DOUBLE SHUFFLE RELATIONS OF DOUBLE ZETA VALUES AND DOUBLE EISENSTEIN SERIES AT LEVEL N

Abstract. In their seminal paper "Double zeta values and modular forms" Gangl, Kaneko and Zagier defined a double Eisenstein series and used it to study the relations between double zeta values. One of their key ideas is to study the formal double space and apply the double shuffle relations. They also proved the double shuffle relations for the double Eisenstein series. More recently, Kaneko and Tasaka extended the double Eisenstein series to level 2, proved its double shuffle relations and studied the double zeta values at level 2. Motivated by the above works, we define in this paper the corresponding objects at higher levels and prove that the double Eisenstein series at level N satisfies the double shuffle relations for every positive integer N. In order to obtain our main theorem we prove a key result on the multiple divisor functions at level N and then use it to solve a complicated under-determined system of linear equations by some standard techniques from linear algebra.

Contents

1. Introduction

Eisenstein series have played important roles in the study of modular forms and elliptic curves. One of their most important properties is that the constant term of their Fourier series expansion is essentially given by the Riemann zeta values at even weight. In the seminal paper [10] Gangl, Kaneko and Zagier defined a double Eisenstein series and used it to study the relations between double zeta values. One of their key ideas is to study the formal double space and apply the double shuffle relations. They then proved the double shuffle relations for the double Eisenstein series. The double zeta relations have also been considered by Baumard and Schneps in [7] from the point of view of period polynomials and double shuffle Lie algebra defined by Ihara. More recently, Kaneko and Tasaka [13] and Nakamura and Tasaka [16] extended the double Eisenstein series to level 2, proved its double shuffle relations and studied the double zeta values at level 2. Motivated by the above works, we define in this paper the corresponding objects at higher levels and consider the double shuffle relations satisfied by them.

We follow the notation in [10]: for any $m, c \in \mathbb{Z}$ and $\tau \in \mathbb{H}$ (upper half plane), we write $m\tau+c \succ 0$ if $m > 0$ or $m = 0$ and $c > 0$ and $m\tau+c \succ n\tau+d$ if $m\tau+c-(n\tau+d) \succ 0$. For any $\mathbf{a} = (a_1, \dots, a_d) \in (\mathbb{Z}/N\mathbb{Z})^d$ and $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$ we define the multiple Eisenstein series at level N by

$$G_{\mathbf{s}}^{\mathbf{a}}(\tau) = G_{\mathbf{s}}^{\mathbf{a};N}(\tau) = \sum_{\substack{m_1N\tau+c_1 \succ \cdots \succ m_dN\tau+c_d \succ 0 \\ m_j, c_j \in \mathbb{Z},\ c_j \equiv a_j \pmod{N}\ \forall j}} \frac{1}{(m_1N\tau+c_1)^{s_1} \cdots (m_dN\tau+c_d)^{s_d}}. \tag{1}$$

Here, by convention, we often choose $0 \le a < N$ to represent the residue class congruent to a modulo N. It is not too hard to show that the series converges absolutely when $s_1 \ge 3$ and $s_j \ge 2$ for all $j \ge 2$. We will call d the depth and the sum $s_1 + \cdots + s_d$ the weight. At level one case, Gangl, Keneko and Zagier [10] studied the double Eisenstein series and related them to modular form by using the Eichler-Shimura correspondence. In [3] Bachmann generalized this to arbitrary depth and obtained many interesting relations among these and the classical Eisenstein series (and the cusp form Δ) using the double shuffle relations.

The main idea to study the multiple Eisenstein series is by using their Fourier series expansions with the help of the so called multiple divisor functions at level N defined as follows: For $\mathbf{a} = (a_1, \dots, a_d) \in (\mathbb{Z}/N\mathbb{Z})^d$ and $\mathbf{s} = (s_1, \dots, s_d) \in (\mathbb{N} \cup \{0\})^d$

$$\sigma_{\mathbf{s}}^{\mathbf{a}}(m) = \sigma_{\mathbf{s}}^{\mathbf{a};N}(m) = \sum_{\substack{u_1v_1+\cdots+u_dv_d=m \\ u_1 > \cdots > u_d > 0 \\ u_j, v_j \in \mathbb{N}\ \forall j=1,\dots,d}} \eta^{a_1v_1+\cdots+a_dv_d} v_1^{s_1} \dots v_d^{s_d} \tag{2}$$

where $\eta = \eta_N = \exp(2\pi i/N)$ is the primitive Nth root of unity and $v_1, \cdots, v_d$ are positive integers. Obviously, one can recover the classical divisor function by setting $N = d = 1$.

We now briefly describe the content of the paper. In the next section we shall first define the double zeta values at level N and write down explicitly the double shuffle relations satisfied by these values. Then we consider the same problem in the formal vector space corresponding to the double zeta values. As consequences of these double shuffle relations we prove two sum formulas in Theorem 3.3.

By the general philosophy, the constant terms of level N multiple Eisenstein series are given by the level N multiple zeta values. Further, we expect that level N multiple zeta values satisfy the double shuffle relations such as those given in Proposition 3.2. Hence we would like to know if the corresponding level N multiple Eisenstein series also satisfy similar relations. When $N = 1$ this has been studied by Bachmann and Tasaka [5].

The main goal of the paper is to prove Theorem 6.5 which gives the double shuffle relations of double Eisenstein series at level N for every positive integer N. The difficulty in generalizing the known $N = 1$ and $N = 2$ cases to arbitrary levels lies in the fact that there are many choices of the constant terms in the generating function of the double Eisenstein series and the other related series. It turns out that this fact is a consequence of an under-determined system of linear equations with $(3N^2+N)/2$ variables and N^2+N equations. Essentially, we need to show these equations are consistent with each other. For this we need a key result concerning N-th roots of unity and the multiple divisor functions at level N which will be proved in section 7. In the last section, using some standard techniques from linear algebra we prove the solvability of the linear system mentioned above. This enables us to derive our main result Theorem 6.3 which generalizes [10, Theorem 7] and [13, Theorem 3]

2. The multiple zeta values at level N

For any $\mathbf{s}=(s_1,\dots,s_d)\in\mathbb{N}^d$ with $s_1\ge 2$ and $\mathbf{a}=(a_1,\dots,a_d)\in\mathbb{Z}/N\mathbb{Z}$, we define the *multiple zeta values at level* N by

$$\zeta_N^{\mathbf{a}}(\mathbf{s})=\sum_{n_1>\cdots>n_d>0,\ n_j\equiv a_j \pmod N\ \forall j}\frac{1}{n_1^{s_1}\cdots n_d^{s_d}}. \tag{3}$$

These numbers are rational multiples of the multiple Hurwitz zeta values which have been studied by many authors. If we define the multiple polylogarithms

$$Li_{\mathbf{s}}(x_1,\dots,x_d)=\sum_{n_1>\cdots>n_d>0}\frac{x_1^{n_1}\dots x_d^{n_d}}{n_1^{s_1}\cdots n_d^{s_d}}.$$

Then it is not hard to see that

$$\zeta_N^{\mathbf{a}}(\mathbf{s})=\frac{1}{N^d}\sum_{\beta_1=1}^{N}\cdots\sum_{\beta_d=1}^{N}\eta^{-\beta_1a_1-\cdots-\beta_da_d}Li_{\mathbf{s}}(\eta^{\beta_1},\dots,\eta^{\beta_d}). \tag{4}$$

We now restrict ourself to levels at one or two. We remark that our definition of the double zeta value at level two is slightly different from that of [13] since we allow $a_1=a_2=0$ in which case we in fact essentially recover the usual double zeta values (at level 1).

We can also use Chen's iterated integrals to derive formulas similar to (4) which will be useful in the regularization of these values. Let $\omega=\frac{dt}{t}$, $\omega_\alpha^a=\omega(N)_\alpha^a=\frac{(\eta^\alpha t)^{a-1}\,dt}{1-\eta^\alpha t}$ $(1\le\alpha\le N)$ be 1-forms. By the partial fraction expansion

$$\frac{t^{a-1}}{1-t^N}=\frac{1}{N}\sum_{\alpha=1}^{N}\frac{\eta^{-\alpha(a-1)}}{1-\eta^\alpha t}$$

we see immediately that

$$\zeta_N^a(r)=\int_0^1\omega^{r-1}\frac{t^{a-1}\,dt}{1-t^N}\left(\overset{\text{def}}{=}\int\cdots\int_{1>t_1>t_2>\cdots>t_r>0}\frac{dt_1}{t_1}\cdot\frac{dt_2}{t_2}\cdots\frac{dt_{r-1}}{t_{r-1}}\cdot\frac{t_r^{a-1}dt_r}{1-t_r^N}\right) \tag{5}$$
$$=\frac{1}{N}\sum_{\alpha=1}^{N}\int_0^1\omega^{r-1}\frac{\eta^{-\alpha(a-1)}\,dt}{1-\eta^\alpha t}=\frac{1}{N}\sum_{\alpha=1}^{N}\eta^{-\alpha a}Li_r(\eta^\alpha).$$

Furthermore,

$$\begin{aligned}\zeta_N^{a,b}(r,s) &= \int_0^1 \omega^{r-1}\frac{t^{a-b-1}\,dt}{1-t^N}\omega^{s-1}\frac{t^{b-1}\,dt}{1-t^N} \\ &= \frac{1}{N^2}\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\int_0^1 \omega^{r-1}\frac{\eta^{-\alpha(a-b-1)}\,dt}{1-\eta^\alpha t}\omega^{s-1}\frac{\eta^{-\beta(b-1)}\,dt}{1-\eta^\beta t} \qquad (6)\\ &= \frac{1}{N^2}\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\eta^{-\alpha(a-b)-\beta b} Li_{r,s}(\eta^\alpha,\eta^{\beta-\alpha})\end{aligned}$$

Our first result is the following explicit form of double shuffle relations.

Proposition 2.1. *For positive integers $r,s\geq 2$ and integers $a,b\in\mathbb{Z}/N\mathbb{Z}$, we have*

$$\begin{aligned}\zeta_N^a(r)\zeta_N^b(s) &= \zeta_N^{a,b}(r,s)+\zeta_N^{b,a}(s,r)+\delta_{a,b}\zeta_N^a(s+r) \qquad (7)\\ &= \sum_{i=0}^{s-1}\binom{r+i-1}{r-1}\zeta_N^{a+b,b}(r+i,s-i)+\sum_{j=0}^{r-1}\binom{s+j-1}{s-1}\zeta_N^{a+b,a}(s+j,r-j) \\ &= \sum_{\substack{i+j=r+s\\ i\geq 2, j\geq 1}}\left(\binom{i-1}{r-1}\zeta_N^{a+b,b}(i,j)+\binom{i-1}{s-1}\zeta_N^{a+b,a}(i,j)\right),\end{aligned}$$

where $\delta_{a,b}$ is the Kronecker symbol, namely, $\delta_{a,b}=1$ if $a=b$ and $\delta_{a,b}=0$ if $a\neq b$.

Proof. The first equality is clear by the definition (3). The second equality follows immediately from the shuffle product formula of iterated integrals [6, (1.5.1)]:

$$\int_0^1 \omega_1\cdots\omega_r\int_0^1\omega_{r+1}\cdots\omega_{r+s}=\sum_\sigma\int_0^1\omega_{\sigma(1)}\cdots\omega_{\sigma(r+s)},$$

where σ ranges over all shuffles of type (r,s), i.e., permutations σ of $r+s$ letters with $\sigma^{-1}(1)<\cdots<\sigma^{-1}(r)$ and $\sigma^{-1}(r+1)<\cdots<\sigma^{-1}(r+s)$. □

3. Double zeta space at level N

Now we introduce the level N version of the *formal double zeta space* studied in [10] as follows. Let $k\geq 2$ and $\mathcal{DZ}(N)_k$ be the $\mathbb{Q}$-vector space spanned by the formal symbols $Z_{r,s}^{a,b}=Z(N)_{r,s}^{a,b}$, $P_{r,s}^{a,b}=P(N)_{r,s}^{a,b}$, and $Z_k^a=Z(N)_k^a$ $(r,s\geq 1, r+s=k, a,b\in\mathbb{Z}/N\mathbb{Z})$ with the set of relations

$$P_{r,s}^{a,b}=Z_{r,s}^{a,b}+Z_{s,r}^{b,a}+\delta_{a,b}Z_{s+r}^a=\sum_{\substack{i+j=k\\ i,j\geq 1}}\left(\binom{i-1}{r-1}Z_{i,j}^{a+b,b}+\binom{i-1}{s-1}Z_{i,j}^{a+b,a}\right) \qquad (8)$$

for $r,s\geq 1$, $r+s=k$. Namely,

$$\mathcal{DZ}(N)_k=\frac{\mathbb{Q}\langle Z(N)_{r,s}^{a,b},P(N)_{r,s}^{a,b},Z(N)_k^a:\ a,b\in\mathbb{Z}/N\mathbb{Z},\ r,s\geq 1,\ r+s=k\rangle}{\mathbb{Q}\langle\text{relations }(8)\rangle}.$$

Clearly

$$\mathcal{DZ}(N)_k = \frac{\mathbb{Q}\langle Z(N)^{a,b}_{r,s}, Z(N)^a_k : \ a,b \in \mathbb{Z}/N\mathbb{Z}, \ r,s \geq 1, \ r+s=k\rangle}{\mathbb{Q}\langle \text{relations (9)}\rangle},$$

where the defining relations are (dropping the dependence on N)

$$Z^{a,b}_{r,s} + Z^{b,a}_{s,r} + \delta_{a,b} Z^a_k = \sum_{\substack{i+j=k\\ i,j\geq 1}} \left(\binom{i-1}{r-1} Z^{a+b,b}_{i,j} + \binom{i-1}{s-1} Z^{a+b,a}_{i,j} \right). \tag{9}$$

Recall that we often choose $0 \leq a < N$ to represent the residue class congruent to a modulo N. Observe that when residue class $\bar{0}$ appears in any conditions involving gcd we should use N to represent it. For example, $\gcd(0,0) = \gcd(0,N) = \gcd(N,N) = N$. We now may define $\mathcal{PDZ}(N)_k$, the formal double zeta space of *pure level* N by restricting $Z^{b,a}_{s,r}$ to

$$\Omega(N) = \{(a,b) : 0 \leq a,b < N, \gcd(a,b,N) = 1\}$$

and Z^a_k to $\{a : 1 \leq a < N, \gcd(a,N) = 1\}$ in the above. This is well-defined since if $\gcd(a,b,N) = 1$ then $\gcd(a+b,a,N) = \gcd(a+b,b,N) = 1$.

