BORCHERDS PRODUCTS EVERYWHERE

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Abstract. We prove the Borcherds Products Everywhere Theorem, Theorem 6.6, that constructs holomorphic Borcherds Products from certain Jacobi forms that are theta blocks without theta denominator. The proof uses generalized valuations from formal series to partially ordered abelian semigroups of closed convex sets. We present nine infinite families of paramodular Borcherds Products that are simultaneously Gritsenko lifts. This is the first appearance of infinite families with this property in the literature.

1. Introduction

This article studies Borcherds Products on groups that are simultaneously orthogonal and symplectic, the paramodular groups of degree two. This work began as an attempt to make Siegel paramodular cusp forms that are simultaneously Borcherds Products and Gritsenko lifts. On the face of it, this phenomenon may seem the least interesting type of a Borcherds product but it is the only known way to control the weight of a constructed Borcherds product. Additionally, for computational purposes, a paramodular form that is both a Borcherds product and a Gritsenko lift is very useful; such a form has simple Fourier coefficients because it is a lift and a known divisor because it is a Borcherds product. In the case of weight 3, a Borcherds product gives the canonical divisor class of the moduli space of $(1, t)$-polarized abelian surfaces. Therefore the construction of infinite families of such Siegel paramodular forms is interesting for applications to algebraic geometry. At the end of this article (see §8) we give nine infinite families of modular forms, including a family of weight 3, which are simultaneously Borcherds Products and Gritsenko lifts. *This is the first appearance of such infinite families in the literature.*

All these Borcherds products are made by starting from certain special Jacobi forms that are theta blocks without theta denominator.

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Theorem 1.1 gives a rather unexpected and surprising way to construct holomorphic Borcherds products starting from theta blocks of positive weight. As it is rather easy to search for theta blocks, we call this the Borcherds Products Everywhere Theorem. The proof uses the theory of Borcherds products for paramodular forms as given by Gritsenko and Nikulin [13], the recent theory of theta blocks due to Gritsenko, Skoruppa and Zagier [15], and a theory of generalized valuations on rings of formal series presented here in section 4. Let η be the Dedekind Eta function and ϑ be the odd Jacobi theta function and write $\vartheta_\ell(\tau, z) = \vartheta(\tau, \ell z)$. The most general theta block [15] can be written $\eta^{f(0)} \prod_{\ell \in \mathbb{N}} (\vartheta_\ell/\eta)^{f(\ell)}$ for a sequence $f : \mathbb{N} \cup \{0\} \to \mathbb{Z}$ of finite support. Here we consider only theta blocks *without theta denominator,* meaning that f is nonnegative on $\mathbb{N}$. Theorem 6.6 is a more detailed version of the main theorem but here is one suitable for this Introduction; the essential point is that the Borcherds Products we construct are holomorphic, not just meromorphic.

Theorem 1.1. *Let $v, k, t \in \mathbb{N}$. Let ϕ be a holomorphic Jacobi form of weight k and index t that is a theta block without theta denominator and that has vanishing order v in $q = e^{2\pi i \tau}$. Then $\psi = (-1)^v \frac{\phi|V_2}{\phi}$ is a weakly holomorphic Jacobi form of weight 0 and index t and the Borcherds lift of ψ is a holomorphic paramodular form of level t and some weight $k' \in \mathbb{N}$. For even v, it suffices for these conclusions that the theta block ϕ without theta denominator be weakly holomorphic. If $v = 1$ then $k = k'$ and the first two Fourier Jacobi coefficients of the Borcherds lift of ψ and the Gritsenko lift of ϕ agree.*

All Borcherds Products that are also Gritsenko lifts of theta blocks without theta denominator are necessarily generated in the manner of the above theorem but other holomorphic Borcherds Products can arise by the same process. In [13], Gritsenko and Nikulin point out that the leading Fourier Jacobi coefficient of a Borcherds Product, $\mathrm{Borch}(\Psi)$, is a theta block ϕ and that when the Borcherds Product is also a Gritsenko lift we have $\Psi = -\frac{\phi|V_2}{\phi}$. Gritsenko and Nikulin gave many examples of paramodular forms that are simultaneously multiplicative (Borcherds) and additive (Gritsenko) lifts, for both trivial and nontrivial characters of the paramodular group. In this article we consider only the case of trivial character. Here we follow their line of thought and, beginning with a theta block ϕ without theta denominator that is a Jacobi form of positive vanishing order $v = \mathrm{ord}_q \phi$, show that $\Psi = (-1)^v \frac{\phi|V_2}{\phi}$ is a weakly holomorphic weight zero Jacobi form with integral and positive singular part. Hence, $\mathrm{Borch}(\Psi)$ is a holomorphic paramodular form. Proving that the character is trivial and determining the symmetry or antisymmetry require some combinatorics.

We note that the techniques developed to prove Theorem 1.1 may be applied to construct antisymmetric Siegel paramodular forms that are new eigenforms for all the Hecke operators. These results will be presented in our next article.

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2. Examples, especially of identities $\mathrm{Grit}(\phi) = \mathrm{Borch}\big(-(\phi|V_2)/\phi\big)$

The paramodular group $K(N)$ has a normalizing involution μ_N and a Borcherds product is a μ_N eigenform, see §3. In the following examples, we decompose $M_k(K(N))$ into a direct sum of μ_N eigenspaces. We write $M_k(K(N)) = M_k(K(N))^+ \oplus M_k(K(N))^-$, where $M_k(K(N))^\epsilon = \{f \in M_k(K(N)) : f|\mu_N = \epsilon f\}$ and similarly for cusp forms.

We will need the following criterion for cuspidality: For $k < 12$ and p prime, elements of $M_k(K(p))^\epsilon$ whose Fourier expansion has the constant term zero are actually in $S_k(K(p))^\epsilon$. To give a proof, consider the Witt map $W : M_k(K(N)) \to M_k(\mathrm{SL}_2(\mathbb{Z})) \otimes M_k(\mathrm{SL}_2(\mathbb{Z}))|\left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right)$, defined by $(Wf)(\tau,\omega) = f\left(\begin{smallmatrix} \tau & 0 \\ 0 & \omega \end{smallmatrix}\right)$. Let $E_k \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ be the Eisenstein series; for $k < 12$, the Witt image of f is $a\left(\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right); f\right) E_k \otimes E_k|\left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right)$. An f without a constant term satisfies $Wf = 0$ and hence $\Phi(f) = 0$, where Φ is Siegel's map. For prime level p, the only other 1-cusp is represented by μ_p and so, when f is also a μ_p eigenform, we have $\Phi(f|\mu_p) = 0$ as well and f is a cusp form. In particular, since $M_2(\mathrm{SL}_2(\mathbb{Z})) = \{0\}$, we always have $M_2(K(p)) = S_2(K(p))$.

N=1. To construct holomorphic Borcherds Products in $S_{k'}((K(N))$, we use a theta block $\eta^u \prod_i \vartheta_{d_i}$ with $d_1^2 + \cdots + d_\ell^2 = 2N$. In level one, for example, the only choice is $1^2 + 1^2 = 2$. Each η contributes a vanishing of $q^{1/24}$ and each ϑ contributes $q^{3/24}$, so that $\phi_{10} = \eta^{18}\vartheta_1^2 \in J_{10,1}^{\mathrm{cusp}}$ has the lowest weight possible since the vanishing order must be a positive integer. It is well known that $S_{10}(\Gamma_2) = \mathbb{C}\Psi_{10}$ is one dimensional and that Igusa's form Ψ_{10} is the Saito-Kurokawa lift of ϕ_{10} as well as a Borcherds Product, $\mathrm{Borch}(\psi)$, which was found by Gritsenko and

Nikulin in [11] where $\psi = -\left(\phi_{10}|V_2\right)/\phi_{10} = 20+2\zeta+2\zeta^{-1}+\cdots \in J_{0,1}^{\mathrm{weak}}$. Historically, this was the first example of Theorem 1.1.

Constructing holomorphic Borcherds Products is usually considered a delicate task because the Fourier coefficients of the singular part of the lifted weakly holomorphic Jacobi form ψ must be mainly positive, but even in level one we can easily construct an infinite family of examples. Set $\Delta = \eta^{24} \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ and for $v \in \mathbb{N}$ define

$$\phi_v = \Delta^{v-1}\eta^{18}\vartheta_1^2 \in J_{12v-2,1}^{\mathrm{cusp}} \text{ and } \psi_v = (-1)^v \frac{\phi_v|V_2}{\phi_v} \in J_{0,1}^{\mathrm{w.h.}}.$$

By Theorem 6.6, $\mathrm{Borch}(\psi_v) \in M_{k_v}(\Gamma_2)$ for some $k_v \in \mathbb{N}$. The second case is particularly interesting. The odd weight form $\mathrm{Borch}(\psi_2)$ vanishes to order 89 on the reducible locus, $\mathrm{Hum}\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} = \Gamma_2 \cdot \{z = 0\}$, and so is divisible by Ψ_{10}^{44}; this leaves a form of weight 35. Therefore, we have $\mathrm{Borch}(\psi_2) = \Psi_{10}^{44}\Psi_{35}$ and a direct proof of the existence of a cusp form of weight 35 in level one. The Borcherds product of Ψ_{35} was found in [12]. Table 1 presents results for small vanishing order v.

Table 1. Weight of $\mathrm{Borch}(\psi_v)$ and multiplicity on $\mathrm{Hum}\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$.

v	k_v	Multiplicity on $\mathrm{Hum}\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$
1	10	2
2	475	89
3	25228	4628
4	1409686	255902
5	81089336	14628136
6	4752949680	853836720
7	282277652800	50558528960
8	16928371578075	3025267676505
9	1022835157543260	182473970938500
10	62169320884762434	11075646070708830

The next series of examples for $N = 2$, 3, 4 and 5 are related to reflective modular forms whose divisors are determined by integral reflections in the corresponding projective stable integral orthogonal group $P\widetilde{O}^+(U \oplus U \oplus \langle -2N \rangle) \cong PK(N)$ where U is the hyperbolic plane, i.e., the even unimodular lattice of signature $(1, 1)$. For the classification of all reflective paramodular forms see [14]. In particular, in [13] one finds the analogues of the Igusa modular forms Ψ_{10} and Ψ_{35} for the levels $N = 2$, 3 and 4.

N=2. For $K(2)$, we can pick $\phi_{8,2} = \eta^{12}\vartheta_1^4 \in J_{8,2}^{\text{cusp}}$ and, setting $\psi_{8,2} = -\frac{\phi_{8,2}|V_2}{\phi_{8,2}}$, get a Borcherds Product with zero constant term, $\text{Borch}(\psi_{8,2}) \in S_8(K(2))$, with representative singular part and divisor

$$\text{sing}\,(\psi_{8,2}) = 16 + 4\zeta + 4\zeta^{-1}, \quad \text{Div}\,(\text{Borch}(\psi_{8,2})) = 4\,\text{Hum}\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}.$$

We use the alternate notation, $\mathcal{H}_N(r_o^2 - 4Nn_om_o, r_o)$, for Humbert surfaces $\text{Hum}\begin{pmatrix} n_o & r_o/2 \\ r_o/2 & Nm_o \end{pmatrix} = K(N)^+\{\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) \in \mathcal{H}_2 : n_o\tau + r_oz + Nm_o\omega = 0\}$, see [13] for details. Since $\phi_{8,2}$ has order of vanishing $v = 1$, Theorem 6.6 tells us that the first two Fourier Jacobi coefficients of $\text{Borch}(\psi_{8,2})$ and $\text{Grit}(\phi_{8,2})$ are equal. Moreover, a part of the divisor of the Gritsenko lifting is induced by the divisor of the lifted Jacobi form, see [13]. Namely, if the Jacobi form has multiplicity m on d-torsion then the lift has multiplicity m on $\mathcal{H}_N(d^2, d)$. Therefore, $\text{Div}(\text{Grit}(\phi_{8,2})) \supseteq \text{Div}(\text{Borch}(\psi_{8,2}))$ and the two forms coincide due to the Koecher principle. This divisor argument together with the Witt map tells us that the space $S_8(K(2))$ is one dimensional. In Ibukiyama and Onodera [17], the ring structure of $M(K(2))$ was given and a generator F_8 of $S_8(K(2))$ was constructed as a polynomial in the thetanullwerte. Thus $\text{Grit}(\phi_{8,2}) = \text{Borch}(\psi_{8,2}) = F_8$ gives three very different constructions of the same modular form. Also, $\phi_{11,2} = \eta^{21}\vartheta_2 \in J_{11,2}^{\text{cusp}}$ gives $\text{Borch}(\psi_{11,2}) \in S_{11}(K(2))$, with representative singular part and divisor

$$\text{sing}\,(\psi_{11,2}) = 22 + \zeta^2 + \zeta^{-2},\ \text{Div}\,(\text{Borch}(\psi_{11,2})) = \mathcal{H}_2(4, 2) + \mathcal{H}_2(1, 1).$$

Again, comparing the divisors and the first Fourier Jacobi coefficient, $\text{Borch}(\psi_{11,2})$ and $\text{Grit}(\psi_{11,2})$ are equal. Furthermore, $S_{11}(K(2)) = \mathbb{C}G_{11}$, for a generator G_{11} constructed from thetanullwerte. The forms F_8 and G_{11} are the cusp forms of lowest weight in the plus and minus spaces of the involution μ_2, respectively, see [17].

N=3. For $K(3)$, we can pick $\phi_{6,3} = \eta^6\vartheta_1^6 \in J_{6,3}^{\text{cusp}}$ and $\psi_{6,3} = -\frac{\phi_{6,3}|V_2}{\phi_{6,3}}$ and get a Borcherds Product with a zero constant term in its Fourier expansion, $\text{Borch}(\psi_{6,3}) \in S_6(K(3))$, and with representative singular part and divisor

$$\operatorname{sing}(\psi_{6,3}) = 12 + 6\zeta + 6\zeta^{-1}, \quad \operatorname{Div}(\operatorname{Borch}(\psi_{6,3})) = 6\mathcal{H}_3(1,1).$$

As above, according to the divisor principle, we have $\operatorname{Borch}(\psi_{6,3}) = \operatorname{Grit}(\phi_{6,3})$. The same argument shows that this is the only Siegel cusp form for $K(3)$ of weight 6, a fact first proved in [18]. Similarly, we have $\phi_{9,3} = \eta^{15}\vartheta_1^2\vartheta_2 \in J_{9,3}^{\text{cusp}}$ and $\operatorname{Borch}(\psi_{9,3}) = \operatorname{Grit}(\phi_{9,3})$ spans $S_9(K(3))$, with representative singular part and divisor

$$\operatorname{sing}(\psi_{9,3}) = 18 + \zeta^2 + 2\zeta + 2\zeta^{-1} + \zeta^{-2},$$
$$\operatorname{Div}(\operatorname{Borch}(\psi_{9,3})) = \mathcal{H}_3(4,2) + 3\mathcal{H}_3(1,1).$$

N=4. For $K(4)$, $\phi_{4,4} = \vartheta_1^8 \in J_{4,4}$ and $\operatorname{Borch}(\psi_{4,4}) \in M_4(K(4))$ satisfy

$$\operatorname{sing}(\psi_{4,4}) = 8 + 8\zeta + 8\zeta^{-1}, \quad \operatorname{Div}(\operatorname{Borch}(\psi_{4,4})) = 8\mathcal{H}_4(1,1).$$

The Borcherds Product, $\operatorname{Borch}(\psi_{4,4})$, is our first example of a noncusp form. The Jacobi form $\phi_{4,4}$ is not a cusp form but this does not affect the divisor argument. Thus $\operatorname{Borch}(\psi_{4,4}) = \operatorname{Grit}(\phi_{4,4})$. Also we have $\phi_{7,4} = \eta^9\vartheta_1^4\vartheta_2 \in J_{7,4}^{\text{cusp}}$ and $\operatorname{Borch}(\psi_{7,4}) \in S_7(K(4))$ satisfies

$$\operatorname{sing}(\psi_{7,4}) = 14 + \zeta^2 + 4\zeta + 4\zeta^{-1} + \zeta^{-2},$$
$$\operatorname{Div}(\operatorname{Borch}(\psi_{7,4})) = \mathcal{H}_4(4,2) + 5\mathcal{H}_4(1,1).$$

Also, $\phi_{10,4} = \eta^{18}\vartheta_2^2 \in J_{10,4}^{\text{cusp}}$ and $\operatorname{Borch}(\psi_{10,4}) \in S_{10}(K(4))$ satisfy

$$\operatorname{sing}(\psi_{10,4}) = 20 + 2\zeta^2 + 2\zeta^{-2},$$
$$\operatorname{Div}(\operatorname{Borch}(\psi_{10,4})) = 2\mathcal{H}_4(4,2) + 2\mathcal{H}_4(1,1).$$

In both cases the Gritsenko lift of ϕ is equal to the Borcherds Product for ψ according to the divisor argument. From [19] we have the dimensions $\dim S_7(K(4)) = 1$ and $\dim S_{10}(K(4)) = 2$.

