



Valentin Blomer · Djordje Milićević

Kloosterman sums in residue classes

Received July 19, 2012 and in revised form May 20, 2013

Abstract. We prove upper bounds for sums of Kloosterman sums against general arithmetic weight functions. In particular, we obtain power cancellation in sums of Kloosterman sums over arithmetic progressions, which is of square-root strength in any fixed primitive congruence class up to bounds towards the Ramanujan conjecture.

Keywords. Kloosterman sums, Kuznetsov formula, arithmetic progressions, Linnik’s conjecture

1. Introduction

The distribution of values of complete exponential sums is of central interest in number theory and arithmetic geometry. In particular, many arithmetic problems can be transformed into bounding sums of Kloosterman sums. While Weil’s bound gives the best possible estimate for individual Kloosterman sums $S(m, n, c)$, one can often use the Bruggeman–Kuznetsov formula to obtain additional savings from the sum over the modulus c . Starting with the ground-breaking work of Deshouillers–Iwaniec [4], this has been a recurring theme in analytic number theory—see e.g. [3, 5, 8] for some spectacular examples. Using the Bruggeman–Kuznetsov formula for the congruence subgroup $\Gamma_0(q)$, one can require additional divisibility conditions on the modulus c . It is a routine exercise to obtain non-trivial bounds for sums of the type

$$\sum_{c \equiv 0 \pmod{q}} \frac{S(m, n, c)}{c^{1/2}} f_\infty\left(\frac{c}{X}\right), \quad (1.1)$$

where $f_\infty : (0, \infty) \rightarrow \mathbb{C}$ is an appropriate fixed weight function, m, n are positive integers, and X is a large parameter. It is much less of a routine exercise to obtain results of the same quality when the congruence condition $c \equiv 0 \pmod{q}$ is replaced with $c \equiv a$

V. Blomer: Mathematisches Institut, Bunsenstr. 3-5, D-37073 Göttingen, Germany;
e-mail: blomer@uni-math.gwdg.de

D. Milićević: Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany;
current address: Department of Mathematics, Bryn Mawr College, 101 North Merion Avenue, Bryn Mawr, PA 19010, U.S.A.; e-mail: dmilicevic@brynmawr.edu

Mathematics Subject Classification (2010): Primary 11L05, 11F72; Secondary 11F30, 11K36, 11L07

(mod q) for some $(a, q) = 1$. The difficulty here lies in the fact that there is no obvious subgroup of $\mathrm{SL}_2(\mathbb{Z})$ where the set of lower left entries of its elements is the set of $c \equiv a \pmod{q}$, and it is also not obvious how to use different cusps of $\Gamma_0(q)$ to encode the congruence condition. It is therefore not clear if spectral theory can provide any non-trivial information even for a fixed progression such as $c \equiv 2 \pmod{5}$. In this article we show that nevertheless the problem can be solved in full generality in the framework of certain congruence subgroups $\Gamma_1(Q)$ (more precisely, for some divisors Q of q^2).

In fact, we will consider the following more general setup. Let q be an arbitrary positive integer. For a function $f : (\mathbb{Z}/q\mathbb{Z})^* \rightarrow \mathbb{C}$ we denote by \widehat{f} its q -Mellin transform

$$\widehat{f}(\chi) = \frac{1}{\phi(q)^{1/2}} \sum_{c(q)}^* \bar{\chi}(c) f(c)$$

where χ is a Dirichlet character modulo q . It satisfies the inversion formula

$$f(c) = \frac{1}{\phi(q)^{1/2}} \sum_{\chi(q)} \widehat{f}(\chi) \chi(c). \quad (1.2)$$

We write $\|\widehat{f}\|_1 := \sum_{\chi(q)} |\widehat{f}(\chi)|$, and we lift f to a function on integers $c \in \mathbb{Z}$ with $(c, q) = 1$ in the obvious way. We think of f as an *arithmetic weight function* against which we want to sum Kloosterman sums. The original problem of Linnik [20] is concerned with the magnitude of a variant of (1.1) with $q = 1$ and f_∞ a sharp cutoff function (see [18], [21]). Weighting contributions of various $S(m, n, c)$ by an arithmetic weight $f(c)$ is the natural adelic counterpart of this question, with various moduli c entering with weights according to their position relative to various p -adic neighborhoods in addition to the archimedean ones. See [7] for a very interesting discussion of arithmetic weights in a different context.

Our principal result is the following theorem.

Theorem 1. *Let m, n, q be positive integers and $X \geq 1$. Let $f : (\mathbb{Z}/q\mathbb{Z})^* \rightarrow \mathbb{C}$, and let $f_\infty : (0, \infty) \rightarrow \mathbb{C}$ be a smooth, compactly supported function. Then uniformly in $mn \leq X^2$ one has*

$$\sum_{(c, q)=1} \frac{S(m, n, c)}{c^{1/2}} f(c) f_\infty\left(\frac{c}{X}\right) \ll_{f_\infty, \varepsilon} X^{1/2+2\theta} \|\widehat{f}\|_1 (mnq)^\varepsilon \quad (1.3)$$

for any $\varepsilon > 0$ and any admissible exponent θ for the Generalized Ramanujan Conjecture for the places dividing mn and the archimedean place.

The large range $mn \leq X^2$ of uniformity, also known as the ‘‘Linnik range’’ [21], is natural and will become apparent in (4.2) below. One can also treat the complementary range, but then the analysis of the relevant integral transforms changes. We remark that, if one assumes the Selberg eigenvalue conjecture, then the factor $X^{2\theta}$ in Theorem 1 can be replaced with $(mn)^\theta$. Currently, the best available result toward the Generalized Ramanujan Conjecture is $\theta = 7/64$, due to Kim and Sarnak [17].

The norm $\|\widehat{f}\|_1$ that appears in Theorem 1 satisfies a general (and generally sharp) estimate $\|\widehat{f}\|_1 \leq \phi(q)^{1/2} \|f\|_2$ by the Cauchy–Schwarz inequality. A particularly interesting class of arithmetic weights f (for q prime) comes from algebraic geometry (e.g. as Frobenius trace functions of perverse l -adic sheaves [7]); see [16] for bounds on various norms of the corresponding \widehat{f} in terms of the “conductor” of the associated sheaf.

In the proof of Theorem 1 we develop a generalized version of the Kuznetsov formula which encodes an additional arithmetic weight and furnishes an exact spectral decomposition of the left-hand side of (1.3). We state this spectral formula in Section 6 as Theorem 4 after we have developed the necessary notation. Several important ingredients in the proof of Theorem 1 may be noteworthy, including the encoding of arbitrary arithmetic weights using twisted Kloosterman sums in Section 2, a more general treatment of newforms along the lines of [15] in Section 3, and a completely explicit computation of the Fourier coefficients (also at ramified places) of Eisenstein series of general central character in Section 5.

As a particular application of Theorem 1, we can choose f to be the characteristic function of the arithmetic progression $c \equiv a \pmod{q}$, where $(a, q) = 1$, and obtain the following variant of (1.1), a version of Linnik’s conjecture [20] in arithmetic progressions:

Corollary 2. *Under the same assumptions as in Theorem 1 one has*

$$\sum_{c \equiv a \pmod{q}} \frac{S(m, n, c)}{c^{1/2}} f_\infty\left(\frac{c}{X}\right) \ll_{f_\infty, \varepsilon} X^{1/2+2\theta} q^{1/2} (mnq)^\varepsilon.$$

Using Weil’s bound $|S(m, n, c)| \leq \tau(c)(m, n, c)^{1/2} c^{1/2}$ [14, Corollary 11.12] individually, one obtains an upper bound $\ll_{m, n} X^\varepsilon (1 + X/q)$ in the situation of Corollary 2. We see that, for fixed m, n , Corollary 2 is non-trivial for $q \ll X^{(1-4\theta)/3-\varepsilon}$, the quality of cancellation obtained being uniform across all primitive classes $a \pmod{q}$.

The dependence on f_∞ in Theorem 1 is completely explicit in the proof (see (4.9)–(4.11) below), but it is somewhat convoluted, so that we decided not to display it in the main theorem. Suffice it to say that Theorem 1 is a very close non-archimedean analogue of the classical situation where f is trivial, but f_∞ is oscillating, and our method puts the archimedean and non-archimedean aspects on an equal footing. In particular the appearance of norms of \widehat{f} on the right-hand side of (1.3) is rather natural as a comparison with (4.9) shows.

As an example of this parallelism in Theorem 1, if we take f_∞ fixed and $f(c) = e(ac/q)$ for some $(a, q) = 1$ and q prime, then $\|\widehat{f}\|_1 = q + O(1)$. On the other hand, if we take $q = 1$, but a similarly oscillating weight function $f_\infty(x) = w(x)e(qx)$ for some fixed, smooth, compactly supported weight function w and some real number $q > 1$, then a stationary phase computation shows that the (usual) Mellin transform of f_∞ satisfies $\widehat{f}_\infty(c + it) \ll_c q^{-1/2} (1 + |t|/q)^{-10}$, and hence (4.9) gives the bound $X^{1/2+2\theta} q (mnq)^\varepsilon$.