Since both sides of (9) are invariant under $(a,b;r,s) \leftrightarrow (b,a;s,r)$ we may just take $r \leq s$. Thus for even k the group $\mathcal{DZ}(N)_k$ has $(k-1)N^2 + N$ generators and $kN^2/2$ relations. Hence

$$\dim \mathcal{DZ}(N)_k \geq \frac{(k-2)N^2 + 2N}{2}.$$

Similarly, for even k the group $\mathcal{PDZ}(N)_k$ has $(k-1)|\Omega(N)| + \varphi(N)$ generators and $(k-1)(|\Omega(N)| + \varphi(N))/2$ relations. Hence

$$\dim \mathcal{PDZ}(N)_k \geq \frac{(k-1)(|\Omega(N)| - \varphi(N))}{2} + \varphi(N). \tag{10}$$

Remark 3.1. *(a). Notice that the double zeta space* $\mathcal{DZ}(2)_k$ *in* [13] *is our* $\mathcal{PDZ}(2)_k$*. (b). The bound in* (10) *is not sharp. For example, when* $(N,k) = (3,4)$ *we have 24 generators and only 13 independent relations instead of 15. So* $\dim \mathcal{PDZ}(3)_4 = 11 > 24 - 15$.

Note that the relations (8) (as well as (9)) correspond to those in Proposition 2.1 when $r,s \geq 2$, under the correspondences

$$Z(N)^{a,b}_{r,s} \longleftrightarrow \zeta^{a,b}_N(r,s), \quad Z(N)^a_k \longleftrightarrow \zeta^a_N(k), \quad P(N)^{a,b}_{r,s} \longleftrightarrow \zeta^a_N(r)\zeta^b_N(s),$$

the binomial coefficients for $i = 1$ on the right vanish in both (8) and (9). For our later applications it is convenient to allow the "divergent" $Z(N)^{a,b}_{1,k-1}$ and $P(N)^{a,b}_{1,k-1}$ etc., and in fact the double shuffle relations in Proposition 2.1 can be extended for $r = 1$ or $s = 1$ by using a suitable regularization procedure for $Li^{ш}_{1,k-1}(1,\eta)$ etc. developed in [2] which was motivated by [11]. For a comprehensive treatment of the general multiple zeta values of level N, please see our paper [17]. Specifically, in our current situation we can define the following renormalized values. Let T be a formal variable,

- Note that $Li_1^*(1)=\zeta_*(1)=T$ and $Li_1^{\text{ш}}(1)=\zeta_{\text{ш}}(1)=T$. By (4) and (5), for $a\in\mathbb{Z}/N\mathbb{Z}$
$$\zeta^a_{N;*}(1)=\zeta^a_{N;\text{ш}}(1)=\frac{1}{N}\left(T+\sum_{\alpha=1}^{N-1}\eta^{-a\alpha}Li_1(\eta^\alpha)\right). \tag{11}$$
- By (7), for $s\ge 2$ and $a,b\in\mathbb{Z}/N\mathbb{Z}$
$$\zeta^{a,a}_{N;*}(1,s)=\frac{1}{N}\left(T+\sum_{\alpha=1}^{N-1}\eta^{-a\alpha}Li_1(\eta^\alpha)\right)\zeta^a_N(s)-\zeta^{a,a}_N(s,1)-\zeta^a_N(s+1),$$
$$\zeta^{a,b}_{N;*}(1,s)=\frac{1}{N}\left(T+\sum_{\alpha=1}^{N-1}\eta^{-a\alpha}Li_1(\eta^\alpha)\right)\zeta^b_N(s)-\zeta^{b,a}_N(s,1)\ \text{if}\quad a\ne b,$$
- By (4), for $a,b\in\mathbb{Z}/N\mathbb{Z}$
$$\begin{aligned}\zeta^{a,b}_{N;*}(1,1)=&\frac{1}{N^2}\left(\frac{T^2}{2}+\sum_{\beta=1}^{N-1}\eta^{-b\beta}\big(TLi_1(\eta^\beta)-Li_{1,1}(\eta^\beta,\eta^{-\beta})\big)\right.\\&\left.+\sum_{\alpha=1}^{N-1}\sum_{\beta=1}^{N}\eta^{-a\alpha-b\beta}Li_{1,1}(\eta^\alpha,\eta^{\beta-\alpha})\right).\end{aligned}$$
- By (6), for $a,b\in\mathbb{Z}/N\mathbb{Z}$
$$\zeta^{a,b}_{N;\text{ш}}(1,s)=\frac{1}{N^2}\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\eta^{-\alpha(a-b)-\beta b}Li^{\text{ш}}_{1,s}(\eta^\alpha,\eta^{\beta-\alpha})$$
where $Li^{\text{ш}}_{1,1}(1,1)=\frac{1}{2}T^2$, $Li^{\text{ш}}_{1,s}(\eta^\alpha,\eta^{\beta-\alpha})=Li_{1,s}(\eta^\alpha,\eta^{\beta-\alpha})$ for all $\alpha\ne\bar{0}\in\mathbb{Z}/N\mathbb{Z}$, and
$$Li^{\text{ш}}_{1,s}(1,\eta^\beta)=TLi_s(\eta^\beta)-\sum_{t=2}^{s}Li_{t,s+1-t}(1,\eta^\beta)-Li_{s,1}(\eta^\beta,\eta^{-\beta})\quad\forall(s,\beta)\ne(1,\bar{0}).$$

The equations in Proposition 2.1 are valid for all $r,\ s\ge 1$. Here we have used the fact that for $\alpha,\beta\in\mathbb{Z}/N\mathbb{Z}$ ($\alpha\ne\bar{0}$) we have
$$Li_1(\eta^\alpha)=\int_0^1\frac{\eta^\alpha\,dt}{1-\eta^\alpha t},\quad Li_{1,1}(\eta^\alpha,\eta^{\beta-\alpha})=\int_{1>t_1>t_2>0}\frac{\eta^\alpha\,dt_1}{1-\eta^\alpha t_1}\frac{\eta^\beta\,dt_2}{1-\eta^\beta t_2}. \tag{12}$$

With the above regularized values we can now extend Proposition 2.1 to these cases. For $r\ge 2$ and $s\ge 1$ we set $\zeta^a_{N;*}(r)=\zeta^a_{N;\text{ш}}(r)=\zeta^a_N(r)$ and $\zeta^{a,b}_{N;*}(r,s)=\zeta^{a,b}_{N;\text{ш}}(r,s)=\zeta^{a,b}_N(r,s)$.

Proposition 3.2. *For positive integers $r,s\ge 1$ and $a,b\in\mathbb{Z}/N\mathbb{Z}$, we have*
$$\zeta^a_{N;*}(r)\zeta^b_{N;*}(s)=\zeta^{a,b}_{N;*}(r,s)+\zeta^{b,a}_{N;*}(s,r)+\delta_{a,b}\zeta^a_{N;*}(s+r)$$
$$\zeta^a_{N;\text{ш}}(r)\zeta^b_{N;\text{ш}}(s)=\sum_{\substack{i+j=r+s\\ i,j\ge 1}}\left(\binom{i-1}{r-1}\zeta^{a+b,b}_{N;\text{ш}}(i,j)+\binom{i-1}{s-1}\zeta^{a+b,a}_{N;\text{ш}}(i,j)\right),$$

where we set $\binom{0}{0} = 1$.

Proof. We only need to check the relations in the case when $r = 1, s \geq 2$ or $r \geq 2, s = 1$ or $r = s = 1$. These follows directly from the definitions and the stuffle and shuffle relations among Li_1 and $Li_{1,1}$. We leave the details to the interested reader. $\square$

The following theorem generalizes both a result of [13] and a result of [10].

Theorem 3.3. *Let k be a positive even integer and $a \in \mathbb{Z}/N\mathbb{Z}$. Then*

$$\sum_{1\leq r<k,\ r\ odd} Z^{a,a}_{r,k-r} = \frac{1}{4}\Big(2Z^{N,a}_{1,k-1} + 2Z^{2a,a}_{1,k-1} - Z^a_k + 2\delta_{a,\bar{0}}Z^N_k\Big), \tag{13}$$

$$\sum_{1<r<k,\ r\ even} Z^{a,a}_{r,k-r} = \frac{1}{4}\Big(2Z^{N,a}_{1,k-1} - 2Z^{2a,a}_{1,k-1} + Z^a_k + 2\delta_{a,\bar{0}}Z^N_k\Big). \tag{14}$$

Proof. Consider the generating functions

$$\mathcal{Z}^{a,b}_k(X,Y) = \sum_{r+s=k} Z^{a,b}_{r,s}X^{r-1}Y^{s-1}.$$

By (9) we see that

$$\mathcal{Z}^{a,b}_k(X,Y) + \mathcal{Z}^{b,a}_k(Y,X) + \delta_{a,b}Z^a_k\frac{X^{k-1}-Y^{k-1}}{X-Y} = \mathcal{Z}^{a+b,b}_k(X+Y,Y) + \mathcal{Z}^{a+b,a}_k(X+Y,X).$$

Set $(X,Y) = (1,0)$ and then $(X,Y) = (1,-1)$ we get, respectively,

$$Z^{a,b}_{k-1,1} + Z^{b,a}_{1,k-1} + \delta_{a,b}Z^a_k = Z^{a+b,b}_{k-1,1} + \sum_{r=1}^{k-1} Z^{a+b,a}_{r,k-r}, \tag{15}$$

$$\sum_{r=1}^{k-1}(-1)^{r-1}\big(Z^{a,b}_{r,k-r} + Z^{b,a}_{r,k-r}\big) + \delta_{a,b}Z^a_k = Z^{a+b,b}_{1,k-1} + Z^{a+b,a}_{1,k-1}. \tag{16}$$

Setting $b = N$ in (15) and $a = b$ in (16) we get

$$\sum_{r=1}^{k-1} Z^{a,a}_{r,k-r} = Z^{N,a}_{1,k-1} + \delta_{a,\bar{0}}Z^N_k, \tag{17}$$

$$2\sum_{r=1}^{k-1}(-1)^{r-1}Z^{a,a}_{r,k-r} + Z^a_k = 2Z^{2a,a}_{1,k-1}. \tag{18}$$

By adding (resp. subtracting) twice of (17) to (resp. from) (18) we obtain (13) and (14). $\square$

Remark 3.4. *Part 1) of* [13, Theorem 1] *follows from the special case of $N = 2$ and $a = 1$ of our theorem. By taking $N = a = 1$ in the theorem we obtain* [10, Theorem 1].

Next we describe the linear relations among $Z^{a,b}_{i,j}$'s using some homogeneous polynomials.

Proposition 3.5. *Let $k \ge 2$ be a positive integer. Let $c_{i,j}^{a,b} \in \mathbb{Q}$ for all $i, j \in \mathbb{N}$ and $a, b \in \mathbb{Z}/N\mathbb{Z}$. Then the following two statements are equivalent:*

(i) *The relation*

$$\sum_{0\le a\le b<N} \sum_{i+j=k} c_{i,j}^{a,b} Z_{i,j}^{a,b} \equiv 0 \pmod{\mathbb{Q}\langle Z_k^a : a \in \mathbb{Z}/N\mathbb{Z}\rangle}$$

holds in $\mathcal{DZ}(N)_k$. Here and in the rest of the paper, $\sum_{i+j=k}$ means $\sum_{i+j=k, i,j\ge 1}$.

(ii) *There exist some homogeneous polynomials $F_{a,b} \in \mathbb{Q}[X, Y]$ $(0 \le a \le b < N)$ of degree $k-2$ such that*

$$\sum_{0\le a\le b<N} F_{a,b}(X_b, Y_a) + F_{a,b}(Y_b, X_a) - F_{a,b}(X_{a+b} + Y_b, X_{a+b}) - F_{a,b}(X_{a+b}, X_{a+b} + Y_a) = \sum_{a,b\in\mathbb{Z}/N\mathbb{Z}} \sum_{i+j=k} \binom{k-2}{i-1} c_{i,j}^{a,b} X_a^{i-1} Y_b^{j-1}. \quad (19)$$

Further, the following two statements are equivalent:

(iii) *The relation*

$$\sum_{a,b\in\Omega(N), a\le b} \sum_{i+j=k} c_{i,j}^{a,b} Z_{i,j}^{a,b} \equiv 0 \pmod{\mathbb{Q}\langle Z_k^a : 1 \le a < N, \gcd(a, N) = 1\rangle}$$

holds in $\mathcal{PDZ}(N)_k$.