N=5. (see [13, §4.3]). For $K(5)$, $\phi_{5,5} = \eta^3\vartheta_1^6\vartheta_2 \in J_{5,5}^{\text{cusp}}$ and $\operatorname{Borch}(\psi_{5,5}) \in S_5(K(5))$ satisfy

$$\operatorname{sing}(\psi_{5,5}) = 10 + \zeta^2 + 6\zeta + 6\zeta^{-1} + \zeta^{-2},$$
$$\operatorname{Div}(\operatorname{Borch}(\psi_{5,5})) = \mathcal{H}_5(4,2) + 7\mathcal{H}_5(1,1).$$

Next, $\phi_{8,5} = \eta^{12}\vartheta_1^2\vartheta_2^2 \in J_{8.5}^{\text{cusp}}$ and $\operatorname{Borch}(\psi_{8,5}) \in S_8(K(5))$ satisfy

$$\operatorname{sing}(\psi_{8,5}) = 16 + 2\zeta^2 + 2\zeta + 2\zeta^{-1} + 2\zeta^{-2} + 2q\zeta^5 + 2q\zeta^{-5},$$
$$\operatorname{Div}(\operatorname{Borch}(\psi_{8.5})) = 2\mathcal{H}_5(5,5) + 2\mathcal{H}_5(4,2) + 4\mathcal{H}_5(1,1).$$

These Borcherds Products are cusp forms because their Fourier expansions have a constant term of zero. In both these cases we have the equality of $\operatorname{Borch}(\psi)$ and $\operatorname{Grit}(\phi)$. The first case follows from the divisor argument or the dimension $\dim S_5(K(5)) = 1$ from [18]. The divisor argument does not apply to the second case because of the terms $2q\zeta^5 + 2q\zeta^{-5}$; for this, we may refer ahead to section 8 or appeal to Table 3 of [19], which shows that $S_k(K(5))^\epsilon$, for $\epsilon = (-1)^k$, is determined by the Fourier Jacobi coefficients of indices 5 and 10 for $k < 12$.

A basis of reflective Jacobi forms for all possible N was given in [14]. From these we obtain many other identities between $\mathrm{Borch}(\psi)$ and $\mathrm{Grit}(\phi)$, the most important of which are the modular forms having multiplicity one on their divisors. In these cases, the Fourier expansion determines the generators and relations of a Lorentzian Kac–Moody super Lie algebra. The next example is of a different nature.

N=37. We turn to a favorite Jacobi form and prove something new about it. Let $J_{2,37}^{\mathrm{cusp}} = \mathbb{C}f$, where f is the Jacobi cusp form of weight two and smallest index introduced in [4] where a table of its Fourier coefficients was given. We will prove that:

$$\forall\, n, r \in \mathbb{Z}, \quad \sum_{\alpha \in \mathbb{Z}} c\left(6\alpha^2 + n\alpha, 30\alpha + r; f\right) = 0. \tag{1}$$

In [15], f is shown to be the theta block $\eta^{-6}\vartheta_1^3\vartheta_2^3\vartheta_3^2\vartheta_4\vartheta_5$. The vanishing order is one and by Theorem 6.6, setting $\psi = -(f|V_2)/f$, we have a holomorphic Borcherds lift, $\mathrm{Borch}(\psi) \in S_2(K(37))$, that shares its first two Fourier Jacobi coefficients with $\mathrm{Grit}(f) \in S_2(K(37))$. In [21] it was shown that for primes $p < 600$, if $p \notin \{277, 349, 353, 389, 461, 523, 587\}$ then the weight two paramodular cusp forms are spanned by Gritsenko lifts, that is, $S_2(K(p)) = \mathrm{Grit}\left(J_{2,p}^{\mathrm{cusp}}\right)$. Thence $S_2(K(37))$ is one dimensional and we see that the Gritsenko lift of f is a Borcherds Product as well. The singular Fourier coefficients of ψ are represented by

$$\mathrm{sing}(\psi) = q^6\zeta^{30} + 4 + 3\zeta + 3\zeta^2 + 2\zeta^3 + \zeta^4 + \zeta^5.$$

Thus the divisor of $\mathrm{Borch}(\psi) = \mathrm{Grit}(f)$ is

$$\begin{aligned}\mathrm{Div}\,(\mathrm{Borch}(\psi)) = {} & \mathrm{Hum}\left(\begin{smallmatrix} 6 & 15 \\ 15 & 37 \end{smallmatrix}\right) + \mathrm{Hum}\begin{pmatrix} 0 & 5/2 \\ 5/2 & 37 \end{pmatrix} + \mathrm{Hum}\left(\begin{smallmatrix} 0 & 2 \\ 2 & 37 \end{smallmatrix}\right) \\ & + 2\,\mathrm{Hum}\begin{pmatrix} 0 & 3/2 \\ 3/2 & 37 \end{pmatrix} + 4\,\mathrm{Hum}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 37 \end{smallmatrix}\right) + 10\,\mathrm{Hum}\begin{pmatrix} 0 & 1/2 \\ 1/2 & 37 \end{pmatrix}.\end{aligned}$$

Thus $\mathrm{Grit}(f)$ vanishes on the Humbert surface in $K(37)^+\backslash\mathcal{H}_2$,

$$\mathrm{Hum}\left(\begin{smallmatrix} 6 & 15 \\ 15 & 37 \end{smallmatrix}\right) = K(37)^+\{\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) \in \mathcal{H}_2 : 6\tau + 30z + 37\omega = 0\}.$$

In terms of Fourier coefficients, write

$$\mathrm{Grit}(f)\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) = \sum_{\alpha,\beta,\gamma \in \mathbb{Z}} a\left(\begin{pmatrix} \gamma & \beta/2 \\ \beta/2 & 37\alpha \end{pmatrix}; \mathrm{Grit}(f)\right) q^\gamma \zeta^\beta \xi^{37\alpha},$$

where $q = e(\tau)$, $\zeta = e(z)$ and $\xi = e(\omega)$. Substitution for ξ using the relation $q^6\zeta^{30}\xi^{37} = 1$ gives

$$\sum_{\alpha,\beta,\gamma\in\mathbb{Z}} a\left(\begin{pmatrix} \gamma & \beta/2 \\ \beta/2 & 37\alpha \end{pmatrix}; \mathrm{Grit}(f)\right) q^{\gamma-6\alpha}\zeta^{\beta-30\alpha} = 0,$$

or, setting $n = \gamma - 6\alpha$ and $r = \beta - 30\alpha$,

$$\forall n, r \in \mathbb{Z}, \quad \sum_{\alpha\in\mathbb{Z}} a\left(\begin{pmatrix} n+6\alpha & (r+30\alpha)/2 \\ (r+30\alpha)/2 & 37\alpha \end{pmatrix}; \mathrm{Grit}(f)\right) = 0.$$

Using $a\left(\begin{pmatrix} n & r/2 \\ r/2 & Nm \end{pmatrix}; \mathrm{Grit}(f)\right) = \sum_{\delta\in\mathbb{N}:\delta|(n,r,m)} \delta^{k-1} c\left(\frac{nm}{\delta^2}, \frac{r}{\delta}; f\right)$, for the Fourier coefficients of the Gritsenko lift, we obtain

$$\forall n, r \in \mathbb{Z}, \quad \sum_{\alpha\in\mathbb{Z}} \ \sum_{\delta\in\mathbb{N}:\,\delta|(n+6\alpha,\, r+30\alpha,\, \alpha)} \delta c\left(\frac{(n+6\alpha)\alpha}{\delta^2}, \frac{r+30\alpha}{\delta}; f\right) = 0.$$

This may be reduced, by induction on $\gcd(n, r)$, to the case $\delta = 1$, which is equation (1), as claimed. A direct proof of equation (1) was shown to us by D. Zagier, and we leave this as a challenge to the reader.

It would be desirable to have a direct proof of $\mathrm{Borch}(\psi) = \mathrm{Grit}(\phi)$ in general when the order of vanishing of the theta block ϕ is one, instead of relying on information about modular forms available in specific cases. Indeed this, as presented in Conjecture 8.1, would be used as an additional tool in the investigation of modular forms. We, in fact, prove Conjecture 8.1 in many cases; namely, weights $4 \le k \le 11$. The examples of this current section for weights $k \ge 4$ are thus merely the first instances in infinite families of Borcherds Products that are also Gritsenko lifts. As weight one paramodular forms with trivial character vanish, this leaves only the cases of weights 2 and 3 open, see §8 for further examples and discussion.

3. Siegel Modular forms, Jacobi forms and liftings

Let $\mathrm{Sp}_n(\mathbb{R})$ act on the Siegel upper half space $\mathcal{H}_n$ by linear fractional transformations. Let $V_n(\mathbb{R})$ be the Euclidean space of real n-by-n symmetric matrices with inner product $\langle A, B\rangle = \mathrm{tr}(AB)$, and extend this product $\mathbb{C}$-linearly to $V_n(\mathbb{C})$. Let Γ be a subgroup of projective rational elements of $\mathrm{Sp}_n(\mathbb{R})$ commensurable with $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$. We write $M_k(\Gamma, \chi)$ for the $\mathbb{C}$-vector space of Siegel modular forms of weight k and character χ with respect to Γ. These are holomorphic functions $f : \mathcal{H}_n \to \mathbb{C}$ that transform with respect to $\sigma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$ by the factor of automorphy $\mu_{\mathrm{Siegel}}^k \chi$ where $\mu_{\mathrm{Siegel}}(\sigma, \Omega) = \det(C\Omega + D)$. Using the slash notation, $(f|_k\sigma)(\Omega) = \mu_{\mathrm{Siegel}}(\sigma, \Omega)^{-k} f(\sigma\cdot\Omega)$, we have $f|_k\sigma = \chi(\sigma)f$ for all $\sigma \in \Gamma$. For $n = 1$, we additionally require boundedness at the cusps, which is redundant for $n \geq 2$ by the Koecher Principle. The space of meromorphic f satisfying this automorphy condition is denoted by $M_k^{\mathrm{mero}}(\Gamma, \chi)$. The space of cusp forms is defined, using Siegel's Φ map, as $S_k(\Gamma, \chi) = \{f \in M_k(\Gamma, \chi) : \forall \sigma \in \Gamma_n, \Phi(f|_k \sigma) = 0\}$. A Siegel modular form has a Fourier expansion of the form $f(\Omega) = \sum_T a(T) e\left(\langle \Omega, T \rangle\right)$, where T runs over $\mathcal{X}_n^{\mathrm{semi}}(\mathbb{Q})$, the semidefinite elements of $V_n(\mathbb{Q})$, and where $e(z) = e^{2\pi i z}$. For cusp forms we may restrict the T to $\mathcal{X}_n(\mathbb{Q})$, the definite elements of $V_n(\mathbb{Q})$. The *principal* congruence subgroups are $\Gamma_n(N) = \{\sigma \in \Gamma_n : \sigma \equiv I_{2n} \mod N\}$. We will be concerned with degree $n = 2$ and the *paramodular* group of level N:

$$K(N) = \begin{pmatrix} * & N* & * & * \\ * & * & * & */N \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Q}), \qquad * \in \mathbb{Z},$$

which is isomorphic to the the integral symplectic group of the skew symmetric form with the elementary divisors $(1, N)$, see [5] and [6]. Fourier expansions of paramodular forms sum over $T \in \mathcal{X}_2^{\mathrm{semi}}(N) = \{\left(\begin{smallmatrix} a & b \\ b & Nc \end{smallmatrix}\right) \in \mathcal{X}_2^{\mathrm{semi}}(\mathbb{Q}) : a, 2b, c \in \mathbb{Z}\}$. The paramodular group K(N) is not maximal in the real symplectic group $\mathrm{Sp}_2(\mathbb{R})$ of rank 2, see [9] for a complete description of its extensions. In particular, for any natural number $N > 1$ the paramodular group $K(N)$ has a normalizing involution μ_N given by $\mu_N = \begin{pmatrix} F_N^* & 0 \\ 0 & F_N \end{pmatrix}$, where $F_N = \frac{1}{\sqrt{N}}\left(\begin{smallmatrix} 0 & 1 \\ -N & 0 \end{smallmatrix}\right)$ is the Fricke involution, and we will frequently use the group $K(N)^+$ generated by $K(N)$ and μ_N. We let $\chi_F : K(N)^+ \to \{\pm 1\}$ be the nontrivial character with kernel $K(N)$ and observe that $M_k(K(N)) = M_k(K(N)^+) \oplus M_k(K(N)^+, \chi_F)$ is the decomposition into plus and minus μ_N-eigenspaces.

The following definition of Jacobi forms, see [13], is equivalent to the usual one [4]. The only difference is that the book of Eichler and Zagier does not address Jacobi forms of half-integral index, which play a rather important role in [13]. Consider two types of elements in Γ_2,

$$h = \begin{pmatrix} 1 & 0 & 0 & v \\ \lambda & 1 & v & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

for $\lambda, v, \kappa \in \mathbb{Z}$, and for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Let the subgroup of Γ_2 generated by the h be called the Heisenberg group $H(\mathbb{Z})$. The character $v_H : H(\mathbb{Z}) \to \{\pm 1\}$ is defined by $v_H(h) = (-1)^{\lambda v + \lambda + v + \kappa}$. The second type constitute a copy of $\mathrm{SL}_2(\mathbb{Z})$ inside Γ_2. This copy of $\mathrm{SL}_2(\mathbb{Z})$ and $H(\mathbb{Z})$ and $\pm I_4$ generate a group inside Γ_2 equal to

$$P_{2,1}(\mathbb{Z}) = \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Z}), \qquad * \in \mathbb{Z}.$$

The character v_H extends uniquely to a character on $P_{2,1}(\mathbb{Z})$ that is trivial on the copy of $\mathrm{SL}_2(\mathbb{Z})$. Likewise, the factor of automorphy of the Dedekind Eta function

$$\mu_\eta\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau\right) = \frac{\eta\left(\frac{a\tau+b}{c\tau+d}\right)}{\eta(\tau)} = \sqrt{c\tau+d}\,\epsilon\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right),$$

extends uniquely to a factor of automorphy on $P_{2,1}(\mathbb{Z}) \times (\mathcal{H}_1 \times \mathbb{C})$ that is trivial on $H(\mathbb{Z})$ and we use this extension as the definition of the multiplier $\epsilon : P_{2,1}(\mathbb{Z}) \to e\left(\frac{1}{24}\mathbb{Z}\right)$. We also write $\epsilon = \epsilon_\eta$ for clarity.

