DERIVED CATEGORIES OF KEUM'S FAKE PROJECTIVE PLANES

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Abstract. We conjecture that derived categories of coherent sheaves on fake projective n-spaces have a semi-orthogonal decomposition into a collection of $n+1$ exceptional objects and a category with vanishing Hochschild homology. We prove this for fake projective planes with non-abelian automorphism group (such as Keum's surface). Then by passing to equivariant categories we construct new examples of phantom categories with both Hochschild homology and Grothendieck group vanishing.

Keywords: derived categories of coherent sheaves, exceptional collections, fake projective planes

Mathematics Subject Classication 2010: 14F05, 14J29

1. Introduction and statement of results

A fake projective plane is by definition a smooth projective surface with minimal cohomology (i.e. is the same cohomology as that of $\mathbb{CP}^2$) which is not isomorphic to $\mathbb{CP}^2$. Such surface is necessarily of general type. The first example of a fake projective plane was constructed by Mumford [43] using p-adic uniformization [20], [44]. Prasad–Yeung [47] and Cartwright–Steger [17] have recently finished classification of fake projective planes into 100 isomorphism classes.

Due to an absense of a geometric construction for fake projective planes there are many open questions about them. Most notably the Bloch conjecture on zero-cycles [8] for fake projective planes is not yet established.

In higher dimension we may call a smooth projective n-dimensional variety X of general type a *fake projective space* if it has minimal cohomology and in addition has the same "Hilbert polynomial" as $\mathbb{P}^n$: $\chi(X, \omega_X^{\otimes l}) = \chi(\mathbb{P}^n, \omega_{\mathbb{P}^n}^{\otimes l})$ for all $l \in \mathbb{Z}$. This definition is consistent with that of fake projective plane since Hodge numbers of a surface determine its Hilbert polynomial.

In fact very little is known about fake projective spaces in higher dimension. Fake projective fourspaces were introduced and studied by Prasad and Yeung in [48].

In this paper we take a perspective that started with a seminal discovery of *full exceptional collections* by Beilinson on projective spaces [7] and Bondal [12], Bondal-Kapranov [13], and Bondal-Orlov [14] in general.

Let X be a smooth projective variety. It has been questioned whether an exceptional collection in the derived category of coherent sheaves $\mathcal{D}^b(X)$ which spans the Grothendieck group $K_0(X)$ or the Hochschild homology $HH_*(X)$ is full, that is generates $\mathcal{D}^b(X)$. More generally one may ask whether the only admissible [13] subcategory $\mathcal{A} \subset \mathcal{D}^b(X)$ which has vanishing Grothendieck group or Hochschild homology is $\mathcal{A} = 0$.

Recently there has been a series of examples [10, 1, 24, 25, 11, 41] showing that this is in fact not the case. In this paper we call a non-zero admissible subcategory $\mathcal{A}$ of the derived category of coherent sheaves $\mathcal{D}^b(X)$ an *H-phantom* if $HH_\bullet(\mathcal{A}) = 0$ and a *K-phantom* if $K_0(\mathcal{A}) = 0$.

Date: February 20, 2015.

We propose the following conjecture which we see of great importance towards understanding the nature of fake projective spaces.

Conjecture 1.1. *Assume that X is an n-dimensional fake projective space with canonical class divisible by $(n+1)$. Then for some choice of $\mathcal{O}(1)$ such that $\omega_X = \mathcal{O}(n+1)$, the sequence*

$$\mathcal{O}, \mathcal{O}(-1), \ldots, \mathcal{O}(-n)$$

is an exceptional collection on X.

Corollary 1.2. *Fake projective spaces as in Conjecture 1.1 admit an H-phantom admissible subcategories in their derived categories $\mathcal{D}^b(X)$.*

See Section 3 for the proof of this Corollary.

We prove Conjecture 1.1 for fake projective planes admitting an action of the non-abelian group G_{21} of order 21. According to the Table given in the Appendix there are 6 such surfaces: there are three relevant groups in the table and there are two complex conjugate surfaces for each group [35].

Theorem 1.3. *Let S be one of the six fake projective planes with automorphism group of order 21. Then $K_S = \mathcal{O}(3)$ for a unique line bundle $\mathcal{O}(1)$ on S. Furthermore $\mathcal{O}$, $\mathcal{O}(-1)$, $\mathcal{O}(-2)$ is an exceptional collection on S.*

Most of Section 4 deals with the proof of this Theorem, which relies on the holomorphic Lefschetz fixed point formula applied to the three fixed points of an element of order 7 as in Keum's paper [34].

Soon after the previous version of this paper appeared as a preprint, Najmuddin Fakhruddin gave a proof of Conjecture 1.1 for those fake projective planes that admit 2-adic uniformization [22], in particular for Mumford's fake projective plane. These cases are disjoint from the ones that satisfy the assumptions of our Theorem 1.3.

For S as in Theorem 1.3 we consider equivariant derived categories $\mathcal{D}^b_G(S)$ for various subgroups $G \subset G_{21}$. A good reference for equivariant derived categories and their semi-orthogonal decompositions is A.Elagin's paper [21].

We denote the descent of the H-phantom $\mathcal{A}_S$ to $\mathcal{D}^b_G(S)$ by $\mathcal{A}^G_S$ (see Section 5 or [21] for details).