(iv) *There exist some homogeneous polynomials $F_{a,b} \in \mathbb{Q}[X, Y]$ $(a, b \in \Omega(N)$ and $a \le b)$ of degree $k-2$ such that*

$$\sum_{a,b\in\Omega(N), a\le b} F_{a,b}(X_b, Y_a) + F_{a,b}(Y_b, X_a) - F_{a,b}(X_{a+b} + Y_b, X_{a+b}) - F_{a,b}(X_{a+b}, X_{a+b} + Y_a) = \sum_{a,b\in\Omega(N)} \sum_{i+j=k} \binom{k-2}{i-1} c_{i,j}^{a,b} X_a^{i-1} Y_b^{j-1}. \quad (20)$$

Proof. For any fixed $a, b \in \mathbb{Z}/N\mathbb{Z}$ we take $F_{a,b}(X, Y) = \binom{k-2}{r-1} X^{r-1} Y^{s-1}$ $(r + s = k)$ and $F_{c,d}(X, Y) = 0$ for all $(c, d) \ne (a, b)$. Then the expansion of the left hand side of (20) determines the values $c_{i,j}^{a,b}$ uniquely such that (9) holds which implies (i). In fact, when $a \ne b$ we obtain an exact equation in (i). Since any relation of the form in (i) in $\mathcal{DZ}(N)_k$ should come from a linear combination of (9) with various choices of $(a, b) \in (\mathbb{Z}/N\mathbb{Z})^2$ modulo $\mathbb{Q}\langle Z_k^a : a \in \mathbb{Z}/N\mathbb{Z}\rangle$ and any homogeneous polynomial is a linear combination of monomials of the form $F_{a,b}(X, Y)$, the equivalence of (i) and (ii) follows immediately. Similar arguments clearly shows the equivalence of (iii) and (iv). $\square$

Remark 3.6. *Proposition* 3.5 *generalizes* [10, Proposition 2.2(i)(ii)] *and* [13, Lemma 1]. *In fact, when $N = 2$ we can take $(a, b) = (0, 1), (1, 1)$ in (iii) and (iv) of Proposition* 3.5 *then we see that $F = F_{0,1}$ and $G = F_{1,1}$ in* [10, Proposition 2.2(ii)] *with relabeling of X's and Y's as follows: $X_0^i Y_1^j \to X_1^i Y_1^j$, $X_1^i Y_0^j \to X_2^i Y_2^j$, and $X_1^i Y_1^j \to X_3^i Y_3^j$.*

4. Fourier series expansion of the double Eisenstein series at level N

In this section we will describe a procedure to find the Fourier series expansion of double Eisenstein series. This can be generalized to larger depths. Similar to the notation used in [3] and [10], for any $a \in \mathbb{Z}/N\mathbb{Z}$ and positive integer s set

$$\Psi_s^a(\tau) = \Psi_s^{a;N}(\tau) = \sum_{c \equiv a \,(\mathrm{mod}\, N), c \in \mathbb{Z}} \frac{1}{(\tau + c)^s} \quad \forall s \geq 2,$$
$$\Psi_1^a(\tau) = \Psi_1^{a;N}(\tau) = \lim_{M \to \infty} \sum_{c \equiv a \,(\mathrm{mod}\, N), |c| < M} \frac{1}{\tau + c}.$$

Then we have

Lemma 4.1. *Let $q = e^{2\pi i \tau}$ and $\eta = \eta_N = \exp(2\pi i/N)$. Then for any $a \in \mathbb{Z}/N\mathbb{Z}$ and $s \in \mathbb{N}$, we have*

$$\Psi_s^a(N\tau) = \begin{cases} -\dfrac{\pi i}{N} - \dfrac{2\pi i}{N} \displaystyle\sum_{n \geq 1} \eta^{an} q^n, & \text{if } s = 1; \\ \dfrac{(-2\pi i)^s}{N^s (s-1)!} \displaystyle\sum_{n \geq 1} n^{s-1} \eta^{an} q^n, & \text{if } s \geq 2. \end{cases} \tag{21}$$

Proof. The well-known Lipschitz formula implies that for all $k \geq 2$

$$\sum_{n \in \mathbb{Z}} \frac{1}{(x+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n \geq 1} n^{k-1} e^{2\pi i n x}.$$

Thus by setting $x = \tau + a/N$ we get

$$\Psi_k^a(N\tau) = \sum_{n \in \mathbb{Z}} \frac{1}{(N\tau + nN + a)^k} = \frac{(-2\pi i)^k}{N^k (k-1)!} \sum_{n \geq 1} n^{k-1} \eta^{an} e^{2\pi i n x},$$

as desired.

Now we deal with the special case when $s = 1$. In the Summation Theorem [15, p. 305] we take $f(z) = 1/(z + \tau')$ where $\tau' = (\tau + a)/N$. Then we get

$$\lim_{M \to \infty} \sum_{c \equiv a \,(\mathrm{mod}\, N), |c| < M} \frac{1}{\tau + c} = \frac{1}{N} \lim_{M \to \infty} \sum_{n=-M}^{M} \frac{1}{\tau' + n} = -\mathrm{Res}_{z = -\tau'} \left(\frac{\pi \cot \pi z}{z + \tau'} \right).$$

Since

$$\cot \pi z = (-i) \frac{1 + e^{2\pi i z}}{1 - e^{2\pi i z}} = -i - 2i \sum_{n \geq 1} e^{2\pi i n z}$$

we have

$$\Psi_1^a(N\tau) = \frac{\pi}{N} \cot \left[\pi \left(\tau + \frac{a}{N} \right) \right] = -\frac{\pi i}{N} - \frac{2\pi i}{N} \sum_{n \geq 1} \eta^{an} q^n.$$

This completes the proof of the lemma. $\square$

Corollary 4.2. *For any* $\mathbf{a} = (a_1, \dots, a_d) \in (\mathbb{Z}/N\mathbb{Z})^d$ *and* $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$ *set*

$$g_{\mathbf{s}}^{\mathbf{a}}(\tau) = g_{\mathbf{s}}^{\mathbf{a};N}(\tau) = \sum_{m_1 > \cdots > m_d > 0} \prod_{j=1}^{d} \Psi_{s_j}^{a_j}(m_j N\tau). \tag{22}$$

Then we have

$$g_{\mathbf{s}}^{\mathbf{a}}(\tau) = \frac{(-2\pi i)^{|\mathbf{s}|}}{N^{|\mathbf{s}|}(\mathbf{s}-1)!} \sum_{n=1}^{\infty} \sigma_{\mathbf{s}-\mathbf{1}}^{\mathbf{a}}(n) q^n, \tag{23}$$

where $|\mathbf{s}| = s_1 + \cdots + s_d$, $(\mathbf{s}-1)! = \prod_{j=1}^{d}(s_j - 1)!$, *and* $\mathbf{s} - \mathbf{1} = (s_1 - 1, \dots, s_d - 1)$.

It is now easy to decompose the level N Eisenstein series (of one variable) into the following form:

$$G_r^a(\tau) = \zeta_N^a(r) + g_r^a(\tau) \tag{24}$$

for any positive integer $r \geq 3$ and $a \in \mathbb{Z}/N\mathbb{Z}$. So we can define two extensions of $G_r^a(\tau)$ as follows.

Definition 4.3. *Let* $\sharp = ш$ *or* $*$. *For all* $s \geq 1$, *we define*

$$G_{s;\sharp}^a(\tau) = \zeta_{N;\sharp}^a(s) + g_s^a(\tau). \tag{25}$$

Notice that the definition is independent of whether $\sharp = ш$ or $*$ by (11).

The following theorem is the key to the double shuffle relations satisfied by the double Eisenstein series at level N.

Theorem 4.4. *The Fourier series expansion of* $G_{r,s}^{a,b}(\tau)$ *for* $r \geq 3, s \geq 2$ *is given by*

$$\begin{aligned} G_{r,s}^{a,b}(\tau) &= \zeta_N^{a,b}(r,s) + g_r^a(\tau)\zeta_N^b(s) + g_{r,s}^{a,b}(\tau) \\ &\quad + \sum_{\substack{h+p=r+s \\ h\geq 1, p \geq \min\{r,s\}}} \zeta_N^{a-b}(p) \left[(-1)^s \binom{p-1}{s-1} g_h^a(\tau) + (-1)^{p-r} \binom{p-1}{r-1} g_h^b(\tau) \right]. \end{aligned} \tag{26}$$

Proof. Our proof follows the lines of that of [10, Theorem 6]. We decompose $G_{r,s}^{a,b}(\tau)$ into the sum of the following four types: (1) $m = n = 0$, (2) $m > n = 0$, (3) $m = n > 0$, and (4) $m > n > 0$.

(1) $m = n = 0$. It gives rise to exactly

$$\sum_{\substack{c > d > 0 \\ c \equiv a,\, d \equiv b \,(\mathrm{mod}\, N)}} \frac{1}{c^r d^s} = \zeta_N^{a,b}(r,s).$$

(2) $m > n = 0$. We are looking at

$$\sum_{\substack{m>0,\ d>0 \\ c\equiv a,\, d \equiv b \,(\mathrm{mod}\, N)}} \frac{1}{(mN\tau + c)^r d^s} = \sum_{m>0} \Psi_r^a(mN\tau) \sum_{\substack{d>0 \\ d\equiv b \,(\mathrm{mod}\, N)}} \frac{1}{d^s} = g_r^a(\tau)\zeta_N^b(s).$$

(3) $m = n > 0$. Then we write

$$\sum_{\substack{m>0\\ c\equiv a,\, d\equiv b\,(\mathrm{mod}\, N)}} \frac{1}{(mN\tau+c)^r(mN\tau+d)^s} = \sum_{m>0} \Psi^{a,b}_{r,s}(mN\tau).$$

Next we compute $\Psi^{a,b}_{r,s}(\tau)$. Using the partial fraction

$$\frac{1}{(\tau+c)^r(\tau+d)^s} = \sum_{\substack{h+p=r+s\\ h\geq 1, p\geq \min\{r,s\}}} \left[\frac{(-1)^s\binom{p-1}{s-1}}{(c-d)^p(\tau+c)^h} + \frac{(-1)^{p-r}\binom{p-1}{r-1}}{(c-d)^p(\tau+d)^h}\right],$$

we obtain

$$\begin{aligned}
\Psi^{a,b}_{r,s}(\tau) &= \sum_{\substack{c>d\\ c\equiv a,\, d\equiv b\,(\mathrm{mod}\, N)}} \frac{1}{(\tau+c)^r(\tau+d)^s}\\
&= \sum_{\substack{h+p=r+s,\, c>d\\ c\equiv a,\, d\equiv b\,(\mathrm{mod}\, N)}} \left[(-1)^s\binom{p-1}{s-1}\frac{1}{(c-d)^p(\tau+c)^h} + (-1)^{p-r}\binom{p-1}{r-1}\frac{1}{(c-d)^p(\tau+d)^h}\right]\\
&= \sum_{\substack{h+p=r+s\\ h\geq 1, p\geq \min\{r,s\}}} \zeta^{a-b}_N(p)\left[(-1)^s\binom{p-1}{s-1}\Psi^a_h(\tau) + (-1)^{p-r}\binom{p-1}{r-1}\Psi^b_h(\tau)\right].
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{\substack{m>0\\ c\equiv a,\, d\equiv b\,(\mathrm{mod}\, N)}} \frac{1}{(mN\tau+c)^r(mN\tau+d)^s} &= \sum_{m>0} \Psi^{a,b}_{r,s}(mN\tau)\\
&= \sum_{\substack{h+p=r+s\\ h\geq 1, p\geq \min\{r,s\}}} \zeta^{a-b}_N(p)\left[(-1)^s\binom{p-1}{s-1}g^a_h(\tau) + (-1)^{p-r}\binom{p-1}{r-1}g^b_h(\tau)\right].
\end{aligned}$$

Here the special case $h = 1$ has to be treated carefully by using (21).

(4) $m > n > 0$. We have

$$\sum_{\substack{m>n>0\\ c\equiv a,\, d\equiv b\,(\mathrm{mod}\, N)}} \frac{1}{(mN\tau+c)^r(nN\tau+d)^s} = g^{a,b}_{r,s}(\tau).$$

The theorem follows by summing up the above four parts. □

Motivated by (26) we now have the extension of the double Eisenstein series of level N to following regularized form.

Definition 4.5. *Let $\sharp = ш$ or $*$. Then for all $r, s \geq 1$, we define*

$$G^{a,b}_{r,s;\sharp}(\tau) = \zeta^{a,b}_{N;\sharp}(r,s) + g^a_r(\tau)\zeta^b_{N;\sharp}(s) + g^{a,b}_{r,s}(\tau)$$

$$+\sum_{\substack{h+p=r+s\\ h\ge 1, p\ge \min\{r,s\}}} \zeta_{N;\sharp}^{a-b}(p)\left[(-1)^s\binom{p-1}{s-1}g_h^a(\tau)+(-1)^{p-r}\binom{p-1}{r-1}g_h^b(\tau)\right]. \quad (27)$$

Remark 4.6. *Unlike Definition* 4.3 *this definition of* $G^{a,b}_{r,s;\sharp}(\tau)$ *depends on the choice of the regularization scheme* $\sharp$. *Moreover, because of* (11) *the dependence only appears in the constant term* $\zeta^{a,b}_{N;\sharp}(r,s)$.

If $s,r\ge 3$ and $a,b\in\mathbb{Z}/N\mathbb{Z}$, then it is not hard to show

$$G_r^a(\tau)G_s^b(\tau)=G^{a,b}_{r,s}(\tau)+G^{b,a}_{s,r}(\tau)+\delta_{a,b}G^a_{r+s}(\tau) \quad (28)$$

which follows easily by the definition. But

$$G_r^a(\tau)G_s^b(\tau)\ne \sum_{\substack{i+j=r+s\\ i,j\ge 1}}\left(\binom{i-1}{r-1}G^{a+b,b}_{i,j}(\tau)+\binom{i-1}{s-1}G^{a+b,a}_{i,j}(\tau)\right)$$

since the right-hand side has undefined terms. Our goal is to give an extension of these double shuffle relations to the case $r,g\ge 1$ and $(r,s)\ne(1,1)$ by using a complete version of the zeta values and Eisenstein series of level N.

5. Decomposition of the zeta values at level N

In this section we break $\zeta_N^a(s)$ into two parts, one of which is inspired by its complete version defined as follows. For all positive integers n and $a\in\mathbb{Z}/N\mathbb{Z}$, we set

$$\mathfrak{z}_N^a(n)=\frac{1}{2}\sum_{\substack{k\in\mathbb{Z}_{\ne 0}\\ k\equiv a\,(\mathrm{mod}\,N)}}\frac{1}{k^n}=\frac{1}{2}\lim_{M\to\infty}\sum_{\substack{0<|k|<M\\ k\equiv a\,(\mathrm{mod}\,N)}}\frac{1}{k^n} \quad (29)$$

by using the Cauchy principal value. This infinite series converges absolutely for $n\ge 2$ and conditionally for $n=1$. This complete version of $\zeta_N^a(s)$ clearly satisfies the stuffle relations as those given by (7).

Remark 5.1. *When the level* $N=2$, *the decomposition of* $\zeta_N^a(n)$ *corresponds to the decomposition of it into the the Bernoulli number part and non-Bernoulli number part. See* [13, page 1103]. *If* $a\equiv 0 \pmod 2$, *then the non-Bernoulli number part is essentially the Riemann zeta values at odd integers.*

To extract more information from $\mathfrak{z}_N^a(n)$ and find its relation to $\zeta_N^a(n)$ we now recall that the n-th Bernoulli periodic function $\bar{B}_n(x)$, (see [1, p. 267]) has the series expansion

$$\bar{B}_n(x)=-\frac{n!}{(2\pi i)^n}\sum_{k\in\mathbb{Z}_{\ne 0}}\frac{e^{2\pi ikx}}{k^n}=-\frac{n!}{(2\pi i)^n}\lim_{M\to\infty}\sum_{0<|k|<M}\frac{e^{2\pi ikx}}{k^n}, \quad (30)$$

which converges absolutely for $n\ge 2$ and conditionally for $n=1$. It is related to the Bernoulli polynomials by $\bar{B}_n(x)=B_n(\{x\})$ where $\{x\}$ is the fractional part of x, except for $n=x=1$ when $B_1(\{1\})=B_1(0)=-1/2$ while

$$\bar{B}_1(1)=-\frac{1}{2\pi i}\lim_{M\to\infty}\sum_{0<|k|<M}\frac{1}{k}=0 \quad (31)$$

by symmetry. The following identify motivates our definition of the constant term of the generating series for the level N Eisenstein series.