For $m \in \mathbb{Q}$, $a, b, 2k \in \mathbb{Z}$, consider holomorphic $\phi : \mathcal{H}_1 \times \mathbb{C} \to \mathbb{C}$, such that the modified function $\tilde{\phi} : \mathcal{H}_2 \to \mathbb{C}$, given by $\tilde{\phi}\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) = \phi(\tau, z)e(m\omega)$, transforms by the factor of automorphy $\mu_{\mathrm{Siegel}}^k \epsilon^a v_H^b$ for $P_{2,1}(\mathbb{Z})$. We always select holomorphic branches of roots that are positive on the purely imaginary elements of the Siegel half space. We necessarily have $2k \equiv a \equiv b \mod 2$ and $m \geq 0$ for nontrivial ϕ. Such ϕ have Fourier expansions $\phi(\tau, z) = \sum_{n,r \in \mathbb{Q}} c(n, r; \phi) q^n \zeta^r$, for $q = e(\tau)$ and $\zeta = e(z)$. We write $\phi \in J_{k,m}^{\mathrm{w.h.}}(\epsilon^a v_H^b)$ if, additionally, the support of ϕ has n bounded from below, and call such forms *weakly holomorphic.* We write $\phi \in J_{k,m}^{\mathrm{weak}}(\epsilon^a v_H^b)$ if the support of ϕ satisfies $n \geq 0$; $\phi \in J_{k,m}(\epsilon^a v_H^b)$ if $4mn - r^2 \geq 0$; $\phi \in J_{k,m}^{\mathrm{cusp}}(\epsilon^a v_H^b)$ if $4mn - r^2 > 0$. Similar definitions are made for subgroups. For example, $\phi \in J_{k,m}(\Gamma(N), \epsilon^a v_H^b)$ means that we demand the automorphy of $\tilde{\phi}$ for $P_{2,1}(\mathbb{Z}) \cap \Gamma_2(N)$ and demand the corresponding support condition at each cusp.

For basic examples of Jacobi forms we make use of the Dedekind Eta function $\eta(\tau) = q^{\frac{1}{24}} \prod_{n \in \mathbb{N}}(1 - q^n)$ and the odd Jacobi theta function:

$$\begin{aligned} \vartheta(\tau, z) &= \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{(2n+1)^2}{8}} \zeta^{\frac{2n+1}{2}} \\ &= q^{\frac{1}{8}} \left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}\right) \prod_{j \in \mathbb{N}} (1 - q^j \zeta)(1 - q^j \zeta^{-1})(1 - q^j). \end{aligned}$$

We have $\vartheta \in J_{\frac{1}{2},\frac{1}{2}}^{\mathrm{cusp}}(\epsilon^3 v_H)$, $\eta \in J_{\frac{1}{2},0}^{\mathrm{cusp}}(\epsilon)$ and $\vartheta_\ell \in J_{\frac{1}{2},\frac{1}{2}\ell^2}^{\mathrm{cusp}}(\epsilon^3 v_H^\ell)$, where $\vartheta_\ell(\tau, z) = \vartheta(\tau, \ell z)$ and $\ell \in \mathbb{N}$, compare [13].

We now report on the existence and uniqueness of the characters, denoted $\epsilon^a \times v_H^b : K(N)^+ \to e(\frac{\mathbb{Z}}{12})$, whose restriction to $P_{2,1}(\mathbb{Z})$ is $\epsilon^a v_H^b$ and whose value on μ_N is one. It follows from [5] that the extended paramodular group $K(N)^+$ is generated by μ_N and $P_{2,1}(\mathbb{Z})$. Thus any character $\chi : K(N)^+ \to e(\mathbb{Q})$ is determined by its value on μ_N and its restriction to $P_{2,1}(\mathbb{Z})$. For the existence, a result of [10] is that the abelianization of $K(N)^+$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/Q\mathbb{Z}$, where $Q = \gcd(2N, 12)$. The character v_F is an element of order two. Furthermore, there is a character of order Q that has a restriction to $P_{2,1}(\mathbb{Z})$ given by $\epsilon^{24/Q} v_H^{2N/Q}$ and a value of one on μ_N. Accordingly, for $a, b \in \mathbb{Z}$, the character $\epsilon^a \times v_H^b$ exists precisely when there is a $j \in \mathbb{Z}$ such that $a \equiv j \frac{24}{\gcd(2N,12)} \mod 24$ and $b \equiv j \frac{2N}{\gcd(2N,12)} \mod 2$. These brief considerations suffice, since we eventually prove that the paramodular forms considered here have trivial character.

Let $\phi \in J_{k,t}^{\text{w.h.}}$ be a weakly holomorphic Jacobi form. Recall the level raising Hecke operators $V_\ell : J_{k,t}^{\text{w.h.}} \to J_{k,t\ell}^{\text{w.h.}}$ from [4], page 41. These operators stabilize both $J_{k,t}$ and $J_{k,t}^{\text{cusp}}$ and have the following action on Fourier coefficients:

$$c\left(n, r; \phi | V_m\right) = \sum_{d \in \mathbb{N}:\, d | (n,r,m)} d^{k-1} c\left(\frac{nm}{d^2}, \frac{r}{d}; \phi\right).$$

Given any $\phi \in J_{k,t}^{\text{w.h.}}$, we may consider the following series

$$\mathrm{Grit}(\phi)\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) = \delta(k) c(0,0;\phi) G_k(\tau) + \sum_{m \in \mathbb{N}} \left(\phi | V_m\right)(\tau, z) e(mt\omega)$$

where $\delta(k) = 1$ for even $k \geq 4$ and $\delta(k) = 0$ for all other k, and $G_k(\tau) = (2\pi i)^{-k}(k-1)!\, \zeta(k) + \Sigma_{n \geq 1} \sigma_{k-1}(n) e(\tau)$ is the Eisenstein series of weight k.

Theorem 3.1. ([5], [6]) *For $\phi \in J_{k,t}$, the series* $\mathrm{Grit}(\phi)$ *converges on $\mathcal{H}_2$ and defines a holomorphic function* $\mathrm{Grit}(\phi) : \mathcal{H}_2 \to \mathbb{C}$ *that is an element of $M_k\left(K(t)^+, \chi_F^k\right)$. This is a cusp form if $\phi \in J_{k,t}^{\text{cusp}}$.*

The paramodular form $\mathrm{Grit}(\phi)$ is called the *Gritsenko lift* of the Jacobi form ϕ and defines a linear map $\mathrm{Grit} : J_{k,t} \to M_k\left(K(t)^+, \chi_F^k\right)$. Forms with character χ_F^k are called *symmetric,* with character χ_F^{k+1}, *antisymmetric.* Gritsenko lifts are hence symmetric. Antisymmetric forms are usually harder to construct. A different type of lifting construction is due to Borcherds (see [1]) via his theory of infinite products in many variables on orthogonal groups. The divisor of a Borcherds Product is supported on rational quadratic divisors. In the case of the

Siegel upper half plane of degree two, these rational quadratic divisors are the so-called Humbert modular surfaces.

Definition 3.2. *Let $N \in \mathbb{N}$. For $n_o, r_o, m_o \in \mathbb{Z}$ with $m_o \geq 0$ and* $\gcd(n_o, r_o, m_o) = 1$*, set* $T_o = \begin{pmatrix} n_o & r_o/2 \\ r_o/2 & Nm_o \end{pmatrix}$ *such that* $\det(T_o) < 0$*. We call*

$$\mathrm{Hum}(T_o) = K(N)^+\{\Omega \in \mathcal{H}_2 : \langle \Omega, T_o \rangle = 0\} \subseteq K(N)^+ \backslash \mathcal{H}_2.$$

a Humbert modular surface.

From [9] we have that a Humbert surface $\mathrm{Hum}(T_o)$ only depends upon two pieces of data: the discriminant $D = r_o^2 - 4Nm_o n_o$ and $r_o \mod 2N$. We may use this data to parameterize Humbert surfaces; write $\mathcal{H}_N(D, r) = \mathrm{Hum}(T_o)$ for any T_o of the form $\begin{pmatrix} n_o & r_o/2 \\ r_o/2 & Nm_o \end{pmatrix}$, with $\gcd(n_o, r_o, m_o) = 1$ and $m_o \geq 0$, satisfying $-\det(2T_o) = D$ and $\langle T_o, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle \equiv r \mod 2N$. For convenience, we extend the notation $\mathcal{H}_N(D, r)$ to be empty when no such T_o exists.

The original Borcherds construction [1] used the Fourier coefficients of vector valued modular forms and was written using the Fourier expansion at a 0-dimensional cusp of an orthogonal modular variety. A variant of Borcherds Products proposed by Gritsenko and Nikulin, see [11] and [13], was based on the Fourier expansion at a 1-dimensional cusp. The difference between these two approaches was explained in [8] for the Borcherds modular form $\Phi_{12} \in M_{12}(O^+(II_{2,26}))$. For the proof of our main Theorem 6.6 we will use

Theorem 3.3. ([11], [13], [8]) *Let $N, N_o \in \mathbb{N}$. Let $\Psi \in J_{0,N}^{\mathrm{w.h.}}$ be a weakly holomorphic Jacobi form with Fourier expansion*

$$\Psi(\tau, z) = \sum_{n,r \in \mathbb{Z}:\, n \geq -N_o} c\,(n, r)\, q^n \zeta^r$$

and $c(n, r) \in \mathbb{Z}$ for $4Nn - r^2 \leq 0$. Then we have $c(n, r) \in \mathbb{Z}$ for all $n, r \in \mathbb{Z}$. We set

$$24A = \sum_{\ell \in \mathbb{Z}} c(0, \ell); \quad 2B = \sum_{\ell \in \mathbb{N}} \ell c(0, \ell); \quad 4C = \sum_{\ell \in \mathbb{Z}} \ell^2 c(0, \ell);$$

$$D_0 = \sum_{n \in \mathbb{Z}:\, n<0} \sigma_0(-n) c(n, 0);\ k = \frac{1}{2} c(0, 0);\ \chi = (\epsilon^{24A} \times v_H^{2B}) \chi_F^{k+D_0}.$$

There is a function $\mathrm{Borch}(\Psi) \in M_k^{\mathrm{mero}}(K(N)^+, \chi)$ *whose divisor in* $K(N)^+ \backslash \mathcal{H}_2$ *consists of Humbert surfaces* $\mathrm{Hum}(T_o)$ *for* $T_o = \begin{pmatrix} n_o & r_o/2 \\ r_o/2 & Nm_o \end{pmatrix}$ *with* $\gcd(n_o, r_o, m_o) = 1$ *and* $m_o \geq 0$*. The multiplicity of* $\mathrm{Borch}(\Psi)$ *on*

$\mathrm{Hum}(T_o)$ *is* $\sum_{n\in\mathbb{N}} c(n^2 n_o m_o, n r_o)$. *In particular, if* $c(n,r) \geq 0$ *when* $4Nn - r^2 \leq 0$ *then* $\mathrm{Borch}(\Psi) \in M_k\left(K(N)^+, \chi\right)$. *In particular,*

$$\mathrm{Borch}(\Psi)(\mu_N\langle\Omega\rangle) = (-1)^{k+D_0}\,\mathrm{Borch}(\Psi)(\Omega), \ \text{for } \Omega \in \mathcal{H}_2.$$

For sufficiently large λ, *for* $\Omega = \left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) \in \mathcal{H}_2$ *and* $q = e(\tau)$, $\zeta = e(z)$, $\xi = e(\omega)$, *the following product converges on* $\{\Omega \in \mathcal{H}_2 : \mathrm{Im}\,\Omega > \lambda I_2\}$*:*

$$\mathrm{Borch}(\Psi)(\Omega) = q^A \zeta^B \xi^C \prod_{\substack{n,r,m\in\mathbb{Z}:\, m\geq 0,\ \text{if } m=0 \text{ then } n\geq 0 \\ \text{and if } m=n=0 \text{ then } r<0.}} \left(1 - q^n \zeta^r \xi^{Nm}\right)^{c(nm,r)}$$

and is on $\{\Omega \in \mathcal{H}_2 : \mathrm{Im}\,\Omega > \lambda I_2\}$ *a rearrangement of*

$$\mathrm{Borch}(\Psi) = \left(\eta^{c(0,0)} \prod_{\ell\in\mathbb{N}} \left(\frac{\tilde{\vartheta}_\ell}{\eta}\right)^{c(0,\ell)}\right) \exp\left(-\,\mathrm{Grit}(\Psi)\right).$$

Remarks: This last representation of $\mathrm{Borch}(\Psi)$ gives an experimental algorithm for the construction of Borcherds products. It gives the first two Fourier Jacobi coefficients of $\mathrm{Borch}(\Psi)$: the first one is a theta block $\Theta = \eta^{c(0,0)} \prod (\vartheta_\ell/\eta)^{c(0,\ell)}$ and the second is the product $-\Theta\Psi$. As is standard, the convergence of an infinite product on $\mathcal{H}_2$ is not defined to mean that the sequence of partial products has a limit; rather, it means that for each $\Omega \in \mathcal{H}_2$, some tail of the product has a sequence of partial products with a nonzero limit. The next proposition is essential in the proof of the Theorem just stated, see [13].

Proposition 3.4. *Continuing with the notation of Theorem 3.3, set* $D_1 = \sum_{n,r\in\mathbb{Z}:\, n<0} \sigma_1(-n)\, c(n,r;\Psi)$. *We have* $tA - tD_1 - C = 0$.

By using multiplicative Hecke operators, one can show that paramodular Borcherds Products satisfy special identities, see Theorem 3.3 and the identity (3.25) in [13]. Heim and Murase have proven a converse, that these special multiplicative identities in fact characterize Borcherds Products among automorphic forms, see [16].

4. Generalized Valuations

Let R be a ring and G an abelian semigroup. A map $\nu : R\backslash\{0\} \to G$ satisfies the *valuation property* on R if

$$\nu(fg) = \nu(f) + \nu(g)$$

for all nontrivial $f, g \in R$; we call such ν a *generalized valuation,* When G is also partially ordered, one could potentially ask for the additional property: for all $y \in G$, $y \geq \nu(f)$ and $y \geq \nu(g)$ imply $y \geq \nu(fg)$, but this property plays no direct role for us here. What is important is that certain rings of formal series admit generalized valuations into partially ordered abelian semigroups of closed convex sets.

For a simple example of a generalized valuation, consider the ring $R_n = \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ of Laurent polynomials in n variables. Let $f = \sum_{I \in \mathbb{Z}^n} a(I) x^I \in R_n$, where we use the multi-index notation $x^I = \prod_{i=1}^n x_i^{I_i}$, and define the support of f as $\mathrm{supp}(f) = \{I \in \mathbb{Z}^n : a(I) \neq 0\}$. For $S \subseteq \mathbb{R}^n$, denote the *convex hull* of S by $\mathrm{conv}(S) = \{\sum_{i=1}^{\text{finite}} \alpha_i s_i \in \mathbb{R}^n : s_i \in S,\ \alpha_i \geq 0,\ \sum_{i=1}^{\text{finite}} \alpha_i = 1\}$. Define the generalized valuation

$$\begin{aligned} \nu_{\text{poly}} : R_n \setminus \{0\} &\to \text{ Closed convex subsets of } \mathbb{R}^n, \\ f &\mapsto \mathrm{conv}\left(\mathrm{supp}(f)\right). \end{aligned}$$

The closed convex subsets of $\mathbb{R}^n$ are a partially ordered abelian semigroup under pointwise addition $K_1 + K_2 = \{x + y \in \mathbb{R}^n : x \in K_1 \text{ and } y \in K_2\}$ and inclusion $K_1 \subseteq K_2$. The map ν_{poly} is indeed a generalized valuation and selected examples will convince the reader that this is not altogether trivial.