In Section 5 we prove the following statements:

Proposition 1.4. *Let S be a fake projective plane with automorphism group G_{21}. For any $G \subset G_{21}$, $\mathcal{A}^G_S$ is an H-phantom.*

Proposition 1.5. *Let Z be the minimal resolution of S/G where $G = \mathbb{Z}/3 \subset G_{21}$. Then Z is a fake del Pezzo surface of degree three, that is Z is a surface of general type with $p_g(Z) = q(Z) = 0$, $K^2 = 3$. Furthermore, Z admits an H-phantom in its derived category.*

Finally, in the four cases when $G \supset \mathbb{Z}/7$ *and* surface S/G is simply-connected, we show that the H-phantom $\mathcal{A}^G_S$ is also a K-phantom:

Proposition 1.6. *Let S be a fake projective plane with automorphism group G_{21}. In the notation of* [47, 17] *and the Appendix assume that the class of S is either $(\mathbb{Q}(\sqrt{-7}), p = 2, \mathcal{T}_1 = \{7\})$ or $\mathcal{C}_{20}$. Let $G = \mathbb{Z}/7 \subset G_{21}$ or $G = G_{21}$. Then $\mathcal{A}^G_S$ is a K-phantoms subcategory of $\mathcal{D}^b_G(S)$.*

The paper is organized as follows. In Section 2 we give some basic results on fake projective planes. In Section 3 we recall the necessary definitions, make some remarks on Conjecture 1.1 and deduce Corollary 1.2 from Conjecture 1.1. In Section 4 we prove Theorem 1.3 and in Section 5 we prove Propositions 1.4 - 1.6.

In the Appendix we give a table of arithmetic subgroups $\Pi \subset PSU(2,1)$ giving rise to fake projective planes and the corresponding automorphism and first homology groups. These results are taken from the computations of Cartwright and Steger[18].

There has been considerate interest in relating the derived category and algebraic cycles or more generally, motives [45], [42], [25]. Thus in particular one may ask whether studying the derived category of a fake projective plane S may be helpful to proving the Bloch conjecture for S.

On the other hand, such a connection would be helpful in giving geometric meaning to the phantom categories.

We thank Igor Dolgachev, Dima Panov and Duco van Straten for helpful discussions on geometry of fake projective planes. We thank Donald Cartwright, Philippe Eyssidieux and Bruno Klingler, Gopal Prasad, Sai Kee Yeung, for answering our questions about classification and construction of fake projective planes and fourspaces.

This paper is related to the other paper of the present authors [23], in which we study n-dimensional varieties admitting a full exceptional collection of length $n+1$.

2. Generalities on fake projective planes

From the point of view of complex geometry fake projective planes were studied by Aubin [4] and Yau [50], who proved that any such surface S is uniformized by a complex 2-ball $\mathbb{B} \subset \mathbb{C}^2$:

$$S \simeq \mathbb{B}/\Pi, \quad \Pi \subset PU(2,1).$$

Hence by Mostow's rigidity theorem S is determined by its fundamental group $\Pi = \pi_1(S)$ uniquely up to complex conjugation; Kharlamov–Kulikov [35] showed that the conjugate surfaces are not isomorphic. Further Klingler [36] and Yeung [51] proved that $\pi_1(S)$ is a torsion-free co-compact *arithmetic* subgroup of $PU(2,1)$. Finally such groups have been classified by Cartwright–Steger [17] and Prasad–Yeung [47]: there are 50 explicit subgroups and so all fake projective planes fit into 100 isomorphism classes. We give the classification table of fake projective planes in the Appendix.

Even though all known constructions of the fake projective planes are analytic, some geometric properties can be extracted from the fundamental group $\Pi = \pi_1(S)$. Firstly, we obviously have

$$H_1(S,\mathbb{Z}) = \Pi/[\Pi,\Pi].$$

Furthermore, we have

$$\mathrm{Aut}(S) = N(\Pi)/\Pi$$

where $N(\Pi)$ is the normalizer of Π in $PU(2,1)$.

Lemma 2.1. *Let S be a fake projective plane with no 3-torsion in $H_1(S,\mathbb{Z})$. Then there exists a unique (ample) line bundle $\mathcal{O}(1)$ such that $K_S \cong \mathcal{O}(3)$.*

Proof. First note that the torsion in $\mathrm{Pic}(S) = H^2(S,\mathbb{Z})$ is isomorphic to $H_1(S,\mathbb{Z})$ by the Universal Coefficient Theorem, hence $\mathrm{Pic}(S)$ has no 3-torsion by assumption.

By Poincare duality $\mathrm{Pic}(S)/tors \cong H^2(S,\mathbb{Z})/tors$ is a unimodular lattice, therefore there exists an ample line bundle L with $c_1(L)^2 = 1$. Now $K_S - 3c_1(L) \in \mathrm{Pic}(S)$ is torsion which can be uniquely divided by 3. □

Lemma 2.2. *Let S be a fake projective plane with an automorphism group G_{21} of order 21. Then there is a G_{21}-equivariant line bundle $\mathcal{O}(1)$ and an isomorphism of equivariant line bundles $K_S \cong \mathcal{O}(3)$.*

Proof. As follows from the classification of the fake projective planes by Prasad-Yeung and Cartwright–Steger [47, 17, 18], the order of the first homology group of the six fake projective planes with automorphism group G_{21} is coprime to 3 (see the Table in the Appendix). Therefore by Lemma 2.1 we have

$$K_S = \mathcal{O}(3).$$

for a line bundle $\mathcal{O}(1)$.