Proposition 5.2. *For all $n \geq 1$ and $a \in \mathbb{Z}/N\mathbb{Z}$, we have*

$$\mathfrak{z}_N^a(n) = -\frac{(2\pi i)^n}{2N \cdot n!} \sum_{l=1}^{N} \exp\left(-\frac{2\pi i l a}{N}\right) \bar{B}_n\left(\frac{l}{N}\right). \tag{32}$$

Proof. This follows quickly from the identity

$$\sum_{l=1}^{N} \eta_N^{lm} = \begin{cases} N, & \text{if } N|m; \\ 0, & \text{otherwise,} \end{cases}$$

for any integer m. □

Corollary 5.3. *Let $1 \leq a \leq N$. Then for all $r \geq 1$*

$$\sum_{l=1}^{N} \sin\left(\frac{2\pi la}{N}\right) \bar{B}_{2r}\left(\frac{l}{N}\right) = \sum_{l=1}^{N} \cos\left(\frac{2\pi la}{N}\right) \bar{B}_{2r+1}\left(\frac{l}{N}\right) = 0.$$

Proof. Notice that under $a \to N-a$ the left-hand side of (32) is invariant if n is even and changes the sign if n is odd. It also follows from the fact that $\bar{B}_{2r}(1-x) = \bar{B}_{2r}(x)$ and $\bar{B}_{2r+1}(1-x) = -\bar{B}_{2r+1}(x)$ for all $r \geq 0$. □

The following corollary provides the exact relation between ζ_N^a and $\mathfrak{z}_N^a$.

Corollary 5.4. *For all positive integers n and $a \in \mathbb{Z}/N\mathbb{Z}$, we have*

$$\mathfrak{z}_N^a(n) = \frac{1}{2}\Big(\zeta_{N;\sharp}^a(n) + (-1)^n \zeta_{N;\sharp}^{-a}(n)\Big),$$

where $\sharp = ш$ or $*$, and when $n = 1$ the right-hand side is defined by (11).

Proof. Suppose $n \geq 2$ first. Then we can break the sum in (29) into two parts, one with positive indices which produces $(2\pi i)^{-n}\zeta_N^a(n)$ and the other negative which leads to $(-2\pi i)^{-n}\zeta_N^{-a}(n)$. For $n = 1$ the corollary follows easily from Proposition 5.2 by using the fact that

$$Li_1(-e^{2\pi i\theta}) - Li_1(e^{2\pi i\theta}) = 2\pi i\theta - \pi i = 2\pi i\bar{B}_1(\theta) \quad \forall \theta \in (0,1).$$

Notice that the $l = N$ term in the sum on the right-hand side of (32) vanishes by (31). We leave the details to the interested reader. □

Proposition 5.2 leads us to the following definition if we follow the guideline that the constant term of the multiple Eisenstein series (even for the regularized values) should be closely related to the multiple zeta values, at any level.

Definition 5.5. *For $n > 0$ and $1 \leq a \leq N$ we define*

$$\beta_n^a = \beta_n^{a;N} = -\frac{1}{2N \cdot n!} \sum_{l=1}^{N} \exp\left(-\frac{2\pi i l a}{N}\right) B_n\left(\frac{l}{N}\right). \tag{33}$$

It is clear from Corollary 5.4, Proposition 5.2, and (31) that

$$\beta_n^a = (2\pi i)^{-n} \mathfrak{z}_N^a(n) + \frac{\delta_{n,1}}{4N}. \tag{34}$$

Let $\sharp = ш$ or $*$. For all $r \geq 1, s \geq 1$ and $a, b \in \mathbb{Z}/N\mathbb{Z}$, we define

$$\tilde{g}_s^a(q) = (2\pi i)^{-s} g_s^a(\tau), \quad \bar{\beta}_r^a = \frac{(2\pi i)^{-r}}{2}\Big(\zeta_{N;\sharp}^a(r) - (-1)^r \zeta_{N;\sharp}^{-a}(r)\Big) - \frac{\delta_{r,1}}{4N},$$
$$I_{r,s}^{a,b}(q) = \tilde{g}_r^a(q)\bar{\beta}_s^b + \sum_{\substack{h+p=r+s\\ h\geq 1}} \bar{\beta}_p^{a-b}\left[(-1)^s\binom{p-1}{s-1}\tilde{g}_h^a(q) + (-1)^{p-r}\binom{p-1}{r-1}\tilde{g}_h^b(q)\right].$$

Notice that the quantities defined above are independent of whether $\sharp = ш$ or $*$ according to (11). The following result generalizes [13, Lemma 3].

Proposition 5.6. *For any integer $k \geq 2$ and $a, b \in \mathbb{Z}/N\mathbb{Z}$, define the generating series*

$$\mathfrak{I}_k^{a,b}(X,Y) = \sum_{r+s=k} I_{r,s}^{a,b}(q) X^{r-1} Y^{s-1}.$$

If $k \geq 3$ then

$$\sum_{\substack{h+p=k\\ h\geq 1}} \Big(X^{h-1}Y^{p-1}\tilde{g}_h^a(q)\bar{\beta}_p^b + Y^{h-1}X^{p-1}\tilde{g}_h^b(q)\bar{\beta}_p^a\Big)$$
$$= \mathfrak{I}_k^{a,b}(X,Y) + \mathfrak{I}_k^{b,a}(Y,X) = \mathfrak{I}_k^{a+b,a}(X+Y,X) + \mathfrak{I}_k^{a+b,b}(X+Y,Y). \tag{35}$$

If $k = 2$ (i.e., $r = s = 1$), then we have

$$\tilde{g}_1^a(q)\bar{\beta}_1^b + \tilde{g}_1^b(q)\bar{\beta}_1^a = I_{1,1}^{a+b,b} + I_{1,1}^{a+b,a}. \tag{36}$$

Proof. We first see that

$$\mathfrak{I}_k^{a,b}(X,Y) = \sum_{r+s=k} I_{r,s}^{a,b}(q) X^{r-1}Y^{s-1}$$
$$= \sum_{\substack{h+p=k\\ h\geq 1}} \tilde{g}_h^a(q)\bar{\beta}_p^b X^{h-1}Y^{p-1} + \sum_{h+p=k} \bar{\beta}_p^{a-b}(p)\cdot$$
$$\left[\tilde{g}_h^a(q)\left(\sum_{r+s=k}(-1)^s\binom{p-1}{s-1}X^{r-1}Y^{s-1}\right) + \tilde{g}_h^b(q)\left(\sum_{r+s=k}(-1)^{p-r}\binom{p-1}{r-1}X^{r-1}Y^{s-1}\right)\right]$$
$$= \sum_{\substack{h+p=k\\ h\geq 1}} \tilde{g}_h^a(q)\bar{\beta}_p^b X^{h-1}Y^{p-1}$$
$$+ \sum_{h+p=k} \bar{\beta}_p^{a-b}\Big(\tilde{g}_h^b(q)Y^{h-1}(X-Y)^{p-1} - \tilde{g}_h^a(q)X^{h-1}(X-Y)^{p-1}\Big) \tag{37}$$

from binomial expansion. Now by considering even and odd p we see that the sum term of (37) has no contribution in $\mathfrak{I}_k^{a,b}(X,Y) + \mathfrak{I}_k^{b,a}(Y,X)$. The last equality of the

(35) is straight-forward so we omit its proof. Finally, (36) follows easily by direct computation. □

6. Double shuffle relations of double Eisenstein series at level N

In this section we are going to define three power series $E_{r,s}^{a,b}(q), P_{r,s}^{a,b}(q)$ and $E_k(q)$ which, together with $I_{r,s}^{a,b}(q)$ for the first one, are complementary to the double zeta values, product of the zeta values and the zeta values at level N, respectively. The latter values, essentially the constant terms of the corresponding Eisenstein series, satisfy the double shuffle relations in Proposition 2.1.

Write $q = \exp(2\pi i\tau)$ and define

$$\begin{aligned}
\tilde{\Psi}_{\mathbf{s}}^{\mathbf{a}}(q) &= \tilde{\Psi}_{\mathbf{s}}^{\mathbf{a};N}(q) = (2\pi i)^{-|\mathbf{s}|}\Psi_{\mathbf{s}}^{\mathbf{a};N}(N\tau), \quad \tilde{g}_{\mathbf{s}}^{\mathbf{a}}(q) = \tilde{g}_{\mathbf{s}}^{\mathbf{a};N}(q) = (2\pi i)^{-|\mathbf{s}|}g_{\mathbf{s}}^{\mathbf{a}}(\tau),\\
(\tilde{g}_k^a)'(q) &= (\tilde{g}_k^{a;N})'(q) = -\sum_{n=1}^{\infty} n\tilde{\Psi}_{k+1}^a(q^n), \quad \Big(f_k'(q) = \frac{q}{Nk}\frac{d}{dq}f_k(q)\Big)\\
\beta_{r,s}^{a,b}(q) &= \tilde{g}_r^a(q)\beta_s^b + \sum_{i+j=r+s}\beta_i^{a-b}\left[(-1)^s\binom{i-1}{s-1}\tilde{g}_j^a(q) + (-1)^{i-r}\binom{i-1}{r-1}\tilde{g}_j^b(q)\right],\\
\varepsilon_{r,s}^{a,b}(q) &= \delta_{r,2}(\tilde{g}_s^b)'(q) - \delta_{r,1}(\tilde{g}_{s-1}^b)'(q) + \delta_{s,1}\big((\tilde{g}_{r-1}^a)'(q) + \tilde{g}_r^a(q)\big) + N\delta_{r,1}\delta_{s,1}\gamma_N^{a,b}(q),
\end{aligned}$$

where $\gamma_N^{a,b} = \gamma_N^{a,b}(q)$ can be defined by the procedure to be outlined in Proposition 8.1. Further we set

$$f_2^a = f_2^a(q) = \frac{1}{N^2}\sum_{n,u=1}^{\infty}\eta^{au}nq^{nu} = \frac{1}{N^2}\sum_{m=1}^{\infty}\kappa_1^a(m)q^m, \text{ where } \kappa_1^a(m) = \sum_{nu=m}\eta^{au}n. \tag{38}$$

Remark 6.1. *(i). To save space, in the rest of the paper we will always suppress the dependence on q in the q-series $\gamma_N^{a,b}(q)$, $f_2^0(q)$, etc. Of course, they all depend on N.*

(ii). We will see that the definition of $\gamma_N^{a,b}$ is not unique. For example, for small levels we may define $\gamma_N^{a,b}$ explicitly as follows. For $N=1$: $\gamma_1^{0,0} = f_2^0 = \tilde{g}_2^0$. For $N=2$ we can set

$$\gamma_2^{0,0} = f_2^0 = \tilde{g}_2^0, \quad \gamma_2^{0,1} = \gamma_2^{1,0} = 0, \quad \gamma_2^{1,1} = \tilde{g}_2^1 - f_2^1. \tag{39}$$

When $N=2$ our choice of the above are different from that of [13] *(see Remark 8.4). For $N=3$ we may define*

$$\begin{aligned}
&\gamma_3^{a,a} = \tilde{g}_2^a - f_2^a \quad \text{and} \quad \gamma_3^{a,0} = f_2^0 + f_2^a - \gamma_3^{a,a} \quad \text{for } a = 1,2,\\
&\gamma_3^{0,0} = \tilde{g}_2^0, \quad \gamma_3^{0,1} = -\gamma_3^{1,0}, \quad \gamma_3^{0,2} = -\gamma_3^{2,0}, \quad \gamma_3^{1,2} = \gamma_3^{2,1} = 0.
\end{aligned} \tag{40}$$

Definition 6.2. *We define*

$$\begin{aligned}
E_k^a(q) &= E_k^{a;N}(q) = \begin{cases} \tilde{g}_k^a(q), & \text{if } k > 2;\\ 0, & \text{if } k \le 2;\end{cases}\\
E_{r,s}^{a,b}(q) &= E_{r,s}^{a,b;N}(q) = \tilde{g}_{r,s}^{a,b}(q) + \beta_{r,s}^{a,b}(q) + \frac{\varepsilon_{r,s}^{a,b}(q)}{2N}, \quad r,s \ge 1;\\
P_{r,s}^{a,b}(q) &= P_{r,s}^{a,b;N}(q) = \tilde{g}_r^a(q)\tilde{g}_s^b(q) + \beta_r^a\tilde{g}_s^b(q) + \beta_s^b\tilde{g}_r^a(q)
\end{aligned}$$

$$+\frac{\delta_{r,2}(\tilde{g}^b_s)'(q)+\delta_{s,2}(\tilde{g}^a_r)'(q)}{2N}+\delta_{r,1}\delta_{s,1}\lambda_N^{a,b}, \quad r,s\geq 1.$$

The quantities $\lambda_N^{a,b}=\lambda_N^{a,b}(q)=\lambda_N^{b,a}(q)$ *will be defined by the procedure to be outlined in Proposition* 8.1 *together with* $\gamma_N^{a,b}$*'s.*

For example, to be compatible with (39) we set

$$\lambda_2^{0,0}=\lambda_2^{1,0}=0, \quad \lambda_2^{1,1}=-f_2^1, \tag{41}$$

and to be compatible with (40) we set

$$\lambda_3^{0,0}=\lambda_3^{1,0}=\lambda_3^{2,0}=\lambda_3^{2,1}=0, \quad \lambda_3^{1,1}=-f_2^1, \quad \lambda_3^{2,2}=-f_2^2. \tag{42}$$

Roughly speaking, the double Eisenstein series $G_{r,s}^{a,b}(\tau)$ is given by the sum of $\zeta_N^{a,b}(r,s)$ and $(2\pi i)^{r+s}E_{r,s}^{a,b}(q)$ (similar for depth one Eisenstein series $G_r^a(\tau)$) while the product $G_r^a(\tau)G_s^b(\tau)$ is given by the sum of $\zeta_N^a(r)\zeta_N^b(s)$ and $(2\pi i)^{r+s}P_{r,s}^{a,b}(q)$. By Proposition 2.1, the zeta function part already satisfies the double shuffle relations. So to prove that similar relations hold for Eisenstein series it suffices to prove the next result which generalizes [13, Lemma 4].