Proposition 4.1. *Let $f, g \in R_n$ be nontrivial Laurent polynomials. We have $\nu_{poly}(fg) = \nu_{poly}(f) + \nu_{poly}(g)$.*

Proof. We have $\mathrm{supp}(fg) \subseteq \mathrm{supp}(f) + \mathrm{supp}(g)$ directly from the definition of polynomial multiplication and taking convex hulls gives one containment $\nu_{\text{poly}}(fg) \subseteq \mathrm{conv}\left(\mathrm{supp}(f) + \mathrm{supp}(g)\right) = \nu_{\text{poly}}(f) + \nu_{\text{poly}}(g)$. The other containment uses the Krein-Milman Theorem: a compact set K in a Euclidean space V is the convex hull of its extreme points $E(K)$; an extreme point of K being, by definition, a point of K that is not in the interior of any line segment contained in K, see [24], page 167. Let I_o be an extreme point of $\nu_{\text{poly}}(f) + \nu_{\text{poly}}(g) = \mathrm{conv}\left(\mathrm{supp}(f) + \mathrm{supp}(g)\right)$ so that necessarily $I_o \in \mathrm{supp}(f) + \mathrm{supp}(g)$, see [24], page 165. We will conclude the proof by showing that $I_o \in \mathrm{supp}(fg)$ so that

$$\begin{aligned} \nu_{\text{poly}}(fg) &= \mathrm{conv}\left(\mathrm{supp}(fg)\right) \\ &\supseteq \mathrm{conv}\left(E(\nu_{\text{poly}}(f) + \nu_{\text{poly}}(g))\right) \overset{K.M.}{=} \nu_{\text{poly}}(f) + \nu_{\text{poly}}(g). \end{aligned}$$

If $I_o \notin \mathrm{supp}(fg)$, then the coefficient of x^{I_o} was cancelled in the multiplication of f and g and so there are at least two decompositions

$$I_o = A + B = a + b,$$

with $A, a \in \mathrm{supp}(f)$ and $B, b \in \mathrm{supp}(g)$ and $(A, B) \neq (a, b)$, hence $A \neq a$ and $B \neq b$. Let $m, w \in \nu_{\text{poly}}(f) + \nu_{\text{poly}}(g)$ be defined as $m = A + b$ and $w = a + B$. Note that m, w and I_o are distinct. However, I_o is the midpoint of $\overline{mw} \subseteq \nu_{\text{poly}}(f) + \nu_{\text{poly}}(g)$ and I_o is extreme, a contradiction. $\square$

The "other" property is $\nu_{\text{poly}}(f+g) \subseteq \text{conv}\,(\nu_{\text{poly}}(f) \cup \nu_{\text{poly}}(g))$.

A naive generalization of the valuation property to infinite series is false. For example, $(1+x+x^2+\dots)(1-x) = 1$ but $[0,\infty)+[0,1] = [0,\infty) \neq \{0\}$. From the point of view of convex geometry, the issue here is applying an appropriate generalization of the Krein-Milman Theorem to closed convex subsets C of a euclidean space V. Let $P \subseteq V$ be a subset called *points* and $D \subseteq V$ be a subset called *directions.* Define a generalized notion of convex hull that incorporates directions as well as points by

$$\text{conv}(P;D) =$$
$$\{\sum_{i=1}^{\text{finite}} \alpha_i p_i + \sum_{j=1}^{\text{finite}} \beta_j v_j \in V : \, p_i \in P,\, v_j \in D,\, \alpha_i, \beta_j \geq 0,\, \sum_{i=1}^{\text{finite}} \alpha_i = 1\}.$$

More geometrically, we can say $\text{conv}(P;D) = \text{conv}(P) + \text{cone}_0(D)$, where $\text{cone}(D) = \mathbb{R}_{>0}\,\text{conv}(D)$ and $\text{cone}_0(D) = \mathbb{R}_{\geq 0}\,\text{conv}(D)$ as usual. A *face* of a convex set C is a convex subset $\tilde{C}$ of C such that every closed line segment in C with a relative interior point in $\tilde{C}$ lies entirely in $\tilde{C}$. Thus, the extreme points of C are exactly the zero dimensional faces. If a face $\tilde{C}$ of C is a ray then the parallel ray from the origin is called an *extreme direction.* The *recession cone* of a nonempty convex set C is defined as

$$\text{r-cone}(C) = \{y \in V : C + \mathbb{R}_{\geq 0} y \subseteq C\}.$$

For a nonempty closed convex set, an extreme direction is necessarily a subset of the recession cone, see [24], page 163 or Theorem 8.3 on page 63. For an important example, consider the convex parabolic region $C = \{(x,y) \in \mathbb{R}^2 : y \geq x^2\}$; the recession cone of C consists of the nonnegative y-axis, which is *not* an extreme direction of C. Here is a generalization of the Krein-Milman Theorem which allows unboundedness.

Theorem 4.2. ([24], page 166.) *Let V be a Euclidean space. Let $C \subseteq V$ be a closed convex set containing no lines. Then C is the convex hull of its extreme points and extreme directions.*

For a second example, define a generalized valuation ν_{series} on polar Laurent series of one variable

$$\begin{aligned} \nu_{\text{series}} : \mathbb{C}((x)) \setminus \{0\} &\to \text{ Closed convex subsets of } \mathbb{R}^1, \\ f &\mapsto \text{conv}\,(\text{supp}(f); [0,\infty))\,. \end{aligned}$$

The valuation property of ν_{series} follows from the equality $\nu_{\text{series}}(f) = [\min(\operatorname{supp}(f)), \infty)$, so that the left endpoint is the usual "order of vanishing" valuation of polar Laurent series. In general finite dimension, the defintion

$$\begin{aligned}\nu_{\text{series}} : \mathbb{C}((x_1, \dots, x_n)) \setminus \{0\} \to \ & \text{Closed convex subsets of } \mathbb{R}^n, \\ f \mapsto \ & \operatorname{conv}\left(\operatorname{supp}(f); [0,\infty)^n\right),\end{aligned}$$

satisfies the valuation propetry, as can be proven directly or reduced to the one variable case. Here, the other property is $\nu_{\text{series}}(f+g) \subseteq \operatorname{conv}\left(\nu_{\text{series}}(f) \cup \nu_{\text{series}}(g); [0,\infty)^n\right)$.

For a third example, similar results hold for Siegel modular forms. For $f \in S_k(\Gamma_n)$ with $\operatorname{supp}(f) = \{T \in \mathcal{X}_n : a(T;f) \neq 0\}$ and $\nu_{\text{Siegel}} = \operatorname{Closure}_{V_n(\mathbb{R})}\left(\operatorname{conv}\left(\mathbb{R}_{\geq 1} \operatorname{supp}(f)\right)\right)$, we have the valuation property for nontrivial cusp forms, see [23]. As a remark, $\nu_{\text{Siegel}}(f)$ has the following property, see [20]. If $\det(Y)^{k/2}|f(X+iY)|$ attains its maximum at $X_o + iY_o \in \mathcal{H}_n$, then $\frac{k}{4\pi} Y_o^{-1} \in \nu_{\text{Siegel}}(f)$.

Generalized valuations for Jacobi forms require extra care, being an intermediate case between elliptic and Siegel modular forms. For $N \in \mathbb{N}$, let $R(N) = \mathbb{C}[\zeta^{1/N}, \zeta^{-1/N}]((q^{1/N}))$ be formal polar Laurent-Puiseaux series in two variables, considering q to be the first and ζ the second variable. For $f = \sum_{n,r \in \mathbb{Q}} c(n,r) q^n \zeta^r \in R(N)$ define the support $\operatorname{supp}(f) = \{(n,r) \in (\frac{1}{N}\mathbb{Z})^2 : c(n,r) \neq 0\}$ and

$$\nu_{\text{J}}(f) = \nu_{\text{Jacobi}}(f) = \operatorname{Closure}_{\mathbb{R}^2}\left(\operatorname{conv}\left(\operatorname{supp}(f); [0,\infty) \times \{0\}\right)\right).$$

The map ν_{J} does not have the valuation property on the entire ring $R(N)$ but we regain the valuation property on a subring that allows recession only in the q-direction. Set the ray $\vec{A} = [0,\infty) \times \{0\}$ for brevity. We will require that $\operatorname{supp}(f)$ be contained in a closed convex parabolic region with recession cone $\vec{A}$.

Proposition 4.3. *The set* $R_J(N) = \{f \in R(N) : \exists a \in \mathbb{R}_+, \exists b, c \in \mathbb{R} : \operatorname{supp}(f) \subseteq \{(n,r) \in \mathbb{R}^2 : n \geq ar^2 + br + c\}\}$ *is a ring.*

Proof. This just amounts to the fact that the sum or union of convex parabolic regions with the same recession cone is contained in another such region. □

In $R_J(N)$, the closure in the definition of ν_{J} is redundant.

Lemma 4.4. *For nontrivial* $f \in R_J(N)$*, we have* $E\left(\nu_J(f)\right) \subseteq \operatorname{supp}(f)$ *and*

$$\nu_J(f) = \operatorname{Closure}_{\mathbb{R}^2}\left(\operatorname{conv}\left(\operatorname{supp}(f); \vec{A}\right)\right) = \operatorname{conv}\left(\operatorname{supp}(f); \vec{A}\right).$$

Proof. Let I_o be an exposed extreme point of $\nu_{\mathrm{J}}(f)$; we will show that $I_o \in \mathrm{supp}(f)$. A face $\tilde{C}$ of a convex set C is *exposed* if $\tilde{C}$ is the subset of C where some linear function h attains its minimum on C. Let h be the linear function on $\mathbb{R}^2$ that attains its minimum on $\nu_{\mathrm{J}}(f)$ uniquely at $I_o = (n_o, r_o)$. We have $h(n,r) = \alpha n + \beta r$ for some α, $\beta \in \mathbb{R}$. First, $\alpha \geq 0$ since $(n_o, r_o) + \vec{A} \subseteq \nu_{\mathrm{J}}(f)$ and, additionally, $\alpha > 0$ since $\{I \in \nu_{\mathrm{J}}(f) : h(I) = h(I_o)\} = \{I_o\}$. Because $\alpha > 0$ and $\mathrm{supp}(f)$ is contained in the latttice $(\frac{1}{N}\mathbb{Z})^2$ and $\mathrm{supp}(f)$ is contained in a convex parabolic region with recession cone $\vec{A}$, the set $\mathcal{S}(B) = \{I \in \mathrm{supp}(f) : h(I) \leq B\}$ is finite for any $B > 0$. Therefore $\inf_{I \in \mathrm{supp}(f)} h(I) = \min_{I \in \mathcal{S}(h(I_1))} h(I)$ for any $I_1 \in \mathrm{supp}(f)$. Thus $\rho = \min_{I \in \mathrm{supp}(f)} h(I)$ exists and is attained on a finite set $\mathcal{M} \subseteq \mathrm{supp}(f)$. It follows that $h \geq \rho$ on $\mathrm{conv}(\mathrm{supp}(f); \vec{A})$ and that, by the continuity of h, we have $h \geq \rho$ on the closure $\nu_{\mathrm{J}}(f)$. Thus, the minimum of h on $\nu_{\mathrm{J}}(f)$ equals ρ and the face of $\nu_{\mathrm{J}}(f)$ where h attains its minimum ρ equals $\{I_o\}$ and contains $\mathcal{M}$; therefore $\{I_o\} = \mathcal{M} \subseteq \mathrm{supp}(f)$. Thus the exposed extreme points of $\nu_{\mathrm{J}}(f)$ are contained in $\mathrm{supp}(f)$. By a theorem of Straszewicz, [24], page 167, any extreme point of a closed convex set is the limit of exposed extreme points. Since $\mathrm{supp}(f)$ is contained in a lattice, it is its own closure and we have $E\left(\nu_{\mathrm{J}}(f)\right) \subseteq \mathrm{supp}(f)$, which is the first assertion of this lemma. The extreme directions of $\nu_{\mathrm{J}}(f)$ are contained in $\{\vec{A}\}$ and $\nu_{\mathrm{J}}(f)$ contains no lines, so that by the generalized Krein-Milman Theorem 4.2 we obtain

$$\nu_{\mathrm{J}}(f) = \mathrm{conv}\left(E\left(\nu_{\mathrm{J}}(f)\right); \vec{A}\right) \subseteq \mathrm{conv}\left(\mathrm{supp}(f); \vec{A}\right),$$

which proves the second assertion. □

Theorem 4.5. *For $\nu_J : R_J(N) \setminus \{0\} \to$ Closed convex subsets of $\mathbb{R}^2$, the valuation property $\nu_J(fg) = \nu_J(f) + \nu_J(g)$ holds for all nontrivial $f, g \in R_J(N)$.*

Proof. Since $\mathrm{supp}(fg) \subseteq \mathrm{supp}(f) + \mathrm{supp}(g)$, we use Lemma 4.4 to conclude

$$\begin{aligned}\nu_{\mathrm{J}}(fg) &= \mathrm{conv}\left(\mathrm{supp}(fg); \vec{A}\right)\\ &\subseteq \mathrm{conv}\left(\mathrm{supp}(f) + \mathrm{supp}(g); \vec{A}\right) = \nu_{\mathrm{J}}(f) + \nu_{\mathrm{J}}(g).\end{aligned}$$

To prove $\nu_{\mathrm{J}}(f) + \nu_{\mathrm{J}}(g) \subseteq \nu_{\mathrm{J}}(fg)$, begin by taking an extreme point I_o of $\nu_{\mathrm{J}}(f) + \nu_{\mathrm{J}}(g) = \mathrm{conv}\left(\mathrm{supp}(f) + \mathrm{supp}(g); \vec{A}\right)$. We necessarily have $I_o \in \mathrm{supp}(f) + \mathrm{supp}(g)$ by 18.3.1 of [24], page 165. As in the proof for polynomials, $I_o \in \mathrm{supp}(fg)$ so that $E\left(\nu_{\mathrm{J}}(f) + \nu_{\mathrm{J}}(g)\right) \subseteq \mathrm{supp}(fg)$. Since $\nu_{\mathrm{J}}(f) + \nu_{\mathrm{J}}(g)$ contains no lines, the generalized Krein-Milman

Theorem 4.2 gives

$$\begin{aligned}\nu_{\mathrm{J}}(f)+\nu_{\mathrm{J}}(g) &= \operatorname{conv}\left(E\left(\nu_{\mathrm{J}}(f)+\nu_{\mathrm{J}}(g)\right);\vec{A}\right)\\ &\subseteq \operatorname{conv}\left(\operatorname{supp}(fg);\vec{A}\right)=\nu_{\mathrm{J}}(fg).\end{aligned}$$

□

Lemma 4.6. *The ring $R_J(N)$ contains*

1. *the polar Laurent polynomials, $\mathbb{C}[\zeta^{1/N},\zeta^{-1/N},q^{1/N},q^{-1/N}]$,*
2. *the Fourier expansions of weakly holomorphic Jacobi forms of level N, $J^{\mathrm{w.h.}}_{k,m}(\Gamma(N))$,*
3. *the infinite products of the form $\prod_{j=1}^{\infty}\left(1+q^{j/N}h_j(q^{1/N},\zeta^{1/N})\right)$, where the $h_j\in\mathbb{C}[q,\zeta,\zeta^{-1}]$ are (either trivial or) polynomials of uniformly bounded degree; that is, there exists a $D>0$ such that for all j, $\deg_\zeta h_j=\max\{|r|:\exists n\in\mathbb{Q}:(n,r)\in\operatorname{supp}(h_j)\}\le D$.*

Proof. For item 1, the support $\operatorname{supp}(f)$ of a polynomial f is compact and so is contained in some convex parabolic region with recession cone $\vec{A}$.