An equally robust analogy can be observed in Corollary 2, in which the Kloosterman sums $S(m, n, c)$ are summed over moduli c in a non-archimedean ball *away from 0*. The

archimedean analogue of Corollary 2 is the situation $q = 1$, but with a smooth test function f_∞ with support in a short interval $[1, 1 + q^{-1}]$. Then f_∞ satisfies $\widehat{f}_\infty(c + it) \ll_c q^{-1}(1 + |t|/q^2)^{-10}$, and (4.9) returns the bound $X^{1/2+2\theta} q^{1/2} (mnq)^\varepsilon$, in complete analogy with Corollary 2. This conclusion should be contrasted with a sum over the progression $c \equiv 0 \pmod{q}$, whose analogue would be the substantially easier case of the support of f_∞ in $[0, q^{-1}]$ (or, equivalently, having X/q in place of X). The (perhaps at first counterintuitive) phenomenon that a thinner or shorter sequence gives rise to a harder estimation problem away from the zero class is not uncommon; compare e.g. an application of Voronoi summation to obtain bounds for $\sum_{n \equiv a \pmod{q}} \lambda_f(n) w(n/X)$ for a fixed cusp form f and $q > X^{1/2}$.

The deeper reason for the analogy between the archimedean and non-archimedean aspect is that the spectral decomposition of the sum in (1.3) using the Bruggeman–Kuznetsov formula gives rise to a spectral sum of Maaß forms of a comparable number of terms in both cases: in slightly simplified terms, we obtain in the non-archimedean case a sum with bounded spectral parameter for the group $\Gamma_1(q)$ (of covolume $\asymp q^2$ in Weyl’s law), while in the archimedean case we obtain a sum with spectral parameter of size up to q (containing $\asymp q^2$ terms by Weyl’s law) for $\mathrm{SL}_2(\mathbb{Z})$. In other words, we are expanding in *different directions* of the full automorphic spectrum of $\mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. In both cases we estimate the spectral sum trivially and therefore obtain results of comparable quality.

We note that, as a direct consequence (see [22]) of Corollary 2, we obtain the following equidistribution result for the Dedekind sums $s(d, c)$. For a real number x , let $\langle x \rangle$ denote the fractional part of x .

Corollary 3. *Let q be a natural number, and let a be an integer coprime to q . Then the set*

$$\{(12 \cdot s(d, c)) : d \pmod{c}, c \leq x, c \equiv a \pmod{q}\}$$

becomes equidistributed in $[0, 1)$ as $x \rightarrow \infty$.

Many variations of the present approach are possible. Depending on the application, one can include additional divisibility conditions on c in (1.3) and thereby relax the condition $(a, q) = 1$ in Corollary 2, one can take m and n to be of opposite sign (using the “opposite sign” Kuznetsov formula), or, perhaps most interestingly, one can sum over m and n and prove large sieve type inequalities as in [4]. An investigation of (4.9) and in particular the dependence on f_∞ in (1.3) makes it also possible to replace the smooth summation condition by a sharp cutoff condition $c \leq X$. Finally, the dependence on θ in Theorem 1 can be improved slightly in certain ranges of m, n, q, X using density results for exceptional eigenvalues as in [14, (16.61)]; see also [14, (16.75)]. We leave these and other extensions to the interested reader.

For the rest of paper, implicit constants may depend on ε (whose numerical value may change from line to line), but all other dependencies are explicitly specified.

2. Encoding arithmetic weights

Let χ_1 be a primitive Dirichlet character modulo q_1 , let $m, \delta \in \mathbb{N}$ be positive integers satisfying $(m\delta, q_1) = 1$, and let $h : \mathbb{N} \rightarrow \mathbb{C}$ be a function such that $|h(c)| \ll c^{-3/2-\eta}$ for

some $\eta > 0$. For every c with $q_1 \mid c$, let

$$S_{\chi_1}(m, n, c) = \sum_{d(c)}^* \chi_1(d) e\left(\frac{md + n\bar{d}}{c}\right)$$

be the twisted Kloosterman sum, where $*$ denotes summation restricted to primitive residue classes. By Möbius inversion, we have

$$\sum_{(c, q_1)=1} S_{\chi_1}(m, nq_1^2, q_1\delta c) h(\delta c) = \sum_{d|q_1} \mu(d) \sum_{dq_1|c} S_{\chi_1}(m, nq_1^2, \delta c) h(\delta c/q_1). \quad (2.1)$$

On the other hand, if c is such that $(q_1, \delta c) = 1$, by twisted multiplicativity of Kloosterman sums (see [14, (1.59)]; our case is obtained *mutatis mutandis*) we have

$$S_{\chi_1}(m, nq_1^2, q_1\delta c) = S(m\bar{q}_1, nq_1, \delta c) S_{\chi_1}(m\bar{\delta c}, 0, q_1) = S(m, n, \delta c) \bar{\chi}_1(m) \chi_1(\delta c) \tau(\chi_1)$$

where $\tau(\chi_1)$ is the Gauß sum, and \bar{x} in one of the first two arguments of a Kloosterman sum denotes the multiplicative inverse of x to the respective modulus. Substituting into (2.1), we obtain

$$\sum_c \chi_1(\delta c) S(m, n, \delta c) h(\delta c) = \frac{\chi_1(m)}{\tau(\chi_1)} \sum_{d|q_1} \mu(d) \sum_{dq_1|c} S_{\chi_1}(m, nq_1^2, \delta c) h(\delta c/q_1).$$

Now, let χ be an arbitrary Dirichlet character modulo q , induced by a primitive character χ_1 modulo $q_1 \mid q$, let $(m, q_1) = 1$, and let h be as above. Then

$$\begin{aligned} \sum_{(c, q)=1} \chi(c) S(m, n, c) h(c) &= \sum_{\substack{\delta|q \\ (\delta, q_1)=1}} \mu(\delta) \sum_c \chi_1(\delta c) S(m, n, \delta c) h(\delta c) \\ &= \sum_{\substack{\delta|q \\ (\delta, q_1)=1}} \mu(\delta) \frac{\chi_1(m)}{\tau(\chi_1)} \sum_{d|q_1} \mu(d) \sum_{\delta dq_1|c} S_{\chi_1}(m, nq_1^2, c) h(c/q_1). \end{aligned}$$

We can collapse the double sum over d and δ to a single sum getting an equality

$$\sum_{(c, q)=1} \chi(c) S(m, n, c) h(c) = \frac{\chi_1(m)}{\tau(\chi_1)} \sum_{d|q} \mu(d) \sum_{dq_1|c} S_{\chi_1}(m, nq_1^2, c) h(c/q_1), \quad (2.2)$$

valid for every m with $(m, q_1) = 1$ (and so a fortiori whenever $(m, q) = 1$).

For general m , we put $m' = m/(m, q^\infty)$ and $n' = n(m, q^\infty)$. Since $S(m, n, c) = S(m', n', c)$ for $(c, q) = 1$, (2.2) holds without the condition $(m, q) = 1$ if we replace m and n by m' and n' on the right-hand side; note that $\chi(m') = \chi_1(m')$. We thus obtain the more general equality

$$\sum_{(c, q)=1} \chi(c) S(m, n, c) h(c) = \frac{\chi(m')}{\tau(\chi_1)} \sum_{d|q} \mu(d) \sum_{dq_1|c} S_{\chi_1}(m', n'q_1^2, c) h(c/q_1), \quad (2.3)$$

valid for any positive integers m, n, c and any Dirichlet character χ modulo q . Here, note that $mn = m'n'$.

Equality (2.3) is at the heart of our argument. We think of the left-hand side of this formula as an average of Kloosterman sums in the usual sense (as a smooth sum over c) but additionally weighted with a special arithmetic weight, namely an arbitrary multiplicative character modulo c . Our equality expresses such a twisted average in terms of twisted Kloosterman sums, which we *can* analyze using spectral theory of automorphic forms for $\Gamma_0(dq_1)$ and character χ_1 , i.e. a character of the quotient $\Gamma_1(dq_1)\backslash\Gamma_0(dq_1)$. In this sense, as remarked in the introduction, we solve the problem of bounding Kloosterman sums with arithmetic weights modulo q (with arbitrary weights in the next paragraph) in the framework of suitable congruence subgroups $\Gamma_1(dq_1)$, where $d, q_1 \mid q$. Recall that the additional factor d was inherited simply from Möbius inversion; it will turn out to be of little relevance in the forthcoming asymptotic analysis.