Theorem 6.3. *Let N be a positive integer. Then there are suitable choices of* $\gamma_N^{a,b}$ *and* $\lambda_N^{a,b}$*,* $0\leq a,b<N$*, such that they provide the solution to the linear system*

$$\gamma_N^{a,a}-\lambda_N^{a,a}=\tilde{g}_2^a, \quad \gamma_N^{a,b}+\gamma_N^{b,a}-2\lambda_N^{a,b}=0 \quad \forall a\neq b\in\mathbb{Z}/N\mathbb{Z}, \tag{43}$$

together with

$$\gamma_N^{a+b,a}+\gamma_N^{a+b,b}-2\lambda_N^{a,b}=f_2^a+f_2^b \quad \forall a,b\in\mathbb{Z}/N\mathbb{Z}. \tag{44}$$

Consequently, for all $r,s\geq 1$ *the three power series* $E_{r,s}^{a,b}(q)$*,* $P_{r,s}^{a,b}(q)$ *and* $E_k^a(q)$ *satisfy the double shuffle relation at level N:*

$$P_{r,s}^{a,b}(q)=E_{r,s}^{a,b}(q)+E_{s,r}^{b,a}(q)+\delta_{a,b}E_{r+s}^a(q) \tag{45}$$

$$=\sum_{\substack{i+j=r+s\\ i,j\geq 1}}\left(\binom{i-1}{r-1}E_{i,j}^{a+b,b}(q)+\binom{i-1}{s-1}E_{i,j}^{a+b,a}(q)\right), \tag{46}$$

Proof. It suffices to show that there are suitable choices of $\gamma_N^{a,b}$ and $\lambda_N^{a,b}$ satisfying (43) and (44) such that the generating functions

$$\mathfrak{E}^a(X)=\sum_{k\geq 1}E_k^a(q)X^{k-1},$$

$$\mathfrak{E}^{a,b}(X,Y)=\sum_{r,s\geq 1}E_{r,s}^{a,b}(q)X^{r-1}Y^{s-1},$$

$$\mathfrak{P}^{a,b}(X,Y)=\sum_{r,s\geq 1}P_{r,s}^{a,b}(q)X^{r-1}Y^{s-1},$$

satisfy the double shuffle relation:

$$\mathfrak{P}^{a,b}(X,Y)=\mathfrak{E}^{a,b}(X,Y)+\mathfrak{E}^{b,a}(Y,X)+\delta_{a,b}\frac{\mathfrak{E}^a(X)-\mathfrak{E}^a(Y)}{X-Y}, \tag{47}$$

$$=\mathfrak{E}^{a+b,b}(X+Y,Y)+\mathfrak{E}^{a+b,a}(X+Y,X). \tag{48}$$

We first calculate the generating functions of the above defined power series. Set

$$\begin{aligned}
\tilde{g}^a(X) &= \sum_{k=1}^{\infty} \tilde{g}_k^a(q)X^{k-1} = -\frac{1}{N}\sum_{n=1}^{\infty}\eta^{an}e^{-\frac{nX}{N}}\frac{q^n}{1-q^n},\\
(\tilde{g}^a)'(X) &= \sum_{k=1}^{\infty}(\tilde{g}_k^a)'(q)X^{k-1} = \frac{1}{NX}\left(\sum_{n=1}^{\infty}\eta^{an}e^{-\frac{Xn}{N}}\frac{q^n}{(1-q^n)^2}-N^2f_2^a(q)\right),\\
\tilde{g}^{a,b}(X,Y) &= \sum_{r,s=1}^{\infty}\tilde{g}_{r,s}^{a,b}(q)X^{r-1}Y^{s-1} = \frac{1}{N^2}\sum_{m,n\geq 1}\eta^{am+bn}e^{\frac{-mX-nY}{N}}\frac{q^m}{1-q^m}\frac{q^{m+n}}{1-q^{m+n}},\\
\beta^{a,b}(X,Y) &= \sum_{r,s=1}^{\infty}\beta_{r,s}^{a,b}(q)X^{r-1}Y^{s-1} = (\tilde{g}^b(Y)-\tilde{g}^a(X))\beta^{a-b}(X-Y)+\tilde{g}^a(X)\beta^b(Y),\\
\varepsilon^{a,b}(X,Y) &= \sum_{r,s=1}^{\infty}\varepsilon_{r,s}^{a,b}(q)X^{r-1}Y^{s-1} = X(\tilde{g}^b)'(Y)-Y(\tilde{g}^b)'(Y)+X(\tilde{g}^a)'(X)+\tilde{g}^a(X)\\
&\qquad -(\tilde{g}_0^b)'(q)+(\tilde{g}_0^a)'(q)+N\gamma_N^{a,b}(q).
\end{aligned}$$

Remark 6.4. *(i). Notice that* $f_2^0(q)=\tilde{g}_2^0(q)$ *but in general* $f_2^a(q)\neq\tilde{g}_2^a(q)$.
(ii). Notice also that

$$(\tilde{g}_0^a)'(q) = -\sum_{n=1}^{\infty}n\tilde{\Psi}_1^a(q^n) = \frac{1}{N}\sum_{n=1}^{\infty}n\sum_{u=1}^{\infty}\eta^{au}q^{un} = Nf_2^a(q).$$

Turning back to the proof of Theorem 6.3, by the definition, we have

$$\begin{aligned}
\beta^a(X) &= \sum_{k=1}^{\infty}\beta_k^a X^{k-1} = -\frac{1}{2NX}\sum_{l=1}^{N}\sum_{n=1}^{\infty}\frac{X^n}{n!}e^{-2\pi lai/N}B_n\left(\frac{l}{N}\right)\\
&= -\frac{1}{2NX}\sum_{l=1}^{N}e^{-2\pi lai/N}\left(\frac{Xe^{Xl/N}}{e^X-1}-1\right)\\
&= -\frac{1}{2N}\sum_{l=1}^{N}\frac{e^{(X-2\pi ai)l/N}}{e^X-1}+\frac{\delta_{a,0}}{2X} = -\frac{1}{2N}\frac{1}{e^{(X-2\pi ai)/N}-1}+\frac{\delta_{a,0}}{2X}.
\end{aligned}$$

Then

$$\begin{aligned}
\mathfrak{E}^a(X) &= \tilde{g}^a(X)-X\tilde{g}_2^a(q)-\tilde{g}_1^a(q),\\
\mathfrak{E}^{a,b}(X,Y) &= \tilde{g}^{a,b}(X,Y)+\beta^{a,b}(X,Y)+\frac{1}{2N}\varepsilon^{a,b}(X,Y),\\
\mathfrak{P}^{a,b}(X,Y) &= \tilde{g}^a(X)g^b(Y)+\beta^a(X)\tilde{g}^b(Y)+\beta^b(Y)\tilde{g}^a(X)\\
&\qquad +\frac{1}{2N}(X(\tilde{g}^b)'(Y)+Y(\tilde{g}^a)'(X))+M_N\delta_{a,b}(\delta_{a,0}-1)f_2^a(q).
\end{aligned}$$

Now we compute $\mathfrak{E}^{a,b}(X,Y)+\mathfrak{E}^{b,a}(Y,X)$. Straight-forward computation yields that

$$\begin{aligned}\tilde{g}^{a,b}(X,Y)+\tilde{g}^{b,a}(Y,X)=&\,\tilde{g}^{a}(X)\tilde{g}^{b}(Y)-\frac{\tilde{g}^{a}(X)+\tilde{g}^{b}(Y)}{2N}\\&-\frac{1}{2N}\coth\left(\frac{X-Y-2\pi i(a-b)}{2N}\right)(\tilde{g}^{a}(X)-\tilde{g}^{b}(Y)),\end{aligned}\tag{49}$$

$$\begin{aligned}\varepsilon^{a,b}(X,Y)+\varepsilon^{b,a}(Y,X)=&\,X(\tilde{g}^{b})'(Y)+Y(\tilde{g}^{a})'(X)+\tilde{g}^{a}(X)+\tilde{g}^{b}(Y)\\&+N\gamma_N^{a,b}(q)+N\gamma_N^{b,a}(q).\end{aligned}\tag{50}$$

On the other hand, if we set $\theta=(X-Y-2\pi(a-b)i)/N$ then

$$\begin{aligned}\beta^{a,b}(X,Y)+\beta^{b,a}(Y,X)=&\,(\tilde{g}^{b}(Y)-\tilde{g}^{a}(X))\beta^{a-b}(X-Y)+\tilde{g}^{a}(X)\beta^{b}(Y)\\&-(\tilde{g}^{b}(Y)-\tilde{g}^{a}(X))\beta^{b-a}(Y-X)+\tilde{g}^{b}(Y)\beta^{a}(X)\\=&\frac{\tilde{g}^{b}(Y)-\tilde{g}^{a}(X)}{2N}\left(\frac{1}{e^{-\theta}-1}-\frac{1}{e^{\theta}-1}\right)\\&+\tilde{g}^{a}(X)\beta^{b}(Y)+\tilde{g}^{b}(Y)\beta^{a}(X)-\delta_{a,b}\frac{\tilde{g}^{a}(X)-\tilde{g}^{a}(Y)}{X-Y}\\=&\frac{\tilde{g}^{a}(X)-\tilde{g}^{b}(Y)}{2N}\coth\left(\frac{X-Y-2\pi i(a-b)}{2N}\right)\\&+\tilde{g}^{a}(X)\beta^{b}(Y)+\tilde{g}^{b}(Y)\beta^{a}(X)-\delta_{a,b}\frac{\tilde{g}^{a}(X)-\tilde{g}^{a}(Y)}{X-Y}.\end{aligned}\tag{51}$$

Adding up (49), $\frac{1}{2N}\times$(50) and (51) we can derive (47) quickly if the conditions in (43) are satisfied. Similarly,

$$\begin{aligned}&\tilde{g}^{a+b,a}(X+Y,X)+\tilde{g}^{a+b,b}(X+Y,Y)\\=&\frac{1}{N^2}\sum_{m\neq n\geq 1}\eta^{am+bn}e^{\frac{-mX-nY}{N}}\frac{q^m}{1-q^m}\frac{q^n}{1-q^n},\\=&\tilde{g}^{a}(X)\tilde{g}^{b}(Y)-\frac{1}{N^2}\sum_{n\geq 1}\eta^{(a+b)n}e^{\frac{-(X+Y)n}{N}}\left(\frac{q^n}{(1-q^n)^2}-\frac{q^n}{1-q^n}\right),\\=&\tilde{g}^{a}(X)\tilde{g}^{b}(Y)-\frac{X+Y}{N}(\tilde{g}^{a+b})'(X+Y)-f_2^{a+b}(q)-\frac{1}{N}\tilde{g}^{a+b}(X+Y),\end{aligned}\tag{52}$$

and

$$\begin{aligned}&\varepsilon^{a+b,a}(X+Y,X)+\varepsilon^{a+b,b}(X+Y,Y)\\=&X(\tilde{g}^{b})'(Y)+Y(\tilde{g}^{a})'(X)+2(X+Y)(\tilde{g}^{a+b})'(X+Y)+2\tilde{g}^{a+b}(X+Y)\\&+2Nf_2^{a+b}(q)-Nf_2^{a}(q)-Nf_2^{b}(q)+N\gamma_N^{a+b,a}(q)+N\gamma_N^{a+b,b}(q).\end{aligned}\tag{53}$$

Further

$$\begin{aligned}&\beta^{a+b,a}(X+Y,X)+\beta^{a+b,b}(X+Y,Y)\\=&(\tilde{g}^{b}(Y)-\tilde{g}^{a+b}(X+Y))\beta^{a}(X)+\tilde{g}^{a+b}(X+Y)\beta^{b}(Y)\end{aligned}$$

$$+(\tilde{g}^a(X)-\tilde{g}^{a+b}(X+Y))\beta^b(Y)+\tilde{g}^{a+b}(X+Y)\beta^a(X)$$
$$=\tilde{g}^b(Y)\beta^a(X)+\tilde{g}^a(X)\beta^b(Y). \tag{54}$$

Adding up (52), $\frac{1}{2N}\times$(53) and (54) we can prove (48) if the conditions in (44) are satisfied.

To complete the proof of the theorem we now need to show that the system (43) together with (44) has at least one set of solutions of $\gamma_N^{a,b}$ and $\lambda_N^{a,b}$ $(0\le a,b<N)$ in terms of $f_2^a(q)$ and $\tilde{g}_2^a(q)$ $(0\le a<N)$. Essentially as a linear algebra problem this will be solved in Proposition 8.1 in the last section of this paper. This completes the proof of the theorem. □

Let $\tilde{\zeta}^a_{N;\sharp}(r)=(2\pi i)^{-r}\zeta^a_{N;\sharp}(r)$, $\tilde{\zeta}^{a,b}_{N;\sharp}(r,s)=(2\pi i)^{-r-s}\zeta^{a,b}_{N;\sharp}(r,s)$, and define $\tilde{G}^a_{r;\sharp}(q)$ and $\tilde{G}^{a,b}_{r,s;\sharp}(q)$ similarly.