For item 2 and positive index m, a weakly holomorphic Jacobi form $\phi\in J^{\mathrm{w.h.}}_{k,m}(\Gamma(N))$ has $4mn-r^2$ bounded below for $(n,r)\in\operatorname{supp}(\phi)$ and thus the Fourier expansion of ϕ is in $R_J(N)$. Index $m=0$ is a degenerate case. Here the Fourier expansion depends only upon q and $\nu_{\mathrm{J}}(f)$ is a ray on the q-axis bounded from below.

For the infinite product in item 3, consider an (n,r) in the support. This requires at least $N\frac{|r|}{D}$ factors of the form $q^{j/N}h_j(q^{1/N},\zeta^{1/N})$, which means the power of q is at least $\sum_{j=1}^{N|r|/D} j/N=\frac{1}{2}N\frac{|r|}{D}(N\frac{|r|}{D}+1)\frac{1}{N}$; therefore $n\ge\frac{N}{2D^2}r^2$ and the support is contained in a convex parabolic region with recession cone $\vec{A}$. □

Lemma 4.7. *For an infinite product $\prod_{j=1}^{\infty}\left(1+q^{j/N}h_j(q^{1/N},\zeta^{1/N})\right)$ as in item 3 of Lemma 4.6, we have*

$$\nu_J\left(\prod_{j=1}^{\infty}\left(1+q^{j/N}h_j\right)\right)=\bigcup_{m=1}^{\infty}\nu_J\left(\prod_{j=1}^{m}\left(1+q^{j/N}h_j\right)\right).$$

Proof. To prove "$\supseteq$," take any $m\in\mathbb{N}$ and any $I\in\nu_{\mathrm{J}}(\prod_{j=1}^{m}(1+q^{j/N}h_j)$. Then because $0\in\nu_{\mathrm{J}}(\prod_{j=m+1}^{\infty}(1+q^{j/N}h_j))$, we have

$$I\in\nu_{\mathrm{J}}(\prod_{j=1}^{m}(1+q^{j/N}h_j))+\nu_{\mathrm{J}}(\prod_{j=m+1}^{\infty}(1+q^{j/N}h_j)),$$

which by Theorem 4.5 and Lemma 4.6, items 1 and 3, implies $I \in \nu_{\mathrm{J}}(\prod_{j=1}^{\infty}(1+q^{j/N}h_j))$.

Next we prove "$\subseteq$". Take any $(n,r) \in \operatorname{supp}(\prod_{j=1}^{\infty}(1+q^{j/N}h_j))$. Then it must be that $(n,r) \in \operatorname{supp}(\prod_{j=1}^{Nn}(1+q^{j/N}h_j))$ since the higher factors cannot contribute to a $q^n\zeta^r$ term. Thus

$$\operatorname{supp}\left(\prod_{j=1}^{\infty}\left(1+q^{j/N}h_j\right)\right) \subseteq \bigcup_{m=1}^{\infty} \nu_{\mathrm{J}}\left(\prod_{j=1}^{m}\left(1+q^{j/N}h_j\right)\right)$$

and so

$$\begin{aligned}\nu_{\mathrm{J}}\left(\prod_{j=1}^{\infty}\left(1+q^{j/N}h_j\right)\right) &\subseteq \operatorname{conv}\left(\bigcup_{m=1}^{\infty} \nu_{\mathrm{J}}\left(\prod_{j=1}^{m}\left(1+q^{j/N}h_j\right)\right); \vec{A}\right) \\ &= \bigcup_{m=1}^{\infty} \nu_{\mathrm{J}}\left(\prod_{j=1}^{m}\left(1+q^{j/N}h_j\right)\right)\end{aligned}$$

since the valuation property for ν_{J} on Laurent polynomials shows, as above, that the $\nu_{\mathrm{J}}\left(\prod_{j=1}^{m}\left(1+q^{j/N}h_j\right)\right)$ are nested and since each has recession cone $\vec{A}$. $\square$

We conclude this section with a few remarks and some notation. Let $\mathrm{FS}(\phi)$ denote the Fourier series of a weakly holomorphic Jacobi form and write $\nu_{\mathrm{J}}(\phi)$ for $\nu_{\mathrm{J}}(\mathrm{FS}(\phi))$. For a $\phi \in J_{k,m}^{\text{w.h.}}$, the generalized valuation ord defined in [15] by

$$\operatorname{ord}(\phi; x) = \min_{(n,r)\in\operatorname{supp}(\phi)} (mx^2 + rx + n)$$

is related to $\nu_{\mathrm{J}}(\phi)$ by $\operatorname{ord}(\phi;x) = \min\langle \nu_{\mathrm{J}}(\phi), \left(\begin{smallmatrix} 1 & x \\ x & x^2 \end{smallmatrix}\right)\rangle$. This gives a variant proof of the valuation property of ord proven in [15].

For $\phi \in J_{k,m}^{\text{cusp}}$, let $(\tau_o, z_o) \in \mathcal{H}_1 \times \mathbb{C}$ be the point where the invariant function $v^{k/2}e^{-2\pi m y^2/v}|\phi(u+iv, x+iy)|$ attains its maximum. One can use the techniques of [20] to prove that $\operatorname{ord}(\phi;x) \le \frac{k}{4\pi}\frac{1}{v_o} + m(x - \frac{y_o}{v_o})^2$.

5. Theta Blocks

Theta blocks are the invention of V. Gritsenko, N.-P. Skoruppa and D. Zagier, see [15] for a full treatment. We will only cite the properties we need here. A *theta block* is a function of the form

$$\eta^{f(0)} \prod_{\ell\in\mathbb{N}} \left(\frac{\vartheta_\ell}{\eta}\right)^{f(\ell)}$$

for some sequence $f : \mathbb{N} \cup \{0\} \to \mathbb{Z}$ with finite support. Reference to Theorem 3.3 shows that theta blocks arise naturally as the leading Fourier Jacobi coefficient of a paramodular Borcherds Product.

A theta block is a meromorphic Jacobi form with easily calculable weight, index, multiplier and divisor. The generalized valuation ord, introduced in [15], is also simple to calculate on theta blocks. Let $\mathcal{G} = C^0(\mathbb{R}/\mathbb{Z})^{\text{p.q.}}$ be the additive group of continuous functions $g : \mathbb{R} \to \mathbb{R}$ that have period one and are piecewise quadratic. Define the positive (non-negative) elements in $\mathcal{G}$ to be the semigroup of functions whose values are all positive (non-negative) in $\mathbb{R}$; this makes $\mathcal{G}$ a partially ordered abelian group.

For $\phi \in J_{k,m}(\chi)^{\text{w.h.}}$ and $x \in \mathbb{R}$ define

$$\operatorname{ord}(\phi; x) = \min_{(n,r)\in \operatorname{supp}(\phi)} (n + rx + mx^2).$$

Then $\operatorname{ord} : J_{k,m}(\chi)^{\text{w.h.}} \to \mathcal{G}$, defined by $\phi \mapsto \operatorname{ord}(\phi)$ is a generalized valuation in the sense that it satisfies

$$\operatorname{ord}(\phi_1\phi_2) = \operatorname{ord}(\phi_1) + \operatorname{ord}(\phi_2)$$

on the ring of all weakly holomorphic Jacobi forms and

$$\operatorname{ord}(\phi_1 + \phi_2) \geq \min\left(\operatorname{ord}(\phi_1), \operatorname{ord}(\phi_2)\right)$$

on each graded piece of fixed weight, index and multiplier. The generalized valuation characterizes Jacobi forms from among weakly holomorphic Jacobi forms because a weakly holomorphic Jacobi form ϕ is a Jacobi form if and only if $\operatorname{ord}(\phi) \geq 0$, and is a Jacobi cusp form if and only if $\operatorname{ord}(\phi) > 0$. One can easily test to see whether a theta block ϕ is a Jacobi form by checking the positivity of $\operatorname{ord}(\phi)$, which has the following pleasant formula [15]

$$\forall x \in \mathbb{R},\ \operatorname{ord}\left(\eta^{f(0)} \prod_{\ell\in\mathbb{N}} \left(\frac{\vartheta_\ell}{\eta}\right)^{f(\ell)}; x\right) = \frac{k}{12} + \frac{1}{2}\sum_{\ell\in\mathbb{N}} f(\ell)\bar{B}_2(\ell x),$$

where $\bar{B}_2(x)$ is the periodic extension of the the second Bernoulli polynomial, normalized in the traditional way, $B_2(x) = x^2 - x + \frac{1}{6}$, and

$$\bar{B}_2(x) = B_2(x - \llcorner x \lrcorner) = \sum_{n=1}^{\infty} \frac{\cos(2n\pi x)}{(n\pi)^2}.$$

The following Theorem, stated only for theta blocks without theta denominator, suffices for our needs.

Theorem 5.1 (Gritsenko, Skoruppa, Zagier)**.** *Let $\ell, m \in \mathbb{N}$ and let $u, k \in \mathbb{Z}$. Select $\mathbf{d} = (d_1, \dots, d_\ell) \in \mathbb{N}^\ell$. Define a meromorphic function*

$\mathrm{THBK}(u;\mathbf{d}) : \mathcal{H}_1 \times \mathbb{C} \to \mathbb{C}$ *by*

$$\mathrm{THBK}(u;\mathbf{d})(\tau,z) = \eta(\tau)^u \prod_{i=1}^{\ell} \vartheta(\tau, d_i z).$$

We have $\mathrm{THBK}(u;\mathbf{d}) \in J_{k,m}^{\mathrm{cusp}}$ *(respectively* $J_{k,m}$*) if and only if*

- $2k = \ell + u$,
- $2m = \sum_{i=1}^{\ell} d_i^2$,
- $u + 3\ell \equiv 0 \mod 24$,
- $\frac{k}{12} + \frac{1}{2}\sum_{i=1}^{\ell} \bar{B}_2(d_i x)$ *has a positive (respectively nonnegative) minimum on* $[0,1]$.

For such a theta block, we have the infinite product

$$\begin{aligned}\mathrm{THBK}(u;d_1,\dots,d_\ell) = q^v \prod_{j\in\mathbb{N}}(1-q^j)^{2k} \prod_{i=1}^{\ell}\left(\zeta^{\frac{1}{2}d_i} - \zeta^{-\frac{1}{2}d_i}\right)\\ \cdot \prod_{i=1}^{\ell}\prod_{j\in\mathbb{N}}(1-q^j\zeta^{d_i})(1-q^j\zeta^{-d_i})\end{aligned} \tag{2}$$

where $v = \frac{u+3\ell}{24}$.

6. Proof of Main Theorem

Beginning with a theta block $\phi \in J_{k,t}^{\mathrm{w.h.}}$,

$$\phi(\tau,z) = \mathrm{THBK}(u;d_1,\dots,d_\ell)(\tau,z) = \eta(\tau)^u \prod_{i=1}^{\ell} \vartheta(\tau, d_i z),$$

with order of vanishing in q equal to v, we wish to investigate $\psi = (-1)^v \frac{\phi|V_2}{\phi}$. It is clear that ψ transforms like a Jacobi form of weight zero with the same index as ϕ; we will show that ψ is weakly holomorphic, not just meromorphic. For prime ℓ we have

$$c(n,r;\phi|V_\ell) = c(\ell n, r;\phi) + \ell^{k-1} c(n/\ell, r/\ell;\phi),$$

and for $\ell = 2$, we have

$$(\phi|V_2)(\tau,z) = 2^{k-1}\phi(2\tau,2z) + \frac{1}{2}\left(\phi(\tfrac{1}{2}\tau,z) + \phi(\tfrac{1}{2}\tau+\tfrac{1}{2},z)\right).$$

The following equation for $\psi = (-1)^v \frac{\phi|V_2}{\phi}$ will often be cited.

Proposition 6.1. *Let $u, v, k \in \mathbb{Z}$ and $\ell, d_1, \dots, d_\ell \in \mathbb{N}$. Take a theta block $\phi = \mathrm{THBK}(u; d_1, \dots, d_\ell)$ of order $v = \frac{u+3\ell}{24}$ and weight $k = \frac{\ell+u}{2}$. For $\psi = (-1)^v \frac{\phi|V_2}{\phi}$, we have*

$$\psi(\tau, z) = (-1)^v 2^{k-1} \frac{\phi(2\tau, 2z)}{\phi(\tau, z)} + (-1)^v \frac{1}{2} \left(\frac{\phi(\frac{1}{2}\tau, z)}{\phi(\tau, z)} + \frac{\phi(\frac{1}{2}\tau + \frac{1}{2}, z)}{\phi(\tau, z)} \right) =$$

$$(-1)^v 2^{k-1} q^v \prod_{j \in \mathbb{N}} (1+q^j)^{2k} \prod_{i=1}^{\ell} \left(\zeta^{\frac{1}{2}d_i} + \zeta^{-\frac{1}{2}d_i} \right) \prod_{i=1}^{\ell} \prod_{j \in \mathbb{N}} (1+q^j \zeta^{d_i})(1 + q^j \zeta^{-d_i})$$

$$+ \tfrac{1}{2} q^{-\frac{1}{2}v} \left((-1)^v \prod_{2 \not| j} (1 - q^{\frac{j}{2}})^{2k} \prod_{i=1}^{\ell} \prod_{2 \not| j} (1 - q^{\frac{j}{2}} \zeta^{d_i})(1 - q^{\frac{j}{2}} \zeta^{-d_i}) \right.$$

$$\left. + \prod_{2 \not| j} (1 + q^{\frac{j}{2}})^{2k} \prod_{i=1}^{\ell} \prod_{2 \not| j} (1 + q^{\frac{j}{2}} \zeta^{d_i})(1 + q^{\frac{j}{2}} \zeta^{-d_i}) \right).$$

Proof. This formula follows from Equation (2) and algebra. □

Corollary 6.2. *Let $u, v \in \mathbb{Z}$ and $k, m, \ell, d_1, \dots, d_\ell \in \mathbb{N}$. Let $\phi = \mathrm{THBK}(u; d_1, \dots, d_\ell)$ be a theta block of order $v = \frac{u+3\ell}{24}$ and index m. Then $\psi = (-1)^v \frac{\phi|V_2}{\phi} \in J_{0,m}^{\mathrm{w.h.}}$ and ψ has integral Fourier coefficients.*

Now we examine the support of ψ. The first term in the equation of Proposition 6.1 involves $\frac{\phi(2\tau, 2z)}{\phi(\tau, z)}$ and we consider it separately.