It is easy to see that $\mathcal{O}_S(1)$ is $\mathrm{Aut}(S)$-linearizable if the embedding $N(\Pi) \subset PU(2,1)$ lifts to $SU(2,1)$. Computations of Cartwright and Steger [19] show that this holds for all S unless $\overline{\Gamma}$ lies in classes $\mathcal{C}_2$ or $\mathcal{C}_{18}$. □

Remark 2.3. For any $G \subset \mathrm{Aut}(S)$ we may consider the quotient surface S/G. The fundamental group of S/G is computed as follows. Write S/G as the quotient of $\mathbb{B}$:

$$S/G = \mathbb{B}/\Pi_G, \ \ \Pi_G \subset PU(2,1).$$

Now the fundamental group $\pi_1(S/G)$ is equal to Π_G/E where $E \subset \Pi_G$ is the subgroup generated by elliptic elements, that is elements with fixed points [2].

All those groups for all S and $G \subset \mathrm{Aut}(S)$ were also computed by Cartwright and Steger [19]: the surface S/G is simply-connected in twelve cases, including the four cases of Proposition 1.6.

3. Remarks for Conjecture 1.1

Recall that an *exceptional collection* of length r on a smooth projective variety $X/\mathbb{C}$ is a sequence of objects $E_1, \ldots E_r$ in the bounded derived category of coherent sheaves $\mathcal{D}^b(X)$ such that $\mathrm{Hom}(E_j, E_i[k]) = 0$ for all $j > i$ and $k \in \mathbb{Z}$, and moreover each object E_i is exceptional, that is spaces $\mathrm{Hom}(E_i, E_i[k])$ vanish for all k except for one-dimensional spaces $\mathrm{Hom}(E_i, E_i)$. An exceptional collection is called *full* if the smallest triangulated subcategory which contains it, coincides with $\mathcal{D}^b(X)$.

The sequence $\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)$ is a full exceptional collection on $\mathbb{CP}^n$ [7].

Remark 3.1. It is easy to show (see the proof of Corollary 1.2 below) that the collection $\mathcal{O}, \mathcal{O}(-1), \ldots, \mathcal{O}(-n)$ in Conjecture 1.1 can not be full. It is also clear that for all fake projective planes and all arithmetic fake projective fourspaces there is no full exceptional collection of objects in derived category. Indeed the first homology group $H_1(S, \mathbb{Z})$ is known to be non-zero torsion for fake projective planes and arithmetic fake projective spaces ([47], Theorem 10.1 and [48], Theorem 4), this implies that there is non-trivial torsion in $Pic(X) = H^2(X, \mathbb{Z})$, which is an obstruction to existence of a full exceptional collection (cf [23], Proposition 2.1).

Proof of Corollary 1.2. Assume that $\mathcal{O}, \mathcal{O}(-1), \ldots, \mathcal{O}(-n)$ is an exceptional collection, and consider its right orthogonal $\mathcal{A}$. By results of Bondal and Kapranov [12, 13] the category $\mathcal{A}$ is admissible, and thus we have a semi-orthogonal decomposition:

$$\mathcal{D}^b(X) = \langle \mathcal{O}, \mathcal{O}(-1), \ldots, \mathcal{O}(-n), \mathcal{A} \rangle .$$

Hochschild homology is additive for semi-orthogonal decompositions [33] [39], so $\dim HH_\bullet(\mathcal{A}) = 0$ that is $\mathcal{A}$ has vanishing Hochschild homology.

To show that $\mathcal{A}$ itself is non-vanishing, that is that the collection $\mathcal{O}, \mathcal{O}(-1), \ldots, \mathcal{O}(-n)$ is not full, we can use one of the following arguments. If the collection was full collection then by [15](Theorem

3.4) or [46] (see proof of main theorem) the variety X would be Fano, which contradicts to the general type assumption.

Alternatively we can use Kuznetsov's concept of height and pseudoheight of an exceptional collection ([40], Section 4) to show that $HH^0(\mathcal{A}) \neq 0$. By [40], Corollary 4.6 it suffices to show that for the pseudoheight we have $ph(\mathcal{O}, \mathcal{O}(-1), \ldots, \mathcal{O}(-n)) \geqslant 1$. By Kodaira vanishing for $i < j$ spaces $Ext^k(\mathcal{O}(-i), \mathcal{O}(-j))$ vanish unless $k = n$, so the relative height $e(\mathcal{O}(-i), \mathcal{O}(-j))$ of any two such objects equals n. Thus the minimum in the definition of the pseudoheight is attained at $p = 0$ and the pseudoheight is equal to $n - 1$. $\square$