Theorem 6.5. *Let N be a positive integer. Let $\sharp=ш$ or $*$. Then for all $a,b\in\mathbb{Z}/N\mathbb{Z}$ and $r,s\ge 1$ with $(r,s)\ne 1$, we have*

$$\tilde{G}^a_{r;\sharp}(q)\tilde{G}^b_{s;\sharp}(q)=\tilde{G}^{a,b}_{r,s;*}(q)+\tilde{G}^{b,a}_{s,r;*}(q)+\delta_{a,b}\tilde{G}^a_{r+s;\sharp}(q)+\frac{\delta_{s,1}\tilde{g}^a_r(q)+\delta_{r,1}\tilde{g}^b_s(q)}{2N} \tag{55}$$
$$=\sum_{\substack{i+j=r+s\\ i,j\ge1}}\left(\binom{i-1}{r-1}\tilde{G}^{a+b,b}_{i,j;ш}(q)+\binom{i-1}{s-1}\tilde{G}^{a+b,a}_{i,j;ш}(q)\right)+f^{a,b}_{r,s}(q), \tag{56}$$

where $f^{a,b}_{r,s}(q)=\binom{k-2}{s-1}\big((\tilde{g}^{a+b}_{k-2})'(q)+\tilde{g}^{a+b}_{k-1}(q)\big)/N$ with $k=r+s$. Moreover,

$$\tilde{G}^a_{1;\sharp}(q)\tilde{G}^b_{1;\sharp}(q)=\tilde{G}^{a+b,b}_{1,1;ш}(q)+\tilde{G}^{a+b,a}_{1,1;ш}(q)+\frac{1}{2}(f_2^a+f_2^b)$$
$$+\frac{1}{2N}\Big(2(\tilde{g}^{a+b}_0)'(q)+2\tilde{g}^{a+b}_1(q)-(\tilde{g}^a_0)'(q)-(\tilde{g}^b_0)'(q)\Big). \tag{57}$$

Proof. By Corollary 5.4 and (34), we have

$$\bar{\beta}^a_s+\beta^a_s=\tilde{\zeta}^a_N(s).$$

So by the definitions,

$$\tilde{G}^a_{r;\sharp}(q)=\tilde{\zeta}^a_{N;\sharp}(r)+\tilde{g}^a_r(q), \tag{58}$$
$$\tilde{G}^{a,b}_{r,s;\sharp}(q)=\tilde{\zeta}^{a,b}_{N;\sharp}(r,s)+\tilde{g}^{a,b}_{r,s}(q)+I^{a,b}_{r,s}(q)+\beta^{a,b}_{r,s}(q). \tag{59}$$

Thus

$$\tilde{G}^a_{r;\sharp}(q)\tilde{G}^b_{s;\sharp}(q)+\frac{\delta_{r,2}(\tilde{g}^b_s)'(q)+\delta_{s,2}(\tilde{g}^a_r)'(q)}{2N}$$
$$=\tilde{\zeta}^a_{N;\sharp}(r)\tilde{\zeta}^b_{N;\sharp}(s)+P^{a,b}_{r,s}(q)+\bar{\beta}^a_r\tilde{g}^b_s(q)+\bar{\beta}^b_s\tilde{g}^a_r(q)$$
$$=\tilde{\zeta}^{a,b}_{N;*}(r,s)+E^{a,b}_{r,s}(q)+\tilde{\zeta}^{b,a}_{N;*}(s,r)+E^{b,a}_{s,r}(q)+\delta_{a,b}\Big(\tilde{\zeta}^a_N(r+s)+E^a_{r+s}(q)\Big)+I^{a,b}_{r,s}(q)+I^{b,a}_{s,r}(q)$$
$$=\sum_{\substack{i+j=r+s\\ i,j\ge1}}\left[\binom{i-1}{r-1}\Big(\tilde{\zeta}^{a+b,b}_{N;ш}(i,j)+E^{a+b,b}_{i,j}(q)+I^{a+b,b}_{i,j}(q)\Big)\right.$$

$$+\binom{i-1}{s-1}\Big(\tilde\zeta^{a+b,a}_{N;\text{III}}(i,j)+E^{a+b,a}_{i,j}(q)+I^{a+b,a}_{i,j}(q)\Big)\Big]$$

by Proposition 3.2, Proposition 5.6 and Theorem 6.3. Note that $(r,s)\neq(1,1)$ just because $r+s\geq 3$ when using Proposition 5.6. Now by the definition

$$\varepsilon^{a,b}_{r,s}(q)+\varepsilon^{b,a}_{s,r}(q)=\delta_{s,2}(\tilde g^a_r)'(q)+\delta_{r,2}(\tilde g^b_s)'(q)+\delta_{s,1}\tilde g^a_r(q)+\delta_{r,1}\tilde g^b_s(q).$$

Hence (55) follows from Definition 6.2, (58), and (59). For (56) we need to compute

$$\begin{aligned}
&\sum_{\substack{i+j=r+s\\ i,j\geq 1}}\left[\binom{i-1}{r-1}\varepsilon^{a+b,b}_{i,j}(q)+\binom{i-1}{s-1}\varepsilon^{a+b,a}_{i,j}(q)\right]\\
=&\sum_{\substack{i+j=r+s\\ i,j\geq 1}}\left[\binom{i-1}{r-1}\Big(\delta_{i,2}(\tilde g^b_j)'(q)-\delta_{i,1}(\tilde g^b_{j-1})'(q)+\delta_{j,1}\big((\tilde g^{a+b}_{i-1})'(q)+\tilde g^{a+b}_i(q)\big)\Big)\right.\\
&\qquad\left.+\binom{i-1}{s-1}\Big(\delta_{i,2}(\tilde g^a_j)'(q)-\delta_{i,1}(\tilde g^a_{j-1})'(q)+\delta_{j,1}\big((\tilde g^{a+b}_{i-1})'(q)+\tilde g^{a+b}_i(q)\big)\Big)\right]\\
=&\,\delta_{r,2}(\tilde g^b_s)'(q)+\delta_{s,2}(\tilde g^a_r)'(q)+\left[\binom{k-2}{r-1}+\binom{k-2}{s-1}\right]\Big((\tilde g^{a+b}_{k-2})'(q)+\tilde g^{a+b}_{k-1}(q)\Big)
\end{aligned}$$

where $k=r+s$. This yields (59) immediately.

Finally, (57) follows from direct computation using (36) and (44). This finishes the proof of the theorem. ∎

Remark 6.6. *When $N=1$ Theorem 6.5 reduces to [10, Theorem 7]. When $N=2$ Theorem 6.5 reduces to [13, Theorem 3] with some correction there.*

7. A key relation on multiple divisor functions at level N

In this section we prove a key result on multiple divisor functions at level N, which will be used in the next section.

Let φ be Euler's totient function. We first need a lemma concerning some special power sums of roots of unity.

Lemma 7.1. *Let $N=\prod_{t=1}^r p_t^{k_t}$ and η be a primitive N-th root of unity. For $\alpha_t\leq k_t$, $t=1,\dots,r$ (but $\boldsymbol\alpha=(\alpha_1,\dots,\alpha_r)\neq(k_1,\dots,k_r)$) we write*

$$J(\boldsymbol\alpha)=J_N(\alpha_1,\dots,\alpha_r)=\{1\leq i<N:\ p_t^{\alpha_t}\| i\quad\forall t=1,\dots,r\}.$$

Then for any choice of r-tuple of non-negative integers $(\ell_1,\dots,\ell_r)$ we have

$$\sum_{i\in J(\alpha_1,\dots,\alpha_r)}\eta^{i\prod_{t=1}^r p_t^{\ell_t}}=\begin{cases}0, & \text{if } \ell_t\leq k_t-\alpha_t-2 \text{ for some } t\leq r;\\ \prod_{t\in I}(-p_t^{\ell_t})\prod_{s\notin I}\varphi(p_s^{k_s-\alpha_s}), & \text{if } C_I \text{ holds,}\end{cases}$$

where C_I is the condition that there is $I\subseteq\{1,\dots,r\}$ such that $\ell_t=k_t-\alpha_t-1$ $\forall t\in I$ and $\ell_s\geq k_s-\alpha_s$ $\forall s\notin I$.

Proof. Suppose ξ is a primitive p^k-th root of unity for some prime p. Then for all $\alpha < k$ we have

$$\sum_{p^\alpha \| i, 1\le i<p^k} \xi^i = \sum_{p^\alpha | i, 1\le i<p^k} \xi^i - \sum_{p^{\alpha+1} | i, 1\le i<p^k} \xi^i = \begin{cases} -1, & \text{if } \alpha = k-1; \\ 0, & \text{if } \alpha < k-1, \end{cases} \tag{60}$$

since for any divisor D of p^k we have

$$\sum_{D|i, 1\le i<p^k} \xi^i = \begin{cases} -1, & \text{if } D < p^k; \\ 0, & \text{if } D = p^k. \end{cases}$$

Let $N_t = p_t^{k_t}$ for all $t = 1, \dots, r$. It is well-known that η can be decomposed as $\eta = \prod_{t=1}^r \xi_t'$ where ξ_t' is a primitive N_t-th root of unity for each t. Then $\xi_t = (\xi_t')^{\prod_{s\ne t, 1\le s\le r} p_s^{\ell_s}}$ is still a primitive N_t-th root of unity. By the Chinese Remainder Theorem it is easy to see that

$$\sum_{i\in J(\alpha_1,\dots,\alpha_r)} \eta^{i\prod_{t=1}^r p_t^{\ell_t}} = \prod_{t=1}^r \left(\sum_{i_t \in J_{N_t}(\alpha_t)} \xi_t^{i_t p_t^{\ell_t}} \right).$$

The lemma now follows from (60) and the fact that $|J_{N_t}(\alpha_t)| = \varphi(p_t^{k_t-\alpha_t})$. □

Recall that for a $a \in \mathbb{Z}/N\mathbb{Z}$ we have defined the level N divisor functions $\sigma_1^a(m) = \sum_{nu=m} \eta^{au} u$ and $\kappa_1^a(m) = \sum_{nu=m} \eta^{au} n$.

Theorem 7.2. *Let $N = \prod_{i=1}^r p_i^{k_i}$ where $p_1, \dots, p_r$ are pairwise distinct prime factors of N. Then for all $m \in \mathbb{N}$ we have*

$$\sum_{\gcd(N,i)=1, 1\le i<N} \sigma_1^i(m) - \varphi(N)\sigma_1^0(m) =$$

$$\sum_{I\subseteq\{1,\dots,r\}} \left(\prod_{t\in I} \varphi(p_t^{k_t}) \prod_{s\notin I} p_s^{k_s} \sum_{p_t | i \ \forall t\in I, p_s\nmid i \ \forall s\notin I, 1\le i<N} \kappa_1^i(m) \right). \tag{61}$$

Proof. To save space we put $[r] = \{1, \dots, r\}$. Let $e_t \ge 0$ for all $t \in [r]$ and assume $m = \prod_{t=1}^r p_t^{e_t} \prod_{i=r+1}^R p_i^{e_i}$ where $p_1, \dots, p_R$ are pairwise distinct primes. Set $Q = \prod_{i=r+1}^R (1 + p_i + \dots + p_i^{e_i})$ ($Q = 1$ if none of $p_{r+1}, \dots, p_R$ appears).

If $\ell_t \le e_t$ for all $t \in [r]$ then

$$\sum_{un=m, p_t^{\ell_t} \| u \ \forall t\in[r]} \sum_{i\in J(\alpha_1,\dots,\alpha_r)} \eta^{iu} u = Q \prod_{t=1}^r p_t^{\ell_t} \sum_{i\in J(\alpha_1,\dots,\alpha_r)} \eta^{i\prod_{t=1}^r p_t^{\ell_t}}.$$

If $\alpha_t > k_t$ for some $t \in [r]$ then $\sum_{i\in J(\alpha_1,\dots,\alpha_r)} \sigma_1^i(m) = 0$ by the definition of J. For any partition $[r] = \amalg \boldsymbol{\Lambda} = \Lambda_1 \amalg \Lambda_2 \amalg \Lambda_2$ with $\boldsymbol{\Lambda} = (\Lambda_1, \Lambda_2, \Lambda_3) \ne (\emptyset, \emptyset, [r])$ we write $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \vdash \boldsymbol{\Lambda}$ if $\alpha_t = 0$ for all $t \in \Lambda_1$, $1 \le \alpha_t < k_t$ for all $t \in \Lambda_2$ and $\alpha_t = k_t$ for all $t \in \Lambda_3$. We remove the case $\boldsymbol{\Lambda} = (\emptyset, \emptyset, [r])$ since $(\alpha_1, \dots, \alpha_r) \ne (k_1, \dots, k_r)$. For such

$\boldsymbol{\alpha}$ we have by Lemma 7.1

$$\begin{aligned}\sum_{i\in J(\alpha_1,\dots,\alpha_r)}\kappa_1^i(m)=&Q\prod_{t\notin\Lambda_3}\Big(-p_t^{e_t}+\varphi(p_t^{k_t-\alpha_t})\sum_{\ell_t=k_t-\alpha_t}^{e_t}p_t^{e_t-\ell_t}\Big)\prod_{t\in\Lambda_3}\Big(\sum_{\ell_t=0}^{e_t}p_t^{e_t-\ell_t}\Big)\\=&Q\prod_{t\notin\Lambda_3}\Big(-p_t^{k_t-\alpha_t-1}\Big)\prod_{t\in\Lambda_3}\frac{p_t^{e_t+1}-1}{p_t-1},\\\sum_{i\in J(\alpha_1,\dots,\alpha_r)}\sigma_1^i(m)=&Q\prod_{t\notin\Lambda_3}\Big(-p_t^{2(k_t-\alpha_t-1)}+\varphi(p_t^{k_t-\alpha_t})\sum_{\ell_t=k_t-\alpha_t}^{e_t}p_t^{\ell_t}\Big)\prod_{t\in\Lambda_3}\Big(\sum_{\ell_t=0}^{e_t}p_t^{\ell_t}\Big)\\=&Q\prod_{t\notin\Lambda_3}\Big(-p_t^{k_t-\alpha_t-1}(p_t^{e_t+1}-p_t^{k_t-\alpha_t}-p_t^{k_t-\alpha_t-1})\Big)\prod_{t\in\Lambda_3}\frac{p_t^{e_t+1}-1}{p_t-1},\end{aligned}$$

if $e_t\geq k_t-1$ for all $t\in[r]$. If $e_t<k_t-1$ then

$$\sum_{i\in J(\alpha_1,\dots,\alpha_r)}\kappa_1^i(m)=\sum_{i\in J(\alpha_1,\dots,\alpha_r)}\sigma_1^i(m)=0$$

when $\alpha_t<k_t-e_t-1$ for some $t\in[r]$. Therefore we have:

$$\sum_{i\in J(0,\dots,0)}\sigma_1^i(m)=\begin{cases}0, & \text{if } e_t<k_t-1 \text{ for some } \ t\in[r];\\ Q\prod_{t\notin\Lambda_3}\Big(-p_t^{k_t-1}(p_t^{e_t+1}-p_t^{k_t}-p_t^{k_t-1})\Big)\prod_{t\in\Lambda_3}\dfrac{p_t^{e_t+1}-1}{p_t-1}, & \text{otherwise.}\end{cases}$$