Finally, we encode an arbitrary arithmetic weight $f : (\mathbb{Z}/q\mathbb{Z})^* \rightarrow \mathbb{C}$ and a long-range archimedean cutoff. With f and f_∞ as in the statement of Theorem 1, we use (2.3) with $h(c) = f_\infty(c/X)/c^{1/2}$, multiply by $\widehat{f}(\chi)$, sum over all Dirichlet characters χ modulo q , and use the inversion formula (1.2). Thus we obtain the basic identity

$$\begin{aligned} & \sum_{(c,q)=1} \frac{S(m, n, c)}{c^{1/2}} f(c) f_\infty\left(\frac{c}{X}\right) \\ &= \frac{1}{\phi(q)^{1/2}} \sum_{\chi(q)} \frac{\chi(m') \widehat{f}(\chi)}{\tau(\chi_1)} \sum_{d \mid q} \mu(d) \sum_{dq_1 \mid c} \frac{q_1^{1/2}}{c^{1/2}} S_{\chi_1}(m', n'q_1^2, c) f_\infty\left(\frac{c}{q_1 X}\right). \end{aligned} \quad (2.4)$$

Our identity relates the sum of Kloosterman sums against both a finite and an archimedean test function (essentially an almost arbitrary smooth, compactly supported function on $\mathbb{Q}^\times \setminus \mathbb{A}_0^\times$) to sums which may be treated by the Kuznetsov trace formula for the group $\Gamma_0(dq_1)$. Here we can also see the underlying motivation for the sums appearing in (2.1).

This prepares ground for our principal application, Theorem 1. In the following sections, we will prove that

$$\begin{aligned} \Sigma_{\chi_1}(m, n, d, q, X) &:= \sum_{dq_1 \mid c} \frac{1}{c^{1/2}} S_{\chi_1}(m', n'q_1^2, c) f_\infty\left(\frac{c}{q_1 X}\right) \\ &\ll_{f_\infty} q_1^{1/2} X^{1/2+2\theta} (mnq)^\varepsilon, \end{aligned} \quad (2.5)$$

uniformly in $mn \leq X^2$ and across all $d, q_1 \mid q$ and all primitive characters χ_1 modulo q_1 . Taking this for granted, Theorem 1 follows from trivial estimates. We note that, if one is not interested in uniformity with respect to m, n, q , then [9, Theorem 1] shows directly that

$$\Sigma_{\chi_1}(m, n, d, q, X) \ll_{m,n,q,f_\infty} X^{1/2+2\theta+\varepsilon}.$$

Remark 1 in [9] gives some explicit polynomial dependence on m, n, q , but considerably weaker than required for (2.5). If χ_1 is trivial, one can also read off uniform bounds for (2.5) from [14, (16.72), (16.75)]. Modulo the Ramanujan–Petersson conjecture, these

bounds have an extra factor $(n(m, q^\infty)q_1)^{1/4}$ compared to (2.5). In the following section we will make systematic use of newform theory in order to optimize the dependence on n, m and q_1 . Our general treatment of newforms and Eisenstein series and corresponding bounds, in particular (3.8) and Lemma 1 below, as well as Theorem 4, may be of independent interest.

3. Fourier coefficients of automorphic forms

In this section, which can be read independently of Section 2, we collect facts and conventions about (holomorphic and Maaß) cusp forms and Eisenstein series and present estimates for their Fourier coefficients which will be used in the estimation of Σ_{χ_1} in Section 4.

Let χ_1 be a primitive character modulo q_1 , let $\varkappa = 0$ if χ_1 is even and $\varkappa = 1$ if χ_1 is odd, and let d be a square-free integer. (In our application to (2.5), q_1 and d will have the same meaning as in the rest of the paper.) For a positive integer $k \geq 2$ satisfying $k \equiv \varkappa \pmod{2}$, let $\mathcal{A}_k(dq_1, \chi_1) = S_k(\Gamma_0(dq_1), \chi_1)$ denote the finite-dimensional space of holomorphic weight k cusp forms of level dq_1 and character χ_1 . Here and on, for any r with $q_1 \mid r \mid dq_1$, χ_1 in the notation for a space such as $\mathcal{A}_k(r, \chi_1)$ (or a basis such as \mathcal{B}_k below) stands, strictly speaking, for the induced character $\chi_1\chi_0$, where χ_0 is the principal character modulo r . For simplicity, we suppress χ_0 from notation (but not from computation) as no confusion will arise. Let $\mathcal{A}_\varkappa(dq_1, \chi_1) = L^2_{\text{cusp}}(\Gamma_0(dq_1)\backslash G, \chi_1, \varkappa)$ denote the space of non-holomorphic weight \varkappa cusp forms on $G = \text{SL}_2(\mathbb{R})$ of level dq_1 and character χ_1 , and, for every $t \in (\mathbb{R} \cup [-i/2, i/2])/\{\pm 1\}$, let $\mathcal{A}_\varkappa(dq_1, \chi_1, t) \subset \mathcal{A}_\varkappa(dq_1, \chi_1)$ denote the finite-dimensional space of forms in $\mathcal{A}_\varkappa(dq_1, \chi_1)$ of Laplacian eigenvalue $1/4 + t^2$.

We normalize Hecke operators for both holomorphic and non-holomorphic cusp forms so that the Ramanujan–Petersson conjecture states that the eigenvalues $\lambda_f(p)$ of T_p are bounded by 2 in absolute value (i.e. [6, (6.1)] for Maaß forms and [15, (2.15)] for holomorphic forms of weight k).

With respect to the standard inner product

$$\langle f, g \rangle = \int_{\Gamma_0(dq_1)\backslash \mathfrak{h}} f(z)\bar{g}(z)y^\ell \frac{dx dy}{y^2} \tag{3.1}$$

of level dq_1 , where $\ell = k$ in the space $\mathcal{A}_k(dq_1, \chi_1)$ and $\ell = 0$ in each of the spaces $\mathcal{A}_\varkappa(dq_1, \chi_1)$, we construct specific orthonormal bases $\mathcal{B}_k(dq_1, \chi_1)$ and $\mathcal{B}_\varkappa(dq_1, \chi_1)$ of these spaces as follows. For every r satisfying $q_1 \mid r \mid dq_1$, let $\mathcal{A}_k^*(r, \chi_1)$ denote the space of holomorphic forms orthogonal to all oldforms of level r and character χ_1 . By Atkin–Lehner theory (in particular the multiplicity one principle), we can choose an orthonormal basis $\mathcal{B}_k^*(r, \chi_1)$ of $\mathcal{A}_k^*(r, \chi_1)$ consisting of *newforms*, i.e. eigenforms for *all* Hecke operators T_n ($n \geq 1$) with eigenvalue $\lambda_f(n)$, say (see [14, Section 14.7]). For $f \in \mathcal{B}_k^*(r, \chi_1)$, let $\mathcal{A}_{dq_1/r}(f)$ be the space spanned by the set of shifts $\mathcal{S}_{dq_1/r}(f) := \{f(bz) : b \mid dq_1/r\}$. Then we have (again by multiplicity one) an orthogonal decomposition

$$\mathcal{A}_k(r, \chi_1) = \bigoplus_{q_1 \mid r \mid dq_1} \bigoplus_{f \in \mathcal{B}_k^*(r, \chi_1)} \mathcal{A}_{dq_1/r}(f).$$

One can obtain an orthonormal basis of $\mathcal{A}_{dq_1/r}(f)$ by orthonormalizing the set $\mathcal{S}_{dq_1/r}(f)$; below, we will specify an explicit orthonormal basis $\mathcal{S}_{dq_1/r}^\square(f) = \{f_{(b)}(z) : b \mid dq_1/r\}$. We obtain the requisite orthonormal basis $\mathcal{B}_k(dq_1, \chi_1)$ of the entire space $\mathcal{A}_k(dq_1, \chi_1)$ by taking the union of all these sets:

$$\mathcal{B}_k(dq_1, \chi_1) := \bigsqcup_{q_1 \mid r \mid dq_1} \bigsqcup_{f \in \mathcal{B}_k^*(r, \chi_1)} \mathcal{S}_{dq_1/r}^\square(f).$$

We construct an orthonormal basis $\mathcal{B}_\varkappa(dq_1, \chi_1, t)$ of the space $\mathcal{A}_\varkappa(dq_1, \chi_1, t)$ analogously and write

$$\mathcal{B}_\varkappa(dq_1, \chi_1) = \bigsqcup_t \mathcal{B}_\varkappa(dq_1, \chi_1, t),$$

the union being taken over the spectral resolution of $\mathcal{A}_\varkappa(dq_1, \chi_1)$.