Remark 3.2. 1. Fake projective planes with properties as in Conjecture 1.1 are constructed in [47], 10.4. Choose $\mathcal{O}(1)$ such that $\mathcal{O}(3) = \omega_X$. Then by the Riemann-Roch theorem the Hilbert polynomial is given by

$$\chi(\mathcal{O}(k)) = \frac{(k-1)(k-2)}{2}.$$

Therefore the collection $E_\bullet = (\mathcal{O}, \mathcal{O}(-1), \mathcal{O}(-2))$ is at least numerically exceptional, that is

$$\chi(E_j, E_i) = 0,\ j > i.$$

2. More generally our definition of an n-dimensional fake projective space includes that its Hilbert polynomial is the same as that of a $\mathbb{P}^n$. It follows that if we assume $\omega_X = \mathcal{O}(n+1)$, then we have

$$\chi(\mathcal{O}(k)) = (-1)^n \frac{(k-1)(k-2)\ldots(k-n)}{n!},$$

so that $k = 1, \ldots, n$ are the roots of χ, and the collection

$$\mathcal{O}, \mathcal{O}(-1), \ldots, \mathcal{O}(-n)$$

is numerically exceptional.

3. G.Prasad and S.-K. Yeung informed us that the assumption $\omega_X = \mathcal{O}(5)$ is known to be true for the four arithmetic fake projective fourspaces constructed in [48].

4. Proof of Theorem 1.3

We start with some general observations.

Lemma 4.1 (see [38](Lemma 15.6.2))**.** *Let X be a normal and proper variety, L,L' effective line bundles on X. Let*

$$\phi : H^0(X, L) \otimes H^0(X, L') \to H^0(X, L \otimes L')$$

denote the natural map induced by multiplication. Then

$$\dim Im(\phi) \geqslant h^0(X, L) + h^0(X, L') - 1.$$

Lemma 4.2. *Let S be a fake projective plane satisfying $\omega_S = \mathcal{O}(3)$ for some (ample) line bundle $\mathcal{O}(1)$. Then $h^0(S, \mathcal{O}(2)) \leqslant 2$ and if $H^0(S, \mathcal{O}(2)) = 0$, then*

$$\mathcal{O}, \mathcal{O}(-1), \mathcal{O}(-2)$$

is an exceptional collection on S.

Proof. We have $h^0(S,\mathcal{O}(3)) = h^0(S,\omega_S) = h^{0,2}(S) = 0$, which implies that $H^0(S,\mathcal{O}(1))$ also vanishes. By Kodaira vanishing and Riemann-Roch theorem we have $h^0(S,\mathcal{O}(4)) = 3$, and the inequality $h^0(S,\mathcal{O}(2)) \leqslant 2$ follows from Lemma 4.1.

Finally it follows from the Serre duality and Remark 3.2 (1) that a necessary and sufficient condition for $\mathcal{O}, \mathcal{O}(-1), \mathcal{O}(-2)$ to be exceptional is vanishing of the space of the global sections $H^0(S,\mathcal{O}(2))$. $\square$

We will consider vector spaces $H^*(S,\mathcal{O}(k))$ as G_{21}-representations. We start by describing the group G_{21} and its representation theory. By Sylow's theorems G_{21} admits a unique subgroup of order 7 and this subgroup is normal. We let σ denote a generator of this subgroup. Let τ denote an element of G_{21} of order 3. Conjugating by τ gives rise to an automorphism of $\mathbb{Z}/7 = \langle\sigma\rangle$ and we can choose τ so that

$$\tau^{-1}\sigma\tau = \sigma^2.$$

Thus G_{21} is a semi-direct product of $\mathbb{Z}/7$ and $\mathbb{Z}/3$ and has a presentation

$$G_{21} = \left\langle \sigma, \tau \mid \sigma^7 = 1, \tau^3 = 1, \sigma\tau = \tau\sigma^2 \right\rangle.$$

Using this presentation it is easy to check that there are five conjugacy classes of elements in G_{21}:

$$\begin{gathered} \{1\} \\ [\sigma] = \{\sigma, \sigma^2, \sigma^4\} \\ [\sigma^3] = \{\sigma^3, \sigma^5, \sigma^6\} \\ [\tau] = \{\tau\sigma^k,\ k = 0, \ldots, 6\} \\ [\tau^2] = \{\tau^2\sigma^k,\ k = 0, \ldots, 6\} \end{gathered}$$

and by basic representation theory there exist five irreducible representations of G_{21}. Let $d_1, \ldots, d_5$ be the dimensions of these representations. Basic representation theory also tells us that each d_i divides 21 and that

$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 = 21.$$

Considering different possibilities one finds the only combination $(d_1, d_2, d_3, d_4, d_5) = (1, 1, 1, 3, 3)$ satisfying the conditions above.

It is not hard to check that the character table of G_{21} is the following one:

	1	$[\sigma]$	$[\sigma^3]$	$[\tau]$	$[\tau^2]$
$\mathbb{C}$	1	1	1	1	1
V_1	1	1	1	ω	$\overline{\omega}$
$\overline{V_1}$	1	1	1	$\overline{\omega}$	ω
V_3	3	b	$\overline{b}$	0	0
$\overline{V_3}$	3	$\overline{b}$	b	0	0

In the character table above $\mathbb{C}$ is the trivial one-dimensional representation of G_{21}, and V_i, $\overline{V_i}$ are conjugate pairs of non-trivial irreducible representations of dimension i.