Now we write $\sum'_{\amalg\boldsymbol{\Lambda}=[r]}$ to mean that in the sum $\boldsymbol{\Lambda}=(\Lambda_1,\Lambda_2,\Lambda_3)$ runs through all partitions of $[r]$ into three parts except for $(\emptyset,\emptyset,[r])$. Then

$$\begin{aligned}&\sum_{I\subseteq\{1,\dots,r\}}\left(\prod_{t\in I}\varphi(p_t^{k_t})\prod_{s\notin I}p_s^{k_s}\sum_{p_t\mid i\ \forall t\in I,p_s\nmid i\ \forall s\notin I,1\leq i<N}\kappa_1^i(m)\right)\\=&\sum_{\amalg\boldsymbol{\Lambda}=[r]}{}'\sum_{\boldsymbol{\alpha}\vdash\boldsymbol{\Lambda}}Q\prod_{t\in\Lambda_1}\Big(p_t^{k_t}(-p_t^{k_t-1})\Big)\prod_{t\in\Lambda_2}\Big(\varphi(p_t^{k_t})(-p_t^{k_t-1})\Big)\prod_{t\in\Lambda_3}\varphi(p_t^{k_t})\Big(\frac{p_t^{e_t+1}-1}{p_t-1}\Big)\\=&Q\prod_{t=1}^{r}F_t-Q\prod_{t=1}^{r}\varphi(p_t^{k_t})\frac{p_t^{e_t+1}-1}{p_t-1}\\=&Q\prod_{t=1}^{r}F_t-\varphi(N)\sigma_1^0(m)\end{aligned}$$

where if $e_t<k_t-1$ then

$$F_t=-\left(\sum_{\alpha_t=k_t-e_t-1}^{k_t-1}\varphi(p_t^{k_t})p_t^{k_t-\alpha_t-1}\right)+\varphi(p_t^{k_t})\frac{p_t^{e_t+1}-1}{p_t-1}=0,$$

and if $e_t \geq k_t - 1$ then

$$\begin{aligned} F_t =& - p^{k_t} p^{k_t-1} - \left(\sum_{\alpha_t=1}^{k_t-1} \varphi(p_t^{k_t}) p_t^{k_t-\alpha_t-1} \right) + \varphi(p_t^{k_t}) \frac{p_t^{e_t+1}-1}{p_t-1} \\ =& - p_t^{k_t-1}(p_t^{e_t+1} - p_t^{k_t} - p_t^{k_t-1}). \end{aligned}$$

The theorem now follows at once. □

Corollary 7.3. *Let $N = \prod_{i=1}^r p_i^{k_i}$ where $p_1, \dots, p_r$ are pairwise distinct prime factors of N. Then we have*

$$\sum_{\gcd(N,i)=1, 1\leq i<N} \tilde{g}_2^i(q) - \varphi(N) f_2^0(q) =$$

$$\sum_{I\subseteq\{1,\dots,r\}} \left(\prod_{t\in I} \varphi(p_t^{k_t}) \prod_{s\notin I} p_s^{k_s} \sum_{p_t|i \ \forall t\in I, p_s\nmid i \ \forall s\notin I, 1\leq i<N} f_2^i(q) \right).$$

Example 7.4. *Let p be a prime and the level $N = p^k$. For any $m \in \mathbb{N}$ we have*

$$\sum_{p\nmid i, 1\leq i<N} \tilde{g}_2^i(q) - \varphi(N) f_2^0(q) = \varphi(N) \sum_{p|i, 1\leq i<N} f_2^i(q) + N \sum_{p\nmid i, 1\leq i<N} f_2^i(q).$$

Example 7.5. *Let p_1 and p_1 be two distinct primes and $N = p_1^j p_2^k$. Then we have*

$$\begin{aligned} \sum_{p_1\nmid i, p_2\nmid i, 1\leq i<N} \tilde{g}_2^i(q) - \varphi(N) f_2^0(q) =& \varphi(N) \sum_{p_1p_2|i, 1\leq i<N} f_2^i(q) \\ &+ p_1^j \varphi(p_2^k) \sum_{p_1\nmid \ell, p_2|i, 1\leq i<N} f_2^i(q) + \varphi(p_1^j) p_2^k \sum_{p_1|\ell, p_2\nmid i, 1\leq i<N} f_2^i(q) + N \sum_{p_1\nmid i, p_2\nmid i, 1\leq i<N} f_2^i(q). \end{aligned}$$

8. A linear algebra problem

In this section, using the standard techniques from linear algebra and the key result on the multiple divisor functions at level N proved in the proceeding section we will derive the solvability of a system of linear equations associated with (43) and (44) for every positive integer N. This completes the proof of our main result on the level N Eisenstein series given in Theorem 6.3.

For every positive integer N we let $\nu(N)$ be the number of its positive divisors (including 1 and N itself).

Theorem 8.1. *For every positive integer N the system (43) together with (44) has infinitely many sets of solutions of $\gamma_N^{a,b}$ and $\lambda_N^{a,b} = \lambda_N^{b,a}$ $(0 \leq a, b < N)$ in terms of $f_2^a(q)$ and $\tilde{g}_2^a(q)$ $(0 \leq a < N)$. Moreover one can always choose*

$$\{\gamma_N^{a,b} : 0 \leq b < a < N\} \cup \{\gamma_N^{N-a,N-a} : 1 \leq a \leq N, a|N\} \tag{62}$$

as the $N(N-1)/2 + \nu(N)$ free variables.

Before giving its proof, we first analyze the linear system in Proposition 8.1 using standard techniques from linear algebra. Let $\mathbf{x}_N$ be a column vector with $(3N^2+N)/2$ components whose transpose is

$$
\begin{aligned}
{}^t\mathbf{x}_N = (&\lambda_N^{0,0}, \lambda_N^{0,1}, \lambda_N^{0,2}, \dots, \lambda_N^{N-1,N-1}, \gamma_N^{0,1}, \gamma_N^{0,2}, \gamma_N^{0,3}, \dots, \gamma_N^{N-2,N-1},\\
&\gamma_N^{0,0}, \gamma_N^{1,1}, \gamma_N^{2,2}, \dots, \gamma_N^{N-1,N-1}, \gamma_N^{1,0}, \gamma_N^{2,0}, \gamma_N^{2,1}, \dots, \gamma_N^{N-1,N-2}).
\end{aligned}
$$

Here the rule to list the entries is to use lexicographic order for $\lambda_N^{a,b}$ $(0 \le a \le b < N)$, then $\gamma_N^{a,b}$ $(0 \le a < b < N)$, then $\gamma_N^{a,a}$ $(0 \le a < N)$, and finally $\gamma_N^{a,b}$ $(0 \le b < a < N)$. Then we can rewrite the system (43) together with (44) as follows:

$$
\begin{cases}
\gamma_N^{a+b,b} + \gamma_N^{a+b,a} - 2\lambda_N^{a,b} = f_2^a + f_2^b, \quad \forall 0 \le a \le b < N, & (\mathrm{LS}_1^{a,b})\\
\gamma_N^{a,b} + \gamma_N^{b,a} - \gamma_N^{a+b,a} - \gamma_N^{a+b,b} = -f_2^a - f_2^b, \quad \forall 0 \le a < b < N, & (\mathrm{LS}_2^{a,b})\\
\gamma_N^{a,a} - \gamma_N^{2a,a} = \tilde{g}_2^a - f_2^a, \quad \forall 1 \le a < N, & (\mathrm{LS}_3^{a})
\end{cases}
$$

where the last two families of the equations are obtained by taking the difference of (43) and (44). We then can express this system by a single matrix equation

$$A_N \mathbf{x}_N = \mathbf{b}_N \tag{63}$$

for some matrix A_N of size $(N^2+N-1)\times(3N^2+N)/2$ and a column vector $\mathbf{b}_N$ of length N^2+N whose entries are given in terms of f_2^a's and $\tilde{g}_2^a$'s only. Notice that since (LS_3^0) is trivial the row size is decreased from N^2+N by one. To prove the proposition one thing we need to show is that every row vector in the left null space $\mathcal{N}(A_N)$ of A_N annihilates $\mathbf{b}_N$.

Example 8.2. *When $N=1$ we get the equation*

$$\begin{bmatrix} -2 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1^{0,0} \\ \gamma_1^{0,0} \end{bmatrix} = 2f_2^0.$$

Clearly $\mathcal{N}(A_1)=\emptyset$ and we may choose $\gamma_1^{0,0}$ arbitrarily and then set $\lambda_1^{0,0}=\gamma_1^{0,0}=f_2^0$.

Example 8.3. *When $N=2$ we get the equation*

$$
A_2\mathbf{x}_2 = \begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 & 0\\ 0 & -2 & 0 & 0 & 0 & 1 & 1\\ 0 & 0 & -2 & 2 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}
\begin{bmatrix} \lambda_2^{0,0}\\ \lambda_2^{0,1}\\ \lambda_2^{1,1}\\ \gamma_2^{0,1}\\ \gamma_2^{0,0}\\ \gamma_2^{1,1}\\ \gamma_2^{1,0} \end{bmatrix}
= \begin{bmatrix} f_2^0\\ f_2^0+f_2^1\\ f_2^1\\ \tilde{g}_2^1 - f_2^1\\ -f_2^0 - f_2^1 \end{bmatrix} = \mathbf{b}_2. \tag{64}
$$

Then $\mathcal{N}(A_2)$ is spanned by the vector $\mathbf{n}_2=(0,0,0,0,1,1)$. We see that

$$\mathbf{n}_2\cdot\mathbf{b}_2 = f_2^0 + 2f_2^1 - \tilde{g}_2^1 = 0$$

which follows from Example 7.4 by taking $p=2$ and $k=1$ there. This implies that the system (64) has infinitely many solutions. Setting $\gamma_2^{a,b}=0$ for $1\ge a\ge b\ge 0$ (in fact, *one may choose them arbitrarily as they are free variables) we only need to solve the system*

$$A_2'\mathbf{x}_2 = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_2^{0,0} \\ \lambda_2^{0,1} \\ \lambda_2^{1,1} \\ \gamma_2^{0,1} \end{bmatrix} = \begin{bmatrix} f_2^0 \\ f_2^0 + f_2^1 \\ f_2^1 \\ -f_2^0 - f_2^1 \end{bmatrix} = \mathbf{b}_2'.$$

Here we obtain A_2' from A_2 by removing the penultimate row of A_2 (which is equivalent to removing the equation $\gamma_N^{1,1} - \gamma_N^{0,1} = \tilde{g}_2^1 - f_2^1$), and then removing the last three columns (which is equivalent to setting $\gamma_2^{a,b} = 0$ for $1 \ge a \ge b \ge 0$). Correspondingly, we obtain $\mathbf{b}_2'$ by removing the penultimate entry $\tilde{g}_2^1 - f_2^1$ of $\mathbf{b}_2$. Clearly, this new system has a unique solution which also gives a solution of the original system:

$$\lambda_2^{0,0} = -f_2^0, \quad \lambda_2^{0,1} = \frac{f_2^1 - \tilde{g}_2^1}{2}, \quad \lambda_2^{1,1} = -\tilde{g}_2^1, \quad \gamma_2^{0,1} = 2\lambda_2^{0,1}, \quad \gamma_2^{a,b} = 0 \quad \forall 1 \ge a \ge b \ge 0.$$

We can also check that (39) *with* (41) *provides another set of solution of* (64).

Remark 8.4. *In an email Kaneko and Tasaka pointed out to us that* [13, (19)] *should be corrected as follows:*

$$\alpha_1 = \bar{g}_0^{\mathbf{o}}(q), \quad \alpha_2 = -\alpha_1, \quad \alpha_3 = 2\bar{g}_0^{\mathbf{o}}(q) + \bar{g}_0^{\mathbf{e}}(q).$$

Together with their choice $\lambda_2^{0,1} = \lambda_2^{1,0} = \lambda_2^{1,1} = 0$ given in [13, Theorem 3] *we find the following solution to (64):*

$$\gamma_2^{0,0} = f_2^0, \quad \gamma_2^{0,1} = f_2^1, \quad \gamma_2^{1,0} = -f_2^1, \quad \gamma_2^{1,1} = \tilde{g}^1, \quad \lambda_2^{0,0} = \lambda_2^{0,1} = \lambda_2^{1,1} = 0.$$

since we have the following correspondence between their notation and ours:

$$\begin{aligned} \alpha_1 &\longleftrightarrow 2\gamma_2^{0,1}, \quad \alpha_2 \longleftrightarrow 2\gamma_2^{1,0}, \quad \alpha_3 \longleftrightarrow 2\gamma_2^{1,1}, \\ \bar{g}_0^{\mathbf{e}} &\longleftrightarrow 2f_2^0, \quad \bar{g}_0^{\mathbf{o}} \longleftrightarrow 2f_2^1, \quad g_2^1 \longleftrightarrow \tilde{g}_2^1. \end{aligned}$$

Example 8.5. *Similarly, when $N = 3$ we see that the $\mathcal{N}(A_3)$ is spanned by the vector* $\mathbf{n}_3 = (0,0,0,0,0,0,1,1,1,1,1)$ *and*

$$\mathbf{n}_3 \cdot \mathbf{b}_3 = g_2^1 + g_2^2 - 2f_2^0 - 3f_2^1 - 3f_2^2 = 0$$

which follows from Example 7.4 *by taking $p = 3$ and $k = 1$ there. Further we can obtain A_3' from the 11×15 matrix A_3 by removing the row of A_3 corresponding the equations $\gamma_N^{2a,a} - \gamma_N^{a,a} = f_2^a - \tilde{g}_2^a$ for $a = 2$ and then removing the 10th and the last 4 columns (which is equivalent to setting $\gamma_2^{a,b} = 0$ for all $2 \ge a \ge b \ge 0$ with $(a,b) \ne (1,1)$). In this way we find the following solution:*

$$\begin{aligned} &\lambda_3^{0,0} = -f_2^0, \quad \lambda_3^{0,1} = \frac{\tilde{g}_2^1 - f_2^0 - 2f_2^1}{2}, \quad \lambda_3^{1,1} = -f_2^1, \quad \lambda_3^{1,2} = \frac{\tilde{g}_2^2}{2}, \\ &\lambda_3^{0,2} = -\frac{f_2^0 + f_2^2}{2}, \quad \lambda_3^{2,2} = 2\lambda_3^{1,2}, \quad \gamma_3^{0,1} = \tilde{g}_2^1 - f_2^0 - 2f_2^1, \quad \gamma_3^{0,2} = -f_2^0 - f_2^2, \\ &\gamma_3^{1,2} = f_2^2 - \tilde{g}_2^2, \quad \gamma_3^{1,1} = \tilde{g}_2^1 - f_2^1, \quad \gamma_3^{a,b} = 0 \ \forall a \ge b, (a,b) \ne (1,1). \end{aligned}$$

Of course, this solution is not unique. For instance, we checked that (40) *with* (42) *provides another set of solution of the system* $A_3\mathbf{x}_3 = \mathbf{b}_3$.

For other levels $N \le 80$ we carried out similar computations by Maple and verified that $\mathcal{N}(A_N)$ always annihilates $\mathbf{b}_N$ using Corollary 7.3. To prove the general case, we need two results concerning dimensions.

Proposition 8.6. *The dimension of the left null space of* $\mathcal{N}(A_N)$ *satisfies*

$$\dim \mathcal{N}(A_N) \ge \nu(N) - 1.$$

Moreover, for every vector $\mathbf{n} \in \mathcal{N}(A_N)$ *we have* $\mathbf{n} \cdot \mathbf{b}_N = 0$.