Lemma 6.3. *Let $\phi = \mathrm{THBK}(u; d_1, \dots, d_\ell)$ be a theta block. Then*

$$\nu_J\left(\frac{\phi(2\tau, 2z)}{\phi(\tau, z)}\right) = \nu_J(\phi).$$

Proof. First we note the equality of the Jacobi valuations of the *atoms:* $\nu_{\mathrm{J}}(1 - q^j \zeta^d) = \nu_{\mathrm{J}}(1 + q^j \zeta^d)$. We use Proposition 6.1 to write

$$\begin{aligned}
&\nu_{\mathrm{J}} \left(\prod_{j \in \mathbb{N}} \left((1 + q^j)^{2k} \prod_{i=1}^{\ell} (1 + q^j \zeta^{d_i})(1 + q^j \zeta^{-d_i}) \right) \right) \\
&\qquad = \bigcup_{m=1}^{\infty} \nu_{\mathrm{J}} \left(\prod_{j=1}^{m} \left((1 + q^j)^{2k} \prod_{i=1}^{\ell} (1 + q^j \zeta^{d_i})(1 + q^j \zeta^{-d_i}) \right) \right) \\
&\qquad = \bigcup_{m=1}^{\infty} \nu_{\mathrm{J}} \left(\prod_{j=1}^{m} \left((1 - q^j)^{2k} \prod_{i=1}^{\ell} (1 - q^j \zeta^{d_i})(1 - q^j \zeta^{-d_i}) \right) \right) \\
&\qquad = \nu_{\mathrm{J}} \left(\prod_{j \in \mathbb{N}} \left((1 - q^j)^{2k} \prod_{i=1}^{\ell} (1 - q^j \zeta^{d_i})(1 - q^j \zeta^{-d_i}) \right) \right).
\end{aligned}$$

The first and third equalities follow from Lemma 4.7. The second equality follows from the valuation property from Theorem 4.5 on Laurent polynomials and the equality of the valuations of the atoms. Adding $\nu_{\mathrm{J}}\left(q^v\prod_{i=1}^{\ell}(\zeta^{\frac{1}{2}d_i}+\zeta^{-\frac{1}{2}d_i})\right)=\nu_{\mathrm{J}}\left(q^v\prod_{i=1}^{\ell}(\zeta^{\frac{1}{2}d_i}-\zeta^{-\frac{1}{2}d_i})\right)$ to the left and right hand sides, in $R_J(24)$ say, and applying Theorem 4.5 completes the proof. $\square$

We use some combinatorial lemmas, set $\mathcal{L}=\{-\ell,\dots,-1,1,\dots,\ell\}$.

Lemma 6.4. *Let $d_1,\dots,d_\ell\in\mathbb{N}$. Set $e_i=\mathrm{sgn}(i)d_{|i|}$ for $i\in\mathcal{L}$. Fix $a\in\mathbb{N}$. Then we have*

$$\sum_{m_1,\dots,m_a\in\mathcal{L}\colon -\ell\le m_1<\dots<m_a\le\ell}(e_{m_1}+\cdots+e_{m_a})^2=\binom{2\ell-2}{a-1}\sum_{i\in\mathcal{L}}e_i^2.$$

Proof. When the lefthand side above is expanded, we have

(1) For each $1\le i\le\ell$, e_i^2 and e_{-i}^2 occur $\binom{2\ell-1}{a-1}$ times.
(2) For each $1\le i\le\ell$, e_ie_{-i} and $e_{-i}e_i$ each occur $\binom{2\ell-2}{a-2}$ times.
(3) For $1\le i,j\le\ell$ with $i\ne j$, we have e_ie_j, $e_{-i}e_{-j}$, e_ie_{-j}, and $e_{-i}e_j$ each occurs $2\binom{2\ell-2}{a-2}$ times.

Adding up each category, since the cases within item (3) cancel themselves out, we get that

$$\sum_{m_1,\dots,m_a\in\mathcal{L}\colon -\ell\le m_1<\cdots<m_a\le\ell}(e_{m_1}+\cdots+e_{m_a})^2$$
$$=\sum_{i=1}^{\ell}(2\binom{2\ell-1}{a-1}-2\binom{2\ell-2}{a-2})\sum_{i=1}^{\ell}e_i^2.$$

Then the result follows from $\binom{2\ell-1}{a-1}-\binom{2\ell-2}{a-2}=\binom{2\ell-2}{a-1}$. $\square$

Lemma 6.5. *Let $d_1,\dots,d_\ell\in\mathbb{N}$. Set $e_i=\mathrm{sgn}(i)d_{|i|}$ for $i\in\mathcal{L}$. Fix $b_1,\dots,b_\beta\in\mathbb{N}$. For any subset $S\subseteq\mathcal{L}$, denote $e_S=\sum_{i\in S}e_i$. Then*

$$\sum_{S_1,\dots,S_\beta}(\sum_{i=1}^{\beta}e_{S_i})^2 \text{ is an integral multiple of } \sum_{i\in\mathcal{L}}e_i^2,$$

where the outer sum on the lefthand side is over all $S_1,\dots,S_\beta$, where each $S_i\subseteq\mathcal{L}$ with $|S_i|=b_i$.

Proof. We proceed by induction on β. The case $\beta=1$ follows from Lemma 6.4. So assume $\beta>1$. The key is that

$$\sum_{S_1,\dots,S_\beta}(\sum_{i=1}^{\beta}e_{S_i})^2=\sum_{S_1,\dots,S_\beta}(e_{S_1}+\cdots+e_{S_{\beta-1}}-e_{S_\beta})^2.$$

The reason for this is that $S \mapsto -S$ gives a involution on subsets $S \subseteq \mathcal{L}$ of any fixed size and satisfies $e_{-S} = -e_S$. Then adding the above righthand side to the lefthand side yields

$$2 \sum_{S_1,\dots,S_\beta} (\sum_{i=1}^{\beta} e_{S_i})^2 = \sum_{S_1,\dots,S_\beta} \Big(2(e_{S_1} + \dots + e_{S_{\beta-1}})^2 + 2e_{S_\beta}^2 \big)$$

The result follows by induction. □

Theorem 6.6. (Borcherds Products Everywhere) *Fix $\ell \in \mathbb{N}$ and $u \in \mathbb{Z}$ with $\ell + u$ even. Let $d_1, \dots, d_\ell \in \mathbb{N}$ with $d_1 + \dots + d_\ell$ even. Assume that $v = \frac{1}{24}(u + 3\ell) \in \mathbb{N}$. If we set $k = \frac{1}{2}(\ell + u)$ and $t = \frac{1}{2}(d_1^2 + \dots + d_\ell^2)$ then we have $\phi = \eta^u \prod_{i=1}^{\ell} \vartheta_{d_i} \in J_{k,t}^{\text{mero}}$. If v is odd, additionally assume that $\phi \in J_{k,t}$. For $\psi = (-1)^v \frac{\phi|V_2}{\phi}$, we have the following:*

1. *$\psi \in J_{0,t}^{\text{w.h.}}$ and $c(n,r;\psi) \geq 0$ for all (n,r) with $4tn - r^2 \leq 0$.*
2. *There is a $k' \in \mathbb{N}$ such that* $\mathrm{Borch}(\psi) \in M_{k'}(K(t))$ *is a holomorphic Borcherds product with trivial character.*
3. $\mathrm{Borch}(\psi)$ *is antisymmetric when v is an odd power of two and otherwise symmetric.*
4. *If $v = 1$, then* $\mathrm{Borch}(\psi) \in M_k(K(t))$ *and* $\mathrm{Borch}(\psi)$ *and $Grit(\phi)$ have the same first and second Fourier Jacobi coefficients.*

Proof. That ψ is weakly holomorphic and has integral Fourier coefficients was proven in Corollary 6.2. We show that the Fourier coefficients of singular indices are nonnegative. Consider the formula for ψ from Proposition 6.1. Consider the case when v is odd. Since $\phi \in J_{k,t}$, we have $4tn - r^2 \geq 0$ for $(n,r) \in \mathrm{supp}(\phi)$ and the same inequality hold on the convex hull $\nu_{\mathrm{J}}(\phi) = \mathrm{conv}(\mathrm{supp}(\phi); \vec{A})$. By Lemma 6.3, $\mathrm{supp}(\frac{\phi(2\tau,2z)}{\phi(\tau,z)}) \subseteq \nu_{\mathrm{J}}(\phi)$ also does not contain any (n,r) with $4tn - r^2 < 0$. When v is even, all the coefficients of $\phi(2\tau,2z)/\phi(\tau,z)$ are nonnegative anyhow. The remaining terms in this equation are

$$\begin{aligned}
&\frac{(-1)^v}{2} \left(\frac{\phi(\frac{1}{2}\tau, z)}{\phi(\tau, z)} + \frac{\phi(\frac{1}{2}\tau + \frac{1}{2}, z)}{\phi(\tau, z)} \right) \\
&\quad = \frac{1}{2} q^{-\frac{1}{2}v} \Bigg((-1)^v \prod_{2 \nmid j} (1 - q^{\frac{j}{2}})^{2k} \prod_{i=1}^{\ell} \prod_{2 \nmid j} (1 - q^{\frac{j}{2}} \zeta^{d_i})(1 - q^{\frac{j}{2}} \zeta^{-d_i}) \\
&\qquad + \prod_{2 \nmid j} (1 + q^{\frac{j}{2}})^{2k} \prod_{i=1}^{\ell} \prod_{2 \nmid j} (1 + q^{\frac{j}{2}} \zeta^{d_i})(1 + q^{\frac{j}{2}} \zeta^{-d_i}) \Bigg).
\end{aligned}$$

It is clear that the nonzero coefficients are all positive because they are coefficients of a formal series of the form $f(q^{1/2}, \zeta) \pm f(-q^{1/2}, \zeta)$, where f

has all nonnegative coefficients. The only terms of the singular part of ψ that might have negative coefficients come from the first term where a monomial $q^n\zeta^r$ with $4tn - r^2 = 0$ might be supported. Therefore, if the multiplicity $\sum_{\lambda\in\mathbb{N}} c(\lambda^2 n_o m_o, \lambda r_o)$ of $\mathrm{Borch}(\psi)$ on $\mathrm{Hum}\left(\begin{smallmatrix} n_o & r_o/2 \\ r_o/2 & tm_o \end{smallmatrix}\right)$ has a negative summand, $c(\lambda_1^2 n_o m_o, \lambda_1 r_o) < 0$, for some $\lambda_1 \in \mathbb{N}$, then $4t(\lambda_1^2 n_o m_o) - (\lambda_1 r_o)^2 = 0$ and $\mathrm{Hum}\left(\begin{smallmatrix} n_o & r_o/2 \\ r_o/2 & tm_o \end{smallmatrix}\right)$ is empty. Thus $\mathrm{Borch}(\psi)$ is a holomorphic Borcherds product by Theorem 3.3.

To prove that the character is actually trivial, we use the notation

$$A = \tfrac{1}{24}\sum_{r\in\mathbb{Z}} c(0,r;\psi), \quad C = \tfrac{1}{4}\sum_{r\in\mathbb{Z}} r^2 c(0,r;\psi),$$
$$D_1 = \sum_{n<0, r\in\mathbb{Z}} \sigma_1(-n)\, c(n,r;\psi).$$

First, we prove $t|C$. From Proposition 6.1, noting that $v > 0$ by assumption here, we have that $c(0,r;\psi)$ is the coefficient of $q^{\frac{1}{2}v}\zeta^r$ of

$$\prod_{2\not| j}(1+q^{\frac{j}{2}})^{2k}\prod_{i=1}^{\ell}\prod_{2\not| j}(1+q^{\frac{j}{2}}\zeta^{d_i})(1+q^{\frac{j}{2}}\zeta^{-d_i}).$$

Let $e_i = \mathrm{sgn}(i)d_{|i|}$ for $i \in \mathcal{L} = \{-\ell, \dots, -1, 1, \dots, \ell\}$. Then $c(0,r;\psi)$ is the coefficient of $q^{\frac{1}{2}v}\zeta^r$ in

$$\prod_{2\not| j}(1+q^{\frac{j}{2}})^{2k}\prod_{i\in\mathcal{L}}\prod_{2\not| j}(1+q^{\frac{j}{2}}\zeta^{e_i}).$$

Let a_j be the number of factors of $q^{j/2}$ selected from the product, and b_j be the total number of $q^{j/2}\zeta^{e_{-\ell}}, \dots, q^{j/2}\zeta^{e_{-1}}, q^{j/2}\zeta^{e_1}, \dots, q^{j/2}\zeta^{e_\ell}$ selected from the product. Denote the set

$$\mathcal{S}_v = \{(a_1, a_3, a_5, \dots; b_1, b_3, \dots : a_i, b_i \in \mathbb{N}\cup\{0\}, \sum_{i=1}(a_i+b_i)i = v\}.$$

For $S \subset \mathcal{L}$, denote $e_S = \sum_{i\in S} e_i$. Then $4C$ is the sum over each of the above elements of $\mathcal{S}_v$ of the sum

$$\left(\binom{2k}{a_1}\binom{2k}{a_3}\cdots\right)\sum_{(S_1,S_3,\dots):|S_i|=b_i}\left(\sum_i e_{S_i}\right)^2 \tag{3}$$

where each $S_i \subseteq \mathcal{L}$. By Lemma 6.5, the above Equation (3) is an integral multiple of $\sum_{i\in\mathcal{L}} e_i^2$ This proves that $4C$ is a multiple of $\sum_{i\in\mathcal{L}} e_i^2$. Since $4t = \sum_{i\in\mathcal{L}} e_i^2$, this proves $t|C$. In particular, C is integral and this shows that $2B \equiv 2C \equiv 0 \mod 2$.

It is clear that $D_1 \in \mathbb{Z}$. By Proposition 3.4, we have $tA - tD_1 - C = 0$, and we deduce that $A \in \mathbb{Z}$. This says the Borcherds product is a paramodular form with a trivial character on $K(t)$.

Finally, for item (4), When $v = 1$, we can say more. From the formula for ψ, we see

$$\psi = 2k + \sum_{i=1}^{\ell} (\zeta^{d_i} + \zeta^{-d_i}) + q(\cdots) + q^2(\cdots) + \cdots .$$

so that

$$\begin{aligned} A &= \tfrac{1}{24} \sum_{r \in \mathbb{Z}} c(0, r; \psi) = \tfrac{1}{24}(2k + 2\ell) = 1, \\ C &= \tfrac{1}{4} \sum_{r \in \mathbb{Z}} r^2 c(0, r; \psi) = \tfrac{1}{4} \sum_{i=1}^{\ell} 2d_i^2 = \tfrac{1}{4} 2(2t) = t, \\ D_1 &= \sum_{n<0, r \in \mathbb{Z}} \sigma_1(-n)\, c(n, r; \psi) = 0. \end{aligned}$$

The equation $c(0, 0; \psi) = 2k$ says the weight of $\operatorname{Borch}(\psi)$ is k. By Theorem 3.3 the first Fourier Jacobi coefficient of $\operatorname{Borch}(\psi)$ is

$$\eta^{c(0,0;\psi)} \prod_{r \in \mathbb{N}} \left(\frac{\vartheta_r}{\eta} \right)^{c(0,r;\psi)} = \eta^{2k} \prod_{i=1}^{\ell} \left(\frac{\vartheta_{d_i}}{\eta} \right),$$

which is exactly ϕ. In view of the formula

$$\operatorname{Borch}(\psi) = \tilde{\phi} \exp\left(-\operatorname{Grit}(\psi)\right) = \phi \xi^C \left(1 - \psi \xi^t + \frac{1}{2!} \psi | V_2 \xi^{2t} + \cdots \right),$$

the second Fourier Jacobi coefficient of $\operatorname{Borch}(\psi)$ is $\phi(-\psi) = \phi | V_2$, which is the second Fourier Jacobi coefficient of $\operatorname{Grit}(\phi)$. This completes the proof, except for item (3), which we postpone until the next section. □

7. Proof of Symmetry and Antisymmetry

This section is devoted to the proof that a paramodular Borcherds Product, constructed as in Theorem 6.6, is antisymmetric if and only if the vanishing order of the theta block is an odd power of two. One can glimpse this in Table 1, where odd weights occur only at vanishing orders 2 and 8. In Table 1, for the special case of level one, antisymmetric and odd weight are equivalent. The following theorem gives the general result on the parity of D_0 in terms of the hypotheses of Theorem 6.6 and thereby completes the proof of item (3) in that theorem.