We use the notation:

$$\omega = e^{\frac{2\pi i}{3}}$$
$$\xi = e^{\frac{2\pi i}{7}}$$

and

$$b = \xi + \xi^2 + \xi^4 = \frac{-1+\sqrt{-7}}{2}.$$

Explicitly V_1 and $\overline{V_1}$ are one-dimensional representations restricted from $G_{21}/\langle\sigma\rangle = \mathbb{Z}/3$. V_3 and $\overline{V_3}$ are three-dimensional representations induced from $\mathbb{Z}/7$: $\rho : G_{21} \to GL(V_3)$ is given by matrices

$$\rho(\sigma) = \begin{pmatrix} \xi & & \\ & \xi^2 & \\ & & \xi^4 \end{pmatrix} \quad \rho(\tau) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and $\overline{V_3}$ is its complex conjugate.

Lemma 4.3. *$H^0(S, \mathcal{O}(4))$ is a 3-dimensional irreducible representation of G_{21} (and thus is isomorphic to V_3 or $\overline{V_3}$).*

Proof. We show that the trace of an element $\sigma \in G_{21}$ of order 7 acting on $H^0(S, \mathcal{O}(4))$ is equal to b or $\bar{b}$. This is sufficient since if $H^0(S, \mathcal{O}(4))$ were reducible it would have to be a sum of three one-dimensional representations and the character table of G_{21} shows that in this case the trace of σ on $H^0(S, \mathcal{O}(4))$ would be equal to 3.

By [34], Proposition 2.4(4) σ has three fixed points P_1, P_2, P_3. Let τ be an element of order 3. τ does not stabilize any of the P_i's, since a tangent space of a fixed point of G_{21} would give a faithful 2-dimensional representation of G_{21} which does not exist as is seen from its character table.

Thus P_i's are cyclically permuted by τ. We reorder P_i's in such a way that

$$\tau(P_i) = P_{i+1 \bmod 3}. \tag{4.1}$$

We apply the so-called Holomorphic Lefschetz Fixed Point Formula (Theorem 2 in [3]) to σ and line bundles $\mathcal{O}(k)$:

$$\sum_{p=0}^{2} (-1)^p \, Tr\big(\sigma|_{H^p(S,\mathcal{O}(k))}\big) = \sum_{i=1}^{3} \frac{Tr(\sigma|_{\mathcal{O}(k)_{P_i}})}{(1-\alpha_1(P_i))(1-\alpha_2(P_i))} \tag{4.2}$$

where $\alpha_1(P_i)$, $\alpha_2(P_i)$ are inverse eigenvalues of σ on T_{P_i}:

$$det(1 - t\sigma_*|_{T_{P_i}}) = (1 - t\alpha_1(P_i))(1 - t\alpha_2(P_i)).$$

$\alpha_j(P_i)$ are 7-th roots of unity. We let $\alpha_j := \alpha_j(P_1)$, $j = 1, 2$. Using (4.1) and commutation relations in G_{21} we find that

$$\alpha_j(P_{i+1}) = \alpha_j(P_i)^2$$

so that

$$\begin{aligned} \alpha_j(P_1) &= \alpha_j \\ \alpha_j(P_2) &= \alpha_j^2 \\ \alpha_j(P_3) &= \alpha_j^4. \end{aligned}$$

To find the values of α_j we apply (4.2) with $k = 0$:

$$1 = \frac{1}{(1-\alpha_1)(1-\alpha_2)} + \frac{1}{(1-\alpha_1^2)(1-\alpha_2^2)} + \frac{1}{(1-\alpha_1^4)(1-\alpha_2^4)}. \tag{4.3}$$

All $\alpha_j(P_i)$ are 7-th roots of unity and it turns out that up to renumbering the only possible values of $\alpha_j(P_i)$ which satisfy (4.3) are

$$\begin{aligned} (\alpha_1(P_1), \alpha_2(P_1)) &= (\xi, \xi^3) \\ (\alpha_1(P_2), \alpha_2(P_2)) &= (\xi^2, \xi^6) \\ (\alpha_1(P_3), \alpha_2(P_3)) &= (\xi^4, \xi^5) \end{aligned}$$

or their complex conjugate in which case we would get b instead of $\bar{b}$ for the trace below.

It follows that $Tr(\sigma|_{K_{S,P_i}}) = Tr(\sigma|_{\mathcal{O}(3)_{P_i}})$ is equal to ξ^4, ξ, ξ^2 for $i = 1, 2, 3$ respectively. Dividing by 3 modulo 7 we see that $Tr(\sigma|_{\mathcal{O}(k)_{P_i}})$ is equal to ξ^{6k}, ξ^{5k}, ξ^{3k} for $i = 1, 2, 3$ respectively.

We use (4.2) for $k = 4$ (note that $H^p(S, \mathcal{O}(4)) = 0$ for $p > 0$ by Kodaira vanishing):

$$Tr\big(\sigma\big|_{H^0(S,\mathcal{O}(4))}\big) = \frac{\xi^3}{(1-\xi)(1-\xi^3)} + \frac{\xi^6}{(1-\xi^2)(1-\xi^6)} + \frac{\xi^5}{(1-\xi^4)(1-\xi^5)} = \bar{b}$$

□

Proof of Theorem 1.3. According to Lemma 4.2, it suffices to show that $H^0(S, \mathcal{O}(2)) = 0$.