We write the Fourier expansion of a modular form f as

$$\begin{aligned} f(z) &= \sum_{n \geq 1} \rho_f(n) n^{k/2} e(nz) && \text{for } f \in \mathcal{A}_k(dq_1, \chi_1), \\ f(z) &= \sum_{n \neq 0} \rho_f(n) W_{\frac{n}{|n|}, \frac{\varkappa}{2}, it_f}(4\pi|n|y) e(nx) && \text{for } f \in \mathcal{A}_\varkappa(dq_1, \chi_1, t_f), \end{aligned} \quad (3.2)$$

where $W_{\alpha, \beta}$ is the Whittaker function. Let f be a newform of level r and character χ_1 . The Fourier coefficients are related to the Hecke eigenvalues as

$$\sqrt{n} \rho_f(n) = \rho_f(1) \lambda_f(n) \quad (3.3)$$

for all $n \in \mathbb{N}$. Moreover, if f is L^2 -normalized with respect to (3.1), then we have the following bounds that are essentially due to Hoffstein and Lockhart (upper bound) and Iwaniec (lower bound):

$$|\rho_f(1)|^2 = \begin{cases} \frac{\cosh(\pi t_f)}{dq_1(1 + |t_f|)^\varkappa} (dq_1(1 + |t_f|))^{o(1)}, & f \in \mathcal{A}_\varkappa^*(r, \chi_1, t_f), \\ \frac{(4\pi)^{k-1}}{dq_1 \Gamma(k)} (kdq_1)^{o(1)}, & f \in \mathcal{A}_k^*(r, \chi_1) \end{cases} \quad (3.4)$$

(see [11, (30)–(31)], [12]). (Compare the slightly different normalization in [11, Section 2.2].) We recall the Hecke relations

$$\begin{aligned} \lambda_f(p^{\alpha+1}) &= \lambda_f(p) \lambda_f(p^\alpha) - \chi_1(p) \chi_0(p) \lambda_f(p^{\alpha-1}) && \text{for all primes } p \text{ and integers } \alpha \geq 1, \\ \chi_1(p) \bar{\lambda}_f(p) &= \lambda_f(p) && \text{for primes } p \nmid r \end{aligned} \quad (3.5)$$

(see e.g. [14, p. 371] or [6, p. 520]), as well as the bounds

$$|\lambda_f(p)| \leq \begin{cases} p^\theta + p^{-\theta}, & p \nmid r, \\ 1, & p \mid r; \end{cases} \quad (3.6)$$

cf. [19, Theorem 1.1 ii) and iii)] (as well as our normalization (3.2) and (3.3)) for the latter bound, which holds verbatim for Maaß forms.

For our principal estimation in the next section, we want to be completely explicit in our construction of the basis $\mathcal{S}_{dq_1/r}^\square(f)$ and compute the Fourier coefficients of the various $f_{(b)}$. For notational simplicity we consider only the holomorphic case; the Maaß case is identical upon replacing k with 0 throughout the argument. We follow closely the argument in [15, Section 2] (see also [1, Section 3]), which requires only minor modification. Let $f \in \mathcal{B}_k^*(r, \chi_1)$ and recall that dq_1/r (and thus each of its divisors b) is squarefree. Define the arithmetic function

$$v(b) := b \prod_{p|b} \left(1 + \frac{\chi_0(p)}{p}\right)$$

where χ_0 is the principal character modulo r , and write $f|_b(z) := b^{k/2} f(bz)$. Starting from the expression

$$\langle E(\cdot, s) f(b_1 \cdot), f(b_2 \cdot) \rangle = \int_{\Gamma_0(dq_1) \backslash \mathfrak{h}} E(z, s) f(b_1 z) \bar{f}(b_2 z) y^k \frac{dx dy}{y^2}$$

where $E(z, s)$ is the standard weight 0 non-holomorphic Eisenstein series of level dq_1 , unfolding using (3.5) to explicitly evaluate multiplicatively shifted convolution L -series as a product of Euler factors

$$\sum_{\alpha=0}^{\infty} \lambda_f(p^{\alpha+1}) \bar{\lambda}_f(p^\alpha) p^{-\alpha s} = \lambda_f(p) (1 + \chi_0(p) p^{-s})^{-1} \sum_{\alpha=0}^{\infty} |\lambda_f(p^\alpha)|^2 p^{-\alpha s},$$

and evaluating residues at $s = 1$, we find as in [15, Lemma 2.4] that

$$\langle f|_{b_1}, f|_{b_2} \rangle = \frac{\bar{\lambda}_f(b') \lambda_f(b'')}{v(b') v(b'')} (b' b'')^{1/2} \langle f, f \rangle, \quad b' = \frac{b_1}{(b_1, b_2)}, \quad b'' = \frac{b_2}{(b_1, b_2)}.$$

The Maaß case is identical except that slightly different special functions occur in the unfolding step (see [6, Section 19]). With this at hand, proceeding as in [15, p. 75], we find that the forms

$$f_{(b)}(z) = \left\{ b \prod_{p|b} \left(1 - \frac{p |\lambda_f(p)|^2}{(p + \chi_0(p))^2}\right)^{-1} \right\}^{1/2} \sum_{c\ell=b} \frac{\mu(c) \bar{\lambda}_f(c)}{v(c)} \ell^{(k-1)/2} f(\ell z) \quad (3.7)$$

form an orthonormal basis of $\mathcal{A}_{dq_1/r}(f)$; this is the basis $\mathcal{S}_{dq_1/r}^\square(f)$ of our choice. (Taking inverses in the product above is justified by (3.6) with any $\theta < 1/2$.) Hence for every $f \in \mathcal{B}_k^*(r, \chi_1)$ or $f \in \mathcal{B}_{\neq}^*(r, \chi_1, t)$, every $n \in \mathbb{N}$ and every $b | dq_1/r$, and with $f_{(b)}$ as in (3.7), we have

$$\begin{aligned} \sqrt{n} \rho_{f_{(b)}}(n) &= \left\{ b \prod_{p|b} \left(1 - \frac{p |\lambda_f(p)|^2}{(p + \chi_0(p))^2}\right)^{-1} \right\}^{1/2} \sum_{c\ell=b} \frac{\mu(c) \bar{\lambda}_f(c)}{v(c)} \rho_f(1) \lambda_f\left(\frac{n}{\ell}\right) \\ &\ll b^\varepsilon \sum_{\ell|b} \frac{\ell |\lambda_f(b/\ell)|}{b^{1/2}} \left| \rho_f(1) \lambda_f\left(\frac{n}{\ell}\right) \right|, \end{aligned} \quad (3.8)$$

with the convention that $\lambda_f(x) = 0$ for $x \notin \mathbb{N}$. Here, we have only used so far that (3.6) holds with any $\theta < 1/2$, and $|\rho_f(1)|$ can be further estimated by (3.4).

We proceed to a discussion of the Eisenstein spectrum, which is parametrized by singular cusps \mathfrak{a} . Write $Q = dq_1$. Recall that a cusp \mathfrak{a} for a group Γ is called *singular* with respect to a multiplier system ϑ on Γ if $\vartheta(\gamma) = 1$ for all γ in the stabilizer $\Gamma_{\mathfrak{a}} \subset \Gamma$. For a cusp \mathfrak{a} for the group $\Gamma = \Gamma_0(Q)$, denoting by $\sigma_{\mathfrak{a}}$ a *scaling matrix* (that is, a matrix such that $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \{\pm\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} : k \in \mathbb{Z}\}$), \mathfrak{a} is a singular cusp for a character χ modulo a divisor of Q if¹

$$\chi\left(\sigma_{\mathfrak{a}}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\sigma_{\mathfrak{a}}^{-1}\right) = 1.$$

As usual, we interpret χ as a character on $\Gamma_0(Q)$ via $\chi(\gamma) = \chi(d) = \bar{\chi}(a)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

For a singular cusp \mathfrak{a} , we consider the weight \varkappa Eisenstein series

$$\begin{aligned} E_{\mathfrak{a},\chi_1}(z, s) &= \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \bar{\chi}_1(\gamma) j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(z)^{-\varkappa} \Im(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s \\ &= \sum_{\tau \in \Gamma_{\infty} \backslash \sigma_{\mathfrak{a}}^{-1}\Gamma} \bar{\chi}_1(\sigma_{\mathfrak{a}}\tau) j_{\tau}(z)^{-\varkappa} \Im(\tau z)^s, \end{aligned} \quad (3.9)$$

where $j_{\tau}(z) = (\tilde{c}z + \tilde{d})/(|\tilde{c}z + \tilde{d}|)$ for $\tau = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$ is the usual multiplier, and $\tau = \sigma_{\mathfrak{a}}^{-1}\gamma$. This series converges absolutely for $\Re s > 1$, and it has a meromorphic extension to all of \mathbb{C} . Eisenstein series have a Fourier expansion similar to Maaß forms, which at the point $1/2 + it$ is given by

$$E_{\mathfrak{a},\chi_1}(z, 1/2 + it) = C_{\mathfrak{a},\chi_1,t}(z) + \sum_{n \neq 0} \rho_{\mathfrak{a},\chi_1}(n, t) W_{\frac{n}{|n|}, \frac{\varkappa}{2}, it}(4\pi|n|y)e(nx). \quad (3.10)$$

In Section 5, we specify a full list of inequivalent singular cusps \mathfrak{a} in (5.1), explicitly compute the Fourier coefficients $\rho_{\mathfrak{a},\chi_1}(n, t)$ of the Eisenstein series $E_{\mathfrak{a},\chi_1}(z, 1/2 + it)$ in (5.3), and prove the following uniform upper bound, which will be used in our principal estimation in the next section.