Theorem 7.1. *Let $k, u \in \mathbb{Z}$ and $v, \ell, t, d_1, \ldots, d_\ell \in \mathbb{N}$. Assume that $2k = \ell + u$, $2t = d_1^2 + \cdots + d_\ell^2$ and $24v = u + 3\ell$, so that $\phi = \eta^u \prod_{i=1}^{\ell} \vartheta_{d_i} \in J_{k,t}^{\text{mero}}$. For $\psi = (-1)^v \frac{\phi|V_2}{\phi}$, let $D_0 = \sum_{n \in \mathbb{Z}: n<0} \sigma_0(-n) c(n, 0; \psi)$. We have*

$$D_0 \equiv 1 \mod 2 \iff \exists \text{ odd } \beta \in \mathbb{N} : v = 2^\beta.$$

Equivalently, Borch(ψ) *is antisymmetric if and only if v is an odd power of two.*

If $n < 0$ then only the second term in Proposition 6.1 can contribute to $c(n, 0; \psi)$. Thus $c(n, 0; \psi)$ is the coefficient of $q^{n+\frac{1}{2}v}\zeta^0$ in

$$\prod_{2 \not| j} (1 + q^{\frac{j}{2}})^{2k} \prod_{i=1}^{\ell} \prod_{2 \not| j} (1 + q^{\frac{j}{2}} \zeta^{d_i})(1 + q^{\frac{j}{2}} \zeta^{-d_i}).$$

Set $e_i = \operatorname{sgn}(i) d_{|i|}$ for $i \in \mathcal{L} = \{-\ell, \ldots, -1, 1, \ldots, \ell\}$. Then $c(n, 0; \psi)$ is the coefficient of $q^{n+\frac{1}{2}v}\zeta^0$ in

$$\prod_{\text{odd } j \in \mathbb{N}} \left((1 + q^{\frac{j}{2}})^{2k} \prod_{i \in \mathcal{L}} (1 + q^{\frac{j}{2}} \zeta^{e_i}) \right). \tag{4}$$

We multiply out the infinite product (4) to an infinite sum. For odd j, let a_j be the number of factors of $q^{j/2}$ selected from the product, and b_j be the total number of $q^{j/2}\zeta^{e_{-\ell}}, \ldots, q^{j/2}\zeta^{e_{-1}}, q^{j/2}\zeta^{e_1}, \ldots, q^{j/2}\zeta^{e_\ell}$ selected from the product. Knowing a_j determines the contribution of the $q^{j/2}$ factors but, for fixed b_j, we still need to sum over all subsets $S_j \subseteq \mathcal{L}$ with $|S_j| = b_j$ in order to get the exponent of ζ.

Definition 7.2. *For $h \in \mathbb{Z}$, define the set $\mathcal{T}_h = \mathcal{T}(h)$ by*

$$\mathcal{T}_h = \{(a_1, a_3, a_5, \ldots) \in \prod_{\text{odd } i \in \mathbb{N}}^{\infty} (\mathbb{N} \cup \{0\}) : \sum_{\text{odd } i \in \mathbb{N}} i a_i = h\}.$$

For $S \subset \mathcal{L}$, denote $e_S = \sum_{i \in S} e_i$. Then multiplying out (4) gives

$$\sum_{A,B \in \mathbb{N} \cup \{0\}} \sum_{\substack{(a_1, a_3, \ldots) \in \mathcal{T}_A, \\ (b_1, b_3, \ldots) \in \mathcal{T}_B}} \left(\binom{2k}{a_1} \binom{2k}{a_3} \cdots \right) q^{\frac{A+B}{2}} \sum_{\substack{S_1, S_3, \ldots \subseteq \mathcal{L}: \\ |S_i| = b_i}} \prod_{\text{odd } i \in \mathbb{N}} \zeta^{e_{S_i}}.$$

In order to grab the coefficients of the monomials with ζ^0, define

$$Z(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $n < 0$, we get that $c(n, 0; \psi)$ is the total sum of

$$(5) \quad \sum_{\substack{A,B\in\mathbb{N}\cup\{0\}:\\ A+B=v+2n}} \sum_{\substack{(a_1,a_3,\dots)\in\mathcal{T}_A,\\ (b_1,b_3,\dots)\in\mathcal{T}_B}} \left(\tbinom{2k}{a_1}\tbinom{2k}{a_3}\cdots\right) \sum_{\substack{S_1,S_3,\dots\subseteq\mathcal{L}:\\ |S_i|=b_i}} Z\left(\sum_{\text{odd } i\in\mathbb{N}} e_{S_i}\right).$$

A series of simplifications will show that the parity of D_0 only depends upon v. For $S \subseteq \mathcal{L}$, we have $e_{-S} = -e_S$ and if, in the final summation, we pair $(S_1, S_3, \dots)$ with $(-S_1, -S_3, \dots)$ when these are distinct, then the sum for $c(n, 0; \psi)$ modulo two may be restricted to sum only over S_i with $S_i = -S_i$. Hence $c(n, 0; \psi)$ is congruent to

$$\sum_{\substack{A,B\in\mathbb{N}\cup\{0\}:\\ A+B=v+2n}} \sum_{\substack{(a_1,a_3,\dots)\in\mathcal{T}_A,\\ (b_1,b_3,\dots)\in\mathcal{T}_B}} \left(\tbinom{2k}{a_1}\tbinom{2k}{a_3}\cdots\right) \sum_{\substack{S_1,S_3,\dots\subseteq\mathcal{L}:\\ |S_i|=b_i\\ \text{and } -S_i=S_i}} Z\left(\sum_{\text{odd } i\in\mathbb{N}} e_{S_i}\right).$$

Restricting to subsets S_i with $S_i = -S_i$ is the same as summing over S_i of the form $T\cup(-T)$, where $T \subseteq \{1, 2, \dots, \ell\}$; and in each such case we have $e_{S_i} = e_{T\cup(-T)} = 0$. Thus we have

$$\sum_{\substack{S_1,S_3,\dots\subseteq\mathcal{L}:\\ |S_i|=b_i\\ \text{and } -S_i=S_i}} Z\left(\sum_{\text{odd } i\in\mathbb{N}} e_{S_i}\right) = \sum_{\substack{T_1,T_3,\dots\subseteq\{1,2,\dots,\ell\}:\\ |T_i|=\frac{1}{2}b_i}} 1 = \prod_{\text{odd } i\in\mathbb{N}} \binom{\ell}{\frac{1}{2}b_i}$$

and this is zero if any b_i is odd. Thus we may restrict the summation for $c(n, 0; \psi)$ to those $(b_1, b_3, \dots) \in \mathcal{T}_B$ where every b_i is even. The same is actually true for the a_i because $\binom{2k}{a_i} \equiv 0 \mod 2$ when a_i is odd. The proof is that, by definition, $\binom{2k}{a} = \frac{2k}{a}\binom{2k-1}{a-1}$ and if a is odd then $\binom{2k}{a_i} \equiv 2k\binom{2k-1}{a-1} \equiv 0 \mod 2$. Furthermore, for even a, we have $\binom{2k}{a} \equiv \binom{k}{a/2} \mod 2$. The proof is to ignore all the odd factors in

$$\binom{2m}{2j} = \frac{(2m)(2m-1)2(m-1)(2m-3)2(m-2)\cdots}{(2j)(2j-1)2(j-1)(2j-3)2(j-2)\cdots},$$

and to cancel a 2 for each even factor, leaving $\binom{m}{j}$. Let $\bar{A} = \frac{1}{2}A$, $\bar{B} = \frac{1}{2}B$, $\bar{a}_i = \frac{1}{2}a_i$ and $\bar{b}_i = \frac{1}{2}b_i$. For negative n, we have

$$c(n, 0; \psi) \equiv \sum_{\substack{\bar{A},\bar{B}\in\mathbb{N}\cup\{0\}:\\ \bar{A}+\bar{B}=\frac{1}{2}v+n}} \sum_{\substack{(\bar{a}_1,\bar{a}_3,\dots)\in\mathcal{T}_{\bar{A}},\\ (\bar{b}_1,\bar{b}_3,\dots)\in\mathcal{T}_{\bar{B}}}} \left(\tbinom{k}{\bar{a}_1}\tbinom{k}{\bar{a}_3}\cdots\right)\left(\tbinom{\ell}{\bar{b}_1}\tbinom{\ell}{\bar{b}_3}\cdots\right).$$

We reorganize this summation by setting $r_i = \bar{a}_i + \bar{b}_i$ for odd i so that $(r_1, r_3, \dots) \in \mathcal{T}(\bar{A} + \bar{B}) = \mathcal{T}(\frac{1}{2}v + n)$. For negative n, we have

$$c(n,0;\psi) \equiv \sum_{(r_1,r_3,\dots)\in\mathcal{T}(\frac{1}{2}v+n)} \prod_{\text{odd } i\in\mathbb{N}} \left(\sum_{\bar{a}_i+\bar{b}_i=r_i} \binom{k}{\bar{a}_i}\binom{\ell}{\bar{b}_i} \right).$$

Now we make use of the binomial convolution identity, valid for n_1, n_2, $k \in \mathbb{N} \cup \{0\}$,

$$\sum_{j=0}^{k} \binom{n_1}{k-j}\binom{n_2}{j} = \binom{n_1+n_2}{k}.$$

The proof is to count the number of size k subsets of $n_1 + n_2$ items by breaking the count into cases according to the number of elements in the subset that are from the first n_1 items and the number that are from the second n_2 items. Thus,

$$c(n,0;\psi) \equiv \sum_{(r_1,r_3,\dots)\in\mathcal{T}(\frac{1}{2}v+n)} \prod_{\text{odd } i\in\mathbb{N}} \binom{k+\ell}{r_i}$$

and remembering the theta block satisfies $k + \ell = 12v$, we have

$$c(n,0;\psi) \equiv \sum_{(r_1,r_3,\dots)\in\mathcal{T}(\frac{1}{2}v+n)} \prod_{\text{odd } i\in\mathbb{N}} \binom{12v}{r_i}.$$

This shows that the parity of D_0 depends only on v and concludes the first series of reductions. At this point, one might finish by computing any example for each v. It is just as easy, however, to continue with the formula at hand, which amounts at least formally to a specific example. According to the definition $D_0 = \sum_{n\in\mathbb{Z}:n<0} \sigma_0(-n)c(n,0;\psi)$, we need only consider n such that $\sigma_0(-n) \not\equiv 0 \mod 2$; this condition holds if and only if $-n$ is a square. So we consider, for $n > 0$,

$$c(-n^2,0;\psi) \equiv \sum_{(r_1,r_3,\dots)\in\mathcal{T}(\frac{1}{2}v-n^2)} \prod_{\text{odd } i\in\mathbb{N}} \binom{12v}{r_i}.$$

We will show that $c(-n^2,0;\psi) \not\equiv 0 \mod 2$ implies that $n = 2^{\frac{\beta-1}{2}}m$ for some odd $\beta, m \in \mathbb{N}$. Let $v = 2^\beta w$, where $w \in \mathbb{N}$ is odd. For $n > 0$,

$$c(-n^2,0;\psi) \equiv \sum_{(r_1,r_3,\dots)\in\mathcal{T}(\frac{1}{2}v-n^2)} \prod_{\text{odd } i\in\mathbb{N}} \binom{2^{\beta+2}\,3w}{r_i}.$$

As before, if $\binom{2^{\beta+2}3w}{r_i}$ is odd then $2^{\beta+2}|r_i$, so we may restrict this sum to r_i that are divisible by $2^{\beta+2}$. Let $\bar{r}_i = r_i/2^{\beta+2}$, so that

$$c(-n^2,0;\psi) \equiv \sum_{(\bar{r}_1,\bar{r}_3,\dots)\in\mathcal{T}(\frac{\frac{1}{2}v-n^2}{2^{\beta+2}})} \prod_{\text{odd } i\in\mathbb{N}} \binom{3w}{\bar{r}_i}.$$

If this summation is not empty, then we have $2^{\beta+2}|(\frac{1}{2}v-n^2)$ or, equivalently, $2^{\beta+2}|(2^{\beta-1}w-n^2)$. This implies $2^{\beta-1}|n^2$ and $8|(w-\frac{n^2}{2^{\beta-1}})$. Since w is odd, the integer $\frac{n^2}{2^{\beta-1}}$ is odd, which implies that $\beta-1$ is even and that $n = 2^{\frac{\beta-1}{2}}m$ for some odd $m \in \mathbb{N}$, as claimed. Thus we have

$$c(-n^2,0;\psi) \equiv \sum_{(\bar{r}_1,\bar{r}_3,\dots)\in\mathcal{T}(\frac{w-m^2}{8})} \prod_{\text{odd } i\in\mathbb{N}} \binom{3w}{\bar{r}_i},$$

since $\frac{\frac{1}{2}v-n^2}{2^{\beta+2}} = \frac{2^{\beta-1}w-2^{\beta-1}m^2}{2^{\beta+2}} = \frac{w-m^2}{8}$. If $c(-n^2,0;\psi) \not\equiv 0 \mod 2$ then $w \equiv m^2 \mod 8$, which implies $w \equiv 1 \mod 8$. In this case, let $w = 1+8\mu$ for $\mu \in \mathbb{N}\cup\{0\}$ and $m = 2\lambda+1$ for $\lambda \in \mathbb{N}\cup\{0\}$. In terms of μ and λ, we have $\frac{w-m^2}{8} = \frac{1+8\mu-(2\lambda+1)^2}{8} = \mu - \binom{\lambda+1}{2}$. So

$$c(-n^2,0;\psi) \equiv \sum_{(\bar{r}_1,\bar{r}_3,\dots)\in\mathcal{T}(\mu-\binom{\lambda+1}{2})} \prod_{\text{odd } i\in\mathbb{N}} \binom{3w}{\bar{r}_i}.$$

Since all nonzero $c(-n^2,0;\psi)$ are of this form, we calculate D_0 as

$$\begin{aligned} D_0 &\equiv \sum_{n\in\mathbb{Z}:n<0} \sigma_0(-n)c(n,0;\psi) \equiv \sum_{n\in\mathbb{N}} \sigma_0(n^2)c(-n^2,0;\psi) \\ &\equiv \sum_{n\in\mathbb{N}} c(-n^2,0;\psi) \equiv \sum_{\lambda\geq 0} \sum_{(\bar{r}_1,\bar{r}_3,\dots)\in\mathcal{T}(\mu-\binom{\lambda+1}{2})} \prod_{\text{odd } i\in\mathbb{N}} \binom{3w}{\bar{r}_i}. \end{aligned}$$

Claim: For $v = 2^\beta(1+8\mu)$, $D_0 \equiv 0 \mod 2$ if and only if $\mu \geq 1$.

To prove the easy direction of this claim, note that $\mu = 0$ forces $\lambda = 0$ and all $\bar{r}_i = 0$, so that $D_0 \equiv 1 \mod 2$. For the harder direction, fix $\mu \geq 1$ and $w = 1+8\mu$. Note that $3w = 3+24\mu$.