Let $\delta = h^0(S, \mathcal{O}(2))$. We know that $h^0(S, \mathcal{O}(4)) = 3$, hence it follows from Lemma 4.1 applied to $L = L' = \mathcal{O}(2)$ that $\delta \leqslant 2$. Therefore as a representation of G_{21} the space $H^0(S, \mathcal{O}(2))$ is a sum of 1-dimensional representations and the same is true for $H^0(S, \mathcal{O}(2))^{\otimes 2}$. Since $H^0(S, \mathcal{O}(4))$ is three-dimensional irreducible, this implies that the natural morphism

$$H^0(S, \mathcal{O}(2))^{\otimes 2} \to H^0(S, \mathcal{O}(4))$$

has to be zero by Schur's Lemma. Now again by Lemma 4.1 $H^0(S, \mathcal{O}(2)) = 0$. This finishes the proof. □

5. Proofs of Propositions 1.4 – 1.6

Let $G \subset G_{21}$ be a subgroup. It follows from Theorem 1.3 that

$$\{\mathcal{O}(-j) \otimes V\}_{j=0,1,2;\, V \in IrrRep(G)} \tag{5.1}$$

forms an exceptional collection in the equivariant derived category $\mathcal{D}^b_G(S)$. Indeed we have

$$Ext^p_G(\mathcal{O}(-j) \otimes V, \mathcal{O}(-k) \otimes W) = Ext^p(\mathcal{O}(-j) \otimes V, \mathcal{O}(-k) \otimes W)^G$$

and this group is zero unless $j = k$, $p = 0$ and $V = W$ in which case it is one-dimensional.

We denote by $\mathcal{A}^G_S$ the right orthogonal to the collection (5.1). We now show that $\mathcal{A}^G_S \neq 0$. For that note that for any nonzero object A in $\mathcal{A}_S$ the object

$$\bigoplus_{g \in G} g^* A$$

will have a natural G-linearization so will be a non-zero object in $\mathcal{A}^G_S$, thus $\mathcal{A}^G_S \neq 0$. Alternatively we could show non-vanishing of $\mathcal{A}^G_S$ using Kuznetsov's criterion (cf the second proof of Corollary 1.2) showing that $H^0(\mathcal{A}^G_S) \neq 0$ since the height of our exceptional collection equals 2.

Proof of Proposition 1.4. We denote by Z_G the minimal resolution of S/G. The geometry of Z_G has been carefully studied by Keum [34]: if $|G| = 7$ or $|G| = 21$ then Z_G is an elliptic surface of Kodaira dimension $\varkappa(Z_G) = 1$ (Dolgachev surface), if $|G| = 3$ then Z_G is a surface of general type $\varkappa(Z_G) = 2$. In each case we compare the equivariant derived category $\mathcal{D}^b_G(S)$ to $\mathcal{D}^b(Z_G)$.

The stabilizers of the fixed points of G action are cyclic and we use [27] or [32] to obtain the semi-orthogonal decomposition

$$\mathcal{D}^b_G(S) \simeq \big\langle \mathcal{D}^b(Z_G), E_1, \ldots, E_{r_G} \big\rangle$$

where r_G is the number of non-special characters of the stabilizers [27].

Note that $p_g(Z_G) = q(Z_G) = 0$, therefore

$$\dim HH_*(\mathcal{D}^b(Z_G)) = \dim H^*(Z_G, \mathbb{C}) = \chi(Z_G).$$

We list $\chi(Z_G)$ as well as other relevant invariants in the table:

G	$\#IrrRep(G)$	$Sing(S/G)$	r_G	$\chi(Z_G)$	$\varkappa(Z_G)$
1	1	$\emptyset$	0	3	2
$\mathbb{Z}/3$	3	$3 \times \frac{1}{3}(1,2)$	0	9	2
$\mathbb{Z}/7$	7	$3 \times \frac{1}{7}(1,3)$	9	12	1
G_{21}	5	$3 \times \frac{1}{3}(1,2) + \frac{1}{7}(1,3)$	3	12	1

In the second column of the table we list the number or irreducible representations of each group G. In the third column we describe the singularities of the quotient surface S/G, using the notation $\frac{1}{n}(1,a)$ for each quotient singularity with local model $\mathbb{C}^2/\mathbb{Z}_n$ with the action given by $(\varepsilon, \varepsilon^a)$, where ε is the nth root of unity.

As already mentioned above r_G is the sum of non-special characters of the stabilizers at fixed points: $\frac{1}{3}(1,2)$ fixed points don't contribute to r_G whereas each $\frac{1}{7}(1,3)$ fixed point has 3 non-special characters.

It follows from the table that in each case we have

$$3 \cdot \#IrrRep(G) = \chi(Z_G) + r_G$$

This implies that the number of exceptional objects in (5.1) matches $\dim HH_*(\mathcal{D}^b_G)$, and therefore in each case $\mathcal{A}^G_S$ is an H-phantom. $\square$

Remark 5.1. One can give an alternative proof of Proposition 1.4 using orbifold cohomology. Baranovsky [5] proved an analogue of Hochschild–Kostant–Rosenberg isomorphism for orbifolds. His result implies that (total) Hochschild homology $HH_*(\mathcal{D}^b_G(S))$ is isomorphic as a non-graded vector space to the (total) orbifold cohomology

$$H^*_{orb}(S/G,\mathbb{C}) = \Big(\bigoplus_{g\in G} H^*(S^g,\mathbb{C})\Big)_G = \bigoplus_{[g]\in G/G} H^*(S^g,\mathbb{C})^{Z(g)}.$$

Here S^g is the fixed locus of $g \in G$, $Z(g)$ is the centralizer, $[g]$ is the conjugacy class of g, and $(\cdot)_G$ denotes coinvariants.

