0$ (see [HL10, Section 4.6] for the notion of quasi-universal families). If we denote by \mathcal{F}_0 the stable sheaf represented by the point $s(0) \in \overline{M}_0$, then by the definition of the quasi-universal family, we see $\mathcal{F}'_0 \cong \mathcal{F}_0^{\oplus r}$.

Therefore we see that \mathcal{F}_0 is torsion free, ${}^0(\mathcal{F}_0, \mathcal{F}_0) = 1$, and ${}^2(\mathcal{F}_0, \mathcal{F}_0) = {}^0(\mathcal{F}_0, \mathcal{F}_0 \otimes \omega_{\mathbb{F}_n}) = 0$. Moreover, note that

$$\begin{aligned} \chi(\mathcal{F}_0, \mathcal{F}_0) &= \frac{1}{r^2} \chi(\mathcal{F}'_0, \mathcal{F}'_0) = \frac{1}{r^2} \chi(\mathcal{F}'_K, \mathcal{F}'_K) \\ &= \chi(\mathcal{E}_K, \mathcal{E}_K) = 1 \end{aligned}$$

due to the deformation invariance of the Euler pairing. This implies

$${}^1(\mathcal{F}_0, \mathcal{F}_0) = -\chi(\mathcal{F}_0, \mathcal{F}_0) + {}^0(\mathcal{F}_0, \mathcal{F}_0) + {}^2(\mathcal{F}_0, \mathcal{F}_0) = -1 + 1 + 0 = 0.$$

Therefore \mathcal{F}_0 is an exceptional torsion free sheaf. By standard arguments (see [KO94, Corollary 2.3]), \mathcal{F}_0 has to be a vector bundle. Finally the sequence of equalities

$$[\mathcal{E}_K] = \frac{1}{r}[\mathcal{F}'_K] = \frac{1}{r}[\mathcal{F}'_0] = [\mathcal{F}_0]$$

imply (4.3), since $K_0((\mathbb{F}_n)_K) \cong K_0(\mathbb{F}_m)$ is torsion free. \square

Finally, Proposition 4.1 directly follows from Corollary 4.4 and Lemma 4.6.

Remark 4.7. It is interesting to describe the set of exceptional objects sharing the class in $K_0(\mathbb{F}_n)$. When $n = 2$, because of Remark 1.2, the nontrivial group generated by ‘double spherical twists’

$$D := \langle T_\alpha^2 \mid \alpha: \text{spherical} \rangle < \text{Auteq } D(\mathbb{F}_2)$$

acts trivially on $K_0(\mathbb{F}_2)$.

Note that the action of D is not free: for example $T_{\mathcal{O}_C(-1)}\mathcal{O} = \mathcal{O}$, since $\mathbb{R}\text{Hom}(\mathcal{O}_C(-1), \mathcal{O}) = 0$. The authors are not sure if the action is transitive on each of such sets or not.

4.2 Numerical transitivity

In this section we consider \mathbb{F}_n for any $n \geq 0$. Let B_4 be the braid group with four strings. It is well known (see [BP93, Proposition 2.1]) that the group $G_4 := \mathbb{Z}^4 \rtimes B_4$ acts on the set of full exceptional collections of a triangulated category \mathcal{T} with $K_0(\mathcal{T}) \cong \mathbb{Z}^4$, via right and left mutations and the shifts. The action descends to $K_0(\mathcal{T})$ (see [GK04, Section 1.5]). Below is a special case of [Nog90, p.216, Corollary].

Proposition 4.8. *Let*

$$\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4 \tag{4.4}$$

be a full exceptional collection of $D(\mathbb{F}_n)$. Then for any full exceptional collection of line bundles

$$S_1, S_2, S_3, S_4, \tag{4.5}$$

there exists an element $g \in G_4$ such that the classes of the members of the mutated collection

$$g(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4)$$

in $K_0(\mathbb{F}_n)$ coincide with those of the chosen collection (4.5).

Proof. Because of the existence of the deformation (4.2), there exists a canonical isomorphism

$$\varphi: K_0(\mathbb{F}_n) \xrightarrow{\cong} K_0((\mathbb{F}_m)_K)$$

which preserves the Euler pairings on both sides.

Consider the deformation of the collections (4.4) and (4.5) to $(\mathbb{F}_m)_K$, which automatically are full exceptional collections on $(\mathbb{F}_m)_K$ by the upper semi-continuity of the cohomology and [KO94, Theorem 6.11]. Denote them by

$$(\mathcal{E}_1)_K, \dots, (\mathcal{E}_4)_K \tag{4.6}$$

and

$$(S_1)_K, \dots, (S_4)_K \tag{4.7}$$

respectively. By the transitivity [GK04, Theorem 4.6.1], we can find an element $g \in G_4$ which sends (4.6) to (4.7).

On the other hand, by definition, the actions of G_4 on the level of K_0 are compatible with the isometry φ . In particular the classes in $K_0(\mathbb{F}_2)$ of the collection

$$g(\mathcal{E}_1, \dots, \mathcal{E}_4)$$

should be the same as those of the collection (4.5). Thus we conclude the proof. \square

Remark 4.9. Let \mathcal{C}' be the set of full exceptional collections of $D(\mathbb{F}_0)$, and $\mathcal{C} := \mathcal{C}'/\mathbb{Z}^4$. The action of B_4 on \mathcal{C} is known to be transitive, but not free. To see this, consider the full exceptional collection $\mathcal{O}, \mathcal{O}(0, 1), \mathcal{O}(1, 0), \mathcal{O}(1, 1)$. If we denote by $\sigma_{23} \in B_4$ the flip of the 2nd and the 3rd threads, then σ_{23}^2 fixes the collection; it is simply because there is no derived morphism between $\mathcal{O}(0, 1)$ and $\mathcal{O}(1, 0)$. Note that, on the other hand, σ_{23}^2 acts nontrivially on the degeneration of the collection to \mathbb{F}_2 .

It is intriguing to understand the stabilizer group of this action, and to see if the action of that stabilizer subgroup on the set of full exceptional collections of $D(\mathbb{F}_2)$ is realized by autoequivalences (4.7) of $D(\mathbb{F}_2)$.

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