Definition 7.3. *For $n,\mu \in \mathbb{Z}$ with $n \geq 0$ and $\mu \geq 1$, define*

$$H(n) = \sum_{(r_1,r_3,\dots)\in\mathcal{T}_n} \prod_{\text{odd } i\in\mathbb{N}} \binom{3+24\mu}{r_i}.$$

Note that we have $D_0 \equiv \sum_{\lambda\geq 0} H\left(\mu - \binom{\lambda+1}{2}\right)$. Consider the generating function for $H(n)$,

$$\sum_{n\geq 0} H(n)q^n = (1+q)^{3+24\mu}(1+q^3)^{3+24\mu}\cdots = \prod_{\text{odd } j\in\mathbb{N}} (1+q^j)^{3+24\mu}.$$

We finally write D_0 in terms of modular forms modulo two:

$$D_0 \equiv \mathrm{Coeff}\left(q^\mu, (\sum_{\lambda\geq 0} q^{\binom{\lambda+1}{2}})(\sum_{n\geq 0} H(n)q^n)\right) \mod 2.$$

Lemma 7.4.

$$\prod_{\text{odd } j\in\mathbb{N}} (1+q^j) \equiv \frac{1}{\prod_{j\in\mathbb{N}}(1-q^j)} \mod 2.$$

Proof.

$$\prod_{\text{odd } j\in\mathbb{N}} (1+q^j) \equiv \frac{\prod_{\text{odd } j\in\mathbb{N}}(1-q^j)\ \prod_{\text{even } j\in\mathbb{N}}(1-q^j)}{\prod_{\text{even } j\in\mathbb{N}}(1-q^j)} = \frac{\prod_{j\in\mathbb{N}}(1-q^j)}{\prod_{j\in\mathbb{N}}(1-q^{2j})}$$
$$= \frac{1}{\prod_{j\in\mathbb{N}}(1+q^j)} \equiv \frac{1}{\prod_{j\in\mathbb{N}}(1-q^j)}.$$

□

This Lemma shows us that

$$\sum_{n\geq 0} H(n)q^n \equiv \frac{1}{\prod_{j\in\mathbb{N}}(1-q^j)^{3+24\mu}} \mod 2.$$

Lemma 7.5.

$$\prod_{j\in\mathbb{N}}(1-q^j)^3 \equiv \sum_{n=0}^{\infty} q^{\binom{n+1}{2}} \mod 2.$$

Proof. The classical Euler identity or the Jacobi triple product identity gives us

$$\eta(\tau)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{(2n+1)^2}{8}}.$$

Consequently we have

$$q^{\frac{1}{8}}\prod_{j\in\mathbb{N}}(1-q^j)^3 \equiv \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}q^{\frac{1}{8}} \mod 2,$$

which proves the lemma. □

Now we may finish the derivation of the parity of D_0.

$$
\begin{aligned}
D_0 &\equiv \mathrm{Coeff}\left(q^{\mu}, (\sum_{\lambda\ge 0} q^{\binom{\lambda+1}{2}})(\sum_{n\ge 0} H(n)q^n)\right) \\
&\equiv \mathrm{Coeff}\left(q^{\mu}, (\prod_{j\in\mathbb{N}}(1-q^j)^3)(\frac{1}{\prod_{j\in\mathbb{N}}(1-q^j)^{3+24\mu}})\right) \\
&= \mathrm{Coeff}\left(q^{\mu}, \frac{1}{\prod_{j\in\mathbb{N}}(1-q^j)^{24\mu}}\right) = \mathrm{Coeff}\left(q^{\mu}, \frac{1}{(\prod_{j\in\mathbb{N}}(1-q^j)^3)^{8\mu}}\right).
\end{aligned}
$$

Now $(\prod_{j\in\mathbb{N}}(1-q^j)^3)^{8\mu} \equiv (\sum_{n=0}^{\infty} q^{\binom{n+1}{2}})^{8\mu}$ and squaring is a linear homomorphism modulo two so that $(\sum_{n=0}^{\infty} q^{\binom{n+1}{2}})^8 \equiv \sum_{n=0}^{\infty} q^{8\binom{n+1}{2}}$. Since we are in the case $\mu \ge 1$, let $\mu = 2^{\alpha}\delta$, where $\delta \in \mathbb{N}$ is odd. Then $\left(\sum_{n=0}^{\infty} q^{8\binom{n+1}{2}}\right)^{\mu} \equiv \left(\sum_{n=0}^{\infty} q^{8\cdot 2^{\alpha}\binom{n+1}{2}}\right)^{\delta}$ is a monic polynomial in $q^{8\cdot 2^{\alpha}}$ and, as such, its reciprocal cannot have a term $q^{2^{\alpha}\delta}$, which is q^{μ}. Thus

$$D_0 \equiv \mathrm{Coeff}\left(q^{\mu}, \frac{1}{\prod_{j\in\mathbb{N}}(1-q^j)^{24\mu}}\right) = 0,$$

when $\mu \ge 1$ and this completes the proof of Theorem 7.1.

8. When do we have $\mathrm{Grit}(\phi) = \mathrm{Borch}\big(-(\phi|V_2)/\phi\big)$?

In order to complete the line of thought that began this research and to completely characterize the paramodular forms that are both Gritsenko lifts of theta blocks without theta denominator and Borcherds Products, it would suffice to prove the following conjecture.

Conjecture 8.1. *Let $\phi \in J_{k,t}$ be a theta block without theta denominator and with vanishing order one in $q = e(\tau)$. Then* $\mathrm{Grit}(\phi) = \mathrm{Borch}(\psi)$ *for* $\psi = -\frac{\phi|V_2}{\phi}$.

We know in the above conjecture that $\mathrm{Borch}(\psi)$ and $\mathrm{Grit}(\phi)$ are both symmetric forms in $M_k(K(t))$ and that they have identical first and second Fourier Jacobi coefficients. The following theorem proves Conjecture 8.1 for weights k satisfying $4 \le k \le 11$. Thus, item (4) of Theorem 6.6 may be strengthened for these weights to assert the complete equality of $\mathrm{Borch}(\psi)$ and $\mathrm{Grit}(\phi)$. The proof proceeds by demonstrating an exhaustive list of examples.

Theorem 8.2. (Theta-products of order one.) *Let $\ell \in \mathbb{N}$ be in the range $1 \le \ell \le 8$, and let $d_1, \dots, d_\ell \in \mathbb{N}$ with $(d_1 + \cdots + d_\ell) \in 2\mathbb{N}$. Then*

Conjecture 8.1 is true for the Jacobi form

$\eta^{3(8-\ell)}\vartheta_{d_1}\cdot\ldots\cdot\vartheta_{d_\ell}\in J_{k,t}$, *where* $k=12-\ell$ *and* $t=(d_1^2+\cdots+d_\ell^2)/2$.

Additionally, this product is a Jacobi cusp form if $\ell<8$ *or if* $\ell=8$ *and* $(d_1\cdot\ldots\cdot d_8)/d^8$ *is even where* $d=(d_1,\ldots,d_8)$ *is the greatest common divisor of the* d_i.

Proof. This theorem is a direct corollary of [7, Theorem 3.2] where it was proved that

$$F_{D_8}=\mathrm{Lift}\,(\vartheta(\tau,z_1)\cdots\vartheta(\tau,z_8))=\mathrm{Borch}\Big(-\frac{(\vartheta(\tau,z_1)\cdots\vartheta(\tau,z_8))|_4V_2}{\vartheta(\tau,z_1)\cdots\vartheta(\tau,z_8)}\Big)$$

for a strongly reflective modular form F_{D_8} of weight 4 in ten variables with respect to $\widetilde{\mathrm{O}}^+(U\oplus U\oplus D_8(-1))$. Similar identities were also obtained for the Jacobi forms $\eta^{3(8-\ell)}\vartheta(\tau,z_1)\cdot\ldots\cdot\vartheta(\tau,z_\ell)$ with respect to the lattice D_ℓ for $\ell>1$. To prove the theorem we have to take the restriction of the last identity for $(z_1,\ldots,z_8)=(d_1z,\ldots,d_8z)$. The fact that the product is a Jacobi cusp form for $\ell=8$ follows from the Fourier expansion of the Jacobi theta-series, see the proof of [10, Lemma 1.2].

For $\ell=1$ our arguments also work. The orthogonal complement of D_7 in D_8 is the lattice $D_1=\langle 4\rangle$ of rank 1. The (quasi) pullback of F_{D_8} to the Siegel upper half plane, defined by the lattice $U\oplus U\oplus\langle 4\rangle$, gives the reflective cusp form $G_{11}=\mathrm{Grit}(\phi_{11,2})\in S_{11}(K(2))$ (see Remark 2 after [7, Corollary 3.5]) discussed in $\mathbf{N}=2$ of section 2 here. We conclude by using the fact that $F\left(\begin{smallmatrix}\tau & z\\ z & \omega\end{smallmatrix}\right)\mapsto F\left(\begin{smallmatrix}\tau & dz\\ dz & d^2\omega\end{smallmatrix}\right)$ defines an injective map $M_k(K(N))\to M_k(K(Nd^2))$ that respects both liftings. $\square$

Theorem 8.2 shows that all examples considered in section 2 for levels $N=1,\ldots,5$ are the first members of *eight* infinite series of Gritsenko's liftings with Borcherds product structure of weight $12-\ell$ for $\ell=1,\ldots,8$. For example, for $\ell=4$ we have $\mathrm{Grit}(\eta^{12}\vartheta_1^4)\in S_8(K(2))$, $\mathrm{Grit}(\eta^{12}\vartheta_1^2\vartheta_2^2)\in S_8(K(5))$, $\mathrm{Grit}(\eta^{12}\vartheta_1^3\vartheta_3)\in S_8(K(6))$ and

$$\mathrm{Grit}(\eta^{12}\vartheta_{d_1}\ldots\vartheta_{d_4})\in S_8(K((d_1^2+\cdots+d_4^2)/2)).$$

We can also construct a *ninth* infinite series of such modular forms of weight 3. Let us take the simplest non-trivial theta blocks, i.e., with a single η factor in the denominator. These are the so-called theta-quarks (see [15] and [3, Corollary 3.9]); for $a,b\in\mathbb{N}$, set

$$\theta_{a,b}=\frac{\theta_a\theta_b\theta_{a+b}}{\eta}\in J_{1,a^2+ab+b^2}(\chi_3),\qquad \chi_3=\epsilon_\eta^8,\qquad \chi_3^3=1.$$

The theta-quark $\theta_{a,b}$ is a Jacobi cusp form if $a \not\equiv b \mod 3$. The following theorem is a direct corollary of [7, Theorem 4.2] about the strongly reflective modular form of weight 3 with respect to $\mathrm{O}^+(2U \oplus 3A_2(-1))$.

Theorem 8.3. *For $a_1, b_1, a_2, b_2, a_3, b_3 \in \mathbb{N}$, we have*

$$\mathrm{Grit}(\theta_{a_1,b_1}\theta_{a_2,b_2}\theta_{a_3,b_3}) = \mathrm{Borch}(\psi) \in M_3(K(t))$$

where $t = \sum_{i=1}^{3}(a_i^2 + a_ib_i + b_i^2)$ and $\psi = -\dfrac{(\theta_{a_1,b_1}\theta_{a_2,b_2}\theta_{a_3,b_3})|V_2}{\theta_{a_1,b_1}\theta_{a_2,b_2}\theta_{a_3,b_3}}$.

This example is very interesting because a paramodular cusp form of weight 3 with respect to $K(t)$ induces a canonical differential form on the moduli space of $(1,t)$-polarized abelian surfaces, see [6]. Therefore the divisor of the modular form in this example gives the class of the canonical divisor of the corresponding Siegel modular 3-fold.

In a subsequent publication, we hope to show that the identity proven as the last example of section 2, $\mathrm{Grit}(\phi_{2,37}) = \mathrm{Borch}(\psi_{2,37})$, is also a member of an infinite family of identities for Siegel paramodular forms of weight 2.

References

[1] Borcherds, R. E.: *Automorphic forms with singularities on Grassmannians*, Inventiones Math. **132** (1998), 491–562.

[2] Cléry, F., Gritsenko, V.: *Siegel modular forms of genus* 2 *with the simplest divisor*, Proc. London Math. Soc. **102** (2011), 1024–1052.

[3] Cléry, F., Gritsenko, V.: *Modular forms of orthogonal type and Jacobi theta-series*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg (2013) 1–33; DOI 10.1007/s12188-013-0080-4.

[4] Eichler M. and Zagier D., *The theory of Jacobi forms,* Progress in Mathematics 55, Birkhäuser Verlag, Berlin 1985.

[5] Gritsenko, V.: *Arithmetical lifting and its applications*, Number Theory. Proceedings of Paris Seminar 1992-93, Cambridge Univ. Press, 1995, 103–126.

[6] Gritsenko, V.: *Irrationality of the moduli spaces of polarized abelian surfaces*, The International Mathematics Research Notices **6** (1994), 235–243.

[7] Gritsenko, V.: *Reflective modular forms in algebraic geometry*, `arXiv:1005.3753` (2010), 28 pp.

[8] Gritsenko, V.: *24 Faces of the Borcherds Modular form* Φ_{12}, arXiv:1203.6503v1 (2012), 14pp.

[9] Gritsenko, V., Hulek, K.: *Minimal Siegel modular threefolds*, Math. Proc. of the Cambridge Philos. Society **123** (1998), 461–485.

[10] Gritsenko, V., Hulek, K.: *Commutator Coverings of Siegel Threefolds*, Duke Math. Jour. **94** (1998), 509–542.

[11] Gritsenko,V., Nikulin, V.: *Siegel automorphic form correction of some Lorentzian Kac–Moody Lie algebras*, Amer. J. Math. **119** (1997), 181–224.

[12] Gritsenko,V., Nikulin, V.: *The Igusa modular forms and "the simplest" Lorentzian Kac–Moody algebras* Matem. Sbornik, **187** (1996), 1601–1643.

[13] Gritsenko, V., Nikulin, V.: *Automorphic Forms and Lorentzian Kac-Moody Algebras, Part II*, International J. Math. **9** (1998), 201–275.
[14] Gritsenko, V., Nikulin, V.: *On classification of Lorentzian Kac–Moody algebras.* Russian Math. Survey **57** (2002), 921–981.
[15] Gritsenko, V., Skoruppa, N. P., Zagier, D.: *Theta Blocks*, Manuscript (2010).
[16] Heim, B., Murase, A.: *A characterization of holomorphic Borcherds lifts by symmetries*, (preprint: March 6, 2012)
[17] Ibukiyama, T., Onodera, F.: *On the Graded Ring of Modular Forms of the Siegel Paramodular Group of Level 2*, Abh. Math. Sem. Univ. Hamburg **67** (1997), 297–305.
[18] Ibukiyama, T.: *On Relations of Dimensions of Automorphic Forms of Sp(2,R) and Its Compact Twist Sp(2) (I)*, Adv. Stud. Pure Math. **7** (1985) Automorphic Forms and Number Theory, 7–30.
[19] Ibukiyama, T., Poor, C., Yuen, D.: *Jacobi forms that characterize paramodular forms*, Abh. Math. Semin. Univ. Hamburg. **83** (No. 1) (2013), 111-128
[20] Poor, C., Yuen, D.: *Linear dependence among Siegel modular forms*, Math. Ann. **318** (2000), 205–234.
[21] Poor, C., Yuen, D.: *Paramodular Cusp Forms*, ArXiv: 0912.0049v1, (2009) 61 pp. (To appear in Math. Comp.)
[22] Poor, C., Yuen, D.: *The Cusp Structure of the Paramodular Groups for Degree Two*, J. Korean Math. Soc. **50** (No. 2) (2013), 445–464.
[23] Poor, C., Shurman, J., Yuen, D.: *Computations with Siegel Modular Forms and an Introduction to the Geometry of Numbers*, (in progress)
[24] Rockafellar, R. T.: *Convex Analysis*, Princeton Landmarks in Mathematics, (1970), 1–451.

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