Lemma 1. *Let m, n, Q be positive integers, let χ_1 be a character modulo Q , let t be a real number, and let $\varkappa \in \{0, 1\}$. Let \tilde{Q} be the smallest positive integer such that $Q \mid \tilde{Q}^2$. Then, for every cusp \mathfrak{a} of $\Gamma = \Gamma_0(Q)$ singular for the character χ_1 , the Fourier coefficients $\rho_{\mathfrak{a},\chi_1}(n, t)$ in the expansion (3.10) of the weight \varkappa Eisenstein series $E_{\mathfrak{a},\chi_1}(z, s)$ defined by (3.9) satisfy*

$$\begin{aligned} \mathcal{E}(m, n, t) &:= \sum_{\mathfrak{a} \text{ singular}} \frac{\sqrt{mn}}{\cosh(\pi t)} \overline{\rho_{\mathfrak{a},\chi_1}(m, t)} \rho_{\mathfrak{a},\chi_1}(n, t) \\ &\ll \frac{1}{(1 + |t|)^{\varkappa}} \frac{(m, \tilde{Q})^{1/2} (n, \tilde{Q})^{1/2}}{\tilde{Q}} (Qmn(1 + |t|))^{\varepsilon}. \end{aligned}$$

¹ See [13, p. 44]; compare with [6, (4.43)]. It is tempting to think of the two elements $\pm\sigma_{\mathfrak{a}}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\sigma_{\mathfrak{a}}^{-1}$ as playing the same role, but this is not quite so, due to the presence of the factor $j_{\tau}(z)^{-\varkappa}$ in the multiplier.

We note that the second fraction does not exceed 1 and that $\tilde{Q} \geq Q^{1/2}$. We postpone this more detailed discussion of the Eisenstein series and the proof of Lemma 1 to the end of the paper. We will not concern ourselves with the constant term $C_{\alpha, \chi_1, t}(z)$ since it does not enter our estimates.

4. Application of the Kuznetsov formula

In this section, we use the Kuznetsov trace formula and the estimates from Section 3, including (3.8) and Lemma 1, to prove our claim (2.5). With notation from Section 2, set

$$g(x) := \left(\frac{4\pi \sqrt{m'n'q_1^2}}{x} \right)^{1/2} f_\infty \left(\frac{4\pi \sqrt{m'n'}}{xX} \right). \quad (4.1)$$

Then $g(x)$ is a smooth function compactly supported on an interval of x satisfying

$$x \asymp \Xi := \sqrt{mn} X^{-1} \leq 1 \quad (4.2)$$

and such that $\|g\|_\infty \asymp (q_1 X)^{1/2} \|f\|_\infty$, and we have

$$c^{1/2} f_\infty \left(\frac{c}{q_1 X} \right) = g \left(\frac{4\pi \sqrt{m'n'q_1^2}}{c} \right).$$

Substituting into the left-hand side of (2.5), we have

$$\Sigma_{\chi_1}(m, n, d, q, X) = \sum_{dq_1|c} \frac{1}{c} S_{\chi_1}(m', n'q_1^2, c) g \left(\frac{4\pi \sqrt{m'n'q_1^2}}{c} \right).$$

The sum on the right-hand side can now be readily transformed with the Kuznetsov formula for the group $\Gamma_0(dq_1)$ and character χ_1 , which we quote from Blomer–Harcos–Michel [2]. We use the usual weight 0 or the weight 1 formula according as χ_1 is even or odd. We define the following integral transforms:

$$\begin{aligned} \dot{g}(k) &= i^k \int_0^\infty J_{k-1}(x) g(x) \frac{dx}{x}, \quad k \in \mathbb{N}, \\ \tilde{g}(t) &= \frac{it^\alpha}{2 \sinh(\pi t)} \int_0^\infty (J_{2it}(x) - (-1)^\alpha J_{-2it}(x)) g(x) \frac{dx}{x}, \quad t \in \mathbb{R} \cup [-i/2, i/2]. \end{aligned}$$

The power series expansion of the Bessel functions [10, 8.440] together with (4.2) yields

$$\dot{g}(k) \ll \|g\|_\infty \Gamma(k)^{-1} \quad (4.3)$$

and

$$\tilde{g}(t) \ll \begin{cases} |t|^\alpha \left(\frac{|\widehat{g}(2it)| + |\widehat{g}(-2it)|}{|t|^{1/2}} + \frac{|\widehat{g}(2+2it)| + |\widehat{g}(2-2it)|}{|t|^{3/2}} + \frac{\|g\|_\infty}{|t|^{5/2}} \right), & |t| \geq 1, \\ \|g\|_\infty \Xi^{-2\theta}, & t \in (-1, 1) \cup [-i\theta, i\theta]. \end{cases} \quad (4.4)$$

Here \widehat{g} denotes the Mellin transform of g , and by (4.1) we have

$$\widehat{g}(s) = (q_1 X)^{1/2} \left(\frac{4\pi \sqrt{mn}}{X} \right)^s \widehat{f}_\infty(1/2 - s). \quad (4.5)$$

The following spectral summation formula holds [2, p. 705]:

$$\sum_{dq_1|c} \frac{1}{c} S_{\chi_1}(m', n' q_1^2, c) g \left(\frac{4\pi \sqrt{m' n' q_1^2}}{c} \right) = \mathcal{H} + \mathcal{M} + \mathcal{E}, \quad (4.6)$$

where

$$\begin{aligned} \mathcal{H} &= \sum_{\substack{k \equiv \varkappa(2), k > \varkappa \\ f \in \mathcal{B}_k(dq_1, \chi_1)}} \dot{g}(k) \frac{(k-1)! \sqrt{m' n' q_1^2}}{\pi (4\pi)^{k-1}} \overline{\rho_f(m')} \rho_f(n' q_1^2), \\ \mathcal{M} &= \sum_{f \in \mathcal{B}_\varkappa(dq_1, \chi_1)} \tilde{g}(t_f) \frac{4\pi \sqrt{m' n' q_1^2}}{\cosh(\pi t_f)} \overline{\rho_f(m')} \rho_f(n' q_1^2), \\ \mathcal{E} &= \sum_{\alpha \text{ singular}} \int_{-\infty}^{\infty} \tilde{g}(t) \frac{\sqrt{m' n' q_1^2}}{\cosh(\pi t)} \overline{\rho_{\alpha, \chi_1}(m', t)} \rho_{\alpha, \chi_1}(n' q_1^2, t) dt. \end{aligned}$$

In comparison with [2], note that we are using the classical parametrization of the Eisenstein spectrum in terms of singular cusps. The formula (4.6) is purely spectral in that $\mathcal{B}_k(dq_1, \chi_1)$ and $\mathcal{B}_\varkappa(dq_1, \chi_1)$ can be arbitrary orthonormal bases of the spaces $\mathcal{A}_k(dq_1, \chi_1)$ and $\mathcal{A}_\varkappa(dq_1, \chi_1)$, respectively. However, we will from now on assume that these bases have been chosen as in Section 3, which will allow us to efficiently estimate the terms on the right-hand side.

We start by bounding \mathcal{M} , the contribution of the Maaß spectrum. Each summand in \mathcal{M} corresponds to a basis vector of the form $f_{(b)}$ for some newform $f \in \mathcal{B}_\varkappa^*(r, \chi_1)$, with $q_1 | r | dq_1$ and $b | dq_1 / r$.

Since $(m', q) = (m', dq_1) = 1$, (3.6) and (3.8) imply

$$\sqrt{m'} \rho_{f_{(b)}}(m') \ll b^{-1/2+\varepsilon} |\rho_f(1) \lambda_f(m'b)| \ll (m')^{\theta+\varepsilon} b^{\theta-1/2+\varepsilon} |\rho_f(1)|.$$

On the other hand, writing $b_0 = (b, r^\infty)$, $b_1 = b/b_0$, $n'_0 = (n' q_1^2, r^\infty)$, and $n'_1 = n' q_1^2 / n'_0$, we have

$$\begin{aligned} \sqrt{n' q_1^2} \rho_{f_{(b)}}(n' q_1^2) &\ll b^{-1/2+\varepsilon} |\rho_f(1)| \sum_{\ell_0 | (b_0, n'_0)} \sum_{\ell_1 | (b_1, n'_1)} \ell_0 \ell_1 \left(\frac{b_1}{\ell_1} \right)^{\theta+\varepsilon} \left(\frac{n'_1}{\ell_1} \right)^{\theta+\varepsilon} \\ &\ll (n'_1)^{\theta+\varepsilon} b^{-1/2+\varepsilon} b_0 b_1^{1-\theta} |\rho_f(1)| \ll (n')^{\theta+\varepsilon} b^{1/2+\varepsilon} |\rho_f(1)|. \end{aligned}$$

Combining these estimates with (3.4), we obtain the following bound for any individual term occurring in the sum for \mathcal{M} :

$$\begin{aligned} \frac{4\pi\sqrt{m'n'q_1^2}}{\cosh(\pi t_f)} \frac{\rho_{f(b)}(m') \rho_{f(b)}(n'q_1^2)}{\rho_{f(b)}(m')} &\ll \frac{(mnb)^\theta}{dq_1(1+|t_f|)^{\varkappa-\varepsilon}} (qmn)^\varepsilon \\ &\ll \frac{(mn)^\theta}{r(1+|t_f|)^{\varkappa-\varepsilon}} (qmn)^\varepsilon. \end{aligned} \quad (4.7)$$

(Note that we can afford to let go of a factor of $b^{1-\theta}$.) Next we use the well-known fact that

$$\#\{f \in \mathcal{B}_{\varkappa}^*(r, \chi_1) : |t_f| \leq T\} \ll (rT^2)^{1+\varepsilon} \quad (4.8)$$

for any r with $q_1 | r$. This weak but very uniform version of Weyl's law follows for instance by combining (3.4) with [14, (16.56)] in the special case $n = 1$.