Proof. Throughout this proof we will drop the subscript N. For each divisor d of N, if $d < N$ we can obtain a vector $\mathbf{n}_N(d) \in \mathcal{N}(A_N)$ by using the following combination of the families of equations in ($\mathrm{LS}_2^{a,b}$) and (LS_3^{a}):

adding ($\mathrm{LS}_2^{a,b}$) with $\gcd(b,N) = d$ and $a = 0$,
adding ($\mathrm{LS}_2^{a,b}$) with $\gcd(a,b,N) = d$ for all $1 \le a < b < N$, and
adding (LS_3^{a}) with $\gcd(a,N) = d$.

This gives rise to the vector $\mathbf{n}_N(d)$ whose entries are either 0 or 1 and whose leading 1 occurs at the position corresponding to the variable $\gamma^{0,d}$. We will show that $\mathbf{n}_N(d) \in \mathcal{N}(A_N)$ and $\mathbf{n}_N(d) \cdot \mathbf{b}_N = 0$ which is equivalent to the fact that LHS $= 0$ and RHS $= 0$, respectively, where

$$\mathrm{LHS} = \sum_{\substack{\gcd(a,N)=d\\ 1\le a<N}} (\gamma^{0,a} - \gamma^{2a,a}) + \sum_{\substack{\gcd(a,b,N)=d\\ 1\le a<b<N}} (\gamma^{a,b} + \gamma^{b,a} - \gamma^{a+b,a} - \gamma^{a+b,b}) \tag{65}$$

and

$$\mathrm{RHS} = \sum_{\substack{\gcd(a,N)=d\\ 1\le a<N}} (\tilde{g}_2^a - f_2^a) - \sum_{\substack{\gcd(a,N)=d\\ 1\le a<N}} (f_2^0 + f_2^a) - \sum_{\substack{\gcd(a,b,N)=d\\ 1\le a<b<N}} (f_2^a + f_2^b).$$

Also notice that the vectors in $\{\mathbf{n}_N(d) : d|N, d < N\}$ are linearly independent since the leading 1 appearing in $\mathbf{n}_N(d)$ corresponds to $\gamma^{0,d}$. This implies that $\dim \mathcal{N}(A_N) \ge \nu(N) - 1$.

By considering (N/d)-th roots of unity (i.e. reducing to level N/d) we may assume without loss of generality that $d = 1$ and we simply write $\mathbf{n}$ for $\mathbf{n}_N(1)$. To show LHS $= 0$ we break the two sums in (65) into the following parts:

(A_a): $\gamma^{0,a}$ from 1st sum,	(B_a): $-\gamma^{2a,a}$, 1st sum,
$(C_1^{a,b})$: $\gamma^{a,b} = \gamma^{N+a,b}$, 2nd sum,	$(C_2^{a,b})$: $\gamma^{b,a}$, 2nd sum,
$(D_1^{a,b})$: $-\gamma^{a+b,a}$, 2nd sum,	$(D_2^{a,b})$: $-\gamma^{a+b,b}$, 2nd sum,
(E_1^{a}): $-\gamma^{0,a}$, 2nd sum where $b = N - a$	(so $a < N/2$),
(E_2^{b}): $-\gamma^{0,b}$, 2nd sum where $a = N - b$	(so $b > N/2$).

Then we obtain the cancelations as follows:

$$\sum_{\gcd(a,N)=1} A_a + \sum_{\gcd(a,N)=1, a<N/2} E_1^a + \sum_{\gcd(b,N)=1, b>N/2} E_2^b = 0$$

$$\sum_{\gcd(a,N)=1} A_a + \sum_{\gcd(a,N)=1, a<N/2} E_1^a + \sum_{\gcd(b,N)=1, b>N/2} E_2^b = 0$$

$$\sum_{\substack{\gcd(a,b,N)=1\\ a<b}} (C_1^{a,b} + C_2^{a,b}) = \sum_{\substack{\gcd(a,b,N)=1\\ 2N>a>b>0}} \gamma^{a,b} = CC_1 + CC_2 + CC_3$$

where

$$CC_1 = \sum_{\substack{\gcd(a,b,N)=1\\ 2N>a=2b>0}} \gamma^{a,b} = -\sum_{\gcd(a,N)=1} B_a,$$

$$CC_2 = \sum_{\substack{\gcd(a,b,N)=1\\ 2N>a>2b>0}} \gamma^{a,b} = \sum_{\substack{\gcd(a,b,N)=1\\ 2N>a>2b>0}} \gamma^{(a-b)+b,b} = -\sum_{\substack{\gcd(a,b,N)=d\\ a<b}} D_1^{a,b}$$

$$CC_3 = \sum_{\substack{\gcd(a,b,N)=1\\ 2N>2b>a>b>0}} \gamma^{a,b} = \sum_{\substack{\gcd(a,b,N)=1\\ 2N>a>2b>0}} \gamma^{(a-b)+b,b} = -\sum_{\substack{\gcd(a,b,N)=d\\ a<b}} D_2^{a,b}.$$

These implies LHS $= 0$ which shows that $\mathbf{n} \in \mathcal{N}(A_N)$.

Now we turn to RHS. Since $N = 1$ case is trivial we now assume $N = \prod_{i=1}^r p_i^{k_i} > 1$ where $p_1, \dots, p_r$ are pairwise distinct prime factors of N. Recall that

$$\text{RHS} = \sum_{\substack{\gcd(a,N)=d\\ 1\le a<N}} (\tilde{g}_2^a - f_2^a) - \sum_{\substack{\gcd(a,N)=d\\ 1\le a<N}} (f_2^0 + f_2^a) - \sum_{\substack{\gcd(a,b,N)=d\\ 1\le a<b<N}} (f_2^a + f_2^b). \tag{66}$$

We want to show that the above expression is exactly equal to the difference of the two sides in Corollary 7.3, which is therefore 0. Clearly the coefficients of $\tilde{g}_2^a$ (=1) and f_2^0 ($=\varphi(N)$) are correct.

Let $\gcd(c, N) = 1$. Now we count how many times f_2^c can appear. Notice that if $a \ne c$ then we have $\gcd(a, c, N) = 1$ and either $a < c$ or $a > c$. Thus the last sum in (66) contributes $N - 1$ copies of f_2^a. Combining this with the one copy from the sum $\sum_{\gcd(a,N)=1}(f_2^0 + f_2^a)$ we see that the coefficient of f_2^a is exactly N.

Now we consider f_2^c with $\gcd(c, N) > 1$. Without loss of generality we assume that there is $1 \le t < r$ such that $p_i | c$ for all $i \le t$ and $p_j \nmid c$ for all $t < j \le r$. Then only the last sum in (66) has nontrivial contributions. In fact, for $1 \le a < N$ we see that $p_i \nmid a$ $(i = 1, \dots, t)$ if and only if $\gcd(a, c, N) = 1$. But obviously the number of such a is given by

$$\prod_{i=1}^t \varphi(p_i^{k_i}) \prod_{j=t+1}^r p_j^{k_j}$$

which agrees with Corollary 7.3. This implies that RHS $= 0$ which shows that $\mathbf{n} \cdot \mathbf{b}_N = 0$.

We have completed the proof of the lemma. $\square$

Proposition 8.7. *The rank of matrix A_N satisfies*

$$\text{rank}(A_N) \ge N^2 + N - \nu(N).$$

Moreover, one may choose the free variables as in (62) *when solving the linear system* (43) + (44) *(or, equivalently, the linear system* $(\mathrm{LS}_1^{a,b}) + (\mathrm{LS}_2^{a,b}) + (\mathrm{LS}_3^{a})$*).*

Proof. We will drop the subscript N again for λ's and γ's. We will prove the lemma by producing the following pivot variables in $\mathbf{x}_N$:

$$\mathcal{S} = \{\lambda^{a,b} : 0 \le a \le b < N\} \cup \{\gamma^{a,b} : 0 \le a < b < N\} \cup \{\gamma^{a,a} : 1 \le a < N, (N-a) \nmid N\}. \tag{67}$$

Easy computation shows that the $|\mathcal{S}| = N^2 + N - \nu(N)$ which yields the lemma immediately.

To streamline our proof we start with some ad hoc terminology. Suppose we have a linear system of variables $x_1, \dots, x_r$ (in this particular order). Then an equation produced from this system by the elementary operations (namely, multiplying an equation by a scalar and adding or subtracting two equations) is called a *pivotal equation* of variable x_i if x_i appears in the equation while none of $x_1, \dots, x_{i-1}$ does. In our situation, our variables λ's and γ's are ordered as in the vector $\mathbf{x}_N$. And clearly $(\mathrm{LS}_1^{a,b})$ provides the pivotal equations of λ's.

We now turn to γ's. We shall produce their pivotal equations by the following steps. We write $\gamma^{a,b} = \cdots$ to mean that the right hand side does not involve any variables from $\mathcal{S}$. In particular, we may omit all $\gamma^{a,b}$ with $a > b$. So by $(\mathrm{LS}_2^{a,b})$ we get the pivotal equation

$$\gamma^{a,b} = \cdots \quad \text{for } 1 \le a < b < N \text{ and } a + b < p. \tag{68}$$

By (LS_3^{a}) we have

$$\gamma^{a,a} = \cdots \quad \text{for } 1 \le a < N/2. \tag{69}$$

To derive pivotal equation $\gamma^{a,a} = \cdots$ for $N/2 < a < N$ with $(N-a) \nmid N$ (thus $a \le N-2$) we first notice that for such a there must be some positive integer $k \le N-2$ such that

$$0 \le \frac{(k-1)N}{k} < a < \frac{kN}{k+1} \le \frac{(N-2)N}{N-1} < N - 1. \tag{70}$$

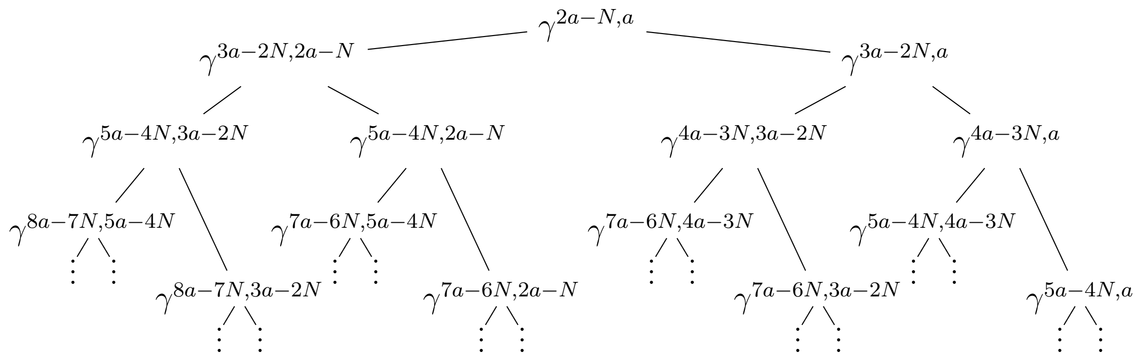
Here a is bounded with strict inequality because if $a = (k-1)N/k$ for some $k > 1$ then $N - a = N/k$ is a divisor of N which is impossible by our assumption. We say such an a satisfying (70) has height $h(a) = k$. If $h(a) = 1$ then we are in the case of (69). If $h(a) = 2$ then $3a < 2N$, i.e. $(2a - N) + a < N$, so using (LS_3^{a}) we have

$$\gamma^{a,a} = \gamma^{2a-N,a} + \cdots = \cdots$$

by (68) since $2a - N < a$. If $h(a) = 3$ then by applying (LS_3^{a}) followed by $(\mathrm{LS}_2^{2a-N,a})$ we get

$$\gamma^{a,a} = \gamma^{2a-N,a} + \cdots = \gamma^{3a-2N,2a-N} + \gamma^{3a-2N,a} + \cdots = \cdots$$

by (68) again since now $4a < 3N$ (and hence $5a < 4N$). Repeating this process for a at higher levels we obtain the following binary tree



In general, this tree is constructed by the following rules: if a node $\gamma^{i,j}$ exists and $i+j>N$ (we call $i+j$ the weight of $\gamma^{i,j}$) then it produces two descendants: $\gamma^{i+j-N,i}$ and $\gamma^{i+j-N,j}$; if $i+j<N$ then it does not have any descendant and therefore becomes a terminal node. It is an easy matter by the induction to show the following properties of this tree:

- Every descendant has smaller weight than its parent. So the tree is finite.
- Every node has the form $\gamma^{ma-(m-1)N,na-(n-1)N}$ for some integers $m>n\geq 1$.
- The weight of every node $\gamma^{ma-(m-1)N,na-(n-1)N}$ satisfies $(m+n)a-(m+n-2)N\neq N$. Otherwise $N-a=N/(m+n)$ is a divisor of N which is impossible.
- Every node $\gamma^{ma-(m-1)N,na-(n-1)N}$ satisfies $ma-(m-1)N\neq 0$. Otherwise $N-a=N/m$ is a divisor of N which is impossible.
- Every node $\gamma^{ma-(m-1)N,na-(n-1)N}$ satisfies $ma-(m-1)N<na-(n-1)N$.

Hence, every terminal node $\gamma^{i,j}$ of the tree satisfies $1\leq i<j$ and $i+j<N$ so it can be canceled by using (68).

To summarize the above, we have produced the pivotal equations for the following

(i) $\gamma^{a,b}=\cdots$ for $0\leq a<b<N$ and $a+b<N$.
(ii) $\gamma^{a,a}=\cdots$ for $1\leq a<N$ with $(N-a)\nmid N$.

Then we may proceed as follows:

(iii) $\gamma^{0,a}=\gamma^{a,a}+\cdots=\cdots$ for $1\leq a<N$ by (LS_3^a) and using (ii).
(iv) $\gamma^{a,b}=\gamma^{0,a}+\gamma^{0,b}+\cdots=\cdots$ for $1\leq a<b<N$ and $a+b=N$ by $(\mathrm{LS}_2^{a,b})$ and (iii).
(v) $\gamma^{a,b}=\gamma^{a+b-N,a}+\gamma^{a+b-N,b}+\cdots=\cdots$ for $0\leq a<b<N$ and $a+b>N$ by $(\mathrm{LS}_2^{a,b})$ and by using the induction on the weight $a+b$ since the weights on the right are strictly smaller than $a+b$.

We now have finished the proof of the lemma. $\square$

Finally, Proposition 8.1 follows from Lemma 8.6 and Lemma 8.7 immediately since
$$\operatorname{rank}(A_N)+\dim\mathcal{N}(A_N)=N^2+N-1.$$

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