For the so-called main sector $[g] = \{id\}$ we have

$$H^*(S,\mathbb{C})^G = H^*(S,\mathbb{C}) = \mathbb{C}^3.$$

For each $g \neq id$ the fixed locus S^g consists of three points, so $H^*(S^g) = \mathbb{C}^3$ and the action of $Z(g) = \langle g \rangle$ on it is trivial, thus each twisted sector is also 3-dimensional.

Taking the sum over all conjugacy classes $[g]$ we obtain

$$\dim HH_*(\mathcal{D}^b_G(S)) = \dim H^*_{orb}(S/G,\mathbb{C}) = 3 \times \#IrrRep(G),$$

which shows that $HH_*(\mathcal{A}^G_S) = 0$.

Proof of Proposition 1.5. In the notation of the previous proof $Z = Z_G$, $G = \mathbb{Z}/3$, and $r_G = 0$ means that

$$\mathcal{D}^b_G(S) \simeq \mathcal{D}^b(Z)$$

in agreement with the derived McKay correspondence [29, 16] which is applicable since S/G has A_2 du Val singularities. It has been already proved above that Z is a surface of general type with $p_g(Z) = q(Z) = 0$ and $\chi(Z) = 9$, and the Noether formula implies that $K_S^2 = 3$. The image of the exceptional collection (5.1) of 9 objects in $\mathcal{D}^b(Z)$ has an H-phantom orthogonal. $\square$

Proof of Proposition 1.6. In the cases under consideration S/G is simply-connected (see Remark 2.3). By a standard argument (e.g. using Van Kampen's theorem as in [6](0.5) or [49](Section 4.1), or more generally see [37](Theorem 7.8.1)) the resolutions Z_G are also simply-connected, in particular $H_1(Z_G, \mathbb{Z}) = 0$.

Then $\mathrm{Pic}(Z_G) = H^2(Z_G, \mathbb{Z})$ is a finitely generated free abelian group. Keum showed in [34] that Kodaira dimension $\varkappa(Z_G) = 1$ (see also Ishida [26]). The Bloch conjecture for Z_G is true[9], so that $CH_0(Z_G) = \mathbb{Z}$. Now by Lemma 2.7 of [24] it follows that $K_0(Z_G)$ is a finitely generated free abelian group, and the same holds for $K_0^G(S) = K_0(\mathcal{D}_G^b(S))$.

The computation of Euler numbers shows that

$$(\text{number of objects in (5.1)}) = \dim HH_*(\mathcal{D}_G^b(S)) = \mathrm{rk}\, K_0(\mathcal{D}_G^b(S)).$$

Finally, the additivity of the Grothendieck group implies that $\mathcal{A}_S^G$ is a K-phantom. □

6. Appendix: Classification table of fake projective planes

We enhance the classification table of the fake projective planes given in [17] which is based on GAP and Magma computer code and its output [18] with the automorphism group $Aut(S)$ and the first homology group $H_1(S, \mathbb{Z})$, which we also take from [18]. There are no original results in this section.

The classification is given by specifying an arithmetic subgroup $\overline{\Gamma} \subset PU(2,1)$, and then each such $\overline{\Gamma}$ may contain several subgroups Π. The group $\overline{\Gamma}$ is described using the following data: l is a totally complex quadractic extension of a totally real field, p is a prime 2, 3 or 5, $\mathcal{T}_1$ is a set of prime numbers (possibly empty). See [17] for details.

N is the index $[\overline{\Gamma} : \Pi]$ and $suf.$ is the suffix (a, b, c, d, e or f) of each group in [18]. G_{21} is the non-abelian group of order 21. In the last column symbol $[n_1, \ldots, n_k]$ denotes the abelian group $(\mathbb{Z}/n_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_k\mathbb{Z})$.