Collecting (4.2), (4.4), (4.5), (4.7) and (4.8), denoting by $\tau(\cdot)$ the divisor function, and using summation by parts, we conclude that

$$\begin{aligned} \mathcal{M} &\ll q_1^{1/2} X^{1/2+2\theta} (mnq)^\varepsilon \sum_{q_1 | r | dq_1} \frac{\tau(dq_1/r)}{r} \left[\#\{f \in \mathcal{B}_{\varkappa}^*(r, \chi_1) : |t_f| < 1\} \cdot \|f_\infty\|_\infty \right. \\ &\quad \left. + \sum_{\substack{f \in \mathcal{B}_{\varkappa}^*(r, \chi_1) \\ |t_f| \geq 1}} \left(\frac{|\widehat{f}_\infty(1/2 \pm 2it_f)|}{|t_f|^{1/2-\varepsilon}} + \frac{|\widehat{f}_\infty(-3/2 \pm 2it_f)|}{|t_f|^{3/2-\varepsilon}} + \frac{\|f_\infty\|_\infty}{|t_f|^{5/2-\varepsilon}} \right) \right] \\ &\ll_{f_\infty} q_1^{1/2} X^{1/2+2\theta} (mnq)^\varepsilon, \end{aligned} \quad (4.9)$$

as required for (2.5).

For the Eisenstein spectrum we replace (4.7) with the bound given in Lemma 1 for $Q = dq_1$. Recalling that $(m', dq_1) = 1$, we obtain a slightly stronger estimate

$$\begin{aligned} \mathcal{E} &= \int_{-\infty}^{\infty} \widehat{g}(t) \mathcal{E}(m', n'q_1^2, t) dt \\ &\ll \frac{q_1^{1/2} X^{1/2}}{(dq_1)^{1/4}} (mnq)^\varepsilon \left(\|f_\infty\|_\infty + \int_{|t| \geq 1} \frac{|\widehat{f}_\infty(1/2 - 2it)|}{|t|^{1/2-\varepsilon}} dt \right) \\ &\ll_{f_\infty} \frac{q_1^{1/2} X^{1/2}}{(dq_1)^{1/4}} (mnq)^\varepsilon. \end{aligned} \quad (4.10)$$

The bound for the holomorphic spectrum \mathcal{H} is along the same lines with (4.3) instead of (4.4), giving

$$\mathcal{H} \ll \|f_\infty\|_\infty \cdot q_1^{1/2} X^{1/2} (mnq)^\varepsilon. \quad (4.11)$$

This completes the proof of Theorem 1.

5. Proof of Lemma 1

It remains to prove Lemma 1. To this end we compute explicitly the Fourier coefficients $\sqrt{n} \rho_{\mathfrak{a}, \chi_1}(n, t)$. This is in principle straightforward, but a bit tedious.

We follow [4], which treats the case when χ_1 is trivial. We first describe a set of (inequivalent) *singular* cusps. Given a divisor $w \mid Q$ and a primitive residue class r modulo $w_Q := (w, Q/w)$, we can always find a residue class $u \pmod{w}$ such that $u \equiv r \pmod{w_Q}$ and $(u, w) = 1$ (since the coprimality condition $(u, p) = 1$ is automatic from $u \equiv r \pmod{w_Q}$ for primes $p \mid w_Q$ and can be imposed by the Chinese Remainder Theorem by requiring, for example, that $u \equiv 1 \pmod{p}$ for $p \nmid w_Q$). Let \mathcal{U}_w be a full set of representatives u of primitive residue classes modulo w_Q chosen so that $(u, w) = 1$ for every $u \in \mathcal{U}_w$. Then a full set of inequivalent cusps for the group $\Gamma = \Gamma_0(Q)$ is given by all fractions $\mathfrak{a} = u/w$, where $w \mid Q$ and $u \in \mathcal{U}_w$ (see [4, Lemma 2.3]). A possible scaling matrix for a cusp \mathfrak{a} is given by (see [4, p. 247])

$$\sigma_{\mathfrak{a}} = \begin{pmatrix} u\sqrt{Q/(w^2, Q)} & 0 \\ w\sqrt{Q/(w^2, Q)} & \frac{1}{u\sqrt{Q/(w^2, Q)}} \end{pmatrix}.$$

We compute that, for every $k \in \mathbb{Z}$,

$$\sigma_{\mathfrak{a}} \begin{pmatrix} 1 & k \\ & 1 \end{pmatrix} \sigma_{\mathfrak{a}}^{-1} = \begin{pmatrix} 1 - kuwQ/(w^2, Q) & ku^2Q/(w^2, Q) \\ -kw^2Q/(w^2, Q) & 1 + kuwQ/(w^2, Q) \end{pmatrix}.$$

Hence the singular cusps $\mathfrak{a} = u/w$ for the character χ_1 of conductor q_1 are given by the condition

$$(w, Q/w) \Big| \frac{Q}{q_1} \Leftrightarrow q_1 \mid [w, Q/w]. \quad (5.1)$$

This description of the set of equivalence classes of singular cusps can also be found in [13, Lemma 13.5].

Recall the definition (3.9) of the Eisenstein series at a singular cusp $\mathfrak{a} = u/w$ as above. A convenient parametrization of the sum on the right-hand side of (3.9) is as follows. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we have

$$\begin{aligned} \sigma_{\mathfrak{a}}^{-1} \gamma &= \sigma_{\mathfrak{a}}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{a}{u\sqrt{Q/(w^2, Q)}} & \frac{b}{u\sqrt{Q/(w^2, Q)}} \\ \frac{(uc - wa)\sqrt{Q/(w^2, Q)}}{(ud - wb)\sqrt{Q/(w^2, Q)}} & \frac{(ud - wb)\sqrt{Q/(w^2, Q)}}{(ud - wb)\sqrt{Q/(w^2, Q)}} \end{pmatrix} \\ &=: \begin{pmatrix} * & * \\ -Cw\sqrt{Q/(w^2, Q)} & D\sqrt{Q/(w^2, Q)} \end{pmatrix}, \end{aligned}$$

where $Cw = -uc + wa$ and $D = ud - wb$ satisfy $\gamma^{-1} \cdot u/w = D/Cw$. In particular, as γ runs through $\Gamma_{\mathfrak{a}} \backslash \Gamma$, the point D/Cw precisely traverses the orbit of the cusp u/w in Γ . By [4, Lemma 3.6] the set of pairs (Cw, D) is characterized by

$$(Cw, D) = 1, \quad (C, Q/w) = 1, \quad CD \equiv u \pmod{(w, Q/w)}.$$

Pairs of integers (C, D) with these properties come in couples $\pm(C, D)$. To each such pair (C, D) with $C > 0$ thus corresponds a unique class $\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma$; its representatives $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ clearly satisfy $a \equiv C \pmod{Q/w}$ and $d \equiv \bar{u}D \pmod{w}$.

By (5.1), the character χ_1 can be induced from the product $\psi_1 \psi_2$, where ψ_1 is a primitive character of some conductor $r_1 \mid w$ and ψ_2 is a primitive character of some

conductor $r_2 \mid Q/w$; in particular, $\chi_1(d) = \psi_1(\bar{u}D)\psi_2(\bar{C})$. Proceeding as in [4, p. 247]² or [6, p. 526] we conclude that for $n > 0$ the n -th Fourier coefficient of $E_{\alpha, \chi_1}(z, 1/2 + it)$ is given by

$$\begin{aligned} \sqrt{n} \rho_{\alpha, \chi_1}(n, t) &= \frac{i^{\varkappa} \pi^s n^{s-1/2}}{\Gamma(s + \varkappa/2)} \left(\frac{(w, Q/w)}{wQ} \right)^s \\ &\quad \times \sum_{(C, Q/w)=1} \frac{\psi_2(C)}{C^{2s}} \sum_{\substack{D(Cw) \\ CD \equiv u \pmod{(w, Q/w)}}}^* \psi_1(u\bar{D}) e\left(-\frac{nD}{Cw}\right), \end{aligned} \quad (5.2)$$

where $s = 1/2 + it$.