l or $\mathcal{C}$	p	$\mathcal{T}_1$	N	$\#\Pi$	$suf.$	$\mathrm{Aut}(S)$	$H_1(S,\mathbb{Z})$
$\mathbb{Q}(\sqrt{-1})$	5	$\emptyset$	3	2	a	$\mathbb{Z}/3\mathbb{Z}$	$[2,4,31]$
		$\emptyset/\{2I\}$			b/b	$\{1\}$	$[2,3,4,4]$
		$\{2\}$	3	1	a	$\mathbb{Z}/3\mathbb{Z}$	$[4,31]$
$\mathbb{Q}(\sqrt{-2})$	3	$\emptyset$	3	2	a	$\mathbb{Z}/3\mathbb{Z}$	$[2,2,13]$
		$\emptyset/\{2I\}$			b/b	$\{1\}$	$[2,2,2,2,3]$
		$\{2\}$	3	1	a	$\mathbb{Z}/3\mathbb{Z}$	$[2,2,13]$
$\mathbb{Q}(\sqrt{-7})$	2	$\emptyset$	21	3	a	$\mathbb{Z}/3\mathbb{Z}$	$[2,7]$
					b	G_{21}	$[2,2,2,2]$
					c	$\{1\}$	$[2,2,3,7]$
		$\{3\}$	3	2	a	$\mathbb{Z}/3\mathbb{Z}$	$[2,4,7]$
					b	$\{1\}$	$[2,2,3,4]$
		$\{3,7\}$	3	2	a	$\mathbb{Z}/3\mathbb{Z}$	$[4,7]$
					b	$\{1\}$	$[2,3,4]$
		$\{7\}$	21	4	a	G_{21}	$[2,2,2]$
					b	$\mathbb{Z}/3\mathbb{Z}$	$[2,7]$
					c	$\mathbb{Z}/3\mathbb{Z}$	$[2,2,7]$
					d	$\{1\}$	$[2,2,2,3]$
		$\{5\}$	1	1	$-$	$\{1\}$	$[2,2,9]$
		$\{5,7\}$	1	1	$-$	$\{1\}$	$[2,9]$
$\mathbb{Q}(\sqrt{-15})$	2	$\emptyset$	3	2	a	$\mathbb{Z}/3\mathbb{Z}$	$[2,2,7]$
					b	$\{1\}$	$[2,2,2,9]$
		$\{3\}$	3	3	a	$\mathbb{Z}/3\mathbb{Z}$	$[2,3,7]$
					b	$\mathbb{Z}/3\mathbb{Z}$	$[2,2,2,3]$
					c	$\mathbb{Z}/3\mathbb{Z}$	$[2,3]$
		$\{3,5\}$	3	3	a	$\mathbb{Z}/3\mathbb{Z}$	$[3,7]$
					b	$\mathbb{Z}/3\mathbb{Z}$	$[2,2,3]$
					c	$\mathbb{Z}/3\mathbb{Z}$	$[3]$
		$\{5\}$	3	2	a	$\mathbb{Z}/3\mathbb{Z}$	$[2,7]$
					b	$\{1\}$	$[2,2,9]$
$\mathbb{Q}(\sqrt{-23})$	2	$\emptyset$	1	1	$-$	$\{1\}$	$[2,3,7]$
		$\{23\}$	1	1	$-$	$\{1\}$	$[3,7]$
$\mathcal{C}_2$	2	$\emptyset$	9	6	a	$(\mathbb{Z}/3\mathbb{Z})^2$	$[2,7]$
					b	$\mathbb{Z}/3\mathbb{Z}$	$[2,7,9]$
					c	$\mathbb{Z}/3\mathbb{Z}$	$[2,9]$
					d	$\mathbb{Z}/3\mathbb{Z}$	$[2,9]$
					f	1	$[2,3,3]$
					g	1	$[2,3,3]$
		$\{3\}$	9	1	$-$	$(\mathbb{Z}/3\mathbb{Z})^2$	$[7]$
$\mathcal{C}_{10}$	2	$\emptyset$	3	1	$-$	$\mathbb{Z}/3\mathbb{Z}$	$[2,7]$
		$\{17-\}$	3	1	$-$	$\mathbb{Z}/3\mathbb{Z}$	$[7]$
$\mathcal{C}_{18}$	3	$\emptyset$	9	1	a	$(\mathbb{Z}/3\mathbb{Z})^2$	$[2,2,13]$
		$\emptyset/\{2\mathrm{I}\}$	1	1	b/d	1	$[2,3,3]$
		$\{2\}$	3	3	a	$\mathbb{Z}/3\mathbb{Z}$	$[2,3,13]$
					b	$\mathbb{Z}/3\mathbb{Z}$	$[2,3]$
					c	$\mathbb{Z}/3\mathbb{Z}$	$[2,3]$
$\mathcal{C}_{20}$	2	$\emptyset$	21	1	$-$	G_{21}	$[2,2,2,2,2,2]$
		$\{3-\}$	3	2	a	$\mathbb{Z}/3\mathbb{Z}$	$[4,7]$
					b	$\{1\}$	$[2,3,4]$
		$\{3+\}$	3	2	a	$\mathbb{Z}/3\mathbb{Z}$	$[4,7]$
					b	$\{1\}$	$[2,3,4]$

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This study (research grant No 14-01-0025) was supported by The National Research University–Higher School of Economics' Academic Fund Program in 2014–2015. S. G. was partially supported by grant MK-1297.2014.1; AG Laboratory NRU-HSE, RF government grant, ag. 11.G34.31.0023; Grant of Leading Scientific Schools (N.Sh. 2998.2014.1). S. G. and L. K. were funded by grants NSF DMS0600800, NSF FRG DMS-0652633, NSF FRG DMS-0854977, NSF DMS-0854977, NSF DMS-0901330, grants FWF P 24572-N25 and FWF P20778, and an ERC grant — GEMIS. E. S. has been supported by the Max-Planck-Institut für Mathematik, SFB 45 Bonn-Essen-Mainz grant and the Hausdorff Institute for Mathematics, Bonn during the Arithmetic and Geometry trimester program 2013.

Part of this work was done in June 2013 during the visit of S. G. to Johannes Gutenberg Universität Mainz, funded by SFB 45. A computation of global sections of line bundles on a surface with torsion of order 64 was funded by grant RScF 14-21-00053.