We transform this expression further into a form which will be convenient for our purposes. We detect the congruence condition in the innermost sum by Dirichlet characters ρ modulo w_Q , getting

$$\sum_{\substack{D(Cw) \\ CD \equiv u \pmod{(w_Q)}}}^* \psi_1(u\bar{D}) e\left(-\frac{nD}{Cw}\right) = \frac{\psi_1(u)}{\phi(w_Q)} \sum_{\rho(w_Q)} \rho(\bar{u}C) \sum_{D(Cw)}^* \bar{\psi}_1 \rho(D) e\left(-\frac{nD}{Cw}\right).$$

Let Q_w be the product of all prime factors $p \mid Q$ such that $p \nmid (Q/w)$. The condition that $(C, Q/w) = 1$ is equivalent to the statement that $C = C_1 C_2$, where $(C_1, Q) = 1$ and $C_2 \mid Q_w^\infty$ (with the correspondence being that $C_2 = (C, Q)$ and $C_1 = C/C_2$). Recalling that the conductor of $\bar{\psi}_1 \rho$ is a divisor of w , the innermost sum over D above equals

$$\begin{aligned} \sum_{d_1(C_2w)}^* \sum_{d_2(C_1)}^* \bar{\psi}_1 \rho(C_1 d_1 + C_2 w d_2) e\left(-\frac{n(C_1 d_1 + C_2 w d_2)}{C_1 C_2 w}\right) \\ = r_{C_1}(n) \bar{\psi}_1 \rho(C_1) S_{\bar{\psi}_1 \rho}(-n, 0, C_2 w), \end{aligned}$$

where $r_{C_1}(n) = \sum_{d(C_1)}^* e(-nd/C_1)$ is the Ramanujan sum. Substituting, the double sum over C and D in (5.2) equals

$$\frac{\psi_1(u)}{\phi(w_Q)} \sum_{\rho(w_Q)} \rho(\bar{u}) \sum_{(C_1, Q)=1} \frac{\rho \psi_2(C_1)}{C_1^{2s}} \sum_{C_2 \mid Q_w^\infty} \frac{\rho \psi_2(C_2)}{C_2^{2s}} r_{C_1}(n) \bar{\psi}_1 \rho(C_1) S_{\bar{\psi}_1 \rho}(-n, 0, C_2 w).$$

The inside sum over C_1 equals

$$\sum_{(C_1, Q)=1} \frac{\bar{\psi}_1 \psi_2 \rho^2(C_1)}{C_1^{2s}} \sum_{\delta \mid (C_1, n)} \mu\left(\frac{C_1}{\delta}\right) \delta = \sum_{\substack{\delta \mid n \\ (\delta, Q)=1}} \frac{\bar{\psi}_1 \psi_2 \rho^2(\delta)}{\delta^{2s-1}} \frac{1}{L(Q)(2s, \bar{\psi}_1 \psi_2 \rho^2)},$$

² There is a small typo in [4, p. 247]: the congruence condition in the rightmost sum in the first display under Lemma 3.6 should be $\delta \gamma \equiv u \pmod{(w, q/w)}$. Also note the additional factor \sqrt{n} in [2, Section 2.1.3] due to the different weight function compared to [4, (1.17)].

where $L^{(Q)}(s, \psi) = \prod_{p \mid Q} (1 - \psi(p)p^{-s})^{-1}$ is the partial L -function. Putting everything together in (5.2), we find that

$$\begin{aligned} \sqrt{n} \rho_{\alpha, \chi_1}(n, t) &= \frac{i^{\varkappa} \pi^{1/2+it} n^{it}}{\Gamma(\frac{1+\varkappa}{2} + it)} \left(\frac{w_Q}{w} \right)^{1/2+it} \frac{\psi_1(u)}{\phi(w_Q)} \sum_{\rho(w_Q)} \frac{\rho(\bar{u})}{L^{(Q)}(1+2it, \bar{\psi}_1 \psi_2 \rho^2)} \\ &\times \sum_{\substack{\delta \mid n \\ (\delta, Q)=1}} \frac{\bar{\psi}_1 \psi_2 \rho^2(\delta)}{\delta^{2it}} \sum_{C_2 \mid Q_w^\infty} \frac{\rho \psi_2(C_2)}{C_2^{1+2it}} S_{\bar{\psi}_1 \rho}(-n, 0, C_2 w). \end{aligned} \quad (5.3)$$

With this explicit computation of the Fourier coefficients $\sqrt{n} \rho_{\alpha, \chi_1}(n, t)$ under our belt, we are now ready to prove Lemma 1. Indeed, we have

$$\mathcal{E}(m, n, t) = \sum_{\substack{w \mid Q \\ q_1 \mid [w, Q/w]}} \sum_{u \in \mathcal{L}_w} \frac{\sqrt{mn}}{\cosh(\pi t)} \overline{\rho_{u/w, \chi_1}(m, t)} \rho_{u/w, \chi_1}(n, t).$$

We insert (5.3), sum over u first, and then estimate trivially, using Stirling's formula, standard lower bounds for $L^{(Q)}(1+it, \psi)$ (including Siegel's bound if ψ is real and $|t| \leq 1$), and Weil's bound for Kloosterman sums. Denoting by $\tau(x)$ the number of divisors of x , we see for $m, n > 0$ that

$$\begin{aligned} \mathcal{E}(m, n, t) &\ll \frac{(mn)^\varepsilon}{\cosh(\pi t) |\Gamma(\frac{1+\varkappa}{2} + it)|^2} \sum_{\substack{w \mid Q \\ q_1 \mid [w, Q/w]}} \frac{w_Q}{w Q \phi(w_Q)} \\ &\times \sum_{\rho(w_Q)} \frac{1}{|L^{(Q)}(1+2it, \bar{\psi}_1 \psi_2 \rho^2)|^2} \sum_{C_2, C'_2 \mid Q_w^\infty} \frac{|S_{\bar{\psi}_1 \rho}(-m, 0, C_2 w)| |S_{\bar{\psi}_1 \rho}(-n, 0, C'_2 w)|}{C_2 C'_2} \\ &\ll \frac{(Qmn(1+|t|))^\varepsilon}{(1+|t|)^\varkappa} \sum_{w \mid Q} \frac{w_Q}{Q} \sum_{C_2, C'_2 \mid Q_w^\infty} \frac{(m, C_2 w)^{1/2} (n, C'_2 w)^{1/2} \tau(C_2 w) \tau(C'_2 w)}{C_2^{1/2} C'_2^{1/2}} \\ &\ll \frac{(Qmn(1+|t|))^\varepsilon}{(1+|t|)^\varkappa} \sum_{w \mid Q} \frac{w_Q (m, w)^{1/2} (n, w)^{1/2}}{Q} \\ &\ll \frac{1}{(1+|t|)^\varkappa} \frac{(m, \tilde{Q})^{1/2} (n, \tilde{Q})^{1/2}}{\tilde{Q}} (Qmn(1+|t|))^\varepsilon, \end{aligned}$$

as desired.

6. A generalized Kuznetsov formula

Combining (2.4), (4.1) and (4.6), we obtain the following new version of the Kuznetsov formula.

Theorem 4. Let $m, n, q \in \mathbb{N}$, $f : (\mathbb{Z}/q\mathbb{Z})^* \rightarrow \mathbb{C}$, and $f_\infty : (0, \infty) \rightarrow \mathbb{C}$ be a smooth, compactly supported function. We keep the notation developed so far. In particular, for a character χ modulo q we write $\varkappa = 0$ if χ is even and $\varkappa = 1$ if χ is odd, and we denote by χ_1 modulo q_1 the underlying primitive character. We write $m' := m/(m, q^\infty)$ and $n' = n(m, q^\infty)$. Then

$$\begin{aligned} & \sum_{(c,q)=1} \frac{S(m, n, c)}{c^{1/2}} f(c) f_\infty(c) \\ &= \sum_{d|q} \mu(d) \sum_{\chi(q)} \sum_{f \in \mathcal{B}_{\varkappa}(dq_1, \chi_1)} \frac{4\pi \sqrt{m'n'q_1^2}}{\cosh(\pi t_f)} \chi(m') \overline{\rho_f(m')} \rho_f(n'q_1^2) F(\chi) F_\infty(t_f) \\ &+ \sum_{d|q} \mu(d) \sum_{\chi(q)} \sum_{\substack{\mathfrak{a} \text{ singular} \\ \text{level } dq_1}} \int_{-\infty}^{\infty} \frac{\sqrt{m'n'q_1^2}}{\cosh(\pi t)} \chi(m') \overline{\rho_{\mathfrak{a}, \chi_1}(m', t)} \rho_{\mathfrak{a}, \chi_1}(n'q_1^2, t) F(\chi) F_\infty(t) dt \\ &+ \sum_{d|q} \mu(d) \sum_{\chi(q)} \sum_{\substack{k \equiv \varkappa(2), k > \varkappa \\ f \in \mathcal{B}_k(dq_1, \chi_1)}} \frac{(k-1)! \sqrt{m'n'q_1^2}}{\pi (4\pi)^{k-1}} \chi(m') \overline{\rho_f(m')} \rho_f(n'q_1^2) F(\chi) F_\infty^*(k) \end{aligned}$$

where

$$\begin{aligned} F(\chi) &= \frac{\widehat{f}(\chi) q_1^{1/2}}{\tau(\chi_1) \phi(q)^{1/2}}, \\ F_\infty(t) &= \frac{\pi^{1/2} i (mn)^{1/4} q_1^{1/2} t^\varkappa}{\sinh(\pi t)} \int_0^\infty (J_{2it}(x) - (-1)^\varkappa J_{-2it}(x)) f_\infty\left(\frac{4\pi \sqrt{mn}}{x}\right) \frac{dx}{x^{3/2}}, \\ F_\infty^*(k) &= (4\pi)^{1/2} i^k (mn)^{1/4} q_1^{1/2} \int_0^\infty J_{k-1}(x) f_\infty\left(\frac{4\pi \sqrt{mn}}{x}\right) \frac{dx}{x^{3/2}}. \end{aligned}$$

Here $\mathcal{B}_{\varkappa}(dq_1, \chi_1)$ and $\mathcal{B}_k(dq_1, \chi_1)$ can be any orthonormal bases of the spaces of non-holomorphic weight \varkappa (holomorphic weight k , respectively) cusp forms of level dq_1 and character χ_1 .

We remark that, despite its appearance, the right-hand side of our generalized Kuznetsov formula is symmetric in m and n . This is especially easy to see for bases $\mathcal{B}_{\varkappa/k}(dq_1, \chi_1)$ consisting of forms f which are eigenforms of Hecke operators T_m with $(m, q) = 1$, so that $\sqrt{m} \rho_f(mn) = \lambda_f(m) \rho_f(n)$ whenever $(m, nq) = 1$; by (3.8), such is the case, in particular, for the special bases constructed in Section 3. In this case, referring also to (3.5), we have

$$\begin{aligned} & \chi(m') \overline{\rho_f(m')} \rho_f(n'q_1^2) \\ &= \sqrt{(mn, q^\infty)/mn} \lambda_f(m/(m, q^\infty)) \lambda_f(n/(n, q^\infty)) \overline{\rho_f(1)} \rho_f((mn, q^\infty)q_1^2). \end{aligned}$$

Our Eisenstein series are in general *not* Hecke eigenfunctions, but the statement of Theorem 4 (being a purely spectral formula) holds also for the basis of Eisenstein series parametrized by pairs of characters as described in [2]; this basis satisfies the usual Hecke relations, and the symmetry in the Eisenstein term can be restored by the same argument.

Acknowledgements. We would like to thank Farrell Brumley for insightful conversations on topics related to this work and Gergely Harcos for helpful feedback. Henryk Iwaniec kindly informed us that in unpublished notes he obtained cancellation in the situation of Corollary 2. Finally we thank the referee for very useful comments.

The first author acknowledges the support by the Volkswagen Foundation and a Starting Grant of the European Research Council. The second author would like to thank the Max Planck Institute for Mathematics in Bonn for their support and excellent research infrastructure at the Institute.

References

- [1] Abbes, A., Ullmo, E.: Comparaison des métriques d'Arakelov et de Poincaré sur $X_0(N)$. *Duke Math. J.* **80**, 295–307 (1995) [Zbl 0895.14007](#) [MR 1369394](#)
- [2] Blomer, V., Harcos, G., Michel, P.: Bounds for modular L -functions in the level aspect. *Ann. Sci. École Norm. Sup.* **40**, 697–740 (2007) [Zbl 1185.11034](#) [MR 2382859](#)
- [3] Bombieri, E., Friedlander, J. B., Iwaniec, H.: Primes in arithmetic progressions to large moduli. *Acta Math.* **156**, 203–251 (1986) [Zbl 0588.10042](#) [MR 0834613](#)
- [4] Deshouillers, J.-M., Iwaniec, H.: Kloosterman sums and Fourier coefficients of cusp forms. *Invent. Math.* **70**, 219–288 (1982/83) [Zbl 0502.10021](#) [MR 0684172](#)
- [5] Duke, W., Friedlander, J. B., Iwaniec, H.: Equidistribution of roots of a quadratic congruence to prime moduli. *Ann. of Math. (2)* **141**, 423–441 (1995) [Zbl 0840.11003](#) [MR 1324141](#)
- [6] Duke, W., Friedlander, J. B., Iwaniec, H.: The subconvexity problem for Artin L -functions. *Invent. Math.* **149**, 489–577 (2002) [Zbl 1056.11072](#) [MR 1923476](#)
- [7] Fouvry, E., Kowalski, E., Michel, P.: Algebraic twists of modular forms and Hecke orbits. [arXiv:1207.0617v1](#) (2012); to appear in *Geom. Funct. Anal.*
- [8] Fouvry, E., Michel, P.: Sur le changement de signe des sommes de Kloosterman. *Ann. of Math. (2)* **165**, 675–715 (2007) [Zbl 1230.11096](#) [MR 2335794](#)
- [9] Goldfeld, D., Sarnak, P.: Sums of Kloosterman sums. *Invent. Math.* **71**, 243–250 (1983) [Zbl 0507.10029](#) [MR 0689644](#)
- [10] Gradshteyn, I. S., Ryzhik, I. M.: *Table of Integrals, Series, and Products*. 6th ed., Academic Press, San Diego, CA (2000) [Zbl 0981.65001](#) [MR 1773820](#)
- [11] Harcos, G., Michel, P.: The subconvexity problem for Rankin–Selberg L -functions and equidistribution of Heegner points. II. *Invent. Math.* **163**, 581–655 (2006) [Zbl 1111.11027](#) [MR 2207235](#)
- [12] Hoffstein, J., Lockhart, P.: Coefficients of Maass forms and the Siegel zero. *Ann. of Math. (2)* **140**, 161–181 (1994) [Zbl 0814.11032](#) [MR 1289494](#)
- [13] Iwaniec, H.: *Topics in Classical Automorphic Forms*. Grad. Stud. Math. 17, Amer. Math. Soc. (1997) [Zbl 0905.11023](#) [MR 1474964](#)
- [14] Iwaniec, H., Kowalski, E.: *Analytic Number Theory*. Colloq. Publ. 53, Amer. Math. Soc., Providence, RI (2004) [Zbl 1059.11001](#) [MR 2061214](#)
- [15] Iwaniec, H., Luo, W., Sarnak, P.: Low lying zeros of families of L -functions. *Inst. Hautes Études Sci. Publ. Math.* **91**, 55–131 (2001) [Zbl 1012.11041](#) [MR 1828743](#)
- [16] Katz, N.: *Moments, Monodromy, and Perversity: a Diophantine Perspective*. Ann. of Math. Stud. 159, Princeton Univ. Press (2005) [Zbl 1079.14025](#) [MR 2183396](#)

-
- [17] Kim, H.: Functoriality for the exterior square of $GL(4)$ and symmetric fourth of $GL(2)$. *J. Amer. Math. Soc.* **16**, 139–183 (2003) [Zbl 1018.11024](#) [MR 1937203](#)
- [18] Kuznetsov, N.: The Petersson conjecture for cusp forms of weight zero and the Linnik conjecture. Sums of Kloosterman sums. *Math. USSR-Sb.* **39**, 299–342 (1981) [Zbl 0461.10017](#) [MR 0568983](#)
- [19] Li, W.: L -Series of Rankin type and their functional equations. *Math. Ann.* **244**, 135–166 (1979) [Zbl 0396.10017](#) [MR 0550843](#)
- [20] Linnik, Y.: Additive problems and eigenvalues of the modular operators. In: *Proc. Int. Congr. Mathematicians (Stockholm, 1962)*, 270–284 (1963) [Zbl 0116.03604](#) [MR 0180541](#)
- [21] Sarnak, P., Tsimmerman, J.: On Linnik and Selberg’s conjecture about sums of Kloosterman sums. In: *Algebra, Arithmetic, and Geometry: in honor of Yu. I. Manin, Vol. II*, *Progr. Math.* 270, Birkhäuser Boston, Boston, MA, 619–635 (2009) [Zbl 1234.11104](#) [MR 2641204](#)
- [22] Vardi, I.: A relation between Dedekind sums and Kloosterman sums. *Duke Math. J.* **55**, 189–197 (1987) [Zbl 0623.10025](#) [MR 0883669](#)